PROCEEDINGS of the FOURTH INTERNATIONAL CONFERENCE for the Psychology of Mathematics Education

Berkeley, California
August 16–17, 1980

Rina Hershkowitz

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for the

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PREFACE

The International Group for the Psychology of Mathematics Education (IGPME) was founded in 1976 at the 3rd International Congress for Mathematics Education. Conferences of the IGPME were held in Utrecht, Osnabrück, and Warwick during the three subsequent years. The Fourth International Conference will take place on the campus of the University of California in Berkeley, immediately following the 4th International Congress for Mathematics Education (ICME) held on that campus.

These Proceedings are being distributed in advance of the conference to permit the participants to study the papers of interest to them before they attend the conference sessions. Speakers are therefore requested NOT to make complete presentations of their papers during the 30 minutes allocated to each contribution. Instead, they are asked to identify the principal issues raised by their papers and to suggest questions for discussion during at least half the time provided.

The contributed papers have been organized for consideration under five general themes:

A. Teaching and Learning Mathematics
B. Investigations of Mathematical Thinking
C. Mathematical Procedures
D. Processes and Strategies in the Mathematics Classroom
E. The U.S.A. National Assessment of Educational Progress

Some papers, which fitted into more than one category, were placed in sessions with other papers that appeared to the editor to raise related issues or questions.

This volume is organized into five sections according to the five themes, with the papers in each theme appearing in the order of presentation. The Table of Contents at the beginning, an alphabetic listing of contributors at the end, and running heads with the authors' names may be used to locate a particular contribution.

In addition to the sessions reported here, the IGPME sponsored major presentations and discussions at the ICME by Richard Skemp and Ephraim Fischbein. These activities will be reported in the Proceedings of ICME IV.

The editor was overwhelmed by the large number of very interesting papers that were submitted. He is grateful for the cooperation of the contributors in furnishing their manuscripts promptly and in the form requested. He is also grateful to Sue Arnold for her careful and speedy preparation of the printer's copy.

The Conference Committee

Robert Karplus, Chair and Editor
Nicloas Herscovics
Richard Lesh
Sigrid Wagner

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CHILDREN'S METHODS OF CHECKING THEIR ARITHMETIC

F. R. Watson, University of Keele, Staffordshire, U.K.

Two related research studies are described. The first, now complete, identifies procedures used by children in checking the correctness of their work in arithmetic (by pencil and paper methods). The second, in progress, will examine the effectiveness of teaching selected checking methods, appropriate for use when a calculator is available.

That checking is 'important', few will disagree; how often it is done, taught or encouraged is a different matter. It might be argued that part of the professional competence of a mathematician is concerned with checking - by testing conjectures and deductions for consistency and reasonableness, examining special cases, etc., we avoid wasting time on falsely based work, and gain reassurance for the next stages of the work. Here we are concerned with routine classroom work rather than research, and although the gulf between these need not be so wide as it is sometimes found to be, that is another theme .....

The two studies to be described here arose from a line of research initiated by M. Ruth Eagle, (Eagle, 1978). They are concerned with checking, by children in the 11-16 age range, of work such as they commonly encounter in school; for the most part they deal with numerical computation, though, as in the first study, items could also involve algebraic or geometrical content. Both studies are qualitative, and involve small samples; they seek to provide some reliable data on an area apparently little explored.

If it is accepted that 'the purpose of numerical computation is to get the right answer', it is clearly vital to know whether or not one's answer is correct, and to have means of amending it if it is not. This is a skill for 'real life', which ought to be taught, or rather learned, in school. How far are such skills taught, as distinct from being picked up 'en route'? How far are they learned - or used - by pupils? What are these skills?

STUDY 1: DESCRIPTION

Merida Lee (Lee, 1979), working with M. Ruth Eagle, carried out a small-scale exploratory study to identify (a) the checking methods which children spontaneously used when asked to judge the correctness of a piece of work, and (b) the methods which they had available, and would use when prompted. She worked with small groups - six groups of three children in each of two schools - who first answered a short written test paper of 15 questions, informally presented, and afterwards discussed their answers (in groups of three). They were supplied with a list of "some possible answers", which included plausible wrong answers (obtained from a pilot run of the test) as well as, usually, the correct one. No instructions on the use of this list were given, and pupils were free to ignore it if they chose; they tended to focus preferentially on the answers they had themselves generated. Their discussions were tape-recorded; the experimenter was present but did not intervene. Later a second similar test was again followed by small group discussion; this time the investigator joined the discussion to prompt consideration of other possible strategies.

In one of the two schools, which used the SMG textbook series (Scottish Mathematics Group, 1968), 9 pupils in each of years 1 and 3 (ages c.12 and 14 years) were involved, three from each of the 'low', 'middle' and 'high' ability categories (as judged by teachers). In the other schools, which used the SMP A-H series (School Mathematics Project, 1968), 18 pupils from forms 2 and 4 (ages 13 and 15 years, approx.) were chosen in a similar way. (It was not intended to make a systematic study of the influence of age, ability and type of course upon the children's use of checking procedures, but it was hoped that the sample would allow a fairly wide and representative range of checking responses to be displayed. In the event, such differences between groups as were noted were unsurprising).

STUDY 1: DISCUSSION

Some of the children (in school 1) were apprehensive about the test, and there was some evidence of 'copied' answers. In the other school, where Miss Lee was already known, they were more at ease. None of the children seemed worried by the tape recorder; often its presence was overlooked.

The main findings are interesting, if depressing:-

(i) There was little spontaneous checking; correctness was usually judged by comparing answers within the group, with unanimity, or forcefulness of presentation, being decisive. Where checking was done, direct repetition of the working was the most common method.

(ii) A wide variety of methods was available with prompting, including some efficient ones which had not been taught. A few methods which had been taught - notably point estimates by approximation - were not popular.

(iii) Reasonableness of answer ("the train can't be going at 1000 mph, so it's wrong"), was not generally considered as a checking criterion. (One child remarked, "If it's a mathematical question they just put any words round it" ..... "I wouldn't get worried if a train was travelling at 100 mph in a maths question" ..... "they do put things like that sometimes").

(iv) Pupil's rating of the importance of checking was low - it was regarded as an end of lesson activity (often, transparently, a 'filler' used by the teacher to keep pupils quiet in the last few minutes of the lesson). It might bring a few more marks in tests - but only when all questions had been completed. Teachers, apparently, often stated that checking was important, and sometimes demanded a check, but seldom checked their own work in public.

(v) A small opportunity sample of five adults was also interviewed individually; rather than being set to complete the test they were asked how they would verify their answers. Again, repeating the working was the most favoured method, and, disappointingly, there was little evidence of checking a method by using a simpler analogous problem eg comparison of

$$1\tfrac{1}{4} \div \tfrac{3}{8} \quad \text{with} \quad 4 \div \tfrac{1}{2}$$

INTERLUDE. THE CALCULATOR: A DISCUSSION OF RESEARCH PROBLEMS

Clearly we cannot generalise on the basis of a small scale study, albeit carefully carried out. However, M. J. Lee's work suggests, in conformity with the experience of many teachers and examiners, that checking is not taken very seriously, and that teachers tend perhaps to pay lip service to the need for it.

Of course, conferring with a colleague is a common and sensible approach to checking - many of us will have done this in preparing our papers for this conference! To repeat working is time-consuming and potentially hazardous, (since the same mistake may be repeated); we will use other methods of checking only if we feel a need to be sure about ehe answer. Evidently, many pupils don't normally feel such a need. We may ask - why should they?

Many everyday numerical situations can be handled by 'order of magnitude' arithmetic. Have I enough cash to pay for petrol, for my lunch? (we still use cash in the UK!), enough in the bank for a new car? How much wall-paper is required for this room? Others demand greater accuracy - measurement of glass for window panes, addition of my mileage claim for travel expenses. In some situations a rough check is enough, in others we need to be sure the answer really is accurate. How does the advent of the electronic calculator affect the balance? It doesn't eliminate the need to check, since we can still make mistakes - but they will be mistakes of a different kind. There may also be a tendency to place too much reliance on the calculator.

What does one mean by a check in a calculator context? (To check a product by pencil and paper 'long multiplication' doesn't seem a sensible idea!) One approach (Thompson, 1980) is to analyse the sort of mistakes which might be made, and then to look for methods of checking which would pinpoint these - or at least signal that an error had been made. Some checks indicate only that an error exists, others carry with them an alternative answer which is (likely to be) the correct one. eg "71x43 = 2414" cannot be correct since the product must end in 3, "43x71 = 3053" gives another (correct) answer. Neither check guards against situations in which one ought to be evaluating 73x41 or 71+43.

Among the possible errors in calculator computations we may list:-

1) Pressing the wrong digit or operator key

358 instead of 359 , 7+8 instead of 7x8

2) Half-pressing, or missing, a key 35x27 instead of 358x27

3) Wrong order (a) of digits 21847 for 28147

(b) of decimal point 21.4 for 2.14

4) Wrong order of operations 3+4-5 instead of 3-4+5

5) 'Losing one's place' eg (a) in adding a column of figures

(b) in evaluating a complicated expression like $\sqrt{4\cdot1(4\cdot1-2\cdot1)(4\cdot1-2\cdot7)(4\cdot1-3\cdot2)}$

6) Using an incorrect algorithm eg "Speed = Distance x time"

cf. 'Is it an add, Miss?' (Brown & Küchemann, 1976)

or using the wrong formula for the surface area of a 2-ended cylinder.

7) 'Not doing what one intends': using an informal plan which is not adhered to. Algorithm not fully formulated, so that it is incorrect (6(b), or imperfectly executed (5(b)).

8) Errors due to round-off/truncation eg $1\frac{3}{7} \times 3 \neq 1$,

$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1000} \neq \frac{1}{1000} + \frac{1}{999} + \ldots + \frac{1}{2} + 1$

An efficient check is one with a high probability of signalling, locating, and, if possible, correcting, an error, (checks can supply at best, only another candidate-answer), without disproportionate additional work (ie of order of magnitude no more than the original).

Considering the list above, we may guard against 1,2,3,and 4 by transformation to an equivalent problem, so that, for example, different digits are entered

eg we may check $1.72^2 = 2.9584$ by $\sqrt{2.9584} = 1.72$.

Less effectively, we may carry out the same operations in another order, eg by using the commutative or associative rules. Rough estimate checks will serve to alert us to errors 2,4,and 3(b) but not to 1 or 3(a).

For type 5, partial answers (subtotals in column addition, separate evaluation of numerator and denominator, etc.) may help. Writing down the algorithm and 'ticking off' the parts as completed may also be useful. (Simple devices such as putting Plastitak on the calculator 'feet', or providing a wide-angle display, may, by freeing a hand for this, do more to help than systematic procedures of checking!)

Errors of type 6 are 'mathematical' rather than computational; common sense checks may help avoid statements like "100 miles in 10 hours gives a speed of 1000 mph".

Those of type 7 may be described as "sloppiness" rather than errors! - the cure is probably to write down the 'plan of campaign'. How to avoid the final category (8) of errors leads to the more complex problems of numerical analysis, which will not be discussed her - (but see Bailey, 1978).

Checking for errors is partly a learned skill but is also a state of mind. How might we encourage checking habits? Perhaps by making checking an explicit activity ("check these computations") and rewarding it in examinations (" ... and explain how you checked your computation"). The example of the teacher is evidently crucial, and habits of checking, like those of explanation, 'setting out', and documentation of computer programming, need to be encouraged. What is the role of understanding in checking? How far are 'rules of thumb' useful? (eg for orders of magnitude: compare the rules about number of figures after the decimal point). Perhaps we should emphasise more that checking is part and parcel of the attitude of the mathematician within his own subject, and of the application of mathematics to the 'real world'.

STUDY 2: DESCRIPTION

Ian Thompson is currently engaged in a small scale study of checking in a calculator context (Thompson, 1980), based on some of the ideas above. Each experimental group of about 30 pupils, aged about 13 years, was given two short computational tests. Test I, (8 questions) was done by pupils without a calculator and was designed, like those of Lee, to discover what checks they were aware of and could use. Test II was done with a calculator, and included more elaborate computations (10 questions). In each test pupils were given an ostensible answer (usually wrong) for each question, and asked to check its correctness, briefly explaining how they did so. (The questions in I were designed with just one checking method in mind to determine whether

pupils knew of this. This was a speed test, so that computation of the correct answer by hand was discouraged. In II they were asked to write down as many different checks as they could think of. Several of the procedures envisaged by Thompson were already in evidence at this stage in test II with one group (together with a few new ones!) Thus checking addition by subtraction, multiplication by division, and use of the commutative and associative rules were fairly frequent. Only the 'mental arithmetic' checks such as 77x999 = 77000 - 77 etc., were absent, (perhaps because the calculator inhibited these?)

The next step was to devise teaching materials which could be used explicitly to teach and practice some of the procedures under consideration, eg commutativity, $a+b=c \Longrightarrow c-a=b$, $9.8^3 \approx 10^3$ $9.8^3 < 10^3$.

These took the form of 14 simple worksheets in which the property or method was first investigated, so that pupils had some experience on which to build, and then explicitly formulated and practiced as a method of checking. At the time of writing, this work is still in progress. The period of teaching involved will be about 10 hours, spaced over several weeks, and the material will be covered in different orders by the two groups. (Simply to ensure that all the material is tried by some pupils rather than investigate order effects). An additional activity, aimed to teach specific checking methods and to promote awareness of the general issues, was a class discussion of ways of using a calculator to add up 15 or 20 three digit numbers. This involved consideration of ideas of grouping and checking partial totals to pinpoint errors efficiently. The intention is later to repeat the tests I and II as post-tests, in the hope that changes in the checking repertoire of pupils will show themselves.

AND NEXT?

Clearly this will leave a great many questions unanswered. What will be the long term effects, if any, on pupils of the teaching of checking methods? Will their attitude to checking, and their use of it, be affected? How can pupils be induced to take a more positive attitude to accuracy? Are these teaching methods effective? - how can they be improved? Which checking methods are most useful/most easily taught/most acceptable to pupils? What is the incidence of different types of error when a calculator is used? Does calculator design influence this? Perhaps the hackneyed words "more research is needed" are justifiable in this context?

References

Bailey, D.E. The effect on the solution of a problem of errors in the calculation Mathematical Gazette, 1978. No. 421 pp. 157 - 164.

Brown, M. & Küchemann, D.E. "Is it an add, miss?" Mathematics in School 5,5 and 6,1., 1976.

Eagle, M. Ruth. Self-appraisal in the learning of mathematics, Proc. 2nd Int. Conf. for Psychol. of Math. Educ., Osnabrucker Schriften zur Mathematik. (Reihe D, Band 1), December1978, pp. 116-119.

Lee, Merida J. How secondary school children check their work in arithmetic: an exploratory study. Unpublished M.Sc. dissertation, University of Keele, 1979.

School Mathematics Project, SMP Books A - H. Cambridge University Press 1968.

Scottish Mathematics Group. Modern Mathematics for Schools Books 1 - 9. London. Blackie & Chambers, 1968.

Thompson, F. Ian. Procedures used by children to check work done with an electronic calculator. Unpublished M.Sc. disseration, University of Keele, 1980.

USING STUDENT'S MISTAKES AS A TEACHING DEVICE

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This is an experimental field study to find out whether students who did their exercises in mathematics by checking other student's solutions will make less mistakes on a teacher made test than students who had to solve the same exercises in the usual way. The study covered parallel classes in grades 7 (upper and lower) 8 and 9 (upper level only). The experimental classes did better in all grades except for the lower level grade 7 where there was no difference. The differences were particularly conspicious for students in the upper halves of the classes and on topics of a more technical algorithmic nature.

ERRORS AS A SOURCE OF FRUSTRATION AND ANXIETY

Any teacher involved in mathematics education is aware of the fact that students make mistakes, and that very often the same student commits the same mistake more than once, even though his teacher has corrected him every time. Teachers are also aware that these mistakes may give rise to frustration and anxiety, particularly so for the interested student. Such errors may cause the student to lose his self assurance and lead him to give up his belief in his ability "to do mathematics".

Why do mistakes cause so much frustration and anxiety? The operating mechanism is mainly the fear of failure. The interested student who had done an exercise, or solved a problem, has invested in this act part of his ego, and therefore the fact that some basic mistake destroys completely his solution is obviously a very frustrating experience. Mathematics is a highly symbolic subject and many interested students have to struggle with it without having obtained enough familiarity with the concepts implied, not enough meaning for the symbols used, and not enough skill with the algorithms used. It is the failure after such a high input of mental energy that causes the frustration.

Things seem to be even worse. The frustration and anger after failure make it impossible for many students to grasp the teacher's correction, and this may,

when the teacher introduces for the first time a variable with a negative sign, -k, he may ask "does this represent a negative number?" Even if some students err and respond "yes, it does" there is not yet any deep ego involvement of the students, and frustration will be minimal. The possible snag in using this approach is related to the problem of using negative instances in the presentation of a concept in the classroom, there is always the danger that some student might store the negative instance as the correct response. It seems reasonable that this approach might be helpful with the more able students, but problematic with the less able ones.

Historically, we might look for a relationship between this approach and the usual fallacies quite famous in the educational literature of mathematics. However, we don't know of any study to support the idea that exposing students to such fallacies diminishes the number of similar mistakes in their performance.

POSITIVE USAGE OF MISTAKES: STUDENTS' MISTAKES AS A TEACHING DEVICE

There is another possibility to lessen the frustration caused by committed mistakes namely to look for possible positive sides one can find in the student's faulty solution. In general this would mean to analyse this solution in class, and then discuss, together with the students, in what cases, if any, this might be a correct solution. Some ideas in this direction can be found in articles by Carmon (1971), Meyerson (1976), Laursen (1978) and Goodman (1979). Neither of them did any empirical evaluation of this approach.

We have emphasized that pointing out mistakes in an exam, or in homework, at an advanced stage of a student's studies of the particular topic, may cause frustration and anxiety, and as a result reduce the amount of learning that takes place. On the other hand it is quite possible that the usual competitive nature of our society may produce a positive motivation for learning if, instead of doing his own exercises, we ask the student to analyse solutions of exercises and problems given by other students not from the same class; some of which might be faulty. Each student's task would be: (i) To discover which solutions, if any, are incorrect; (ii) To point out what the mistake is; (iii) To give a correct solution.

We may expect that this approach will, arouse the natural curiosity of the students, lead to discussion and interaction among them, and therefore lead to more involvement. Also, the fact that the student who produced the solution is

at least partly, contribute to the fact that so many students commit so many times the same mistake again and again. A frustrated student does not have the ability to learn from his own mistakes.

These facts make it imperative to search for ways to eliminate students' mistakes, without causing that much frustration.

Before continuing please notice that in this study we use intermittently the terms error and mistake to mean a student's response to a mathematical exercise or problem, that is not acceptable to a knowledgeable person in this subject.

HOW DO WE CORRECT STUDENTS' MISTAKES

The most usual way observed by the authors in classrooms is the teacher just remarking: "What X has done (said) is wrong, the correct answer is .." or "would you Y give the correct answer ..". In other cases the teacher provides a counter-example which shows that the student's statement is not generally true. As for instance if the student's mistake touches upon the assumption that $x^2 + y^2 = (x + y)^2$ (for all x,y) the teacher may remark "Well, how if $x = 3$ and $y = 4$".

Educators generally assume that these remarks do not improve students responses. With the advent of modern curricula the prevailing attitude was to base instruction and students' understanding on a structural approach. This was based on the assumption that if students understand the structure they will learn to know how this structure implies the mode how certain operations should be applied and, therefore, such an understanding would eliminate mistakes. Alas, these hopes did not materialize. As the study by Tim Grey (preprint, 1976) has shown students committed in 1975 the same mistakes as did students in 1923, only more of these. The reason behind it might be, as we have already said before, that in the modern curricula students do not obtain enough familiarity with the concepts and symbols, nor enough skill with the application of algorithms. Many of the symbolic operations look so similar to one another, that many students transfer operations that are correct in one area to situation in which they are completely wrong.

One remedy that might perhaps help reduce, to some extent, the number of students' mistakes might be to expose them to possible misapplications at the moment when they are presented with a given concept for the first time. For instance,

not from the same class, and nobody knows who he is, will eliminate students' solidarity not to harm other students. We hypothesize that the competitive nature of the student will induce him to pay more attention and not to repeat the mistakes that he himself will discover, or will be pointed out by the teacher when checking homework. We may also hope that the students will transfer the skill of checking finished solutions to their own work. This is the basic idea of the study described in further sections.

THE EXPERIMENT

We decided to run the experiment in mathematics and in another structural subject, namely, formal Hebrew grammar - conjugetions of some classes of Hebrew verbs. To find out whether the J.H. students have the ability to read, understand, and correct other students' work, and what is a good way to organize the worksheets with the solved exercises, we decided to carry out a short pilot study in each subject in two level-one grade 9 classes - one experimental and one control - of about 25 students each. The topic in mathematics was: functions (linear and quadratic) and in grammer: conjugation of Hebrew verbs. The results in mathematics as measured by teacher made tests, showed an average difference of 8 points (on a scale of 100) to the benefit of the experimental group.

On a delayed test, parallel to the previous one, given 3 weeks later, the difference grew to 12 points. Similar results were obtained in Hebrew grammar. In the worksheets for the pilot study we used students' mistakes remembered from previous years.

As a result of these outcomes it was decided to conduct a larger study in grades 7,8 and 9. In grade 7 we decided to investigate at two levels of ability, high and medium. In grade 8 and 9 only at the higher level. To prepare for the major study one year was spent collecting students' mistakes in both subjects, in grades 7,8 and 9. It was also decided to conduct the experiment in each grade on two topics, changing the role of each class, so that a class that served as an experimental group in one topic will become a control group in the other, and vice versa. The same teacher had to teach both classes - experimental and control. All parallel classes were equal in their achievement before the experiment. Among the Hebrew grammar teachers we found only one volunteer, we decided therefore, to conduct the grammar study in two classes only, in grade 9 upper level.

Control groups had to be exposed to the same exercises as those in the experimental group, with the only difference that in the experimental group, the students received worksheets with solved examples, while in the control group the same examples were used with no solutions.

The topics selected for study were:

Grade 7:	(i) Properties of algebraic operations
High and low level	(ii) Operations with signed numbers
Grade 8:	(i) Union and intersection of sets
High level	(ii) Finding solution sets of linear equations.
Grade 9:	(i) Simplification of algebraic expressions
High level	(ii) Correspondence between sets - functions.

RESULTS AND INTERPRETATION

We were essentially interested in getting answers to three questions :
(i) will Junior High students be able at all to cope with carrying out exercises by identification and correction of errors committed by other students?;
(ii) Will students who do their practice in the mode described in(i) perform better on the usual classroom tests than students who practice on the same exercises in the usual way?
(iii) What will be the students' attidude to this moe of practice?

For lack of space we omit the usual tables and move directly to general outcomes and their interpretation.

To question (i): In all level-one classes we did not encounter any problems with the introduction of the experimental mode of doing exercises. The only exception were the lower level grade 7 classes. In these classes we had to spend much more time to explain the new mode of doing exercises. After this the experiment ran quite smoothly in these classes too.

To question (ii): The results in all classes, in mathematics and in Hebrew grammar were in the predicted direction. In mathematics the maximum difference was 13 points (on a scale of 100) on the immediate test and 20 points on the delayed test, and the minimum difference was 1 point on the immediate and 3 points on the delayed test. The drop in achievement between the immediate and the delayed test was in all cases, except one, to the benefit of the experimental

classes. The median difference was 6 points on the immediate test and 12 points on the delayed test.

The results in Hebrew grammar were in all cases in the predicted direction - with a maximum difference of 12 points and a minimum difference of 4 points.

The differences in mathematics were significant ($P<0.05$) in about 40% of the cases. The differences in Hebrew grammar were significant in about 70% of the cases.

A deeper analysis of the individual outcomes in all grades and all classes has shown that a general approach of comparison of averages is misleading. In all grades and classes there was a vast difference between the upper and lower half of the class. Even though the differences were still in the predicted direction in all halves, except for the lower half of the lower level of grade 7, it is the upper half that benefitted most from the experiment. This was true for both, the immediate and delayed tests.

Another interesting outcome of a deeper analysis revealed that the highest differences between the experimental and control classes were obtained in topics of an algorithmic nature such as, simplification of expressions, finding solution sets etc.

To question (iii): Students' attitude was measured by an open ended 5 item questionnaire of the nature : "Would you like to continue doing the exercises in this new mode?" The general response, even of the more able students was, that this mode is much more demanding than the usual one.

CONCLUSIONS

Much more research is needed to be able to suggest more decisive recommendations. Based on the results of this study it seems worthwhile to use a mixed mode of exercises, that is some parts of the usual sets of exercises should be in the form of solved problems some of which are faulty. The task of the learner would be to check the solutions, find the faulty ones, and suggest how these should be corrected. These exercises might contribute most in topics dealing with algorithmic tasks.

REFERENCES

Carman, R.A. Mathematical Mistakes. The Mathematics Teacher, February, 1971, (109-115).

Gray, H.Tim, Error Analysis: An Item Analytic Investigation of Diagnostic Tests. University of Dallas (Preprint).

Meyerson, L.N., Mathematical Mistakes. Mathematics Teaching, 1976 938-40).

Laursen, K.W., Errors in First-Years Algebra. Mathematics Teacher, 1978, (194-195).

Pincus, Morris, If you don't know how children think, how can you help them? Arithmetic Teacher, 1975 (580-585).

LAST MINUTE RESULTS

Observations from exams given after this study was finished showed that students in classes that participated in the experiment tend to check their results significantly more than students in classes that did not participate.

ALGORITHMIC STRUCTURE AND CHILDREN'S ERROR PATTERNS IN ARITHMETIC*

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For this study the universe of whole-number subtraction tasks was partitioned into nearly homogeneous equivalence classes, called "types." A diagnostic instrument consisting of two items of each type was administered to 492 intermediate-grade children in seven schools. It proved possible to define a metric characterizing the "distance" of a child's observed pattern of right and wrong answers from standard decomposition and equal-addition subtraction algorithms. In each school the children's behaviors were closer to the structure of the decomposition algorithms. Students scoring below 75% were given remedial instruction using a decomposition algorithm, followed by a parallel form of the diagnostic test. Those whose initial scores were between 64% and 75% showed a negative and significant correlation ($r = -.57$, $p < .01$) between gain score and original distance from the treatment algorithm. For the students whose initial scores were lower, the correlations were smaller and generally not significant. The findings suggest that among children who do not need extensive remedial work, those whose initial behaviors most conform to the structure of an algorithm are likely to profit most from instruction in that algorithm.

INTRODUCTION

The teaching of algorithms remains an important component of school mathematics. When an algorithm is learned only partially or incorrectly, the deficiency must be detected and repaired if mastery of basic skills is to be achieved. This paper reports a study in the measurement of children's error patterns in subtraction tasks, and the dependence of the success of remedial instruction on the nature of those error patterns.

First we describe the construction of an algorithmically-based diagnostic instrument for subtraction. The method is suitable for generalization to other algorithmic tasks in mathematics. The test is designed for maximum reliability and for large-scale use in group situations; thus detailed individual protocol analyses, valuable as they are, do not enter into the procedure.

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*This report is based on a doctoral dissertation by Jeannine Gramick, 1975, at the University of Pennsylvania, under the direction of Gerald A. Goldin.

Next we introduce a formula for characterizing the "distance" of a child's pattern of right and wrong answers on the test from the "ideal" patterns associated with standard algorithms. This enables us to use the test instrument not only for identifying specific deficiencies, but also for evaluating quantitatively the degree to which the structure of the subject's error patterns does or does not resemble the structure of standard algorithms.

Finally we study the relationship between the "distance" of a child's behavior from a particular subtraction algorithm, and the success of the child during group remedial instruction in that algorithm.

CONSTRUCTION OF THE TEST INSTRUMENT

A universe of content for diagnostic testing can be described by a family of "item forms," each of which defines a class of test items by specifying replacement sets for the variable elements in an otherwise fixed syntactic structure (Osburn, 1968). A series of arithmetic tests has been developed using item forms; proposed forms for subtraction are given in Table 1 (Hively, Patterson, and Page, 1968). This research did not produce the expected result that each item form represented a distinct homogeneous class of behavior. The forms are not mutually exclusive, and do not correspond explicitly to paths through an algorithm. Durnin (1971) studied equivalence classes of subtrac-

Table 1. *Item Form Titles for Subtraction (Hively, Patterson, and Page, 1968)*

1. Basic fact, minuend ≤ 10
2. Subtract 0
3. Answer = 0
4. Basic fact, minuend > 10
5. No borrow, no 0 in answer or problem
6. No borrow, x - 0 fact in problem
7. No borrow, 0 - 0 fact in problem
8. No borrow, x - x fact in problem
9. No borrow, small, unequal lengths
10. No borrow, large, unequal lengths
11. Simple borrow
12. Simple borrow, one-digit subtrahend
13. Simple borrow, one-digit answer
14. Simple borrow, medium
15. Borrow, one-digit from large number
16. Borrow, medium, subtrahend one digit short
17. Borrow, medium, unequal lengths
18. Separated borrows
19. Repeated borrows
20. Borrow across 0
21. Borrow across two 0's
22. Large numbers
23. Horizontal, equal lengths
24. Horizontal, unequal lengths
25. Check
26. Equation, missing addend
27. Equation, missing subtrahend
28. Equation, missing minuend

tion items associated with standard algorithms, intersecting these classes with each other and with some of the Hively-Patterson-Page item forms to obtain "types" more nearly homogeneous than the item forms.* Our procedure represented a refinement of this approach.

First we constructed flow charts for eight distinct subtraction algorithms, five based on the decomposition method and three based on equal-addition. Recall that decomposition requires "borrowing" from the column to the left in the minuend, while in the equal-addition method the digit to the left in the subtrahend is instead increased by one. Figure 1 illustrates flow charts for one algorithm of each kind, omitting output ("write the difference ...") in-

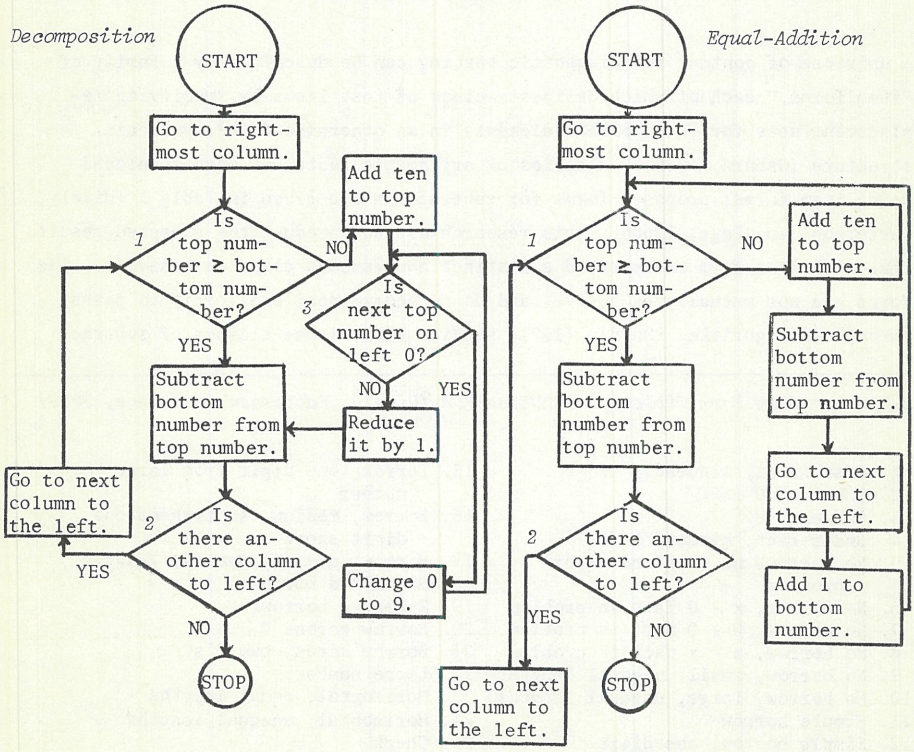
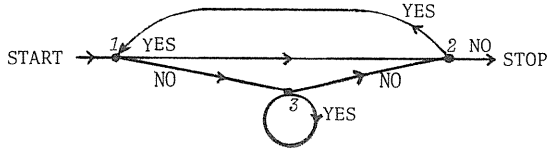


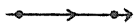
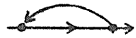
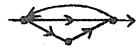
Figure 1. *Flow Charts for Subtraction Algorithms (Gramick, 1975)*

*A class of items is homogeneous if competence on any item in the class implies competence on the others.

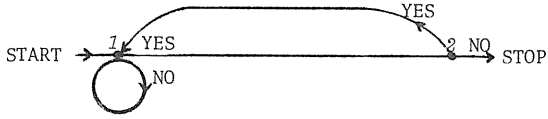
Figure 2. *Directed Graphs and Equivalence Classes for Subtraction Algorithms* (Numerals in italics refer to Figure 1.)

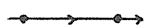
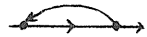
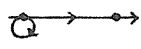
Decomposition:



	Path	Example	Description of Equivalence Class	Types (Table 2)
1.		7 -3	simple fact, minuend less than ten less than ten	--
2.		98 -34	two or more columns, no borrowing	1-4
3.		43 -17	two or more columns; borrowing but not across 0	5-18,22-3,28-30, 36-7,39
4.		304 -147	two or more columns; at least one borrow across 0	19-21,24-7,31-5, 38

Equal-Addition:



	Path	Example	Description of Equivalence Class	Types (Table 2)
1.		7 -3	simple fact, minuend less than ten	--
2.		368 -241	two or more columns, no borrowing	1-4
3.		1563 - 875	two or more columns, consistently irregular ("consistent borrow")	5-7,19-20,22, 24-5,28,31-2,36
4.		8456 -1729	three or more columns, intermittently irregular ("inconsistent borrow")	8-18,21,23,26-7, 29-30,33-5,37-9

structions. For each of the eight algorithms we then examined the distinct *paths* through the flow chart; two paths were considered the same if they traverse the same loops, no matter how many times each loop is traversed. The set of paths for an algorithm *partitions* the universal set of tasks into mutually disjoint equivalence classes, in accordance with the path required to process each task. A pictorial representation by means of a *directed graph* assists in visualizing the various paths. Figure 2 displays the directed graphs for the

algorithms of Figure 1, together with descriptions of the paths and the corresponding equivalence classes. Of course each algorithm leads to a different partitioning of the universal set.

To construct the diagnostic instrument, we intersected the first 22 item forms with each other to obtain 57 mutually exclusive classes of tasks. From this list we eliminated basic subtraction facts, which were considered prerequisite skills for our study; we also decided not to treat 0 as distinct from other numbers in "no borrow" situations, since the prerequisite facts included 0 in all positions. A pilot study with 79 subjects allowed two forms to be combined into a single class when the following conditions suggested homogeneity: (a) there was a logical reason to combine them, (b) the fractions of subjects performing successfully on the two forms were approximately equal, and (c) the fraction of subjects performing *consistently* on the two forms was much closer to 1 than to the fraction expected if the forms were completely independent. After these procedures there remained 28 mutually exclusive classes. These were intersected with the equivalence classes obtained from the eight algorithms to yield the 39 *types* listed in Table 2, which formed the basis for the diagnostic tests. The underlying assumption of our method was that to some extent a subject's cognitive structure can be sorted into those paths through an algorithm which have been mastered and those which have not. The types on the other hand were taken to represent nearly homogeneous classes of items, independent of the particular algorithm used.

Two items of each type were randomly generated and randomly sequenced. The result was a test of 78 items, two each of 39 types; together with a correspondence between the paths of eight algorithms and the types requiring the use of each path. This correspondence is included in Figure 2 for two algorithms. A parallel form was constructed for use as a posttest.

In scoring the test, a subject was considered to "know" a type if *either* of the two items of that type was answered correctly. Under the assumption of homogeneity, let p be the probability that a subject competent on a type misses an item ("clerical error"), and q the probability that a non-competent subject answers an item correctly ("lucky guessing"). Then the probability of misjudging a competent subject is p^2, while the probability of misjudging a non-competent subject is $2q - q^2$. With p not too large (about .10 or .20) and q

Table 2. *Types of Items in the Subtraction Instrument, with Examples*

Key: S = small length (one- or two-digit minuend); M = medium length (three- or four-digit minuend); L = large length (minuend of five or more digits). U = minuend and subtrahend of unequal length; E = minuend and subtrahend of equal length. IC = irregular column (in which the digit from the minuend is less than the digit from the subtrahend). Number of digits in minuend and subtrahend may be indicated in brackets; thus [m4s3] means a four-digit minuend and a three-digit subtrahend. cb = consistent "borrow" (all columns are irregular except the left-most); ib = inconsistent "borrow" (some regular and some irregular columns, excluding the left-most). Illustrative examples are written horizontally to conserve space; these appeared in standard vertical format in the actual test instruments.

Type		*Characteristics*	*Examples*
1	No IC's	S, U	36 - 4
2		S, E	62 - 30
3		M or L, U	33020 - 2020
4		M or L, E	40335 - 10313
5	Simple IC's (exactly one IC; in the column to the left of the IC, the digits are not equal)	S, E [m2s2], 1-digit remainder	36 - 28
6		S, U [m2s1], 2-digit remainder	97 - 9
7		S, E [m2s2], 2-digit remainder	43 - 15
8		M, U [m3s1] or [m4s1]	3070 - 7
9		M, U [m3s2] or [m4s3]	759 - 82
10		M, U [m4s2]	1266 - 37
11		M, E, 1-digit remainder	466 - 458
12		M, E, remainder of 2 or more digits	914 - 241
13		L, U [s1]	503481 - 4
14		L, E, 1-digit remainder	101097 - 101089
15		L, E or U, neither 1-digit subtrahend nor 1-digit remainder	13010 - 10810
16	Separated IC's (non-repeated)	M, E [m4s4] (two or more nonconsecutive IC's; in the columns to the left of each IC, the digits are unequal)	6163 - 5726
17		M, U [m4s3]	9870 - 963
18		L, E or U	522692 - 9505
19	Repeated IC's (non-separated)	M, E, cb always across 0	901 - 285
20		M, E [m4s4], cb across 0 and not across 0	5206 - 4679
21		M, E [m4s4], ib across 0	6905 - 5836
22		M, E, cb not across 0	376 - 198
23		M, E [m4s4], ib not across 0	8359 - 3769
24	(adjacent IC's, or IC with an equal column to its left)	M, U [m3s2] or [m4s3], cb always across 0	5004 - 639
25		M, U [m4s3], cb across 0 and not across 0	4100 - 666
26		M, U [m4s3], ib across 0	2701 - 698
27		M, U [m4s2], ib across 0	4602 - 26
28		M, U [m4s3] or [m3s2], cb not across 0	6634 - 998
29		M, U [m4s3], ib not across 0	1995 - 799
30		M, U [m4s2], ib not across 0	1233 - 87
31		L, E or U, cb always across 0	800005 - 66716
32		L, E or U, cb across 0 and not across 0	93304 - 36476

(continued)

Table 2. *(continued)*

Type		*Characteristics*	*Examples*
33	Repeated IC's (non-separated) *continued*	L, E or U, ib always across 0	14024 - 3192
34		L, E or U, ib across 0 and across two or more consecutive non-0 digits	579201 - 529379
35		L, E or U, ib across 0 and across one non-0 digit	64703 - 1779
36		L, E or U, cb not across 0	77372 - 8787
37		L, E or U, ib not across 0	317552 - 79101
38	Repeated and separated IC's	L, E or U, borrow across 0	10588 - 7629
39		L, E or U, no borrow across 0	51013 - 2904

much less than p (about .05 or less), our criterion is better than requiring both items to be answered correctly, and also better than using only one item of each type.

DEFINING A METRIC BETWEEN ALTERNATIVE ARITHMETIC ALGORITHMS AND OBSERVED BEHAVIORS

A subject's performance on the diagnostic instrument partitions the set of types into two classes: those for which competency is inferred, and those for which it is not. For the subject's behavior to be consonant with the structure of a given algorithm, the subject should be either wholly competent or wholly non-competent for each equivalence class of the algorithm. To the extent that there is competence on some types within an equivalence class and non-competence on others, the subject's behavior cannot be accounted for exclusively by considering the paths of that algorithm--the partition provided by the algorithm does not provide sufficient information.

This idea motivates the following development. Let A and B be distinct algorithms for processing the same universal set U containing N types. If the partition of U associated with A is a refinement of the partition associated with B, we write $A \leq B$; this introduces a partial ordering on the set of algorithms. The finest algorithm F would require a different path for every type, while the coarsest algorithm C would process all types by the same path. Suppose that A partitions the N types into k classes with a_i elements in the ith class, $i = 1, \ldots, k$; where $N = \sum_{i=1}^{k} a_i$; and similarly for B with b_j elements in the jth class, $j = 1, \ldots, m$. The (signed) distance from A to C may be defined as $d(A,C) =$

$\log [N!/a_1!\cdots a_k!]$. Readers familiar with information theory will recognize this expression as the amount of information contained in the partitioning of U by A, or the negative entropy associated with the partition. More generally, with $A \leq B$, $d(A,B) = d(A,C) - d(B,C) = \log [b_1!\cdots b_m!/a_1!\cdots a_k!]$. The base of the logarithm is unimportant; this merely establishes the scale of units.

To measure how close a given algorithm A comes to accounting for the behavior of a subject on the diagnostic test, we intersect the two subsets of U obtained from the subject's performance, with the equivalence classes of A. This yields a partition S of U which is a refinement of the partitioning by A; and $d(S,A)$ is the desired measure. Notice that $d(S,A) = 0$ when the subject has scored consistently within each equivalence class of A. Table 3 shows a sample test score interpretation, in which the distance of a subject's actual behavior from the two algorithms described above is computed.

Table 3. *Sample Interpretation of Test Score, Subject 16 (Gramick, 1975)*
(Types for which the subject was deemed non-competent are circled.)

Decomposition (Algorithm A)

Path 1. --

Path 2. 1 2 3 4

Path 3. 5 6 7 8 9 10 11 12 13 14
15 16 17 18 22 23 (28) 29 30 (36) 37 39

Path 4. (19) (20) (21) (24) (25) (26) (27) (31) (32) (33)
(34) 35 38

$d(S,A)$
$= \log [\frac{4!\ 22!\ 13!}{4!20!2!2!11!}] \approx 4.3$ (base ten)

Equal-Addition (Algorithm B)

Path 1. --

Path 2. 1 2 3 4

Path 3. 5 6 7 (19) (20) 22 (24) (25) (28) (31)
(32) (36)

Path 4. 8 9 10 11 12 13 14 15 16 17
18 (21) 23 (26) (27) 29 30 (33) (34) 35 37 38 39

$d(S,B)$
$= \log [\frac{4!\ 12!\ 23!}{4!4!8!18!5!}] \approx 7.2$ (base ten)

The logarithmic measure reflects our intuitive feeling that the decomposition algorithm better accounts for this subject's error pattern.

METHOD AND FINDINGS

A prerequisite skills test of ten basic subtraction facts was administered to 553 children in grades 3 to 6 in seven Baltimore parochial elementary schools, reflecting a socioeconomic and racial cross-section of the city. 492 students who scored nine or better were given the diagnostic test in four parts within the space of a week, requiring about 15 minutes per part. The pretest data were used to study the reliability of the instrument, to measure the distance of the child-

ren's behaviors from the two algorithms described above, and to identify those children in need of remedial work. All subjects in the study had previously been taught subtraction algorithms based on the decomposition method.

The pretest was split into two halves containing one item of each type. Each half was scored separately, resulting in an A-score and a B-score which could be compared with each other and with the primary score P obtained by the previously discussed method. We found $r_{AB} = .960$, $r_{AP} = .969$, and $r_{BP} = .967$. In order to investigate our measure of distance, we explored the conjecture that these children, who had been taught decomposition algorithms, would prove "closer" in the structure of their error patterns to a decomposition algorithm than to an equal-addition algorithm. The two algorithms of Figure 1 were selected because they have the same number of equivalence classes (4), and approximately the same numbers of types in their respective classes. The mean distance of all subjects from the decomposition algorithm was found to be 2.9; and from the equal-addition algorithm, 4.4. Furthermore, in each of the seven schools the mean distance was smaller for the decomposition algorithm.

Next, 229 children who were deemed competent on 29 or fewer of the 39 types were given remedial instruction in the form of worksheets and lessons taught by one of us (JG). The instruction was based on a decomposition algorithm having five distinct paths, and took place in groups of 6-25 in five periods totaling about 150 minutes, over approximately one week. A parallel form of the diagnostic test was then administered as a posttest in three parts. The administration of the pretest, the treatment, and the posttest in all seven schools was begun in November 1972 and completed by February 1973. Data for 207 students who completed the sequence were included in the analysis.

Based on the number of types correct on the pretest, subjects were sorted into five groups: a low group (0-6), three middle groups (7-12, 13-18, and 19-24), and a high group (25-29). For each group a Pearson product-moment correlation coefficient r_{XY} was computed between (X) the distance from the treatment algorithm as measured by the pretest, and (Y) the gain score from the pretest to the posttest. The null hypothesis of zero correlation was investigated using a two-tailed test (Glass and Stanley, 1970, p. 310), and the results are indicated in Table 4. For the low group, $r = .316$ and $.05 < p < .10$; for the high group, $r = -.567$ and $p \ll .01$; while for the middle groups, $p > .10$. The systematic variance in the high group subjects' gain scores which can be accounted

Table 4. *Initial Distance from the Treatment Algorithm (X) and Gain Score (Y)-- Means and Correlations*

Pretest Score		N	$\overline{X}$	$\overline{Y}$	r_{XY}	df	significance
Low level	0- 6	29	1.2	23.4	.316	27	$.05 < p < .10$
Middle levels	7-12	18	5.0	21.2	-.247	16	$p > .10$
	13-18	23	5.1	16.2	-.132	21	$p > .10$
	19-24	53	4.6	14.4	-.155	51	$p > .10$
High level	25-29	84	1.9	11.7	-.567	82	$p \ll .01$

for by initial distance from the treatment algorithm is approximately $r^2 = .32$. By comparison the variance which could be accounted for by IQ's proved negligible.

DISCUSSION

The theoretical perspectives taken in this research seem widely generalizable, with the potential for quantitative and reliable measurements. Recent work on the role of task variables has focused attention on the need to investigate and control for algorithmic structure in studying mathematical problem-solving behavior (Goldin, 1979). The present paper highlights the complexity of algorithmic variables even in the context of routine arithmetic exercises.

The findings suggest that among children who do not need extensive remedial work, those whose initial error patterns most conform to the structure of an algorithm are likely to profit most from instruction in that algorithm. The possible reversal of this effect for the lowest group of children provides an intriguing subject for future investigation.

References

Durnin, J.H. Assessing behavior potential: a comparison of three methods (Doctoral disseration, Univ. of Pa., 1971). Univ. Microfilms No. 7217343.

Glass, G.V. and Stanley, J.C. Statistical methods in education and psychology. Englewood Cliffs, N.J.: Prentice-Hall, 1970.

Goldin, G.A. Structure variables in problem solving, in Goldin, G.A. and McClintock, C.E. (eds.), Task variables in mathematical problem solving. Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics, and Environmental Educ., 1979

Gramick, J. Should algorithms be taught to conform to observed behaviors? (Doctoral dissertation, Univ. of Pa., 1975). Univ. Microfilms No. DCJ76-03168.

Hively, W. II, Patterson, H.L., and Page, S.H. A universe-defined system of arithmetic tests. Journal of Educ. Measurement, 1968, 5, 275-295.

Osburn, H.G. Item sampling for achievement testing. Educ. and Psych. Measurement, 1968, 95-104.

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SOME COGNITIVE FACTORS AS CAUSES OF MISTAKES IN THE ADDITION OF FRACTIONS

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What happens when a student is asked to use a skill which he has not mastered? The present study analyses this question in the specific context of adding fractions, with the purpose of describing the types of mistakes made by students in grades 7 to 9 and analysing these types in the light of psychological principles that guide the cognition and intellectual activity of the student. Some questions for further investigations are suggested.

§ 1. THE CHALLENGE OF MISTAKES

The guiding idea of this paper is well expressed by Ginsburg (1977): "Children make mistakes because they use faulty rules ... The faulty rules have sensible origins. Children's mistaken procedures are in fact good rules badly applied or distorted to some degree (p. 110) ... Errors result from organized strategies and rules. Children's behavior is meaningful, not capricious" (p. 129).

In a recent review paper, Radatz (1979), has suggested an error classification and their possible causes, giving examples from different mathematical topics. We shall analyse some mistaken procedures used by students when adding fractions. The addition tasks given to our sample belong to the first and second levels of the taxonomy; namely, knowledge and comprehension with the corresponding thought processes - recognition, recall and algorithmic thinking.

Why is adding fractions such a hard problem? Carpenter suggests that "students are not viewing the fractions to be added as representing quantities but see them as four separate whole numbers to be combined in some fashion" (Carpenter et al, 1976, p.138). We would like to suggest more specific reasons for the phenomena, reasons that demonstrate some general principles of cognitive processes, and tie in with some of the work reviewed by Radatz.

§ 2. THE TEST, THE SAMPLE AND THE KEY TO CATEGORIZATION

As part of a wider study (Hershkowitz, Note 1), two open addition exercises were given to 494 Israeli students, age 13-15. They were: $\frac{1}{2} + \frac{1}{3}$ and $\frac{1}{2} + \frac{1}{4}$. (The addition exercise in Carpenter, 1976 was $\frac{1}{2} + \frac{1}{3}$.) Addition of fractions was explicitly taught to these students at the age of 10-11 years.

The answers were analysed with respect to a three step algorithm for the addition of fractions:

1. Find a common denominator.

2. Expand the fractions (in such a way that they will have a common denominator).
3. Add the numerators of the expanded fractions (that now have a common denominator).

With respect to this algorithm three categories of mistakes can be distinguished. Within these three categories some subcategories will be described. All the types of mistake found by Carpenter were found in our sample, but we also identified some additional types, that may have been included by Carpenter under "other unacceptable responses". Our categories are unlikely to cover all the possibilities, if only because mistakes may depend also on the whole numbers constituting the fractions, and we have but two simple exercises.

§ 3. THE CATEGORIES OF MISTAKES

Category I: THE PRE ALGORITHMIC STAGE;
NO INDICATION OF THE COMMON DENOMINATOR ALGORITHM

a) The numerators are multiplied and the denominators are added ($\frac{1}{2} + \frac{2}{3} = \frac{2}{5}$; $\frac{1}{2} + \frac{1}{4} = \frac{1}{6}$).
For $\frac{1}{2} + \frac{1}{4}$ there is another possibility, viz. that the common numerator 1 is taken to be the numerator of the result.

b) The numerators are added and the denominators are multiplied ($\frac{1}{2} + \frac{2}{3} = \frac{3}{6}$).

c) The numerators and the denominators are multiplied respectively ($\frac{1}{2} + \frac{2}{3} = \frac{2}{6}$).

d) The numerators and the denominators of the given fractions are added respectively ($\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$; $\frac{1}{2} + \frac{1}{4} = \frac{2}{6}$).

e) The numerators are added and the denominators ignored ($\frac{1}{2} + \frac{2}{3} = 1 + 2$).

ANALYSIS

It is unlikely that, even if we were to ask, we would get satisfactory answers from the students to the question why they did the addition as they did. Hence we can only speculate as to the causes for the above mistakes (cf. Carpenter (p.138) who writes: "there is no way of knowing exactly what respondents were thinking, it is interesting to speculate on the incorrect strategies being used"). When speculating we will suggest an explanation (or explanations) which seems most reasonable to us. We are aware of the fact that other explanations could be given. We do not rule them out: the question which explanation is right cannot be answered in this study. However, explanations that have applicability to other mathematical domains are preferable.

In (a) and (b) it seems that the student remembers something of the addition algorithm. He remembers that addition and multiplication (or one of the two) appear in the algorithm, but he gets them in the wrong place. This is a very common phenomena (almost as common as forgetting) and it deserves a special name. We will call it *WRONG*

RECONSTRUCTION OF DETAILS. We wish to emphasise the difference between mere reconstruction and "reconstruction of details". In reconstruction of details mainly memory is activated, as opposed to other kinds of reconstruction where other functions (such as logical considerations, analogies, etc.) play a significant role. Also note that, as teachers, we might prefer complete forgetting to wrong reconstruction, but learning has its own rules which are independent of our preferences. In learning, partial forgetting occurs frequently and then the process of reconstruction takes place.

In (c) and (d) it seems reasonable to assume the influence of the multiplication algorithm. In (c) the students used precisely the multiplication algorithm. Assuming that our supposition is correct, we can consider these cases of *MISIDENTIFICATION*, which is quite common in the learning of mathematics. The student uses an algorithm appropriate to one situation in another situation to which it does not apply.

Subcategory (d) can also be considered as occurring under the influence of the multiplication algorithm, if we suppose the student to have made the following *implicit* considerations: "in order to multiply two fractions one should multiply the numerators and denominators respectively. Therefore, in order to add two fractions one should add the numerators and denominators respectively". Considerations like these (even when implicit) will be called a *WRONG ANALOGY*. (Compare also "errors of interference" in Radatz (1979).)

We can suggest yet another approach to explain (d): the student looks at the addition exercise and sees the plus sign and the numbers. He knows that the plus sign means add. He tries to find out what should be added. It is only natural to decide that the denominator should be added to the denominator and the numerator to the numerator. This is natural because in many cases we do add things that belong to the same category. We add apples to apples, chairs to chairs, tens to tens, and hundreds to hundreds. Considerations like these (again, even if they are implicit) can be called a *WRONG INTERPRETATION OF SYMBOLS*. Note that the interpretation of symbols, as well as analogy, is a powerful and useful means of cognition in many situations.

Subcategory (e) is also interesting because it reflects another common tendency of our cognition. When we have a hard task to perform we sometimes concentrate on an easy or familiar part of it. Then, when we finish with this part, we forget about the other parts thinking that we are through with the whole task.

Answers obtained by the above processes are very rarely checked by the student to see if they are right. This is because the student is usually unaware of the fact that he has used a faulty process, and feels sure that he remembers the algorithm correctly. Hence he sees no point in checking the answer. Moreover, even if a student has doubts

)out the correctness of his answer and wants to check it, he very often does not have
ıe means so to do. If you know that $\frac{2}{6} = \frac{1}{3}$ and $\frac{1}{3} < \frac{1}{2}$ and you know the meaning of
ldition, it is easy to see that you are wrong when you get: $\frac{1}{2} + \frac{1}{4} = \frac{2}{6}$. In this
ıse "viewing the fractions as representing quantities" (in Carpenter's words) might
:lp, but it is harder to see that you are wrong when you get $\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$, because it
; less obvious that $\frac{3}{5} < \frac{2}{3}$. Even if the student does have the required knowledge
) check his result, frequently he does not use it or see its relevance to his present
)ntext. This latter phenomena we call *COMPARTMENTATION*. It will be referred to again
ıter.

ıtegory II: THERE IS AN INDICATION THAT THE STUDENT IS ACTIVATED SOMEWHOW BY THE IDEA OF THE COMMON DENOMINATOR BUT THE IDEA OF EXPANDING THE FRACTIONS IS MISSING.

) Subcategories (a) - (d) in Category I, with an additional step - a common fraction line. For example, instead of writing $\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$ the students write $\frac{1}{2} + \frac{2}{3} = \frac{1+2}{5}$.

) A common denominator is obtained by adding all denominators and numerators: the numerators remain untouched and are added to each other at the end ($\frac{1}{2} + \frac{2}{3} = \frac{1+2}{8}$, $\frac{1}{2} + \frac{1}{4} = \frac{1+1}{8}$).

ANALYSIS

ı (a), the common fraction line suggests the schema of the common denominator: often, step in finding the common denominator is the writing of a common fraction line. One ın argue that a student who responded in a form which fits (a) - (d) of Category I, ıy have taken the common denominator step, but in his head. If we know this to be the ıses then we will consider the response as belonging to (a) of Category II. Since, ıwever, we are not concerned with determining the actual process by which a particular ıudent responds, but with the categories of responses, this does not make any difference. The point is that the same mistake can be caused by different cognitive processes also mentioned by Radatz (1979, p.171).

th (a) and (b) in Category II are (more than anything else) *WRONG RECONSTRUCTION OF TAILS*.

tegory III: THERE ARE INDICATIONS OF BOTH THE IDEA OF COMMON DENOMINATOR AND THE IDEA OF EXPANDING THE FRACTIONS.

The common denominator is obtained by adding the denominators. The new numerator is obtained by adding the original numerator and the denominator of the other fraction. In other words, expanding the fraction is considered as adding the same number to numerator and denominator ($\frac{1}{2} + \frac{2}{3} = \frac{4+4}{5}$) .

b) The common denominator is obtained by multiplying the denominators. The new numerator is obtained by adding the original numerator and the denominator of the other fraction, ($\frac{1}{2} + \frac{2}{3} = \frac{4+4}{6}$, $\frac{1}{2} + \frac{1}{4} = \frac{5+3}{8}$) .

c) The common denominator is obtained correctly. The new numerator is obtained by multiplying the numerator and denominator of the same fraction, ($\frac{1}{2} + \frac{2}{3} = \frac{2+6}{6}$, $\frac{1}{2} + \frac{1}{4} = \frac{2+4}{4}$) .

d) The common denominator is obtained correctly. The new numerators are obtained by adding the numerator and denominator in each fraction respectively, ($\frac{1}{2} + \frac{1}{4} = \frac{3+5}{4}$

e) The common denominator in $\frac{1}{2} + \frac{1}{4}$ is chosen to be 4. The new numerator is obtained by multiplying the numerator by the denominator of the other fraction, as if the common denominator was obtained by multiplication, ($\frac{1}{2} + \frac{1}{4} = \frac{4+2}{4}$) .

f) The common denominator in $\frac{1}{2} + \frac{2}{3}$ is chosen to be 3 (the larger number). The new numerator is obtained by adding the numerator and the denominator of the other fraction ($\frac{1}{2} + \frac{2}{3} = \frac{4+4}{3}$) .

ANALYSIS

It is worth mentioning that a student does not necessarily confine himself to one (correct or incorrect) strategy. Thus we found many students whose two responses fel into two different categories above. This is another case of the phenomena of *COMPARTMENTATION* which we mentioned above. Namely the student does not notice the essential contradiction between his two responses.

Subcategories (a) - (d) are the result of *WRONG RECONSTRUCTION OF DETAILS*, but note that in (a) the reconstruction seems reasonable. The only difference between (a) and the right answer is that the fractions were expanded by adding the same number to numerator and denominator, instead of multiplying. Hence, although it looks like a simple memory failure, it may have a logical basis of wrong analogy or wrong generalization (which are quite close to each other). It is permitted to *multiply* the denominator and the numerator by the same (non zero) number. Hence the student might think that it is also permitted to *add* the same number to the denominator and the numerator. His *WRONG RECONSTRUCTION OF DETAILS* is thus logically supported by *WRONG ANALOGY* (or *WRONG GENERALIZATION*).

Subcategory (e) can be considered as a result of both *WRONG RECONSTRUCTION OF DETAILS* and *MISIDENTIFICATION*. The guiding "rule" seems to be a combination of two elements:

(1) if one denominator divides the other, then the common denominator is the greater of the two;

(2) the new numerator is obtained by multiplying the old numerator by the denominator of the other fraction.

Each one, in its proper context, is correct, but by combining the two the student obtains a *WRONG RECONSTRUCTION OF DETAILS*. He also *misidentified* the situation in which the second element may be applied.

In subcategory (f) a *WRONG ANALOGY* probably occurred. Choosing 3, the greater number, to be a common denominator for 2 and 3, is analogous to choosing 4 (again the greater number) to be a common denominator for 2 and 4.

§ 4. A STATISTICAL TABLE

It is interesting to compare the distribution of answers in our sample to that in Carpenter's sample. Table I gives the percentages for three categories of response: the correct answer, the most common mistake (which was the same as in Carpenter's sample) and the category of all other mistakes. Although the percentages for all the other subcategories of mistakes mentioned above are insignificant statistically, these mistakes do occur and we may be able to remove them by studying their psychological origins as indicated above.

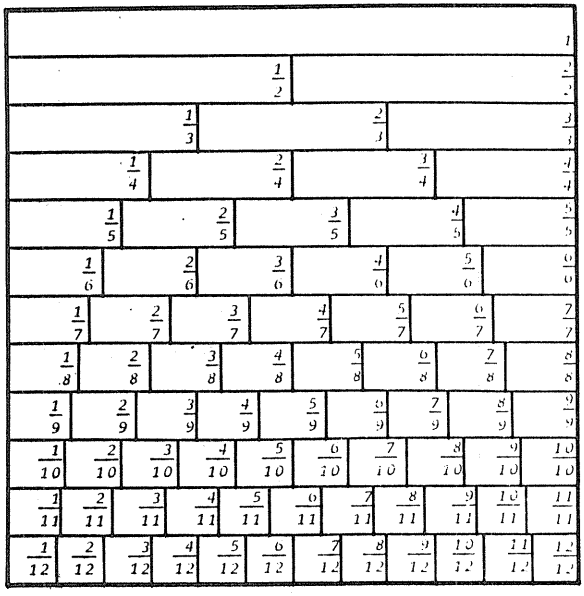
	The correct answer	Subcategory I (d) (The numerators and the denominators are added respectively)	All other mistakes	No answer
$\frac{1}{2}+\frac{1}{4}$	58.3%	18.8%	16.4%	6.5%
$\frac{1}{2}+\frac{2}{3}$	45.7%	24.3%	22.3%	7.7%

§ 5. CONCLUSION

The principles by which cognition acts include *FORGETTING, PARTIAL FORGETTING, MIS-IDENTIFICATION, RECONSTRUCTION OF DETAILS, ANALOGY* and *INTERPRETATION OF SYMBOLS*. The last three have a positive role in learning and should generally be encouraged, however, they can also cause serious mistakes. Our problem is how to exploit these principles to encourage learning and at the same time to avoid their causing mistakes of the types mentioned above. For example, we saw that a popular mistake was $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$ and we suggested that this arose from *WRONG ANALOGY* or *WRONG INTERPRETATION OF SYMBOLS*.

Methods of obviating this type of mistake will need careful investigation, but two which have been found intuitively useful are as follows:

(i) A fraction chart. This idea occurs in the texts of the *Midlands Mathematical Experiment (1967)*. We use it in a remedial program devoted to basic knowledge in fractions.



This chart can be used when the student learns (or relearns) the techniques of expandir comparing and adding fractions. He will thus be encouraged to see fractions "as quantities" as opposed to four separate numbers. When adding fractions he will be encouraged to check his answer against the chart.

(ii) Another suggestion which might prove useful is the suppression of the common fraction line. Our impression is that the common fraction line takes the place of the idea of expanding. (This is a special case of a general phenomena in the learning of mathematics, where a concept or an idea is replaced in the learners cognitive structure by its formal representation. Hence if the concept does not exist and there is only its formal representation, mistakes are very likely to occur.) Therefore we would recommend to avoid writing common fraction lines when adding fractions. For example:

$$\frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6} = 1\frac{1}{6} .$$

The student will then have no formal symbols while doing the calculations and may be forced to think about the ideas that underlie the calculations.

The work described in this paper has suggested some further questions, whose investigation may shed more light on the psychological principles involved in the learning of fraction addition:

1) Does the choice of numbers in an addition exercise influence the type of mistake made?

2) Can we characterise students who are systematic in their mistaken "belief" as opposed to those who "use" more than one type of mistake? (We use here the term "systematic" as defined by Cox (1975).)
3) Does knowledge of the multiplication algorithm hinder mastery of the addition algorithm (see Category I (c) and (d) above)?

Rather than jump on the band waggon of those who recommend the virtual abandonment of fractions in these days of the pocket calculator, we feel that it is worth investigating problems such as those here mentioned, and thus help more students achieve mastery in the manipulation of simple fractions.

REFERENCES

Avital, S.M. and Shettleworth, S.J., Objectives for Mathematics Learning, Ontario Institute for Studies in Education, Bull. No. 3, Toronto, 1968.

Bloom, B.S. (Ed.) The Taxonomy of Educational Objectives, Handbook I, Cognitive Domain, New York, 1956.

Carpenter, T.D., Coburn, T.G., Reys, R.E., Wilson, J.W. Notes from National Assessment: addition and multiplication with fractions, The Arithmetic Teacher, February, 1976, 23, 137-141.

Cox, L.S. Systematic Errors in the Four Vertical Algorithms in Normal and Handicapped Population, Journal for Research in Mathematics Education, November 1975, 6, 202-220.

Ginsburg, H. Children's Arithmetic: the learning process, Van Nostrand Company, 1977.

Midlands Mathematical Experiment, George G. Harrap Co. Ltd., London, Vol. 1, part A, p. 109. Second Edition, 1973.

Nadatz, H. Error Analysis in Mathematics Education, Journal for Research in Mathematics Education, May 1979, 10, 163-171.

NATURAL DEVELOPMENTAL SEQUENCES FOR LEARNING AND TEACHING FRACTIONS

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This paper is about a theoretical analysis of proposed natural sequences designed to assist learners in developing conceptual structures of fractional number ideas. Gradual conceptual growth and development are analyzed through a discussion of conceptual structures, cognitive processes, and cognitive development from constructivistic perspectives. The analysis for one sequence is guided by applying cognitive processes associated with mathematical generalization to the evolution of whole number multiplication, while the analysis for another sequence is guided by the notion of operators and function composition. It is suggested that research about fractions may be improved by keeping in mind an entire sequence rather than restricting attention to small segments; and a plea is made for better methods of measuring conceptual structures.

An instructional area that continues to present difficulties in the mathematics curriculum is the learning and teaching of fractions. Research and assessments continue to show that adolescents and adults have not learned fractional concepts very well (Kieren, 1976b; NEAP, 1979), and the reasons for this general learning and teaching problem are due partly to the complicated nature of the conceptualization of fractional numbers, including their multiple interpretations: part-whole, division, operators, ratios, and measures (Kieren, 1976a, 1976b, Note 1).

Researchers are searching for the most appropriate interpretation to be utilized in the initial phases of fractional number development. Payne (1976) has investigated an instructional sequence called the Initial Fractional Sequence based on the part-whole and measure interpretations as represented by continuous rectangular regions; Kieren (Note 1) and Dienes (1971) have tested an operator interpretation; and Noelting (Note 2; Note 3) has determined stage classifications of fractional development through studying the division and ratio interpretations.

It is still too early to determine the implications of these research projects, but it is generally agreed that students should eventually learn all interpretations and associated interrelationships; however, Payne (1976) reported that conflicts can arise when instruction based on different interpretations are used together; more specifically he found the set model interfered with measurement ideas associated with regions. According to Payne's analysis, "The set

model is very closely related to ratio; in fact, they may be the same thing" (p. 179); Payne concluded that, "it appears that ratio ideas are more readily accommodated to region ideas than conversely" (p. 167). In other words, region ideas should precede ratio ideas. Also, it is generally accepted that learning ratio and proportional ideas require Piaget's formal reasoning stage of cognitive development, and depending on the individual, this stage may develop later than generally thought (Tomlinson-Keasey, 1972).

So it would appear from these research results that a measurement model may be the most appropriate interpretation of fractions for initial experiences, and that a study of ratio should probably occur sometime later. But exactly how should the development of fractional ideas be sequenced from an instructional perspective in order to maximize conceptual growth? Is there a "natural" instructional progression through the different interpretations of fractions based on a natural learning sequence? Kieren (1976b) reported, "Suffice it to say then that a 'natural' developmental sequence for the concept of rational numbers is not known at this stage" (p. 3); so the question remains to be answered: Does a natural developmental sequence exist for fractions, and if so, what is it?

The principal thrust of this paper is to lay a foundation from which natural sequences for learning and teaching fractions may be derived. The view of how students learn is based primarily on the constructivistic theories of Dienes (1961), Piaget (1970) and Skemp (1971, 1979a) all of whom infer that coherent mathematical conceptual structures are built up through an active interaction with appropriate environments. Skemp's (1979a) two-level cybernetic model provides the theoretic setting for concept formations, acquisitions, and conceptual structures; Dienes' (1961) and Skemp's (1971, chap. 4; 1979a, chap. 11) work provide the background for cognitive processes, most notably abstraction and generalization processes at both the functioning (intuititive) and reflective levels of intelligence; and Piaget's developmental psychology provides the foundation for cognitive development.

The major purpose of the natural sequences is to assist the learner in acquiring, maintaining, and extending conceptual structures associated with fractional concepts; and it is important that this goal not be confused with goals associated with developing computational skill and proficiency, though they are certainly related.

The building blocks of conceptual structures are the internalized actions that move from one state of knowledge to another. It is the systematic construction of states of knowledge (concepts), connected or interrelated by actions (other concepts in the form of operators or functions, relations, propositions, and so forth) that comprises a conceptual structure (Mick & Brazier, 1979a). So to develop natural sequences we need to choose the appropriate actions at the outset and then follow them through their various phases of growth and development. Since the need to extend whole number division is a natural launching point for developing fractional numbers, I have chosen the actions associated with whole number multiplication for the beginning context in the development of natural sequences; thus the problem reduces to choosing an interpretation and isolating the specific actions associated within that context.

CONCEPTUAL STRUCTURES

This section is divided according to distinct interpretations of whole number multiplication, each of which is eventually to be transformed into interpretations of fractions. Three conceptual interpretations are identified but only the first two are discussed: operator-state (or state-operator), operator-operator and state-state.

In the operator-state interpretation whole number multiplication is characterized by the action of duplicating sets of objects. For example, 3×4 is either thought of as "three sets of four objects" or "four sets of three objects," but in either case the conceptual roles of the factors are distinct. In the interpretation "three sets of four objects," the factor "four" states the measure of a given set of objects while the factor "three" states how many of these sets are to be constructed or duplicated, and in the resulting static situation "three" can be considered a measure of a set of sets; in other words, there is a hierarchical order to the measures as well as an intervening action. It is proposed that the distinction between these hierarchical measures be designated by calling the lower level measure a state--a state of numerousity of objects; and by calling the higher level measure an operator--the action of duplicating sets. In sum, the entire operation associated with $3 \times 4 = 12$ consists of three distinct actions: (i) choose factor roles, for example, let three play the role of an operator and let four play the role of a state of knowledge; (ii) duplicate three sets of four objects each; and (iii) unite these three sets of four objects into one set of twelve objects.

At this stage the die as been casted; the rest of the evolving sequence grows and develops naturally via the mathematical context and constructivistic principles; one needs only follow the initial actions of whole number multiplication

as they mature, including major reconstructions as necessary. For example, there are two distinct inverse actions depending on whether the divisor plays the role of operator or state: if an operator, then the principal action is sharing or partitioning; if a state, then the principal action is repeated subtraction or measurement.

Using the operator-operator interpretation, Dienes (1971) has written a mathematics curriculum in which he describes a natural passage from multiplication operators to fraction operators, where this passage depends on defining multiplication of whole numbers in terms of unitary replacement operators. For example, multiplying by four is described as a four-for-one machine that replaces each element of some given input set by a set of four objects; so if a set of three objects is fed into this four-for-one multiplication operator, each of the three objects is replaced by a set of four objects; and the resulting output is the union of the three sets of four objects each; this means that the input set of three objects is replaced by a set of twelve objects. The entire operation consists of two distinct actions (Dienes, 1971): "(i) every element of the set of three objects is replaced by a set of four objects, and (ii) these three sets of four objects are united into one set of twelve objects" (p. 86).

So again, having carefully defined explicit actions for whole number multiplication, the die is cast and all further growth and development follows predictably from mathematical and psychological principles. For example, division follows the pattern of the inverse actions of the unitary operators; namely, dividing by four is described by a one-for-four machine that replaces each set of four objects in some given input set by a set of one object.

COGNITIVE PROCESSES

It is proposed that the development of a natural sequence for fractions follows the cognitive processes associated with mathematical generalization (Skemp, 1971 & 1979a; Dienes, 1961; Laenen, 1979). And as used in this paper, mathematical generalization is considered in constructivistic terms as extending and generalizing actions of learners relative to a suitable environment in view of finding a coherent structure (Mick & Brazier, 1979b). It is patterned after the more formal mathematical processes associated with embedding one number system within another.

A brief outline is given here to assist in the continuity of the paper: (1) an action is internalized and applied to similiar examples, (2) the conceptual structure of the operation is analyzed apart from its application, (3) the inverse operation is used to extend the set to include new elements in a manner

accompanied by reflective thought--a state of consciousness is required, (4) having created new elements, the original operation is generalized over this new enlarged set, and (5) reflective and putting-it-all-together processes assist to establish the various interrelationships between concepts.

One of the salient features of Dienes' operator approach is the psychological ease of passing from whole number multiplication to fractions--just a simple manner of combining multiplication and division operators end to end in function composition, thus mostly avoiding the five phase process outlined above. Skemp (1979a), in apparent support of Dienes' position, suggests that if multiplication is learned as the composition of two operations, "...and if properly taught, it is only marginally more difficult to acquire at the outset" (p. 254).

Noelting's empirical-based approach to the development of fractional ideas also differs from the five-phase approach outlined above. From the results of ratio test items completed by students varying in age from 6 to 16 years, Noelting (Note 1) has identified two distinct periods of equilibration for the growth of ratios and fractions: "...one corresponding to preoperational processes leading to the construction of the concept of ratio [and fractions], the other to the operational processes leading to the construction of the Common Denominator and Common Factor algorithms" (p. 265). (The reader is referred to Noelting's studies for an exact accounting of his four-phase description of each equilibration period.)

COGNITIVE DEVELOPMENT

Cognitive development pertains to the natural conceptual acquisition developed through maturation, experience, social interaction, and processes of self-regulation (Piaget & Inhelder, 1969). The extent of this development is usually described by Piaget's sequential developmental stages: sensori-motor, pre-operational, concrete operational, and formal operational.

In developing the proposed natural sequences for fractions, it would be helpful to determine the level of cognitive development necessary to assure successful development at all phases in the evolution of multiplication. For example, how far in the development of fractional concepts can we expect students to go without formal reasoning capabilities? Or more generally, how do cognitive development and conceptual growth of fractions interrelate and assist one another? Does the acquisition of fractional concepts follow Piaget's stage classification of cognitive development (refer to Noelting's studies)?

One cognitive developmental factor needed to complete a mathematical generalization is the ability to reflect on the related cognitive processes and

conceptual structures. For indeed the goal is to develop conceptual structures. Skemp (1979b) describes reflective capability as, "...the ability to make one's own concepts and mental processes objects of conscious attention, perhaps to describe them, perhaps also consciously and deliberately to change them" (p. 201). And since the objects of reflections are conceptual structures, it may be that reflective thinking requires cognitive development to at least the level of formal operations. If so, students in the fifth and sixth grades, for example, may not be very successful in completing a mathematical generalization; and therefore will have difficulties in extending whole number multiplication to fractional numbers as outlined in the five phases.

Another cognitive development factor of importance in acquiring fractional ideas is the ability to understand ratios and proportions. Research (Suydam, 1978) indicates that proportional capabilities are related to formal thought; and if so, then once again it appears that cognitive development to the level of formal operations is necessary for a successful evolution of whole number multiplication to fractional numbers.

In conclusion, two sequences for developing fractions were presented, each claiming to be natural and each differing in important mathematical and psychological ways. A third sequence was alluded to under the state-state classification; and then too, Noelting's work may very well be another candidate for a natural developmental sequence, although I question Noelting's close parallel between ratio and division interpretations. My limited data show distinct strategies and growth patterns between ratio and division--more in line with Payne's analyses; but I offer my observation only in terms of a conjecture for considerably more empirical evidence is needed before reaching a conclusion. In any case I am suggesting that we may sharpen our research studies by studying the whole sequence rather than isolated parts; or when studying isolated parts, design research in the context of the entire sequence. However, developing effective methods to measure conceptual structures may do more than anything else to provide us with a "breakthrough" in research of learning and teaching fractions.

REFERENCE NOTES

1. Kieren, T. E. The rational number construct--its elements and mechanisms. Unpublished manuscript, 1977.

2. Noelting, G. Constructivism as a model for cognitive development and (eventually learning). Unpublished manuscript, 1979. (a) (Available from Universite Laval, Québec).

3. Noelting, G. The qualitative and quantitative aspects in the development of the concept of fraction. Unpublished manuscript, 1979. (b) (Available from Université Laval, Québec).

REFERENCES

Dienes, Z. P. On abstraction and generalization. Harvard Educational Review, 1961, 31, 281-301.

Dienes, Z. P. The elements of mathematics. New York: Herder and Herder, 1971.

Kieren, T. E. On the mathematical, cognitive and instructional foundations of rational numbers. In R. A. Lesh & D. A. Bradbard (Eds.), Number and Measurement. Columbus, OH: ERIC/SMEAC, 1976. (a)

Kieren, T. E. Research on rational number learning. Athens: The Georgia Center for the Study of Learning and Teaching Mathematics, 1976. (b)

Laenen, C. Generalizing as an aspect of mathematical thinking. In David Tall (Ed.), Proceedings of the Third International Conference for the Psychology of Mathematics Education. Coventry, England: Warwick University, 1979.

Mick, H. W., & Brazier, G. D. A logical-cognitive view of learning and teaching mathematical concepts. In D. C. Fuson & W. E. Geeslin (Eds.), Explorations in the Modeling of the Learning of Mathematics. Columbus, OH: ERIC/SMEAC, 1979. (a)

Mick, H. W., & Brazier, G. D. Applying mathematical generalization in the acquisition of conceptual structures in trigonometry. In David Tall (Ed.), Proceedings of the Third International Conference for the Psychology of Mathematics Education. Coventry, England: Warwick University, 1979. (b)

National Assessment of Educational Progress. Changes in mathematical achievement, 1973-79 (Report No. 09-MA-01). Denver: Education Commission of the States, 1979.

Payne, J. N. Review of research on fractions. In R. A. Lesh & D. A. Bradbard (Eds.), Number and Measurement. Columbus, OH: ERIC/SMEAC, 1976.

Piaget, J. Piaget's theory. In P. H. Mussen (Ed.), Carmichael's Manual of Child Psychology (Vol. 1). New York: Wiley, 1970.

Piaget, J., & Inhelder, B. The psychology of the child. New York: Basic Books, 1969.

Skemp, R. R. Intelligence, learning, and action: a foundation for theory and practice in education. New York: Wiley, 1979. (a)

Skemp, R. R. Goals of learning and qualities of understanding. In David Tall (Ed.), Proceedings of the Third International Conference for the Psychology of Mathematics Education. Coventry, England: Warwick University, 1979. (b)

Skemp, R. R. The psychology of learning mathematics. London: Viking-Penquin, 1971.

Suydam, M. N. Review of recent research related to the concepts of fractions and of ratio. In E. Cohors-Fresenborg & I. Wachsmuth (Eds.), Proceedings of the Second International Conference for the Psychology of Mathematics Education. Osnabrück, Germany: Fachbereich Mathematik, Universität Osnabrück, 1978.

Tomlinson-Keasey, C. Formal operations in females from eleven to fifty-four years of age. Developmental Psychology, 1972, 6, 364.

TOWARDS THINKING MATHEMATICALLY

Part 1 - What is a fraction ?

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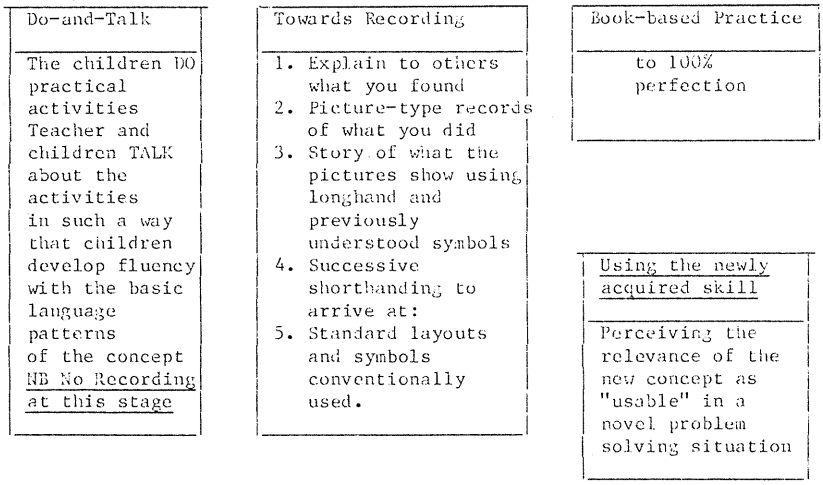
Part 1 puts forward a basic model for teaching mathematical concepts, illustrating its use in the context of naming fractions and identifying equivalent classes of fractions. Children need help drawing out the mathematical implications of several concrete embodiments of a concept. Language development appears to play a crucial part in the process. The model attempts to explain how, as well as indicate the role of teachers' questions.

In their survey of Mathematics teaching in primary schools, Her Majesty's Inspectorate Department of Education and Science (1978) observed that concrete apparatus-based learning experiences were being given to children, but that, in general, teachers failed to draw their pupil's attention to the mathematical implications of those practical activities. There seems to be a missing link in children's minds between the activities they do and the symbol-showing algorithms they are asked to use to produce correct answers. Teachers are usually the first to admit this. What they continually cry out for is some guidance about how to help children make the links between the two... a cry which all too often passes unanswered in the training and education of teachers. The job of an advisory teacher in mathematics in the London Borough of Ealing involved working for extended periods of time alongside teachers in their classrooms attempting to find solutions to their problems of teaching maths concepts to children. Over a period of three years, a pattern began to emerge from all these particular solutions and the teaching model about to be described is largely a distillate of, the stages that, time and again, children seemed to pass through as they moved toward an understanding of the concepts they were working on.

Watching one another teach in this way, it became clear that most of the explaining in maths lessons was being done by the teachers. These explanations were frequently incredibly involved. Just try writing out the explanation required to subtract 274 from 596 and the point is made ! What does not generally seem to be realized is that children need time to develop the language patterns to talk about such processes for themselves. They also need time learning how best to record what they have eventually learnt to say. What is more, children cannot reasonably be expected to attempt such recording until they have become fluent in the use of that language. That, too takes time. It makes sense to be on top of the language first for, if all one's attention is going into explaining something verbally, there is simply no attention left to attend to the task of recording that explanation in symbols.

This all adds up to a teaching model which allows a great deal of time for children to develop mathematical language patterns and to make their own conceptual links. I shall call the model we developed the "Do-and-Talk Model" and it looks like this:



As with all models, this oversimplifies the process. I should like to illustrate its use in the context of learning how to name fractions and I hope, by this means, to give a flavour of what is involved.

Let's think for a moment about the fraction two thirds. There are two basic language patterns a learner needs to be able to explain how he arrives at such a name. One relates to a situation where a whole is cut up into equal parts. A child has to be able to say something like:

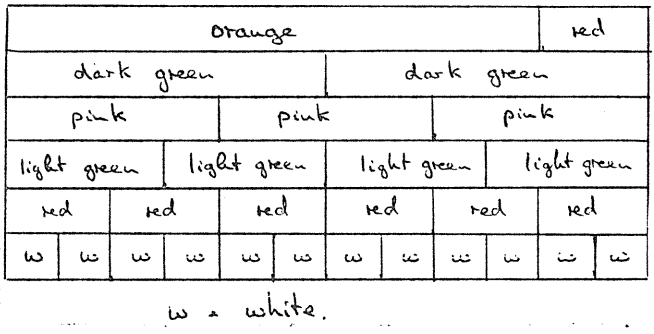
"The whole is cut up into 3 equal bits so each piece is called a third. I then take two of those pieces and that means I've got two thirds".

I think you'll agree, that that is not an example of everyday English. The second language pattern refers to equal subsets of a set of things like, say, 12 marbles. A child has to be able to explain:

"I group the marbles into 3 equal piles with 4 marbles in each (in this case). Each pile is one third of the marbles. If I take two piles, I've got two thirds of all the marbles and that's 8 marbles".

Once again the sentence construction is elaborate to say the least, yet all of it is necessary to explain the notion of "two thirds". The teacher's task, according to the model, is to find a concrete activity which provides something for the children to DO and, therefore, provides a context for the development of these language patterns.....something to TALK about.

One such concrete embodiment for the "parts-of-the-whole" situation involves the use of Cuisenaire rods. Children are asked to select a rod (or combination of rods) as their whole rod and then try to make up as many rows the same length using rods of one colour only. One such pattern made by a child looked like this:



It is important to notice who is doing the thinking in all that follows. The teacher does not use questions that are closed, expecting only an answer. Instead he makes considerable use of open questions because they put the onus to do the thinking onto the children. They also make space for a child to say what he thinks, what he sees, and what he does. Questions like; "Which rod is half of the orange-and-red whole ?" are closed. The reply, "dark-green" is all that's needed. However, questions like, "Can you tell me what you've done and what you've found ?" have quite a different effect. They are open and they require the respondent to talk about his own thoughts and insights.

To demonstrate the kind of interaction that takes place in the "Do-and-Talk" box of the model, imagine starting a discussion with a group of children with such an open-ended question. It is now possible to illustrate how the development of the language patterns for naming fractions takes place. On asking children an open question, they usually begin by hesitantly describing precisely what they see in front of them:

"I put out an orange-and-red rod and I made a dark green row, a pink row, a light green row, etc., etc. There were two dark green rods, three pink ones, four etc. etc."

That's a perfectly sound use of everyday English. It describes what's done and what was found, but it needs considerable development if it is to bring out the mathematics in the situation. The teacher asks another open question, again requiring the child to dig into his own thoughts; "What else can you say about the dark green rod?" he is hoping that the word "half" is somewhere in the archives of the child's mind. It usually is and so also is "quarter". But quite possibly the child might need prompting to say something like:

"Yes, a dark green rod is half the orange-and-red one. And a light green is a quarter".

The reasoning, which enables a child to say which rods are quarters and which are halves is not usually clear enough at this stage to make it possible for him to attach fraction names to the other rods in the pattern. He needs to understand that it is called a half because there are two equal pieces. Put another way, he needs to answer the question: "Why did I call it a half?" and "Why is a light green called a quarter ?" A blank look when asked such a question by the teacher indicates a child's need for

help. This is the point when teachers or another child usually offer an explanation. They say:

"We call the dark green rod a half because we used two of them to make the whole".

They may offer further exemplars like:

"We call the light green a quarter because the whole is broken up into four pieces. The pink is called a third, because there are three pieces..."

Now, without ever giving a child much of a chance to rehearse such an explanation for himself to see how appropriately it describes his own thinking, he is often moved on to a new example:

"So what can you say about the red rod ?"

The child needs an opportunity to develop appropriate language patterns of his own. Asking this question instead gives him the chance to do that rehearsal:

"Well, let's go back. What can you say about the light green rod now ?"

The difficulty with which children stumble over the explanation and the number of trials and rehearsals they seem to need before the language pattern begins to come together testifies to their need for time and space to develop the necessary language. It is not uncommon for children to need three or four weeks working with equipment in this way before they can <u>say</u> things like:

"The dark green rod is a half of the whole orange-and-red because we need two of them. The whole breaks into 3 pinks so each pink is a third. There are 6 reds to make the whole so each red is a sixth of the whole. Three reds are the same length as a half and 6 whites are the same as a half. 1 white is called a twelfth of the whole because there are twelve of them. That means six twelfths are the same as a half and 3 reds are the same....."

Still more time is needed to develop the fluency to make observations about the equivalence of fractions like this:

"A dark green rod is half of the orange-and-red whole.
but it is also the same as a pink and half of a pink.
Another way of saying that is; "a half is the same as
a third and half of a third". That's the same as a third
and a sixth because half of a third is the same as a red
rod which is a sixth. You can see that a half is the same
as three sixths and also the same as two quarters, and the
same as three sixths".

Such statements are eventually made in response to questions from teachers such as; "Can you tell everything you can about each rod ?"
It is important to note too how tied children are at this stage to the equipment. Pack it away and ask them to tell you what they found out about an orange-and-red whole and everything goes out of focus. It's even longer before they look into pictures in their mind and talk from those. One girl who described fluently what she knew about the parts of a particular whole without any equipment to hand was asked what enabled her to say all that. She said:

"Well it's like having a colour television in my head.
I can see what I'm saying !"

That is the cue to move on to the task of recording what a child so clearly sees and can talk about. There is now a confidence and a fluency in the way in which language patterns are used to describe the images on those personal mental screens that makes it possible to attempt to capture what they understand on paper. The fluency to move into the Towards Recording phase of the model is there. It should be clear where the picture records come from - the child's mind ! The stories of what the pictures say will be equally personal. If they said six twelfths are the same as two fourths, that is precisely what they will think of writing. Nevertheless it is surprising how much their thinking "wobbles" under the intellectual strain of trying to record what is otherwise so clearly seen. Frequent attempts to record gradually perfects the process, but, as always happens when we write and say things again and again, the children begin to shorten things and write:

6 twelfths = 2 fourths

and shorter still:

6, 12ths = 2, 4ths

And then, and it seems only then, is it appropriate to introduce the standard mathematical notations of the concept in question. In this paper, it is the notation of fractions and $\frac{6}{12}$ for 6,12ths is appropriate.

The acceptance of such conventional shorthands do, however, have to be negotiated. Children will only accept and use notations when they perceive the relevance of them to their recording. The teacher's task is to wait for that moment.

The importance of allowing children to forge their own links between DOING situations and the standard notations and layouts cannot be over-emphasized. Effectively children are laying tracks between the concrete and the abstract via pictures on their own personal mental screens. The track-laying process is vital because, although the learner may have a new-found confidence to operate symbolically, should the going get rough it is possible for him to track back to picture images or even right back to the concrete apparatus itself to cope with the difficulty.

The time taken to lay these tracks must also not be underestimated. This paper only describes one concrete embodiment of the parts-of-a-whole aspect of fractions. Children will need at least two other embodiments of this situation before they can abstract the concept of naming fractions. In additon they still have the naming of fractions in the context of equal-subsets-of-a-set-of-things to master as well as developing the ability to identify all the equivalent forms of any given fraction.

Department of Education and Science, Primary Education in England: a survey by H.M. Inspectorate of Schools, London H.M.S.O., 1978.

TOWARDS THINKING MATHEMATICALLY

Part II - Combining Fractions

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Taking up where Part I left off, ways of helping children to combine fractions are described. It is argued that the use of language and concrete embodiments is equally salient here. This is exemplified by a brief consideration of the kinds of activities and discussions which might usefully take place in order to foster understanding of the way in which fractions can be combined under multiplication.

Suppose that a teacher has worked with her children in the way described in the previous paper, and now wishes to help them understand how to combine fractions under the four basic operations. There are a number of foundations upon which such understanding can be built. First of all, the children now have a clear grasp of the meaning of the expression $\frac{a}{b}$ and of the equivalence of fractions. These concepts are rooted in a number of varied concrete embodiments, all of which share the idea of some whole entity being divided into a set of equal parts. In one embodiment the whole would be a set of counters divided into b equal subsets of which a were taken for some purpose or other. Another might be a ruler, where upon the whole would be some length or other divided into b subdivisions of which a were required to measure a particular object. Others include a circle divided into equal segments, and the Cuisenaire rods discussed more fully in the previous paper. The other main foundation is a clear grasp of the four operations as they apply to the natural numbers.

By the time that the children have had the learning experiences described in that paper, in the context of several such embodiments, they may well have become so fluent that they can transform any fraction written in standard notation into many equivalent forms without recourse to any concrete context at all. Nevertheless, because this fluency is so rooted in practical commonsense situations which the children have discussed and reasoned about at great length, any slight difficulty can be dealt with

easily and naturally by resorting to a concrete context. The link with everyday reality has not been broken. More importantly, a problem posed in terms of a particular embodiment can be solved as practically or as symbolically as the children wish. By working continually in the "Do and Talk" manner, and progressing through idiosyncratic recording methods to standard notation, the degree of symbolisation is only such that the children can understand. Thus not only do they have the reinforcing experience of successfully solving problems at whatever level of sophistication they can manage but the very problem-solving process itself increases this sophistication at a pace matched to the children concerned. Because of these advantages a strong case can be made for continuing work with fractions along similar lines. The remainder of this paper consists of examples of the sorts of problems that might be used in order to develop fluency with multiplication as it applies to fractions.

In the development of the concepts of fractions and equivalent fractions the word "of" will have arisen naturally in several of the contexts. Children will have made both oral and written statements like the following:

"A half of the counters is the same as two quarters of them"

"Three quarters of that cake is the same as six eighths of it"

"$\frac{3}{8}$ of an inch is the same as $\frac{6}{16}$ of an inch"

Thus the idea of a fraction of a concrete entity of some description will be both familiar and firmly based in reality. To the child, taking a bths of whatever it is means dividing the whole concerned into b equal pieces and then taking a of them. Hence the transition to the idea of fractions of entities which are themselves fractions need not be at all strange, provided that this is handled in practical situations rather than by some appeal to deductive reasoning. For example, suppose that the context is a set of counters, laid out in front of the child.

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Teacher: What would you do to find $\frac{1}{4}$ of the set of counters ?

Child: Put the counters into four equal groups and see how many there were in each.

Teacher: Show me

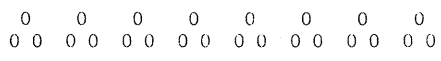
0 0	0 0	0 0	0 0
0 0	0 0	0 0	0 0
0 0	0 0	0 0	0 0

Now what would you do to find $\frac{1}{2}$ of a $\frac{1}{4}$ of the counters ?
Some children will need no further questioning than this and will immediately select one of the quarters and divide it into two equal parts. Others may need some prompting of the form "show me a quarter of the counters. What would you do to find half of that ?" Either way, some approximation to the following dialogue could happen ;

Child: $\frac{1}{2}$ of $\frac{1}{4}$ of the counters is 3 counters

Teacher: What would you do to find out what fraction of all of the counters that is ?

Child: See how many lots of 3's I could make from all of them.



3 counters is $\frac{1}{8}$ of all of them.
After many such doing and talking experiences, children will be able to make written recordings of what they are doing, rather like this:

$\frac{1}{3}$ of the counters is 8 counters

I divided the 8 counters into 4 equal pieces.

There were 2 in each

So $\frac{1}{4}$ of $\frac{1}{3}$ of the counters is 2

Then I counted and there were 12 lots like that in all the counters.

So $\frac{1}{4}$ of $\frac{1}{3}$ of the counters is the same as $\frac{1}{12}$ of them

Children could abbreviate this as and when they were confident to do so, leading eventually to the form:

$$\frac{1}{4} \text{ of } \frac{1}{3} = \frac{1}{12}$$

Once a given child has started off on this line of work, it can become self-supporting. He can arrive at several such results in this empirical manner, either on his own, or as part of a group of children. In a group the children can set each other tasks to do and have the responsibility of checking each other's work. In this way the teacher can have periods of interaction with a small group of children and then leave them to continue independently so that another group can receive particular attention.

After awhile, some children will begin to write statements like

$$\frac{1}{3} \text{ of } \frac{1}{5} = \frac{1}{15}$$

and without any apparent manipulation. When this happens it appears as though they have spotted a pattern in which they are now generalising. This pattern may be entirely the result of their own thinking, be completely copied, or somewhere between. Whichever it is vital questions for the teacher to ask are

"How did you get that ?"

"How can you be sure it works ?"

Only in this way can the teacher be sure that the tracks between the "real world" and the symbols on the paper are securely laid. Furthermore at another level the situation is an example of the general principle of always checking a pattern by testing it out on new examples.

It may well be that some children do not reach the stage of recognising the pattern on their own, or at least not in the time that is available. Some prompting may therefore be necessary. The pitfall to be avoided in such prompting is that the child may see the symbol manipulation as a trick of some sort, not as something that makes practical sense. If the child falls into that particular pit his only hope is to remember that trick - he will have no means of reconstructing it. Thus the prompted child needs most of all to be asked the second question above.

"How can you be sure it works ?"

By ensuring that he can answer this for himself, the teacher is also ensuring that the track leading from the real world situation to the symbolic manipulation is complete for that child too. And because of this very completeness movement backwards and forwards along that track will be possible, and mathematics will not be a collection of arbitrary hard to remember rules. Instead the children will genuinely be moving towards thinking mathematically.

RATIONAL NUMBER IDEAS AND THE ROLE OF REPRESENTATIONAL SYSTEMS

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This paper provides an overview of an NSF funded project, Rational Number Ideas and the Role of Representational Systems (Note 1), which is currently being conducted by Behr (Northern Illinois University), Lesh (Northwestern University), and Post (University of Minnesota). The purposes of the project are to describe: (a) the acquisition of progressively more complex and well-integrated conceptual systems which children use to make rational number judgments, and (b) the role that various representational modes play in this development.

The investigation of rational numbers is important in its own right, because so many of the "trouble spots" in elementary school mathematics are related to rational number ideas. The development of rational number ideas is also viewed as an ideal context in which to investigate general mathematical concept acquisition processes because:

(a) It occurs on the threshhold of a significant period of cognitive reorganization: the transition from concrete to formal operational thinking.

(b) Interesting qualitative transitions occur not only in the structure of the underlying concepts but also in the representational systems used to describe and model these structures. For example, multiplication no longer means repeated addition, $\frac{1}{2} \times \frac{1}{3} = \square$ may be read "$\frac{1}{2}$ of $\frac{1}{3}$" rather than "$\frac{1}{2}$ times $\frac{1}{3}$," etc.

(c) The roles of representational systems are quite differentiated and interact in psychologically interesting ways. Both figurative and operational aspects of tasks are critical. Piaget has focused on the operational aspects of tasks and concepts, using the term "decalage" to refer to tasks which are characterized by the same structure but differing in difficulty. However, for rational number concepts, it is the rule, not the exception,

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that models embodying the same concept vary radically, i.e., by several years, in the ease with which they are understood by children. Therefore, information about how figurative content influences task difficulty is important for those who must select or devise appropriate models to illustrate rational number concepts.

(d) The rational number concept involves a rich set of integrated subconstructs and processes, many of which have been identified in previous research. These are related to a wide range of "elementary but deep" concepts (e.g., measurement, probability, coordinate systems graphing, etc.) which are presupposed by a variety of problem solving situations and are often taken to be "easy," when, in fact, many of these concepts developed rather late in the history of science and are exceedingly unobvious to those who have not assimilated them (Hawkins, 1979). While some rational number concepts are notoriously difficult to change, others are quite easy to alter with only minimal instruction. Many of these concepts require no more than concrete operational abilities, but once acquired, facilitate the acquisition of a broad range of formal operational processes.

This project is concerned not only with what children can do naturally, but also with what they can do accompanied by minimal guidance or following theory-based instruction. The interest is in exploring the "zone of proximal development" of children's rational number concepts (Vygotsky, 1962), not only to describe the "programs" children typically use to process rational number information and interpret rational number situations, but also to describe how these programs change as a result of theory-based instruction. Rather than seeking only to accelerate rational number understandings along narrow conceptual paths, the project is interested in studying the results of broadening and strengthening deficient conceptual systems.

The testing component, which will be described below, includes the development of written tests for grades 2-8, instruction-mediated tests for grades 4-8, and clinical interviews for grades 2-8. The instruction-mediated tests included items which could not be adapted to a pencil and paper format (e.g., items requiring a spoken stimulus or response) but which did not require one-to-one interviews, or which required an integrated teaching/testing format. The instructional component, described in the companion to this paper (Behr, Post, Silver, & Mierkiewicz, 1980), includes a 12-week sequence to teach rational number concepts in grades 4 and 5 and diagnosis and remediation focusing on grades 6 and 7.

The distinctive features of the project include:

(1) The extension of previous logical analyses of rational number concepts, and the models used in teaching them, to include a psychological analysis of those ideas and models and the processes children use in tasks characterized by those ideas and models. The project will focus, in particular, on the role various representational systems play in the acquisition of rational number concepts.

(2) The integration of tasks from previous studies (Karplus, Note 2; Kieren, 1976; Carpenter, Coburn, Reys, & Wilson, 1978; Noelting, 1979) into a single task analysis scheme which will provide a generating scheme to guide both the testing and instructional components of the project and which will facilitate comparisons of the results. The generating scheme will be modified later to include insights derived from the instructional phase of the project. A literature review analyzing and synthesizing results of previous studies dealing with rational numbers and with representational systems will be included.

(3) The production of an integrated instruction and testing program, based on the same conceptual scheme, which will provide a common starting point for future research and development efforts and facilitate relating results of future projects to past work. In particular, the written test should allow future researchers to identify children in grades 2-8 who have interesting problems for follow-up investigation and help identify the interesting problems to be pursued in clinical interviews, remedial instruction, etc.

ANALYSIS OF THE RATIONAL NUMBER CONCEPT

Kieren's (1976) logical analysis of the rational number concept into five sub-constructs (part-whole relationships, measure relationships, ratio, quotient, and operator) provides the groundwork for the preliminary conceptual analysis. Post (1974) and Post and Reys (1979) integrated Kieren's work with a logical analysis of several possible concrete models teachers use in representing rational numbers. Dienes' (1963) mathematical variability principle and perceptual variability principle correspond to these logical and representational analyses, respectively.

In this project, the analyses of Kieren (1976), Novillis (1976), Post and Reys (1979), and Wagner (1975) have been considered in devising a refined logical scheme utilizing the following dimensions to obtain a 4 X 3 X 2 matrix:

Interpretation of $\frac{a}{b}$: fraction, ratio, operator, indicated quotient

Understandings: rational number interpretations, relations, operations

Models: discrete, continuous, order

Levels of these dimensions can be logically subdivided. For example, relations include equivalence, proportions, and comparisons; operations include addition, multiplication, and combining; continuous models include those for length, area, and volume.

This matrix represents a logical rather than psychological analysis of rational numbers because (a) it is derived from alternative logical constructions of the rational number concept simplified to fit the forms that these constructions take when presented to school age children, and (b) it is possible to sort problems into the various categories without actually talking to children or observing their behaviors in relation to the problems. Thus, logical schemes have been used to generate a variety of problem types to be used with children to investigate their behavior. The current project will carry the process through an additional phase, revising the scheme by superimposing a psychological analysis of problem types based on children's thinking.

Three outcomes are anticipated in the revision:

(1) Some cells will be psychologically unimportant or uninteresting and can be discarded. For example, adding ratios is rarely a meaningful task, and real problems of this type seldom occur.

(2) Subdivisions within some cells will become necessary. For example, familiarity versus unfamiliarity of particular concrete models may be important to a child. Or, situations may differ concerning the salience of a, b, or the relationship between them in the ratio a:b; an example in which the relationship is salient is the identification of "miles per hour" as "speed."

(3) Some of the cells will be redefined. For example, while a mathematician might make a logical distinction between unary and binary operations, the meaningful psychological difference to a student may depend upon whether or not the problem situation is inherently bimodal. That is, in a bimodal problem (unary operation), only one of the numbers may be given as an object; the other number may be given verbally, either as a statement about something that happened in the past or as a command to do something in the future.

Some of the psychological dimensions that are hypothesized to affect the acquisition of the rational number concept being used in the project are:

Modes of presentation: concrete manipulatives, pictures, real world,

spoken symbols, written symbols

Quantitative characteristics of the numbers: value of the number $\frac{a}{b}$, size of the numerator, size of denominator, etc.

Types of quantities represented (e.g., miles per hour, pieces of pie)

Salience of parts versus whole (e.g., in circular region problems the whole is implied; in probability situations the whole may never exist)

Concrete versus abstract dimension in relation to concrete and figural models

Static versus dynamic characteristics of models or situations

Familiarity of problem components

Further distinctions can be made within the above dimensions. For example, within the pictorial mode of representation, circular regions may be treated as different from rectangular regions, and may be more useful for the acquisition of part-whole understandings than for addition of fractions.

An additional perspective of this project is the careful consideration of the processes children use in rational number tasks. One important class of processes are the translations among and between the modes of presentation depicted in Figure 1 (Lesh, 1979). A major focus of the project is the role of manipulative materials as facilitators of the acquisition and use of rational number concepts as the child's understanding progresses from concrete to abstract rational number understandings. Psychological analyses have already shown that manipulatives are just one component in the development of representational systems and that the other modes also play a role in the acquisition and use of concepts.

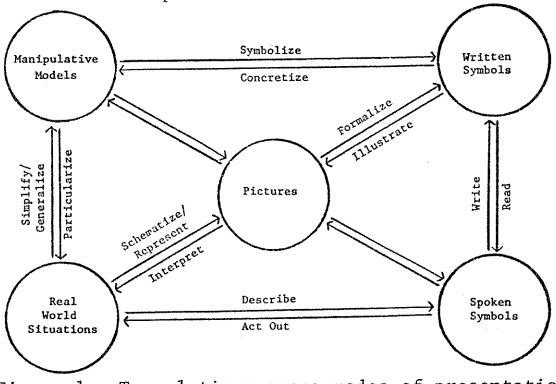

Figure 1. Translations among modes of presentation

Other important processes to be investigated include: the process of representation itself, particularly in relation to problems which are inherently

bimodal (e.g., "Here is a picture of half a pie. If 1/4 of a pie is added, how much pie will there be altogether?"); and transformations within a single mode (e.g., computational processes within the written symbolic mode, partitioning of objects within the concrete manipulative mode).

All three of these points of view--logical, psychological, and process--contributed to the outline of the dimensions considered in the generating schemes for the instruction, testing, and error analysis components of the project.

OVERVIEW OF THE TESTING PROGRAM

The testing program has five overall goals: (1) to provide baseline data on children's understanding of rational numbers across grades two through eight; (2) to provide pre- and post-test measures of children participating in the instructional and remedial phases of the project; (3) to help develop an overall scheme that logically and psychologically analyzes the development of a rational number conceptual structure and that attends to modes of representing rational numbers; (4) to integrate, wherever possible, results from previous rational number tasks and tests of Karplus (Note 2), Kieren (1976), Carpenter, Coburn, Reys, and Wilson (1978), Noelting (1979), and others, into that scheme; and (5) to give future researchers a base from which to assess rational number understanding as it relates to their respective research. Wherever possible, items have been adapted from previous rational number tasks and tests used by the researchers mentioned above, as well as by the NAEP.

Three components comprise the overall testing program: (1) paper and pencil tests; (2) instruction mediated tests; and, (3) clinical interviews. Each component is described briefly below.

The three paper and pencil tests cover three broad content areas: basic rational number concepts, ratios and proportions (including relations among rationals), and operations with rationals. The tests are organized in modules, to facilitate adaptation across grade levels, and to keep the length of each test under thirty minutes. Parallel test forms will serve as pre- and post-instruction measures. A second function of the written tests will be to provide diagnostic information for the identification of areas in which remediation is needed, or to suggest suitable starting points for clinical interviews.

The instruction mediated tests integrate a teaching component for presentation with a paper and pencil response mode to evaluate students in each of

the following areas which cannot be evaluated by a written test alone:

(1) Operators and function machines. Since items on this topic require some preliminary instruction to acquaint children with particular operator embodiments, the teacher will present a brief unit on operators from a prepared script using function machines as visual aids. A written test will accompany this instruction.

(2) Orange juice experiments (based on Noelting, 1979). The teacher will demonstrate the basic situation on which the test items are based: mixtures with two different proportions of orange juice and water are prepared and then compared to determine which should have a stronger orange juice taste. A written test comparing various mixtures then follows, to verify the hierarchy of quantitative distinctions found by Noelting.

(3) Indicated quotients using calculators. A brief lesson will be presented on the use of calculators to find the value of rational numbers as indicated quotients in order to find sums of fractions, followed by a written test.

(4) Translations from the spoken, concrete, and real world modes to written and pictorial modes which were impossible to adapt to a strict pencil and paper format. For example, the teacher can ask the students to write a mathematical sentence involving fractions, to draw a picture to represent it, etc. A word problem can be presented and the students can be asked to make a diagram representing the important relationships in the problem. A real world problem can be presented for solution, using real objects in a demonstration from which the students must extract the question.

The clinical interviews will be modifications of several prototypical interview sequences which can be designed individually based on the information obtained about the student from the paper and pencil and instruction mediated tests. The interview will be between one-half and one hour in length. Part of the time will be spent on a subset of items which can best be administered in the interview setting, e.g., translations from the concrete manipulative to the spoken symbolic mode. The starting point for the remainder of the interview will be determined by the patterns of the child's errors on written tests.

Specifically, the interview items will address three main concerns:

(1) The lemonade experiments of Karplus (Note 2).

(2) Common error patterns that arise on the paper and pencil tests. The interviewer will be prepared to probe, in the manner of the "buggy" programs

of Brown and Burton (1978), the conceptual and/or processing difficulties which may underly a child's errors.

(3) The investigation of the translations between and within modes of representation. For example, the child may be presented the fraction 2/3 in symbolic form and asked to represent it using Cuisenaire rods, parts of a circle, or the number line.

The clinical interview will permit the exploration of non-answer finding stages in problem solving. For example, to find out about a child's mental representation of a problem situation, the interviewer may present a problem and direct the child to pick appropriate concrete materials that would be useful in the representation or solution of the problem, but not to actually solve the problem. Communication skills can be assessed through such activities as asking the child to read a word problem and describe it to the interviewer in such a way that the interviewer can solve it.

SOME ANTICIPATED PROJECT OUTCOMES

The expected products of the project include the literature review and synthesis of past research in the areas of rational numbers and modes of representation; an instructional package for introducing the rational number concept, appropriate to grades four and five; paper and pencil tests for grades 2-8 covering understanding of the fraction concept, ratios and proportions, and addition and multiplication with rationals; instruction mediated tests for grades 2-8 on operators, indicated quotients using hand calculators, proportional reasoning, and several translations among modes, complete with scripts for teachers to use in administering the tests; prototypical clinical interviews to be used to follow up selected students identified from the results of the written tests for either more detailed investigation or remediation; a refined conceptual scheme that will be useful for further research on the acquisition of rational number ideas.

A number of conclusions are also anticipated which will provide some direction for future research and development efforts. Some tentative conclusions that have begun to emerge include the following.

The instructional sequence used in the project, which is profoundly different from current practice, is expected to be very beneficial in facilitating understanding of the rational number concept, relations, and operations. In contrast with typical sequences, which move quickly into relations and operations, the instruction in the project builds on a carefully constructed

understanding of basic concepts and on mode-to-mode translations. As much as two-thirds of the overall sequence is devoted to a thorough treatment of the fraction concept; instruction in relations and operations is expected to require significantly less time following this treatment. It appears that many topics (such as finding equivalent fractions and addition of fractions with like denominators) require a minimum of instructional time because of the children's firm understanding of the basic concepts.

The role of manipulative materials is being investigated beyond "which is best" to consider "best for what purpose and at what stage in the learning cycle." Manipulatives contribute to the understanding of basic concepts, but at some point (for example, by the time students are adding fractions with unlike denominators), they play a different role. The manipulatives become useful in representing the problem, but are not useful in performing operations required to obtain an answer. At that point, the manipulatives help the student represent the problem and relate an answer (obtained by computation with written symbols) back to a real situation.

Problems which are inherently bimodal have been mentioned previously. For example, a student may be presented with a picture of 3/4 of a pie, then told that half was to be taken away, and asked how much pie would be left. Here the problem presentation combines pictorial with spoken symbolic modes. The student may elect to solve the problem in one or the other mode, or in yet a third, such as written symbols. One anticipated conclusion of the project is that children do not usually solve a problem from start to finish in a single mode; several typical patterns for given modes of problem presentation should be describable.

The project is expected to provide information which will help describe qualitative shifts in children's thinking about rational numbers. Major reorganizations in thinking are required when old ideas are given new names and interpretations and vice versa. For example, a child who has come to think of multiplication as repeated addition cannot apply this interpretation to "$\frac{1}{2} \times \frac{1}{3}$." Multiplication, which had previously always been an operation that either "made things bigger" or "kept things the same," must be reinterpreted in light of the fact that multiplication involving fractions less than one "makes things smaller." Tracing the development of children's thinking in relation to these kinds of changes is expected to be very informative. Other hypotheses or anticipated outcomes are described in the companion to this article.

REFERENCE NOTES

1. Behr, M., Lesh, R., & Post, T. The role of manipulative aids in the learning of rational numbers. RISE grant #SED 79-20591, Northern Illinois University.

2. Karplus, R. Early adolescent student reasoning in mathematics. RISE grant #SED 79-18962, The Regents of the University of California, Lawrence Hall of Science, Berkeley.

REFERENCES

Behr, M., Post, T., Silver, E., & Mierkiewicz, D. Theoretical foundations for instructional research on rational numbers. In Proceedings of the International Group for the Psychology of Mathematics Education, Berkeley, California, 1980.

Brown, J. S., & Burton, R. R. Diagnostic models for procedural bugs in basic mathematical skills. Cognitive Science, 1978, 2, 155-192.

Carpenter, T., Coburn, T. G., Reys, R. E., & Wilson, J. W. Results from the first mathematics assessment of the National Assessment of Educational Progress. Reston, Va.: The National Council of Teachers of Mathematics, 1978.

Dienes, Z. P. An experimental study of mathematics learning. London: Hutchinson, 1963.

Hawkins, P. An approach to science education policy. Background paper for a program of research in mathematics and science education. Washington, D. C.: National Science Foundation, 1979.

Kieren, T. E. On the mathematical, cognitive, and instructional foundations of rational numbers. In R. A. Lesh (Ed.), Number and measurement: Papers from a research workshop. Columbus, Ohio: ERIC/SMEAC, 1976.

Lesh, R. Mathematical learning disabilities: Considerations for identification, diagnosis, and remediation. In R. Lesh, D. Mierkiewicz, & M. G. Kantowski (Eds.), Applied mathematical problem solving. Columbus, Ohio: ERIC/SMEAC, 1979.

Noelting, G. The qualitative and quantitative aspects in the development of the concept of fraction (Sharing cakes experiment). Paper given at the NATO International Conference on Intelligence and Learning. York, England, 1979.

Novillis, C. F. An analysis of the fraction concept into a hierarchy of selected subconcepts and the testing of the hierarchical dependencies. Journal for Research in Mathematics Education, 1976, 7, 131-144.

Post, T. R. A model for the construction and sequencing of laboratory activities. Arithmetic Teacher, 1974, 21, 616-622.

Post, T. R., & Reys, R. E. Abstraction, generalization, and the design of mathematical experiences for children. In K. Fuson & W. Geeslin (Eds.), Explorations in the modeling of the learning of mathematics. Columbus, Ohio: ERIC/SMEAC, 1979.

Vygotsky, L. S. Thought and language. Cambridge, Mass.: The M.I.T. Press, 1962.

Wagner, H. A. An analysis of the fraction concept (Doctoral dissertation, Northwestern University, 1975). Dissertation Abstracts International, 1976, 36, 7919A-7920A. (University Microfilms No. 76-13, 111)

THEORETICAL FOUNDATIONS FOR INSTRUCTIONAL RESEARCH ON RATIONAL NUMBERS

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This paper describes theoretical constructs upon which instructional research concerned with the learning of rational number concepts is being based. The theoretical constructs suggest types of manipulative materials, representational modes, and rational number constructs on which instruction is based. An argument is made that the oral mode of representation is an important intermediary between the manipulative and symbolic modes. Concepts from information psychology are discussed which suggest a basis for observation of student performance.

Reviews of research on the use of manipulative aids in learning mathematics give evidence that their use facilitates learning of mathematical skills, concepts and principles (Fennema, 1972; Kieren, 1969; Gerling & Wood, 1976; Payne, 1975; and Suydam & Higgins, 1976). Implications for classroom practice is unclear because questions such as how, why, and with whom various kinds of manipulative aids should be used remain uninvestigated.

The current research results reflect little consideration to psychological variables that potentially affect human learning. The work of the project described herein, concerned with the role of manipulative aids in the learning of rational number concepts, derives its theoretical base from three separate but compatible theories. The theories provide perceptual, instructional and information processing models for development of instructional materials and for observation of instructional effects.

THE PERCEPTUAL MODEL

The organization and selection of instructional materials used in the project employ two variability principles of Z. P. Dienes. The Perceptual Variability (Multiple Embodiment) Principle suggests that conceptual learning is maximized when a concept is represented in a variety of physical contexts. Exposure to

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these various representations which differ in appearance but retain the same conceptual structure, is hypothesized to promote mathematical abstraction.

Similar to the Perceptual Variability Principle in that it encourages multiplicity in patterns of exposure, the Mathematical Variability Principle asserts that if a mathematical concept is dependent upon a number of variables, then systematic variation of these is a prerequisite for effective learning of the concept (Dienes, 1967). This principle is hypothesized to promote generalization of mathematical concepts.

Post (1973, 1979; Reys & Post, 1973) have incorporated these principles into a two dimensional matrix each specific to a particular mathematical concept. The project utilized such a matrix designed for the concept of rational number. The horizontal axis (Mathematical Variability Principle) consists of the five constructs of rational number identified by Kieren (1976); the vertical axis (Perceptual Variability Principle) incorporates the variables of length, area and number and lists manipulative aids upon which project instructional materials are based--counters, paper folding, rods, and the numberline.

THE INSTRUCTIONAL MODEL

Reported studies have generally imposed a linear interpretation on Bruner's three modes of representational thought, i.e., first the enactive phase, then the iconic phase, and finally the symbolic phase. More realistically, these phases should be incorporated into an interactive model. Such a model is presented by Lesh (1979) (and the chapter by Lesh, Landau, and Hamilton this volume) who suggests that certain "translation" processes between and within modes of representation may be important in making an idea meaningful. The modes of representation identified in the model are real world situations, spoken symbols, written symbols, pictures, and concrete models. The first two of these represent an extension of Bruner's model. Of major interest in the project is the hypothesis that it is translations among and within certain of these modes which make ideas meaningful for children.

THE INFORMATION PROCESSING MODEL

Gagné and White (1978) deal generally with the relationship between memory structures and learner performance. They present a theoretical model upon which research on instruction and learning outcomes can be based. Some elaboration of the memory structures of interest to this research project is given in a later section.

RATIONAL NUMBER-- ITS MANY FACETS

Kieren (1976) has identified five subconstructs of rational number--part-

whole, ratio, quotient, operator, measure. Each is important in its own right and in addition contributes to a complete picture of the overall concept. Kieren (1976), Noelting (1979) and Karplus (1970) have all observed the rational number thinking of children at various stages. One concern in this project is development of prototype instructional materials to teach rational number thinking. The current year of our project work has dealt with writing and piloting instructional materials which reflect the rational number subconstructs identified by Kieren. These materials have attempted to emphasize translations within and between modes of presentation (we have elected to use modes of presentation rather than modes of representation because we believe it more clearly conveys the nature of the instructional setting and because it is not possible to previously identify the mode in which an individual encodes a particular experience.)

Initially we conceptualized instruction for rational numbers as suggested in Figure 1.

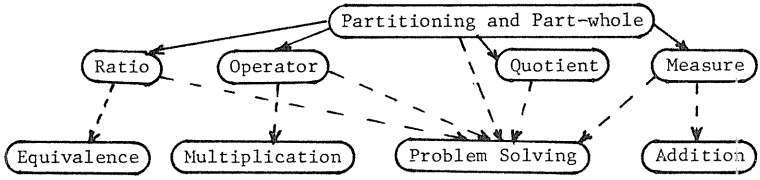

Figure 1. Conceptual Scheme for Instruction on Rational Numbers

The arrows and dashed arrows in Figure 1 are to suggest hypothesized relationships between rational number constructs, relations and operations on rational numbers. The diagram suggests that (1) partitioning and the part-whole construct of rational numbers are basic to learning other constructs of rational number; (2) the ratio construct is most "natural" to maturate the concept of equivalence and nonequivalence; (3) operator and measure constructs are the most pervasive for maturing the understanding of operations of multiplication and addition of rational numbers. The solid arrows suggest a conviction on our part at this point in our work, while the dashed arrows represent more tentative hypotheses. One important expected outcome of our second year of work is the identification of manipulative aids, which are the most "natural" for developing certain rational number concepts, relationships, skills, and understandings.

THE ROLE OF ORAL SYMBOLISM

The written symbolic mode is the dominate mode in which one generally asks children to demonstrate their knowledge of mathematics. The concrete mode is

the dominate mode of presentation. In order for children to meaningfully write and read (decode and comprehend) written mathematical symbolism it is imperative that a strong association between the concrete mode and written symbolic mode be internally established by the learner (Van Engen, 1949). The gap between the concrete and written symbolic modes is greater than earlier perceived (Behr, 1976; Payne, 1975). Persuasive arguments that the oral symbolic mode is an important intermediary can be made.

The project investigators are developing teaching sequences for rational number concepts which emphasize the role of oral verbalization. The teaching sequences require the children to: (1) orally explain and reconstruct the sequential procedure used in the manipulation of the concrete aids; (2) orally verbalize the important mathematical observation to be derived from the objects; (3) record by writing or completing a written English sentence the orally stated mathematical observation. This is done before any attempt is made to introduce mathematical symbolism. In pilot lessons where we have omitted oral activity before going to a written worksheet activity our preliminary observations indicate that children are more likely to have difficulty.

RESEARCH ON READING

A controversial position among researchers in the psychology of reading (Gleitman & Rozin, 1977) is that reading requires explicit and conscious discovery and building from what one already knows implicitly for the sake of speech. The history of wirtten language indicates a clear trend away from attempts to transcribe meanings directly towards the transcription of meaning mediated by sound. There is increased reliance on speech as a mediator between written symbols and meaning. A linguist Charles Fries (1962) suggests that learning to read in one's native language involves transferring from the auditory signs of language signals which children already know to the new visual signals from the same language signals.

Written mathematical symbolism is different from the written symbolism for our native language. The latter is an alphabetic system while mathematical symbolism is not. Mathematical symbolism is an "abbreviation" of written native language. Because of the interrelations between speaking, reading, and writing, it is reasonable to posit that the oral mode of expression stands as an important intermediary between the ideas of mathematics and the written mathematical symbolization of those ideas.

The Soviet psychologist Vygotsky (1976) addresses the question of the interrelationship between spoken and written language. Vygotsky indicates that

> A feature of this system [written language] is that it is a second order symbolism, which gradually becomes direct symbolism. This means that written language consists of a system of signs that designate the sounds and words of spoken language [emphasis ours] which in turn, are signs for real entities and relations. Gradually this intermediate link, spoken language [emphasis ours], disappears and written language is converted into a system of signs that directly symbolize the entities and relations between them (p. 106).

Vygotsky asserts that the primary symbolic representation should be ascribed to speech and on the basis of speech all other sign systems are created.

RESEARCH ON WRITING

A review of research on writing by Bereiter (1980) suggests areas where problems might lie for children to express meaningful associations between mathematical ideas expressed via manipulative aids and via mathematical symbolism. The first deals with limitation on children's information processing capacity. Children cannot integrate all of the skills necessary to present a written record of a thought string and retain the thought string simultaneously. Cognitive demands needed to generate the mathematical ideas and simultaneously record the information exceed a child's information processing capacity.

A second involves a developmental sequence of writing skills in children. The simplest system to produce intelligible writing is referred to as associative writing. An apparent precursor to this is transcribed speech. One researcher (Simon, 1973) observed that children in the first years of schooling typically verbalize or subvocalize what they write. This behavior mainly disappears by the 3rd or 4th grade. This seems to be true for narrative writing and also later for expository writing. The problem of writing mathematical symbolism has not been addressed; however, writing fluency with mathematical symbolism becomes intelligible in many learners even later, if at all.

INFORMATION PROCESSING PSYCHOLOGY

Information processing psychology identifies a number of types of memory structures. Two of particular interest are episodic and semantic memory. Episodic memory receives and stores information about episodes or events which are temporally related as well as the temporal-spatial relations among these events. A perceptual event can be stored in the episodic system solely in terms of its perceptual properties or attributes; moreover, such an event is always stored in terms of its autobiographical reference to contents already existing in the episodic memory store. It is expected that the episodic memory system is susceptible to transformation and loss of information (Tulving, 1972).

A person's episodic memories are located in and refer to his own personal past. It must be translated into the form "I did such and such in such and such a

place at such and such a time" (Tulving, 1972). Thus it seems children need to engage in, rather than only observe the manipulation of objects, in order to form episodic memory structures. Implications regarding the use of concrete aids to present mathematical concepts to children are clear.

Semantic memory is necessary for the use of language. It is the part of memory in which the organized knowledge a person possesses about words and other verbal symbols is stored. Also stored in semantic memory are the meaning and referents of words and verbal symbols as well as relations among the words and referents and about rules, formulas, and algorithms for the manipulation of the symbols, concepts and relations.

The semantic system permits the retrieval of information not directly stored in it; episodic memory permits retrieval only of information entered on an earlier occasion. The episodic memory system does not include the capabilities of inferential reasoning or generalization; these capabilities are part of semantic memory. By relying on semantic memory it is possible for a person to know something he did not specifically learn.

These observations about episodic and semantic memory have relevance to the question of teaching mathematics based on manipulative aids. Episodic experience which a child gains from concrete aids may not provide retrievable knowledge without semantic information about the episodes and about relationships among different episodic experiences. Verbal interaction by the learner with a teacher or peers to observe the similarities and differences among episodic experiences and the materials on which they are based is probably essential.

This latter contention is supported by Vygotsky's analysis of the process of internalization of knowledge. Vygotsky considers the process of internalization as consisting of a series of transformations:

1. An operation that initially represents an external activity is reconstructed and begins to occur internally.
2. An interpersonal process is transformed into an intrapersonal one.
3. The transformation of a interpersonal process into an intrapersonal one is the result of a long series of developmental events.

It is the second step in this process which is significant in the present context. Every function in children's cultural development, according to Vygotsky, appears twice: first on the social level, and later, on the indi-

vidual level. That is, first the function appears at the interpsychological level (between people) and then at the intrapsychological level (within the child). This applies equally to various kinds of functions including voluntary attention, logical memory, and to the formation of concepts.

The example which Vygotsky gave to illustrate this process (the formation of a gesture) does not have an exact analog in the context of learning mathematical concepts from concrete materials. However, the following sequence of steps seems compatible with Vygotsky's concept of the process of internalization of knowledge.

Consider the model of paper folding to show fractions: initially the child through imitating the behavior of the teacher, learns to fold and shade the paper to show equal-size parts and represent fractions. Secondly the child orally tells the sequence of steps and explains how the resulting demonstration exemplifies for him/her the mathematical concept represented. As experience of this type continues, the outer speech of the child used to communicate to a teacher or a peer gradually changes to inner speech. In this way the transformations associated with the concrete aids become internalized.

It is in the context of these empirical and theoretical considerations that our research project will attempt to gain some insights into the question of how interactions among and within modes of representation facilitate the learning of mathematical concepts and operations. Emphasis in our research will be on how meaningful symbolism for mathematical concepts can be developed for children.

REFERENCES

Behr, M. J. Teaching Experiment: The Effect of Manipulatives in Second Grader's Learning of Mathematics: Volume I. PMDC Technical Report No. 11, Tallahassee, Florida, 1976.

Bereiter, C. Development in Writing in Lee W. Gregg and Erwin R. Steinberg (Eds). Hillsdale, N. J. Lawrence Erlbaum Assoc., In press.

Dienes, Z. P. Building Up Mathematics (Revised Edition) London. Hutchinson Educational, 1967.

Fennema, E. H. Models and Mathematics. Arithmetic Teacher 19: 635-640, December 1972.

Fries, C. C. Linguistics and Reading. New York: Holt, Rinehart and Winston, Inc., 1962.

Gagné, R. M. & White, R. T. Memory Structures and Learning Outcomes, Review of Educational Research 48: 187-222, 1978.

Gerling, M. & Wood, S. Literature Review: Research on the Use of Manipulatives in Mathematics Learning. PMDC Technical Report No. 13, Tallahassee, Florida, 1976.

Gleitman, L. R. & Rozin, P. The Structure and Acquisition of Reading I: Relations Between Orthographics and the Structure of Language. In Arthur Reber & Don L. Scarborough (Eds.). Toward a Psychology of Reading. New York: Erlbaum Assoc., 1977.

Karplus, R. & Peterson, R. W. Intellectual Development Beyond Elementary School II: Ratio, a survey. School Science and Mathematics, 1970.

Kieren, T. E. Review of Research on Activity Learning. Review of Educational Research 39 (October 1969): 509-22.

Kieren, T. E. On the Mathematical, Cognitive, and Instructional Foundations of Rational Numbers. In R. Lesh (Ed.) Number and Measurement: Papers from a Research Workshop. ERIC, 1976.

Noelting, G. The Development of Proportional Reasoning and the Ratio Concept (The Orange Juice Experiment), 1979.

Payne, J. N. Review of Research on Fractions in R. Lesh and D. Bradbard (Eds.). Number and Measurement: Papers from a Research Workshop. Columbus, Ohio: ERIC/SMEAC, 1975.

Simon, J. La langue ecrite de l'enfant. Paris: Presses Universitaries de France, 1973.

Suydam, M. & Higgins, J. Final Report: Review and Synthesis of Studies of Activity Based Approaches to Mathematics Teaching. NIE Contract No. 400-75-0063. Submitted to NIE September, 1976.

Tulving, E. Episodic and Semantic Memory. In E. Tulving (Ed.) Organization of Memory, New York: Academic Press, 1972.

Van Engen, H. An Analysis of Learning in Arithmetic I. Elementary School Journal, Feb. 1949, 321-329.

Vygotsky, L. S. Mind in Society. M. Cole, V. John-Steiner, S. J. Cribner, E. Souberman (Eds.). Cambridge, Mass.: Harvard University Press, 1978.

ON THE UNDERSTANDING OF CONCEPTS AND RULES IN SECONDARY MATHEMATICS: SOME EXAMPLES ILLUSTRATING THE DIFFICULTIES

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Samples of less successful students (aged 12 to 16) were tested on the topics of fractions and ratio. The analysis was designed to find the specific difficulties and deficiencies of these children. After it was shown that their understanding is at most instrumental, and not relational, we tried to promote improvement by special questions which are related to the students' problem-solving process. The types of questions are considered by which a higher rate of success can possibly be achieved.

At the present time we are following a line of research at Osnabrück University which has as its aim the description of some details about special difficulties and problems faced by poorer students in using concepts and rules. We intend to make suggestions as to how these deficiencies could perhaps be compensated. The research was partly suggested by some results of the project "Concepts in Secondary Mathematics and Science" (CSMS) of Chelsea College, London University (Hart, 1978, 1979). However, while there the "hierarchies of understanding" in mathematics are emphasized, we try to find the reasons for the inability of so many students to use mathematical concepts and rules. For this it is necessary to analyse the mistakes we find in the students' answers; we use informal interviews as well as the students' written solutions. By this procedure we follow on the one hand Birkhan (1979) and Shulman (1978), and use on the other hand the experience of e.g. Ginsburg (1977) and Schlaak (1968; cf. also Radatz, 1980).

For the first step of our research we gave about 100 students of the 7th year of school a test with fraction problems (these students belonged to the poorer half of their agegroup). The choice of the 7th year of school had the advantage that almost only the long-term effects of learning fractions were to be observed, because the students had not been confronted with fractions for about a year.

Our test was set with several points in mind. It comprised 27 items which can be sorted into three facets:

A. Operations with fractions;

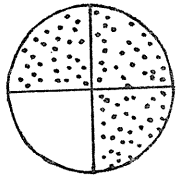
B. Form of representation of problems;

C. Operational forms.

A more exact description of the test and a discussion of the results will be published elsewhere (Hasemann, 1981). In the present paper we examine only two pairs of items.

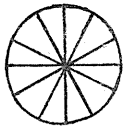
Several items appeared twice in the test, in different forms of representation but either the same or very similar in content. E.g.:

(1) A part of the circle is dotted. Shade in $\frac{1}{6}$ of the dotted part. What fraction of the whole circle have you shaded in?



(2) Calculate: $\frac{1}{6} \cdot \frac{3}{4}$.

(3) Shade in first $\frac{1}{4}$ of the circle and then $\frac{1}{6}$ of the circle as well. What fraction of the circle altogether have you now shaded in?



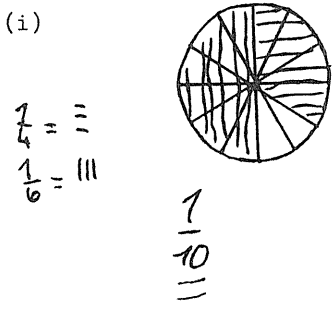
(4) Calculate: $\frac{1}{6} + \frac{1}{3}$.

Both items (1) and (2) also occured in the English test which emanated from the CSMS project, but with quite different results from ours (cf. Hart, 1978): 53% of the English students found the solution of the diagrammatical problem (1), but only 23% could solve the corresponding computational question. Of our students, 52% multiplied $\frac{1}{6}$ and $\frac{3}{4}$ correctly, but only 30% were successful with the diagram. The application of the rule ("numerator times numerator, denominator times denominator") gave hardly any difficulty, but the students found it extraordinarily difficult to recognize a given situation from a diagram, to follow the instructions, and to interpret the results. The problems our students had with diagrams are also evident in the additions item (3), but we also have here (as in (4)) the weakness of the operation: the diagram was shaded in correctly by 29% of the students, but only 14% gave the fraction $\frac{5}{12}$. With item (4) 19% were successful, but 35% calculated $\frac{1}{6} + \frac{1}{3} = \frac{2}{9}$, and for 21% $\frac{1}{6} + \frac{1}{3} = \frac{1}{9}$ is true.

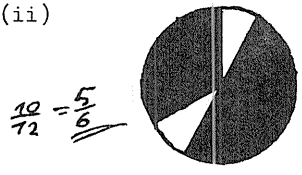
... mistakes of the students are

particular clear from their diagrams. The following figure shows four typical wrong solutions:

(i)



(ii)



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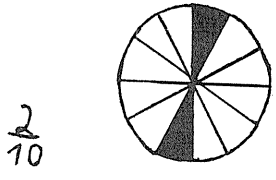
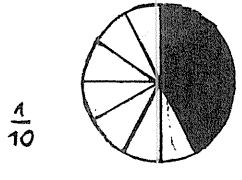


(iv)



Very frequently the students shaded in 4+6 twelfths of the circle instead of $\frac{1}{4} + \frac{1}{6}$ (see (i) and (ii)). Here it is clear that these students are uncertain even of the concept 'fraction'. On the other hand there are students who gave as an answer the fractions $\frac{1}{10}$ or $\frac{2}{10}$ and wrote them beside either no diagram at all, or a wrong one, or even the right one (see (iii) and (iv)). These students apparently preferred to do a calculation; they used a "rule": For some students $\frac{1}{4} + \frac{1}{6} = \frac{1}{10}$ is valid, because $4 + 6 = 10$; and some others calculate $\frac{1}{4} + \frac{1}{6} = \frac{1+1}{4+6}$. Especially remarkable is (iv), as this solution $\frac{1}{10}$ (presumably calculated) was written next to the correct diagram. But we learned from informal interviews with the students that many of them are not unduly concerned at getting different answers from using different methods of solving the problem (as here by calculation and by a diagram). Similar observations were made by Ginsburg (1977) regarding young children's arithmetic.

For the second step of our study we asked ourselves the question

if, and if so how, it is possible to get the students to verify their steps towards solution, or at least to check if their solution is plausible. 'Plausible' means here that the result of the calculation is compatible with the result one would expect in the situation given in the problem. Though if the students are already uncertain in the handling of the concepts they use, they can hardly be in a position to check their results. With e.g. item (3) in (i), the result could seem quite plausible to the student. On the other hand, in (ii) 4 + 6 twelfths are shaded in, but the student has noticed that he has now shaded in $\frac{10}{12} = \frac{5}{6}$ of the circle. Why does he not see his mistake?

In our second test we set word problems on the topic of ratio. In these items the situations described are unambiguous and, because they are taken from ordinary life, thoroughly familiar. With some items we gave diagrams as well, as aids for their solution. The test was a written one, taken by about 60 students, and we also took several informal interviews. The idea with these problems is to stimulate the students to examine their working by additional questions (control-questions), without, however, interfering in the problem-solving process.

The control-questions which are set can be divided into two groups:

A. Questions which relate to a <u>comparison</u> between methods of solving the problem. These are items for which more than one method can be used to get the correct result. The students are asked to explain why (or why not) by different methods they obtain the same result.

Examples:

(5) a) 7 litres of milk cost 10.50 DM. How much milk does one get for 6 DM?

b) In the following arrows diagram the arrangement of the sizes is proportional. Complete the diagram:

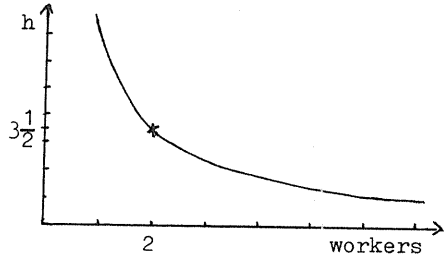
1 l ⟶ 1.50 DM
2 l ⟶ 3.00 DM
3 l ⟶
4 l ⟶
5 l ⟶
6 l ⟶
7 l ⟶

c) Compare your result from a) with the value you can read

in the arrows diagram. Did you get the same result in each case? ☐ yes ☐ no
Why are the results (not) the same?

(6) a) To shovel a heap of sand onto a lorry 2 workers take $3\frac{1}{2}$ hours. How many hours would 7 workers need?

b) See if you can read from the diagram how long 7 workers need to load the sand.



Are both your results the same? ☐ yes ☐ no
Why (not)?

B. Questions which should stimulate the students to formulate an assumption about the solution to the problem, or even to consider whether one is able to solve the problem with the information given. This means that the student must first find the structure of the problem and should then form a strategy for its solution.

Examples:

(7) A bus is hired for an excursion. The driver charges 108.00 DM for a journey of 180 km. For 120.00 DM will the journey be longer or shorter?
☐ longer ☐ shorter
Prove the answer you expect by a calculation.
Are you sure that the answer is correct? ☐ yes ☐ no
Can you check whether your answer really is correct?
☐ yes ☐ no, because ...

(8) 3 bars of chocolate cost 3.00 DM,
5 bars of chocolate cost 4.70 DM,
9 bars of chocolate cost 8.10 DM.
Can you calculate how many bars of chocolate you would get for 6.60 DM? ☐ yes ☐ no Calculation:
Are you sure that the answer is right?
☐ yes ☐ no Why?

From the analysis of the results from these (and similar) pro-

we can draw the following conclusions:
To A: The comparison of two differnt methods of solution for the same problem gives no advantage (at least not for the poorer students), because they usually - as we saw above in the items on fractions - interpret the different methods of solution as separate exercises. Indeed there are students who say that they got the same answer for each of a pair of items (even when that is not so, e.g. with item (5), a) and b)!), but almost always they give as the reason "because I calculated correctly". Most students do not see the significance of the diagram in item (6).

To B: Here the situation is differnt. If the students have reached a conclusion by serious consideration as to the result to be expected (which can be determined by the interviews) then this assumption is very often strong enough to lead them to recognize from an incorrect calculation that the result is wrong. Though indeed this recognition only seldom helps in finding the correct road to solution.

As we have already found, our students use, if it is at all possible, algorithms and rules which they have learnt; and they are certain that they will get the right solution for their problem (even when the algorithm they have used is quite unsuitable for that problem). Most of them are also not in a position to explain why a rule is so and not otherwise, i.e. their understanding is at most instrumental and not relational (cf. Skemp, 1979; Hasemann,1981). For this reason they are not able to check their method of solution, even when they have learnt a test-algorithm: Many students maintain that their answer is correct "because I did the test-algorithm" - even when the result is incorrect. If, on the other hand, the students are required to check whether the result of their calculation is plausible, many of them recognize very well that they may have made a mistake; in any case when they are familiar with the situation given in the problem.

In other words, the rules and algorithms seem to play a too large part in the mathematical thinking of the poorer students. Separated from any plausibility, these rules are used for calculation and for testing results, whereby not even different answers in these complementary processes disturb the pupils.

On the other hand the examination of the results for plausibility, that is, the obtaining of realistic terms, meets with better success, and it remains to be discovered whether the control and test mechanisms of rules and algorithms can be developed to help the students.

References

Birkhan, G. Analyse von Denkfehlern in der Mathematikdidaktik als kognitionspsychologisches Untersuchungsinstrument. Komplexe menschlicher Informationsverarbeitung: Bern, Stuttgart, Wien, Huber, 1979.

Ginsburg, H. Children's Arithmetic: The Learning Process: New York, Van Nostrand, 1977.

Hart, K. Understanding of Fractions in the Secondary School. Proceedings of the Second International Conference for the Psychology of Mathematics Education, 1978, OSM, Reihe D, Band 1, Universität Osnabrück.

Hart, K. Towards a Hierarchy of Understanding Mathematics. Proceedings of the Third International Conference for the Psychology of Mathematics Education, 1979, Warwick University.

Hasemann, K. On the Difficulties with Fractions. Educational Studies in Mathematics, to appear in 1981.

Radatz, H. Fehleranalysen im Mathematikunterricht: Braunschweig, Vieweg, 1980.

Schlaak, G. Fehler im Rechenunterricht: Hannover, Schroedel, 1968.

Shulman, L.S. Strategic Sites for Research on Teaching. Paper presented at AERA Annual Meeting 1978, Toronto.

Skemp, R. Goals of Learning and Qualities of Understanding. Proceedings of the Third International Conference for the Psychology of Mathematics Education, 1979, Warwick University.

EFFECTS OF A PROCESS-ENRICHED APPROACH TO FRACTIONS*

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This study has been extracted from one phase of the Calgary Junior High School Mathematics Project (CJHMP) in which the authors explored the effects of using a process-enriched, student investigation approach to fractions with early adolescents. Two hundred forty-six experimental treatment students received 11 1/2 weeks of process-enriched teaching modelled on South Nottinghamshire Project strategies while two hundred thirty-four control students received regular textbook treatments. In the context of Grade Seven fractions, significant differences were found favouring the process-enriched classes on measures of achievement, attitude, and "general mathematical strategies."

INTRODUCTION

The reports of the National Assessment of Educational Progress (NAEP) in the United States have indicated that the computational skills of thirteen and seventeen-year olds are adequate but their problem solving skills are poor, especially when several steps are required to arrive at a solution or when non-routine applications are involved. There is a real danger that, under current "back-to-basics" emphases, skills of number and symbol manipulation may be stressed so much that the already serious inability of many students to successfully handle multi-step non-routine problems will be further intensified. Answer-getting through straight recall and the application of a one-step algorithm have been found to be the predominant modes for approaching problems at all of the age levels tested by the NAEP (Carpenter, et al, 1975). The use of memorized rules rather than reasoning in mathematics appears to be at the heart of the difficulty. Students who know facts and algorithms but have not learned to apply them in novel situations, eventually end up in a predicament like that of the fifteen-year-old who remarked that he can do any problem (of the kind he has been working on) if he is given a worked-out example, but on exams he frequently doesn't understand what a question is asking for (evidently neither really understanding the terminology nor having the basic concepts firmly anchored in real experience).

*A preliminary, partial reporting of findings from one phase of the Calgary Junior High School Mathematics Project funded jointly by Planning and Research, Alberta Education, and the Calgary Board of Education.

Why are student problem solving skills not more substantial at the secondary school level? Analyses of junior high school mathematics programs have indicated that little attention has been given to the development of general mathematical strategies such as drawing conclusions from mathematical investigations, making generalizations, and giving explanations and proofs (Bell, 1976, p.5.1). An underlying assumption in the planning of the CJHMP project was that improvement in problem solving is more likely to result from developing abilities to employ general strategies in mathematics by using a small number of concretely based investigations than from having pupils do a large number of routine textbook-type exercises and "problems." The experimental teaching methods used in the project have been modelled on the pioneering work of the South Nottinghamshire Project (SNP). The SNP teaching approaches characteristically use a variety of simple concrete materials to pose "well-motivated" problems in which the concrete "props" facilitate understanding the mathematical ideas involved (Bell, 1976, p.5.4). A pupil carries out a mathematical investigation beginning with concrete materials, experimenting, recording what happens, formulating questions, and writing-up accounts of experimental results as well as applying the results to practical situations. Such strategies are not generally embodied in North American school mathematics textbook materials nor do they occur with any frequency in the most common secondary school mathematics mode of instruction:"example--rule--exercises." Given that the need for learning general mathematical strategies is real, could North American mathematics teachers be led to successfully use process-enriched, student-investigation teaching approaches? What effects would a process-enriched teaching approach have on student achievement, attitudes and applications of general mathematical strategies?

THE PROBLEM

One of several phases of the Calgary Junior High School Mathematics Project, this study was designed to address the question:

> Contrasting the scores obtained by Grade Seven students who were taught a unit on fractions by either a process-enriched or a regular textbook approach, are there any significant differences on measures of achievement in fractions, anxiety, enjoyment, or "general mathematical strategies?"

PROCEDURES

The study focused on the Grade Seven unit on Fractions (Rational Numbers) outlined by the Alberta Junior High School Mathematics Curriculum Guide.

There were 246 experimental treatment students in ten Calgary Board of Education classes taught by six teachers, and 236 regular treatment student in nine classes taught by four teachers. The two groups of students represented a stratified sample of Grade Seven (12 to 13-year-old) students matched according to Stanford Achievement Test mathematics scores.

The experimental teaching unit was assembled from resource materials that fit an investigative, process-oriented approach to the teaching and learning of fractions. Using these resource materials, the experimental teachers edited, modified, selected, organized and designed a set of investigations and exercises to cover all of the Fractions topics for Grade Seven. Strategies and materials were drawn mainly from Journey into Maths, TG1 and TG2 (Bell, Wigley, and Rooke, 1978, 1979), Fractions, An Operational Approach (Dienes, 1967), "Constructive Rational Number Tasks" (Kieren, 1979), and School Mathematics Project, Books A and D (SMP, 1971, 1970).

By using practical materials as a springboard for initiating mathematical investigations which encourage drawing conclusions and making abstractions and generalizations, the experimental treatment was designed to provide learning contexts appropriate to every level of aptitude in a mixed ability class.

Prior to the teaching of the Fraction units the following battery of pre-tests was administered to the experimental and regular classes.

1. CSMS "Fractions, Year 1 and 2" test
   - a slightly Canadianized version
   - two parts: problems (word, diagrammatic and computational) and parallel computation exercises
   - for item analyses and discussion, see Hart (1978, pp. 177-183).

2. Brindley's"Attitude towards Fractions" scale
   - modelled after an "Attitude towards Mathematics" scale (Aiken, 1973, pp. 67-71)
   - measures enjoyment and anxiety level towards fractions

3. Bell's "General Mathematical Strategies" test
   - measures ability to give explanations, to follow instructions, to recognize and express relationships, and to generate examples (meeting criteria, finding complete sets, ...) (Bell, 1976, p.A5.13).

4. Kieren's "Fractional Number Thinking Test"
   - constructed for the CJHMP

- assesses Piagetian cognitive level in the context of fractions.

The experimental teachers taught the experimental units to one or more of their classes as part of their regular teaching load for approximately eleven and a half weeks (four of those weeks treated ratios). For roughly the same period of time, the regular teachers taught the same topics in the usual manner following one of the currently authorized textbooks.

While the fraction units were being taught, regular classroom observations were made and the students filled in a questionnaire on their perceptions of the use of materials, student investigations, and various teacher presentations. The preceding information will be used to document differences between the two instructional treatments.

The "CSMS Fractions" test, the "Attitude towards Fractions" scale and the "General Mathematical Strategies" test were also given as post-tests following the completion of the units. On the average approximately 20% of the students were unable to write both pre-and post-tests for any given test instrument.

One-way repeated measures analyses of variance were used on the data to test null hypotheses generated from the question stated under The Problem.

Subsequent to the submission of this paper, it is planned to pursue two-way repeated measures analyses of variance to determine if there were any differential treatment effects among students at concrete, transitional or formal ability levels (in the context of fractions) as classified by Kieren's "Fractional Number Thinking Test."

FINDINGS

As recorded in Table I, statistically significant post-treatment mean differences (probability <0.05), favouring the experimental group, were found in the scores from the following tests: CSMS Fractions (Total), CSMS Fractions (Problems with Facility <0.50), Anxiety towards Fractions, Enjoyment of Fractions, General Mathematical Strategies (Total), and Giving Explanations.

TABLE I

Experimental versus Regular Mean Contrasts with Probability Levels from Repeated Measures Analyses of Variance on CSMS Fractions, Fraction Attitude, and General Mathematical Strategies Scores (Grade Seven)

Test or Subtest		Max. Score	Experimental Mean Score	Regular Mean Score	Probability (Evs R)
Number of Students			N≈207	N≈179	
CSMS Fractions	Pre	84	35.85 (43%)	34.75 (41%)	
	Post	84	51.57 (61%)	47.76 (57%)	<0.01
Problems (Facility <0.50)	Pre	46	9.95 (21%)	9.35 (20%)	
	Post	46	16.40 (36%)	14.39 (31%)	0.01
Computation	Pre	12	3.15 (26%)	2.43 (20%)	
	Post	12	6.68 (56%)	5.68 (47%)	0.34
Attitude to Fractions Anxiety	Pre	25	14.13	14.38	
	Post	25	14.50	13.81	0.048
Enjoyment	Pre	25	17.60	17.85	
	Post	25	18.92	17.91	<0.01
General Math Strategies	Pre	43	11.33 (26%)	12.59 (29%)	
	Post	43	15.43 (36%)	15.42 (36%)	0.03
Giving Explanation	Pre	12	2.31 (19%)	2.88 (24%)	
	Post	12	2.64 (22%)	2.66 (22%)	<0.01
Following Instructions	Pre	6	1.68 (28%)	1.94 (32%)	
	Post	6	2.44 (41%)	2.48 (41%)	0.25
Recognizing Relationships	Pre	10	2.59 (26%)	3.56 (36%)	
	Post	10	4.33 (43%)	4.84 (48%)	0.08
Generating Examples	Pre	15	4.75 (32%)	4.21 (28%)	
	Post	15	6.02 (40%)	5.43 (36%)	0.89

CONCLUSIONS

On the total CSMS Fractions test and the Problems with Facility <0.50 subtest, the modest but statistically significant improvement in the performance of the experimental group, as compared with the regular, was accompanied by no significant difference in computational facility. Apparently, the computational skill of the experimental group did not suffer in spite of teacher-perceived shortages of "practice time." Gains in student ability to reason through the somewhat demanding CSMS Fractions items occurred in spite of the prior unfamiliarity of students and teachers with process-oriented approaches to mathematics teaching and learning, with the inevitable attendant adjustment difficulties.

The regular group demonstrated increased anxiety towards fractions over the treatment period while the experimental group's anxiety decreased (the higher the score the less the anxiety). Significantly greater enjoyment of fractions was reported by the experimental group as contrasted with the regular group.

Although the post-test "General Mathematical Strategies" and "Giving Explanations" mean scores did not differ very much, the experimental group improvements were significantly better. In fact, the regular groups' demonstrated ability to give explanations dropped. The improvement in experimental group ability to give explanations might be attributable to the experiences gained in providing "write-ups" for the investigations they carried out. The "write-ups" were found to be difficult and were generally disliked by the students, probably because of lack of prior experience, but an improvement in explanations does appear to have been achieved.

None of the other General Mathematical Strategies subtest evidenced significant mean differences. In retrospect, they were not as heavily emphasized in the experimental treatment as "giving explanations" was.

Though not reported here, the classroom observation data and Student Questionnaire responses did demonstrate that the experimental teachers were using a teaching method tangibly different from that of the regular teachers. Furthermore, the achievement, attitude, and general mathematical strategies results reported here tend to confirm that something different was happening in the experimental classes.

Relatively favourable improvements in experimental group achievement in fractions, anxiety towards fractions, enjoyment of fractions, application of general mathematical strategies and ability to give explanations, have indicated that a process-enriched, student investigation approach like that of the experimental treatment in this study could be inserted in a regular curriculum to good effect, particularly in the area of encouraging the development of higher mental processes and of more positive attitudes towards mathematical topics.

REFERENCES

Aiken, L. R. Two Scales of Attitude Towards Mathematics. Journal for Research in Mathematics Education. Vol. 5, No. 2, March 1974, 67-71.

Bell, A. W. A Developmental Study of Process Attainment in Mathematics Including the Construction and Investigation of a Process-Oriented Curriculum for the First Secondary Year. Doctoral dissertation, University of Nottingham, 1976.

Bell, Alan, Alan Wigley and David Rooke. Journey into Maths, Teacher's Guides 1 and 2 (The South Nottinghamshire Project). Bishopbriggs, Glasgow: Blackie and Son, 1978, 1979.

Carpenter, Thomas P., et al. Results and Implications of the NAEP Mathematics Assessment: Secondary School. The Mathematics Teacher, 1975, Oct., 453-470.

Dienes, Z. P. Franctions, An Operational Approach. Harlow, Essex: Educational Supply Association, 1967.

Hart, K. The Understanding of Fractions in the Secondary School. Proceedings of the Second International Conference for the Psychology of Mathematics Education, Osnabruck, 1978, 177 to 183.

Kieren, T. E. Constructive Rational Number Tasks. Delta-K, Mathematics Council of the ATA, Edmonton, Alberta, March, May, September, 1979.

The School Mathematics Project. Teacher's Guide for Book A (Metric) ... Book D Cambridge : Cambridge University Press, 1971 ... 1970.

LOGICAL THINKING: IRRELEVANT ATTRIBUTES AND NEGATIVE INSTANCES

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In conjunctive attribute identification tasks, sequences of all positive instances are thought to be favored over mixed sequences of positive and negative instance in laboratory tasks, but in mathematical conjunctive attribute identification tasks, mixed sequences are favored over all positive instances. The subjects were 100 ninth grade algebra students randomly assigned to one of four treatments formed by crossing sequence condition (+ or ±) with frequency of level of irrelevant attributes (.5 or .9). Results support the classical favoring of + over ± for .5 frequency but ± over + for .9 frequency. Prevalent levels of irrelevant attributes are shown to be a critical factor in the effects found for negative instances.

Reviews of research on attribute learning in psychology continue to support sequences of all positive instances over sequences of a mixture of positive and negative instances for the conjunctive rule (Bourne & Dominowski, 1972; Erickson and Jones, 1978). Analysis of frequency differences among dimensions among the positive and negative classes appears to have some promise for explaining differential results for positive and negative instances (Erickson and Jones, 1978). Several studies give evidence for the explanatory power of frequency analysis (Bourne, et al., 1976).

Research with conjunctive concepts in mathematics, in apparent contradiction to the research in psychology, has consistently favored sequences of positive and negative instances over sequences of all positive instances (Shumway, 1971, 1974, 1977). Sowder (1980) has identified the study of negation and negative instances as a critical problem for research in mathematical concept and principle learning.

Paralleling the work of Bourne, et al., (1976), Shumway and White (1977) have also begun examining the potential of frequency analysis to explain the relative roles of positive and negative instances in attribute learning. The present study documents the differential effects of high frequencies (.9) of levels of irrelevant attributes versus equal frequencies (.5) across both positive and negative instances. Bourne, et al., (1976), has manipulated the frequencies of levels of *relevant* attributes among the positive instances and the negative instances. Careful study of mathematical concepts has lead

Shumway and White (1977) to manipulate the frequencies of levels of *irrelevant* attributes across positive and negative instances.

Method.

Subjects. The subjects were 100 ninth grade students enrolled in Algebra I in an upper middle class suburban school during May, 1980. The treatments were administered by the school mathematics department chairman during the mathematics classes of other teachers as a special study on logical thinking. Participation was voluntary and all students in each class participated.

Experimental design. The design was a 2x2 factorial with two sequence conditions (all positive, or mixed positive and negative instances) and two task conditions (.5, or .9 frequency of a level of each irrelevant dimension). The stimulus population consisted of 32 five-letter strings varying in five dimensions of two levels each.

Table 1
Sample Distribution
Frequency

Sample Condition		Sequence	n=	Sequence	n=
	+	1	5	6	5
		2	5	7	5
		3	5	8	5
		4	5	9	5
		5	5	10	5
	±	11	5	16	5
		12	5	17	5
		13	5	18	5
		14	5	19	5
		15	5	20	5

There were five different sequences in each cell and five subjects were given each sequence. In the all positive sequence (+), positive instances were chosen at random, subject to the condition that either both levels of the irrelevant dimensions were equally probable (.5) or that one level of each of the irrelevant dimensions appeared at the probability level of .9. The mixed sequences (±) contained the same first twenty positive instance as the corresponding all positive sequences (1 corresponds with 11, 6 with 16, etc.).

Each positive instance was followed by a randomly generated negative instance to produce the (+) sequences. The mixed sequences were also subject to the condition that either both levels of the irrelevant dimension were equally probable (.5) or that one level of each of the irrelevant dimension appeared at the probability level of .9.

Table 2

Probability of Occurrences of Values of Each Dimension in Positive and Negative Categories

	Dimension and Levels									
	1		2		3		4		5	
Condition and Category	B	C	D	F	G	H	J	K	L	M
Positive .5 Condition	.50	.50	1.00	.00	.50	.50	.00	1.00	.50	.50
Negative	.50	.50	.33	.67	.50	.50	.67	.33	.50	.50
Positive .9 Condition	.10	.90	1.00	.00	.90	.10	.00	1.00	.10	.90
Negative	.10	.90	.33	.67	.90	.10	.67	.33	.10	.90

The stimulus population consisted of 32 letter strings varying in five dimensions of two levels each. The levels of each dimension were B or C, D or F, G or H, J or K, and L or M. Each stimulus was a letter string of five letters. Subjects were asked to identify the instance as positive or negative, were given correct feedback, and then guessed the values which they believed to be the correct values of the relevant attributes. Subjects were told the concept consisted of two letters always present at the same time (conjunction). The concept was D and K. Of the total population of 32 stimuli, 8 are positive and 24 are negative. Subjects were trained with two practice tasks, one of three dimensions and one of four dimensions. A pretest consisting of a five-dimension task was given to determine subjects ability to use information from negative instances as well as positive instances (Solution for the pretest could be obtained in 4 instances using the information from all instances or in 11 instances using only the information contained in the positives). During training, pretest, and task there were six instances per page. Subjects were thus able to look back to as many as five previous instances. Subjects were

not allowed to look back at previous pages. All subjects completed 40 trials (50 trials for sequence #8. Correct solution required 40 trials, 10 extra were added to allow checking). The criterion variable was the number of trials until solution was obtained. Instances were presented as follows:

I think the two letters are:

C D H K L + or -

\+ ____ and ____

Subjects used a card to cover subsequent instances and worked their way down each page, line by line at their own rate.

Results and Discussion

The following table gives the means and standard deviations for trials to solution for each cell on the pretest.

Table 3

Means and Standard Deviations for Pretest Scores by Cell

Frequency

Sequence Condition		.5		.9	
+	M	5.8	M	6.2	
	SD	3.9	SD	3.9	
		n=25		n=25	
±	M	4.5	M	6.8	
	SD	3.5	SD	4.1	
		n=25		n=25	

Because of near significance in the margin comparison between .5 and .9 ($F(1,96) = 3.03$, $p < .085$), analysis of covariance was chosen for the analysis.

Mean trials to criterion, standard deviations and adjusted means were as follows:

Table 4

Means, Standard Deviations and Adjusted Means for Criterion Scores by Cell

Frequency

Sequence Condition		.5		.9	
+	M	4.04		M	14.40
	SD	2.15		SD	12.97
	AM	4.06		AM	14.22
		n=25			n=25
±	M	6.44		M	9.36
	SD	6.09		SD	8.07
	AM	7.17		AM	8.79
		n=25			n=25

The test of nonequality of regression was not significant ($F(3,92) = 1.97$ $p < .124$), the regression was significant ($F(1,95) = 6.89$, $p < .01$), sequence condition (+,±) interacted with frequency (.5,.9) ($F(1,95) = 6.92$, $p < .01$), and the .9 frequency tasks were more difficult than the .5 frequency tasks ($F(1,95) = 12.97$, $p < .001$). Results without the use of covariance were similar.

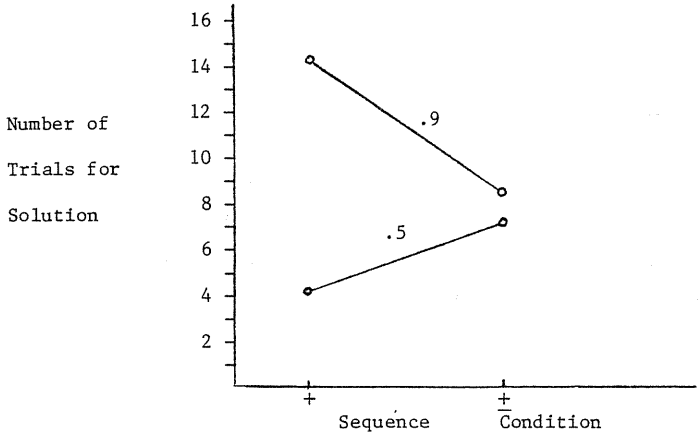

Figure 1

Figure 1 is a plot of the adjusted means for the four cells of the design. The lower line of the graph confirms the usual psychological result for conjunctive attribute learning; namely, sequences of all positive instances are favored over sequences of positive and negative instances. The upper line of the graph shows tasks with high frequency (.9) levels of irrelevant attributes are more difficult and the presence of high frequency (.9) levels of irrelevant attributes confirms the mathematical concept learning task result; namely, sequences of positive and negative instances are favored over sequences of all positive instances. It seems to be clear the effect of negative instances is dramatically affected by the frequency of levels of irrelevant attributes.

The reason for these results seems clear. For example, in the .9 condition, the letter C appears 90% at the time in both the positive and negative instances. Given only positives, it would take many examples before the letter C would change to B and thus allow C to be eliminated (on the average, 9 times out of 10, the letter would be C rather than B). On the other hand, the appearance of C in a negative instance (on the average, 9 times out of 10) would allow hypotheses involving C to be eliminated quickly. Clearly prevalent levels of irrelevant attributes have a profound effect on the role negative instances play in conjunctive attribute identification tasks. We believe mathematics concepts often have prevalent levels of irrelevant attributes present, and are attractive to students because of their frequency. Knowledge of the importance of irrelevant attributes should 1) aid in the applicability of psychological research to mathematical concept learning, 2) clarify the importance of irrelevant attributes and negative instances for research on mathematical concept learning, and 3) suggest attention to irrelevant attributes and negative instances may facilitate the thoughtful teaching of mathematics.

References

Bourne, L. E. Jr. & Dominowski, R. L. Thinking. Review of Psychology (Vol. 23) Palo Alto, Calif.: Annual Reviews, 1972, 105-130.

Bourne, L. E. Jr., Ekstrand, B. R., Lovallo, W. R., Kellogg, R. T., Hiew, C. C. & Yaronsh, R. A. Frequency Analysis of Attribute Identification. Journal of Experimental Psychology: General, 1976, 105, 294-312.

Erickson, J. R. & Jones, M. R. Thinking. Annual Review of Psychology, (Vol. 29), Palo Alto, Calif.: Annual Reviews, 1978, 61-90.

Shumway, R. J. Negative instances and mathematical concept formation: A preliminary study. Journal for Research in Mathematics Education, 1971, 2, 218-227.

Shumway, R. J. Negative instances in mathematical concept acquisition: Transfer effects between the concepts of commutativity and associativity. Journal for Research in Mathematics Education, 1974, 5, 197-211.

Shumway, R. J. Positive versus positive and negative instances in the acquisition of the concepts of distributivity and homomorphism. Journal of Structural Learning, 1977, 4, 331-348.

Shumway, R. J. & White, A. L. Prevalent levels of irrelevant attributes: A potentially important variable in mathematical concept learning. Unpublished manuscript, Ohio State University, 1977.

Sowder, L. Concept and Principle Learning. In R. J. Shumway (Ed.) Research in Mathematics Education, professional reference series, National Council of Teachers of Mathematics, Reston, Va: 1980, 244-285.

NUMBER SENSE AND COMPUTATIONAL SKILLS

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Abstract

We will first describe our model of understanding (instrumental, relational, communicable) which is close to the ideas of Skemp. Communicable understanding includes "prematheматical" activities (see Semadeni) and "logical" understanding (see Skemp). Examples show that many skills in arithmetic can describe either instrumental or communicable understanding (instrumental skills or higher skills). There are two different methods to develop higher skills, the reflective way and the intuitive way.

Part 2 gives a list of various aspects how the hand-held calculator might be used in arithmetic teaching. In part 3 we discuss the possible impact of the calculator on arithmetic skills and abilities. Since the calculator works on a syntactical level we will distinguish digital and analogous data. According to this distinction we have a syntactical number sense (instrumental skills) and a semantical number sense (relational understanding). There are possibilities to develop both by using calculators via the intuitive way of teaching.

Part 4 describes some investigations to back these theoretical considerations. Three investigations were done only for a short period each in one class (grade 3 and grade 4, age 9-10) with the findings that an increase in mental arithmetic is possible by using calculators appropriately (instrumental skills). The fourth investigation is a one-year-study with about 1000 students (grade 3) where the calculator is only used to play three different calculator games. First findings show an increase of semantical number sense (higher skills). A fifth study tries to find out how the students develop guess-and-test strategies.

1. Levels of understanding and related goals

According to the definitions of Skemp we distinguish three levels of understanding. Regarding arithmetic a child has instrumental understanding, if she or he can apply rules or give correct stimulus-response answers without knowing a proof for these skills.

concept or a model about that doing: She or he is convinced of the correctness or the mathematical reasons for that doing. Skemp calls his third level logical understanding. Since we see the difference to relational understanding in "being able to convince other people" we will prefer to say communicable understanding or active or describable or demonstratable understanding. On a higher stage of ability communicable understanding includes Skemp's "logical understanding". But on a lower stage it includes also "premathematical" activities (see Kirsch, 1979; Semadeni).

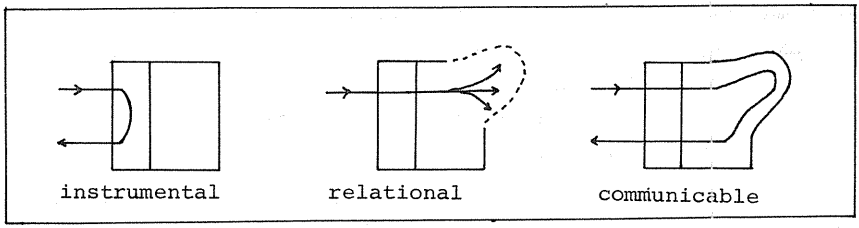

Fig. 1. Three levels of understanding

The related goals of teaching arithmetic are

producing skills (counting, computing, handling algorithms,...)

building up concepts (number line, basic operations,...)

using these concepts (reasoning, applying, argueing, solving word problems,...)

Skills are easy to check or to test. It is also possible to check if a child is able to use a concept or not. But it is not easy to find out if a child has relational understanding. Usually we test communicable understanding. Mistakes or failures herein only say that there is a lack of being able to communicate or a lack of relational understanding. It is easier to test computational skills than to test if a child has a good number sense.

We define number sense as the relational understanding of the "concept of numbers and their relations" in the context of the order of magnitude and the four basic operations. To have a good number sense means to have a good relational understanding of the structures $\mathbb{N}(\leq,+,\cdot)$, $\mathbb{Z}(\leq,+,\cdot)$, $\mathbb{D}(\leq,+,\cdot)$, or $\mathbb{Q}(\leq,+,\cdot)$ ($\mathbb{N}$ for positive integers, $\mathbb{Z}$ for integers, $\mathbb{D}$ for decimals, $\mathbb{Q}$ for rational numbers. We do not consider real numbers.)

Most of the US publishers offer a catalogue "scope and sequence of instructional objectives" for their school books. But many of these catalogues are only a listing of skills. To teach number

list of goals which distinguishes between (instrumental) skills, concepts, abilities, and (higher) skills. We will give some examples.

1.1 Instrumental skills

These skills are possible without "thinking". Only instrumental understanding is necessary. We learn these skills by copying them from other people, in making mistakes and getting immediate feed back, by stimulus-response activities, etc. A lot of training is necessary. Examples:

Counting (learning the number poem, combining rythms with the poem, telling the poem and pointing to the elements of a set,...),

Grouping and regrouping,

Doing the four basic operations with easy numbers (on a certain level, learning them by stimulus response activities),

Addition and multiplication tables (taught mainly by stimulus-response activities),

Paper-and-pencil algorithms for the four basic operations (instrumental skills at least for weak students. The kind of teaching has no big effect on the level of that skill),

Carrying and borrowing (for weak students),

Rounding and approximations,

Computing with fractions (for many students).

1.2 Concepts

In our opinion it is not possible to describe a concept. Having a concept means having the right intuition at the right moment. We see a concept as a very complex network. We can mention different aspects or give a list of important facts or describe parts of a context. We will list some partial aspects of the "concept of numbers and their relations":

Aspects of using positive integers (cardinals, ordinal numbers, counting numbers, numbers as operators, numbers as a sequence of digits for syntactical use or for applying algorithms, pressing calculator buttons, numbers for measuring, numbers for coding like phone or street numbers,...)

Decimal system (counting, grouping, regrouping, calling or writing numbers, measuring, order of magnitude, computing, rounding,...)

Aspects of using fractions (parts of a unit, measuring with parts, fractions as operators, proportions, percentages, decimal notation, fractions as figures of digits and symbols for syntactical use or for applying algorithms,...)

Aspects of ordering numbers (different activities according to different aspects of using numbers, number line, decimal

Aspects of multiplication see fig. 2.

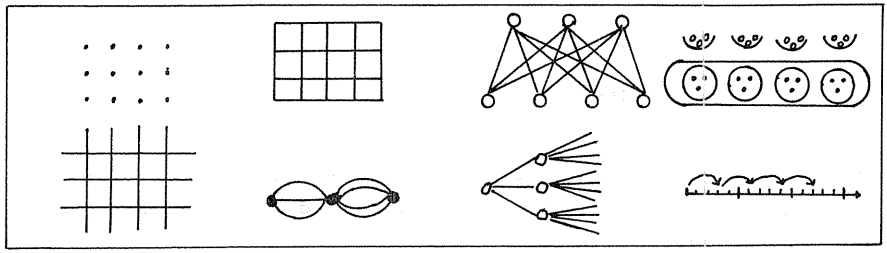

Fig. 2. Aspects of multiplication

1.3 Abilities and higher skills

We consider activities relating to a communicable understanding. Abilities are activities using consciously relational understanding. Higher skills are unconsciously coordinated by relational understanding or are based on the intuition of a concept. Abilities run reflectively and higher skills run intuitively. Instrumental skills and higher skills both run without reflection like stimulus-response reactions (see fig. 3). The difference is the preceding development of a relational under-

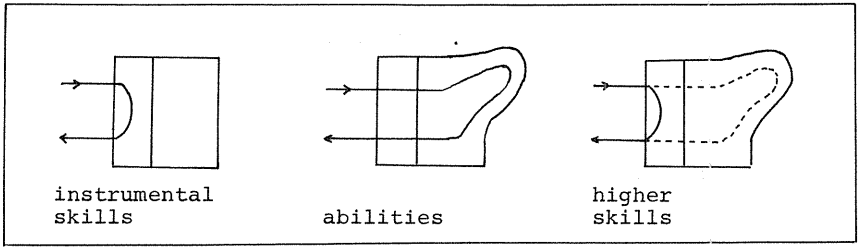

Fig. 3. Abilities and skills

standing. We will give some examples for abilities or higher skills:

Carrying and borrowing,
Doing paper-and-pencil algorithms for the four basic operations,
Doing mental arithmetic,
Computing with fractions,
Estimation,
Working on word problems.

We see two different methods to develop higher skills (see fig. 4.):

(a) The reflective way: Teaching concepts leads to abilities

(b) The intuitive way: Using instrumental skills continuously develops bits of relational understanding which leads to higher skills.

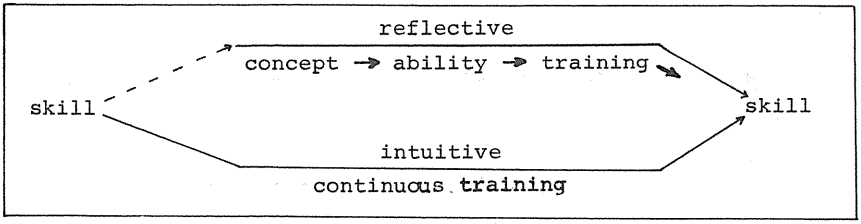

Fig. 4. Concepts for teaching arithmetic

My feeling is that most of the teachers more emphasize (a) than (b), but that children learn most of the arithmetic more via (b) than (a). (Of course there are also teachers who do not teach (a) or (b) at all. They just train instrumental skills.)

2. Calculators and teaching arithmetic

There are some general concepts of teaching arithmetic like traditional math, new math, or back-to-the-basics. All these concepts do not include the electronical hand-held calculator for early grades. Many mathematicians, teachers and parents condemn the calculator as a tool in mathematics education for young children. But these children have access to the calculator and they use it. To solve that conflict some calculator projects in the USA, Great Britain, France, Germany, Sweden and in some other countries try to incorporate the calculator into the curriculum. Analysing the role of the calculator in these projects we find various aspects (Meissner, 1978):

Relief (The calculator relieves the students from mental arithmetic and paper-and-pencil algorithms. They only have to press calculator keys and to read the calculator display.)

Motivation (Using calculators brings fun.)

Checking (After having done mental arithmetic or paper-and-pencil algorithms the calculator is used to check the results.)

Games (The calculator is the tool to play games. Examples: The Big One, Hit the Target,...)

Matches (Who is quicker: Mental arithmetic versus calculator.)

Drill-and-practice (Hide the calculator display, press the buttons, say the result, and look if you are right.)

Operator (Mechanical operating (+4)- or (x3)-machines do not exist. Stretchers and shrinkers as models cannot compute. The calculator solves these problems.)

input.)
Concept building (The calculator is an almost perfect model of the "concept of numbers and their relations". Playing with the calculator allows to discover aspects of that concept.)
Crutch (Handicapped only may have success by using a calculator.)

3. The calculator impact on arithmetic goals

The calculator works on a syntactical level to process sequences of symbols (digits). Using the calculator is to stress the dualism "digital - analogous". Mathematical algorithms and the calculator only process digital data. But real-life-data are analogous. Decimal notation and place value express both digital and analogous data. Therefore the meaning of the data and the interpretation of input and output will become still more important.

According to the dualism "digital - analogous" we distinguish between approximation and estimation. Approximation needs mental arithmetic, number sense, and computational skills. Estimation needs knowledge about the environment and experiences with quantities. Both approximation and estimation are necessary to control the results of a problem. (For example we would rather encourage to "work metric" than to "think metric", since "think metric" is also possible restricted to the digital mode.)

According to the dualism "digital - analogous" we distinguish between syntactical and semantical number sense. Syntactical number sense is immediate knowledge of facts without any interpretation or discussion of the meaning. After the stimulus (problem) the response (result) follows immediately (without any computation). The calculator helps to train syntactical number sense. Semantical number sense needs an interpretation of the data as quantities, knowledge about these quantities, and immediate comparison. Using the calculator in the guess-and-test mode helps to train a feeling for the order of magnitude of numbers, which is a certain aspect of the semantical number sense.

Expanding these ideas we consider the calculator as a function machine to develop a "function sense". The main idea is to work in the guess-and-test mode: The wrong output immediately gives new hints for the next (better) input (see Meissner, 1980). Via stimulus-response the student gets an immediate feed back to analyse the structure of a whole set of data (relations, functions) or to develop understanding of mathematical concepts, which depend on the view of a whole set of data (approximation, continuous limit,

boundary, integral,...).

Summarizing the advantages of using calculators in teaching arithmetic we can say that the calculator helps to train instrumental skills (learning basic computational facts via drill-and-practice). Also by stimulus-response it builds up a syntactical number sense as part of the concept of numbers and their relations and a function sense as part of a semantical number sense.

Summarizing disadvantages of the calculator we must say that the instrumental skill "computing" will decrease. Due to many observations children accept every result from the calculator display without reflection, the ability of having a feeling for the right ball park will decrease.

4. Research

To back our theoretical considerations we have done some investigations. Three of them discuss the development of instrumental skills (mental arithmetic on a stimulus-response level). The fourth investigation tries to find out if we can also develop higher skills with the calculator (aspects of semantical number sense). A fifth investigation tries to explore the process of developing guess-and-test strategies: We try to show that the intuitive way of teaching (see (b) in 1.3) automatically develops bits of relational understanding (function sense).

The main part of the presentation in Berkeley will be dedicated to describe these investigations.

5. References

Kirsch, A. Beispiele für "Prämathematische" Beweise. In Beweisen im Mathematikunterricht, Hölder-Pichler-Tempsky, Wien 1979, 261-274

Meissner, H. Project TIM 5/12 - Taschenrechner im Mathematikunterricht für 5- bis 12-Jährige. In Zentralblatt für Didaktik der Mathematik, 1978, 10, 221-229

Meissner, H. Problem Solving with the One Way Principle. In Proceedings of the Third International Conference for the Psychology of Mathematics Education, 1980, 157-159

Semadeni, Z. The Concept of Prematematics as a Theoretical Background for Primary Mathematics Teaching. Polish Academy of Sciences, Warsaw (preprint)

Skemp, R.R. Goals of Learning and Qualities of Understanding. In Proceedings of the Third International Conference for the Psychology of Mathematics Education, 1980, 197-202

TOWARD A "FINE STRUCTURE" OF EDUCATIONAL THEORY: THE USE OF COMPUTERS TO TEACH MATHEMATICS[1]

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Some mathematics CAI courseware prepared for the PLATO computer-assisted instruction system is presented as an example of design work based on "micro" analysis of cognitive tasks. Several specific examples are given, indicating the use of "cognitive science" conceptualizations as a foundation for the analysis. Some evaluation data is also presented.

I. *Introduction.* In one (obvious) sense, this report deals with the use of computers to teach mathematics, but in a more profound sense it is part of an extended program, undertaken by the Madison Project from 1957 to the present time, to build toward a "fine structure" of relevant educational theory. What is meant by "fine structure" is a focus on events that are short in time and/or highly specific in nature, and for which explanations are sought in terms of, perhaps, a single information-processing act, often by a single human being. The precise "size" of such an event can vary considerably, but is always far smaller than the aggregations of many stimuli, many responses, and many human subjects that are commonly studied in educational R and D. As our main example to illustrate "fine structure" analysis, we shall use the computer assisted instruction ("CAI") courseware for the PLATO computer system, created at the Computer-Based Education Research Laboratory (CERL) at the University of Illinois, by a team of Lesson Designers and computer programmers under the direction of the author. The specific example of this courseware is needed to give credibility to our general claims, but it is the general claims themselves to which we wish to call primary attention.

Very briefly, the courseware is designed for use in grades 3 through 7 (chronological student ages of 8 years old to 12 years old), but extensive portions of it can be used outside this age range, especially for older children. The material was designed for use by computer terminals in the corner of the regular classroom (4 terminals per classroom of 28 students seems optimal), with each child scheduled for half an hour at a terminal each school day, plus whatever outside-of-class time a student desires and can

arrange (and for many students "outside-of-class" time is very extensive; some students arrange to get *three times* as much time on the computer *outside* of class as they do during class; This is possible because there are terminals all over town: in homes, offices, and in special "terminal rooms" that are open 24 hours a day, 7 days a week. All terminals (approximately 1,400 of them) are connected to the same central computer, which recognizes a student, and continues his planned course of study, no matter which terminal he uses to access the computer, or what time of day or night it may be). Further (and more detailed) descriptions of PLATO are given in Davis [1974]; Davis [1980A]; Smith and Sherwood [1976]; Slottow et al. [1977]; and Swinton, Amarel, and Morgan [1978].

II. *The Importance of "Fine Structure" or Microscopic Details.* Very surprisingly, the question most often asked about computer systems seems to be: "can computers teach effectively"? This question is cast in a strikingly inappropriate "aggregated" or "macroscopic" form: it ignores what kind of computer, what content is to be learned, which aspects of the content are to be stressed, what kinds of courseware, what human or other resources are available, and so on. It ignores the detailed information processing that is presumed to occur. The question, despite its persistence, can have no useful answer. Yet it does persist, which is an interesting commentary on the superficiality of much contemporary thought.

We would argue for a *far* more minutely detailed look at the educational use of computers. Consider this first step toward a more "microscopic" theory: from interview studies (Erlwanger, 1973) we knew that many 6th grade students had no idea of the *size* of, say, the decimal 0.7, or of the fraction $\frac{2}{11}$. As a response to this problem, Lesson Designers Sharon Dugdale and David Kibbey created a PLATO lesson (called *Darts*) where balloons are depicted as attached to a vertical number line (locations specified by random numbers), and the student "bursts" the balloons by hitting them (if he can) with a "dart", which is "thrown" across the screen by typing in numbers to indicate where the dart should hit the number line. Hitting the balloons thus depends upon knowing (or discovering) how to name a point that is, say, 2/3 of the way from 6 to 7, and so on.

Clearly, it would be a somewhat more "detailed" or "microscopic" question if we asked whether experience with the lesson *Darts* improved student knowledge of the *size* of fractions and decimals. But even this question is too macroscopic, for PLATO lessons do not stand entirely alone; they are designed

to be used by *teachers,* and teachers can use PLATO lessons in different ways. We report some evaluation data below [more aggregated than we would wish], but for the moment we remark that more than four years of trials make it clear that *Darts* (and some related lessons, such as Painting Rectangles), when used by experienced teachers, can produce a satisfactory knowledge of the *size* of fractions and decimals in almost every student in a typical fourth or fifth grade class. (This is an unusual achievement; few students at this age display such understanding. Cf. Alderman, Swinton, and Braswell, 1979.)

Notice that this example already *begins* the process of disaggregating "macroscopic" events: we are no longer dealing with *any* computer, but with one which must be capable of displaying a picture of balloons, and showing the motion of a dart and the bursting of the balloons. We are no longer talking about "arithmetic" in general, but specifically about knowledge of the *size* of fractions and decimals; and judgements of that size on a linear geometric arrangement. And we have committed ourselves to a more specific use of CAI - not as a totally "stand alone" system (a way in which CAI is often used), but rather as a *tool* to be used under the direction of a teacher, *within* the usual classroom setting.

Although this is far more narrowly defined than the terms of the question "can computers teach?", it obviously still deals with somewhat larger events than "a single act of information processing." A second example moves closer to that "microscopic" goal: the error involved in attempting to solve a quadratic equation

$$x^2 + x - 6 = 24$$

by the incorrect procedure

$$(x - 2)\ (x + 3) = 24$$

$$x - 2 = 24 \qquad x = 26$$

$$x + 3 = 24 \qquad x = 21$$

is, as is well known, remarkably persistent (cf. Davis and McKnight, 1979). Matz (Matz, 1979) has explained this (and many similar errors) by postulating "deeper level rules" that generate "surface level rules": one deeper level rule generalizes from specific numbers to *all* numbers of some (usually ill-defined) class. This is nearly always a *correct* generalization rule, without which one could not learn arithmetic - indeed, the rule works correctly, in the present example, for the "2", the "3", and the "24". Were these numbers "7", "19", and "2" respectively, the problem would still be solvable by some of the same methods. Students who make this present error, however, are

"generalizing" from some illustrative example, such as

$$(x - 5)\ (x + 2) = 0,$$

without noticing that *the zero here is a SPECIAL number, one of the VERY RARE exceptions to the deeper-level rule that generalizes over numbers.*

Clearly, both *task analysis* and *behavioral objectives* represent attempts to move toward a more "microscopic" level of analysis, and have achieved occasional successes in this attempt. The present "cognitive science" analysis is NOT quite the same sort of thing, as one can see most clearly by studying the analysis of the Jamesine Friend "unary addition" error (Davis and McKnight, 1979), or by reading Matz (1980) or Winograd (1971), because the "micro" analyses we speak of look toward single information-processing acts, or something approximating this, using more macroscopic tasks only as expedients when necessary (because no "micro" analysis is available). The difference can be suggested by the comment of Minsky and Papert (1972) that one has traditionally asked why one stick figure looks taller than the other

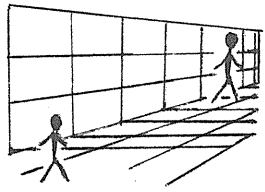

Fig. 1

when, from an information processing perspective, a more fundamental question is: "Why do humans perceive those patterns of line segments and circles as representives of human figures?"

The PLATO courseware is often considered novel because of various innovations in the PLATO hardware. An equally innovative aspect of this courseware is the cognitive science theory that underlies the design of the PLATO lessons.

III. *Some Evaluation Data.* An independent third-party evaluation of this computer-based curriculum has been carried out by Educational Testing Service of Princeton, N.J. Another independent evaluation (using some of the same data) has been carried out by John Gilpin. [Slottow, et al., 1977]

A dozen classrooms of children used the computer lessons, and were matched, student-by-student (on a "matched-pair" basis) with children in classrooms that did not use the computer lessons. A wide range of socio-economic variables were represented among the student population, including also a range of ethnic and racial variables. Various tests were used, but we cite here results on the computations sub-test of the Comprehensive Test of Basic Skills, Level 2, Form R. [For details, see Dugdale and Kibbey, 1977]

During the 1975-76 school year, *grade equivalents* for the students in PLATO and non-PLATO classes were as shown in Table I.

		Fall Administration of CTBS test		Spring Administration of CTBS test		Gain
	N	Mean	S.D.	Mean	S.D.	
PLATO	129	5.0	1.58	6.4	1.70	1.4
non-PLATO	129	5.0	1.62	6.1	1.66	1.1

TABLE I

That is, the addition of the computer lessons shifted the mean gain in G.E. from 1.1 to 1.4 years. (There is less than one chance in a hundred that a difference this large could be due to chance.) The following year the program was continued in these schools, and the gain from fall to spring, averaged over all students using the computer lessons, was 2.0 years (in grade equivalents).

The gains are even more pronounced at the 4th grade level. (At higher grade levels, many students "topped out" on both the courseware and on the tests, so that substantially larger gains in grades 5 and 6 would presumably be possible, given appropriate computer lessons and appropriate tests.) John Gilpin and Sharon Dugdale have devised a method of displaying the year's progress of each individual student; we reproduce in Figure 2 both their data and their arrow display format. Each arrow represents one student: the length of the arrow shows the student's growth (measured in GE by CTBS tests) over one school year.

For our present purposes, we wish to call attention to merely one aspect of the PLATO results: many students proved capable of grade equivalent gains of 3 and 4 years, achieved during one year of school. The attitude data from the

PLATO trials are equally positive.

On every single attitude question used, differences strongly favorable to PLATO were observed [Slottow, et al., 1977]. Pupils were enthusiastic about the mathematics lessons which the computer presented on the TV-like screens, many students sought *extra* sessions, their attitudes toward mathematics improved (as measured by a questionnaire), and so did their attitudes toward their own ability to deal with mathematics. Teacher assessments, though inevitably subjective, were very strongly positive, including even reports that PLATO had decreased anti-social behavior.

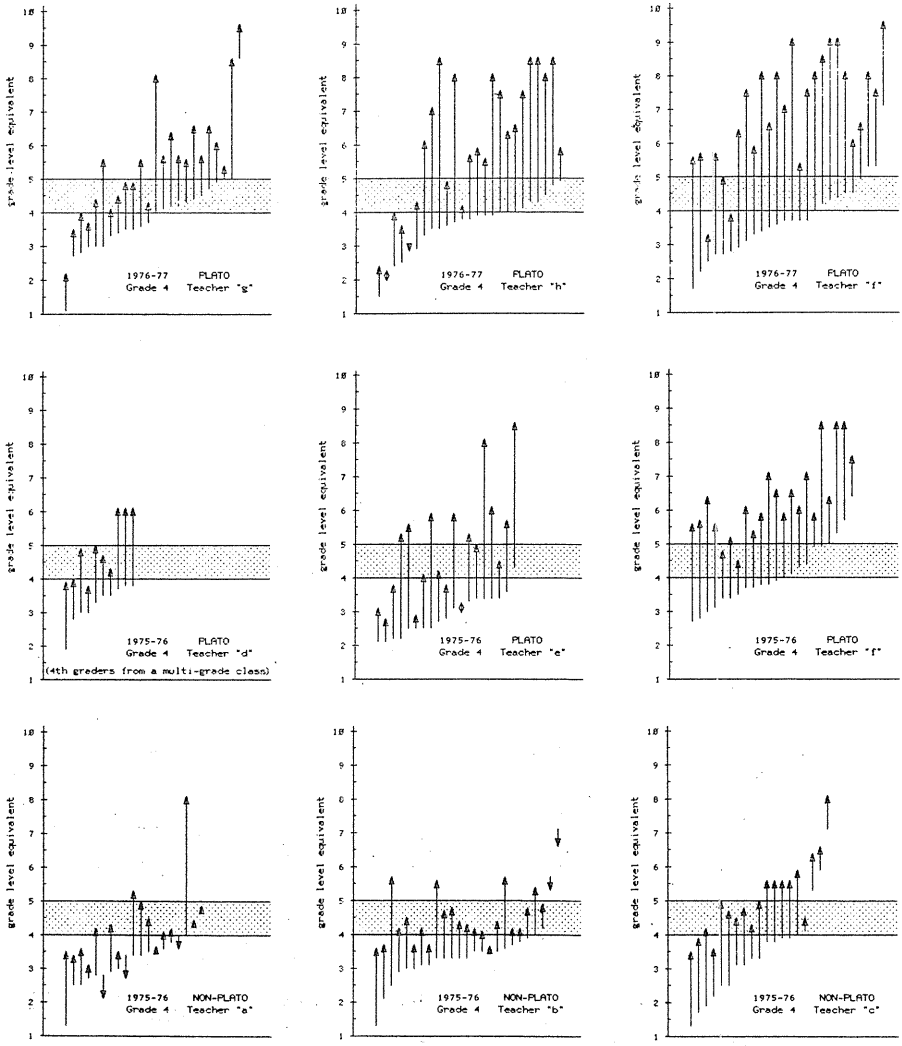

Fig. 2. Changes in arithmetic standing (in grade equivalents) of individuals in PLATO and non-PLATO 4th grade classes. Each vector represents a specific student. The tail of the vector shows the student's grade equivalent on the pretest; the head of the vector shows the student's grade equivalent on the posttest. The shaded area on each graph shows the "expected" growth range of a 4th grade student, from 4.0 to 5.0. The bottom row of graphs corresponds to the 3 non-PLATO classes tested in 1975-76. The middle and top rows are the PLATO classes tested in 1975-76 and 1976-77 respectively. The non-PLATO classes were chosen as comparison classes for the 1975-76 PLATO classes. (No non-PLATO classes were tested in 1976-77.) Teacher "f" taught a PLATO class in each of the two years. PLATO vs. non-PLATO differences have usually shown up most clearly at the 4th grade level. The test used was the Comprehensive Test of Basic Skills, Level 2, Form R, 1968-69 edition.

FOOTNOTE

[1]National Science Foundation Contract C-723, by Grant NSF PES 741, and by Grant NSF SED 79-12740. This paper was prepared for presentation at the Fourth International Conference of the International Group for the Psychology of Mathematics Education, Berkeley, California, August 17, 1980.

BIBLIOGRAPHY

Alderman, Donald L., Spencer S. Swinton, and James S. Braswell. Assessing basic arithmetic skills and understanding across curricula: Computer-assisted instruction and compensatory education. Journal of Children's Mathematical Behavior, 1979, 2 (2), 3-28.

Davis, Robert B. Alternative uses of computers in schools: Cognition vs. natural language statements. Paper presented at session 26.15 of the Annual Meeting of the American Educational Research Association, Boston, Massachusetts, April 10, 1980. [1980A]

Davis, Robert B. Information Processing and the study of algebraic thought. Paper presented at pre-session of the Annual Meeting of the National Council of Teachers of Mathematics, Seattle, Washington, April 15, 1980. [1980B]

Davis, Robert B. What classroom role should the PLATO computer system play, Proceedings of the 1974 national computer conference, (AFIPS Press, 210 Summit Avenue, Montrale, New Jersey 07645), May 1974, 169-173.

Davis, Robert B. and Sharon Dugdale. The use of micro-assessment in CAI lesson design. Journal of Children's Mathematical Behavior, 1976, 1 (supp. no. 1), 85-102.

Davis, Robert B. and Curtis C. McKnight. Modeling the processes of mathematical thinking. Journal of Children's Mathematical Behavior, 1979, 2(2), 91-113.

Dugdale, Sharon and David Kibbey. Elementary mathematics with PLATO. Urbana, Il: Computer-based Education Research Laboratory, University of Illinois. July 1977.

Erlwanger, Stanley H. Benny's conception of rules and answers in IPI mathematics. Journal of Children's Mathematical Behavior, 1973, 1(2), 7-26.

Matz, Marilyn. Towards a computational model of algebraic competence. Masters thesis, Massachusetts Institute of Technology, June 1980.

Matz, Marilyn. Underlying mechanisms of bugs in algebraic solutions. Paper presented at session 10.07, San Francisco meeting of the American Educational Research Association, April, 1979.

Minsky, Marvin, and Seymour Papert. Artificial intelligence memo no. 252. Cambridge, MA: Massachusetts Institute of Technology Artificial Intelligence Laboratory, January 1, 1972.

Slottow, Gene, Robert B. Davis, et. al. Final report: Demonstration of the PLATO IV computer-based education system. Jan. 1, 1972 - June 30, 1976. Final report of NSF contract C-723. Urbana, Illinois: Computer-based Education Research Laboratory, University of Illinois (March 1977).

Smith, Stanley and Bruce A. Sherwood. Educational uses of the PLATO computer system. Science, 1976, 192, 344-352.

Swinton, Spencer S., Marianne Amarel, and Judith A. Morgan. The PLATO elementary demonstration - educational outcome evaluation. Vols. I and II. Princeton, N.J.: Educational Testing Service, Nov. 1978.

Winograd, Terry. Procedures as a representation for data in a computer program for understanding natural language. Ph.D. thesis. Cambridge, MA: Massachusetts Institute of Technology, Artificial Intelligence Laboratory, Feb. 1971.

A DEVELOPMENTAL THEORY OF INSTRUCTION

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A method of instruction is described which is based on a contemporary theory of cognitive development. It is shown that the method produces good retention, reasonable transfer, and a low rate of conceptual errors. To the extent that such results are not currently being produced in the area of mathematics, it is suggested that the method might be a worthwhile one to explore in this domain.

The stages of reasoning which elementary school children exhibit in thinking about Piaget's Balance Beam problem are as follows. At the age of 3 or 4, when asked which side of a balance beam will go down, children make a very global evaluation of each side, often basing their answer on the perceptual characteristics which the weights on each side exhibit. At the age of 4½ to 6, they carefully quantify the number of weights on each side, and pick the side with the greater number. At the age of 7 or 8, they still quantify the number of weights on each side. However, if the number of weights on each side is equal, they now predict that the weight which is at the greater distance from the fulcrum will go down. At the age of 9 or 10, they succeed in effecting a primitive compensation between the weight and distance. Either they make their decision on the basis of the dimension of greater difference, or else they form some mental sum of the total weight and distance on each side. These latter strategies do not always yield the correct answer, of course, since they are not yet based on proportional reasoning. Nevertheless, the strategies do work in the vast majority of cases, and represent the pinnacle of what Piaget has labelled "concrete operational reasoning".

If the sort of progression which is observed on Piaget's Balance Beam problem were unique to that problem, it would be of limited interest. In fact, however, this is not the case. Precisely the same progression is observed on a wide variety of other problems during the same age range (Case, 1978a). Even the rate of progression appears to be quite similar. (Case, 1978a, Siegler 1978) Children's progress in thinking about the balance scale may therefore be taken as typifying the progress in their reasoning which take place during the elementary school period.

What sort of changes in children's cognitive structures does this progress in reasoning entail? The type of structural analysis I shall present is drawn

from the information processing tradition. In this tradition, an executive control structure is defined as a procedure for getting a subject from an unsatisfactory initial situation, to a more satisfactory terminal situation, via a sequence of intermediate mental operations. As such, any executive control structure may be analized in terms of three components.: (1) a representation of the problem situation, (2) a representation of the objective, and (3) a representation of the steps for achieving that objective. In the case of the balance beam, these three components might be as follows:

PROBLEM SITUATION

. A mental plan or blueprint (scheme) representing the fact that a balance beam with weights has been placed in front of the child.

OBJECTIVE

. A scheme representing the goal to be achieved, namely to predict which side of the balance will go down when the arms are released.

STRATEGY

A scheme representing the steps to be taken, namely scanning each arm of the balance, to see which weight looks heavier.

Since the above schemes will appear in all the subsequent analyses I shall present. I shall list them in abbreviated form as follows:

Executive Control Structure: Stage 1: (3-4 years)

PROBLEM SITUATION

. Balance Beam with Weights

OBJECTIVE

. Find side which will go down

STRATEGY

1. Scan each weight to see if one is heavy. If so, pick it.

The executive control structure which children develop during the next stage may be represented in a similar format as follows.

Executive Control Structure: Stage 2 (4½-6 years)

PROBLEM SITUATION

. Balance Beam with weights

. Different Number of weights on each side

OBJECTIVES

. Find side which will go down

. Find side with greater quantitative "pull"

STRATEGY

1. Scan each weight, to see if one is heavier, if so, pick it (if not, go to 2).

2. Count weight on each side, label side with large number as heavier (Return to 1).

Note that each component of the structure now contains one additional element. In effect, a new "loop" has been added, consisting of a representation of a new problem feature, a representation of a new goal which is occasioned by this feature, and a representation of a new step or set of steps (subroutine) for reaching this goal. I have used an indented notation in listing the objectives, to indicate a relationship of subordination. The child still has the same "top level" objective. However, he now has a lower level objective as well, which is subordinate to his initial one. In terms of the child's mental functioning, what this subordination of goals implies is that the child must store his top level objective in working memory, while he devotes his attention to the subordinate one. Whatever the mental load of the first strategy, then, the load of the second strategy will be at least one unit higher.

Executive Control Structure: Stage 3 (7-8 years)

PROBLEM SITUATION

. Balance Beam with Weights

. Different Number of Weights on each side

. Each weight at a different distance from the fulcrum

OBJECTIVES

. Find side which will go down

. Find side with greater quantitative pull

. Find side with greater distance from the fulcrum

STRATEGY

1. Scan each weight to see if one is heavy. If so pick it (If not, go to step 2)
2. Count weight on each side, label one with greater number as heavy (if unequal, return to step 1. If equal, go to step 3).
3. Count number of distance pegs from fulcrum. Label weight at greater distance as having heavier pull. Return to step 1.

Note once again that this structure preserves much of the pre-existing structure, but incorporate a new loop into it. This new loop consists of the representation of a new problem feature, the representation of a new subgoal and the representation of a new subroutine for reaching it. The addition of a new loop also adds a unit of load to the child's working memory. It adds this load since it requires that some additional pointer be stored, indicating the intention to execute the new routine. It continues to add a unit after the routine has been executed, since the product of the routine must be

stored for subsequent integration with the product of other subroutines.

If the above analysis accurately reflects the changes which take in the child's cognitive structure, a question which naturally arise is how one might facilitate these changes. Developmentalists have generally agreed that progress can be facilitated by presenting children with problems one level beyond their current level. They have disagreed, however, as to how much progress may be expected when this is done. Many developmentalists are quite pessimistic. This is because of the fact that the changes which take place in one executive structure appear to be so closely linked to the parallel changes which take place in a wide variety of others. If one sees this linkage as being due to the existance of some underlying "structure of the whole", it is hard to see how any isolated intervention could have much impact.

In my own work, however, I have followed a suggestion made by Pascual-Leone, and shown that the general factor which links specific executive control structures is not the existence of some general "structure of the whole". Rather, it is the existence of a limited working memory, one which can only hold so many newly established subgoals (or products of previous operations) in mind, while executing whatever operation it is working on currently. As long as one insures that the number of such schemes (or "pointers") which a child must hold in mind does not exceed his available capacity, there appears to be no limit to the number of new loops which can be incorporated.

If one adopts "executive facilitation" as one's instructional objective, how might one proceed? As a minimum, I believe, one would have to go through the following three steps.

1. The Establishment of an Executive Hierarchy in the Problem Domain. The first step does not involve instruction at all. It involves analysis. The object of this step is to discover the sequence of executive structures which children bring to the domain spontaneously (or which they might be enabled to bring to the domain) and to specify these structures explicitly in the form of an executive hierarchy. Note that this sort of hierarchy differs from the standard learning hierarchy described by Gagné, in that the top level goal remains the same from the simplest to the most complex structure.

Both rational and empirical methods can be useful in specifying such a hierarchy. One can begin by laying out a sequence of problems which one thinks require increasingly sophisticated structures. One can then give them to a

group of children who have had some exposure to the domain, and who are of different ages, and note whether or not the problems form an empirical scale. At this point error analysis becomes valuable. One can examine the most common errors which children make on various problems, and interview them to determine what sort of reasoning has gone into producing these errors; then modify one's hierarchy appropriately. A procedure for this sort of analysis has been presented elsewhere (Case, 1978b).

The result of one's endeavour at this first step should be the description of a nested set of executive control structures on the one hand, and a nested set of problems on the other. Siegler (1978) has provided an illustration of how this sort of analysis can be conducted for the balance beam task, and shown that the following four classes of problems form a well ordered empirical scale (i) Problems where a weight is placed on only one side. (ii) Problems where two weights are placed at equal distances from the fulcrum, and one weight is composed of a slightly greater number of units than the other. (iii) Problems where the weights are equal, and placed at different distances from the fulcrum. (iv) Problems where both weight and distance vary, in opposite directions.

2. Determination of Students' Current Level in the Executive Hierarchy. The second step is to determine the executive level of the students whom one will be teaching. Once a domain has been thoroughly mapped out, this may be done by presenting the children with a test which contains the various problem types which have been identified, and then determining the level at which the children consistently began to give incorrect answers. If a domain has been mapped out only tentatively, one might again wish to examine children's errors and to interview them before proceeding.

3. Facilitation of Progress from One Level of the Hierarchy to the Next. The final step is to facilitate children's progress from their existing level of functioning to the desired level in a series of steps. Each step may be broken down into a series of substeps, as follows: (i) Design a situation where children will understand the top level goal of the task, and can determine on their own the adequacy of their current strategy for reaching it, (ii) present the children with a few problems which are solvable by means of the executive structure which they currently have available. (iii) Advance to problems which are not solvable by means of the current executive control structure and which are one level above it. (iv) Help the student to notice the new feature in these problems, which he has not taken into account and which is leading him into error (v) Help the student to develop a new sub-

routine for dealing with this new feature. (vi) Provide a period of practice so that the student will no longer need to store the intention to execute this subroutine, but consolidate it as an automatic part of his procedure.

As an illustration, suppose that students were assessed as having a level 2 executive structure on the balance beam task, and that the aim was to help them develop a level 4 executive structure. One might go about this in the following manner. First, one might place a balance beam in front of the students, and give them several example problems where the only factor which varied was weight. The students would succeed at these problems, and realize that the behaviour of the balance beam, once the arms were released, constituted a check on the adequacy of their strategy. Then one could introduce them to a level 3 problem, where the weight was the same but distance varied. Their prediction would be that the two arms would balance, and they could be shown that this prediction was not confirmed. With a series of prompts, they could be led to notice that the distance also varied, and to incorporate a subroutine into their existing strategy to check for this feature. The practice period would consist of a series of level 1, 2, and 3 problems intermixed.

Once the child had consolidated a subroutine for checking distance from the fulcrum as well as the number of weights, one could introduce him to a level 4 problem, where the weight and distance factors both varied, but acted in opposite directions. Once again, he could be led to see that his current strategy was inadequate, by being shown that it frequently led him to make incorrect predictions. Then he could be led to a simplified procedure for taking both dimensions into account simultaneously, such as counting down the weights and along the distance on each (thus in effect generating a total). Once again, a period of practice would be necessary, in which problems from levels 1, 2, 3 and 4 were intermixed.

Although it may not be obvious, the above procedure should have the effect of reducing the working memory load of executing a level 4 strategy considerably. In the early phases, the instructor should function as an "external executive", removing some of the planning burden from the child. As the child becomes more familiar with the task, he should be able to take on this burden himself, since practice should make it lighter. The counting procedure should also act as memory aid for the child, since it whould remove the burden of having to store one number on each side while computing a second.

Rita Watson and I have actually tried out an instructional program such as that described above. The instruction was presented during two 15 minute sessions, and the results were encouraging. On a transfer test given two weeks after the instruction, by an experimenter who had not been present during the training, not one of the control subjects passed any of the level 3 or level 4 problems. By the contrast, the experimental group passed 75% of these problems. Even their errors were encouraging, since they were no longer conceptual in nature, but mechanical (eg. errors in counting).

We have used a similar method of executive facilitation with a variety of other developmental tasks, including maze tracing, quantity estimation, conservation, and control-of-variables. (Case, 1978b) In each case the results have been the same. We have shown (i) excellent retention, (ii) good transfer, and (iii) a reduced incidence of conceptual as opposed to mechanical errors. My suspicion is that one reason we have obtained these three results is that we have enabled the child to keep his conceptual understanding of the problem closely integrated with his strategy for solving it. To the extent that this is a problem in the area of mathematics instruction, then, and to the extent that results such as these cited are not being achieved, it seems to me that the method of executive facilitation might prove a useful one to explore.

The papers which follow represent two efforts in this direction, both of which I think show considerable promise. As is the case with any creative application, the papers not only show how the method might be applied in a new domain, they also contribute toward clarifying the nature of the method itself.

References

Case R. Intellectual development from birth to adulthood: A Neo-Piagetian interpretation. In R. Siegler (Ed.) Young children's thinking: What Develops? Hillsdale, N.J. Lawrence Erlbaum, 1978a.

Case, R. A developmentally based theory and technology of instruction. Review of Educational Research, 1978b, 48, 439-469.

Siegler, R.S. The origins of scientific reasoning. In R.S. Siegler (Ed.) Children's Thinking: What Develops? Hillsdale, N.J. Lawrence Erlbaum Associates 1978a.

A DEVELOPMENTALLY BASED APPROACH TO THE TEACHING OF PROPORTIONALITY

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This study was designed to compare three instructional methods for teaching an equivalence proportionality task to groups of normal 4th and 5th graders and math disabled 6th and 7th graders. The methods were derived from a standard textbook curriculum, Gagné's cumulative learning theory and Case's neo-Piagetian instructional principles. 27 subjects of each type were trained in one of the three methods and given immediate and one-month delayed post-tests. The results showed that both experimental groups outperformed the standard curriculum groups. Among disabled subjects the Case group was superior to the Gagné group on the delayed test only. However, many of the normal Gagné subjects induced the Case strategy while forgetting that of the Gagne training. Thus, it was suggested that the Case training is more durable than the Gagné training for the disabled students, though the two may be equally effective for teaching normal children who can generalize proportionality strategies more easily.

Despite recent advances in research in mathematics instruction, children are still largely taught with curricula based on the formal structure of mathematics with no account taken of their level of cognitive development. Moreover, there is a strong suggestion in the literature that a) children's level of cognitive development affects the ease with which they can learn basic arithmetic concepts and b) children who experience difficulty only in mathematics may be functioning at a lower level of cognitive development in the quantitative domain than other children. If this is the case, it seems likely that instructional approaches which can lessen the developmental burden placed on children's cognitive systems might be able to do a more effective job of teaching some children than current approaches.

Two instructional technologies reduce the developmental load of the learner in some fashion. Gagné (1976), while not referring to developmental theory directly, suggests lessening the amount of new information a child must deal with at any one time. Since children's ability to coordinate novel information has been shown to be related to their stage of cognitive development (Case, 1974), Gagné's approach may be considered to have a developmental component.

Case (1978) also focuses on reducing the information load of the learner, but considers, as well, a) the child's ability to understand the conceptual core of a learning task and b) the oversimplified solution strategies which he/she may develop when unable to understand the concept fully.

This study was designed to test the following hypotheses: (1) both Gagné's and Case's technologies should be more effective in teaching math disabled or cognitively immature children than conventional classroom technology, (2) of the two developmental technologies, Case's should be the more effective, since its consideration of developmental limitations is more explicit and extensive. In order to test these propositions, three instructional approaches were applied to the task of teaching the solution of simple proportion problems with one missing element. Two types of children, those too young to have received instruction in the task and those who had already received instruction, but who had failed to learn it, were selected because of developmental limitations.

The first of the three training procedures was individual instruction from a standard fifth grade California State Mathematics textbook. The text used a variety of proportionality situations accompanied by pictures to instruct the student in the cross-multiplication and division algorithm.

The Gagné based instruction was constructed on the principles of the cumulative learning theory of which the learning hierarchy is the central figure. This instruction was based on a 16 step learning hierarchy, also designed to teach the cross-multiplication and division algorithm (Anderson, 1979). Two different contexts with corresponding manipulatives were used to illustrate each step of the hierarchy. Both of these were mixture problems, one of discrete quantity (two types of nuts), the other of continuous quantity (lemon juice and water). Steps included teaching the meaning of complete and incomplete rates, representing them as equivalent fractions, and finding the missing element by means of the algorithm.

Neo-Piagetian theory provided a basis for Case to describe developmentally based instruction (1978). He has identified a number of instructional principles and procedures to implement them. The focus here is in developing an appropriate executive scheme while maximizing both the familiarity of the situation in which the task must be learned and the salience of the relevant cues, and minimizing the number of items of information that must

be coordinated. The following are substeps suggested for effecting this process: a) the initial teaching situation must be made as familiar and cognitively simple as possible, b) problems should be designed to show the learner where his incorrect strategy fails and thereby set up cognitive conflict, c) the instructor should show the learner why his incorrect strategy fails, by drawing attention to ignored cues, d) the instructor should demonstrate a more adequate strategy, e) the new strategy should be consolidated and extended through practice, feedback and gradual addition of complexities as the strategy becomes automatized.

The Case instruction began with an intuitively simple proportionality situation: matching pieces of gum and boxes. The basic strategy taught was recognizing and replicating a one-to-many correspondence in order to find the missing quantity to preserve the equivalence of the two ratios. The instruction followed a hierarchy which included matching a given unit ratio (1:x) to find a missing element, repeating a unit ratio, reducing a given ratio to a unit ratio and applying those three strategies to non-integral ratios. After each step was mastered, a new complexity was added by combining the basic strategies. The complete instructional sequence was repeated with the more abstract juice mixture problem (c.f. Noelting, 1975). Manipulatives were used to verify correct use of the executive strategy or to challenge use of an incorrect one.

METHOD

The subjects were selected from an upper middle class elementary and junior high district in the San Francisco Bay Area. Twenty-seven fourth and fifth graders ("normal" learners) who had not previously been exposed to proportionality problems were selected at random and assigned to one of the three instructional groups. An equal number of sixth and seventh graders who had been taught but not mastered the proportionality problems were selected from a pool of "disabled" learners who scored poorly on standardized math achievement tests. These students were also randomly assigned to the three training groups. Basic knowledge of multiplication, division and fraction concepts was insured by a pretest.

Two post-tests were administered to each subject, the first two days after training and the second a month later. There were 20 problems of four types on each test: six juice equivalence problems, similar to those used in the Gagné and Case Training, six juice comparison problems, a far transfer task, five word problems, as in the standard curriculum training, and three Mr. Tall/

Mr. Short problems (Karplus, Karplus and Wollman, 1974), another equivalence task. All subjects were encouraged to use the techniques that they had learned. Manipulative aids were readily available. Significant dependent variables were the total scores and scores on the four subparts of each post-test.

In addition to the principal hypotheses mentioned above for both normal and disabled learners, it was hypothesized that both normal and disabled learners regardless of training method would perform equally well. Five contrasts of interest for each variable were tested: disabled vs. normal, and within each learner type, standard curriculum vs. experimental methods, and Gagné vs. Case. Total probability of Type 1 error was .15 for each variable (.05 for learner effects, .025 for each of the four method contrasts).

RESULTS AND DISCUSSION

Table 1 shows mean total scores and percentage of mastery (arbitrarily set at 65% of the problems) by training method and learner type for each post-test.

Table 1

Mean Total Scores and Percentage of Subjects Achieving Mastery on Proportionality Problems

Learner Group (Grade)	Training Method	Immediate Post-test Mean Total Score (Max = 20)	Immediate Post-test % Mastery	Delayed Post-test Mean Total Score (Max = 20)	Delayed Post-test % Mastery
4-5	Standard	10.78	33	9.78	33
	Gagné	14.78	67	15.44	78
	Case	15.22	78	17.78	100
6-7	Standard	11.22	44	8.67	11
	Gagné	13.78	56	11.22	33
	Case	15.00	78	16.11	100

Among both normal and disabled learners the experimental group significantly outperformed the standard groups on both post-tests. Contrary to prediction, there were no differences between the Gagné and Case groups on either post-test for normal 4th and 5th graders, or for disabled 6th and 7th graders on the immediate post-test. However, the disabled Case group performed better than the disabled Gagné group on the delayed post-test. Regardless of method there were no significant differences in performance between learner types on

the immediate post-test, but the normal learners performed significantly better than the disabled learners on the delayed test.

Further analysis showed that standard-experimental group differences on the total score could be attributed to differences on the juice equivalence problems for both grade levels on the immediate post-test, to all four types of problems for grades 4 and 5 on the delayed post-test and to all but the Mr. Tall problems for the 6th and 7th grades on that post-test.

In comparing immediate and delayed post-test performance differences, an important result emerged: the normal children improved over time after either experimental training, but the disabled children improved only if they had had the Case training.

Investigation of solution strategies sheds light on the differential effects of the training on the two learner types and on near and far transfer problems. For both learner types standard group subjects were limited in their method for solving both similar (verbal) and transfer problems. The majority of their correct responses were obtained by proportional reasoning that they had induced themselves and not by the algorithm.

A similar pattern was found among the Gagné subjects. On the immediate post-test some 50% of correct responses were obtained by the instructional method. On the delayed test, though many of the younger subjects had forgotten the algorithm, their performance improved because they used proportional logic correctly. It may be assumed that something in the training triggered a proportionality scheme in these able math students. On the other hand, the disabled Gagné subjects who forgot the algorithm reverted back to the more developmentally elementary additive logic.

For both learner types the Case groups demonstrated approximately 75% use of the instructional technique, showing no decrease over time. Even for the disabled learners there was some consolidation of the instruction that allowed it to be recalled and applied.

Thus, while there was some retention and transfer of the instructional techniques in all groups, they occured more frequently in the Case groups than in the other two. The similarity of strategies induced intuitively by the algorithm groups to that taught to the Case group implies that understanding of

proportionality may best be conveyed through the "one-to-many" rather than by the traditional algorithm, however thoroughly taught.

It is also interesting to note that almost all subjects in all groups showed some inconsistency in their reasoning. Not being at the stage of operational thought, they were unable to examine their own thinking to identify the two forms of reasoning that they employed. From time to time a subject would vacillate uncomfortably between an additive and proportional strategy in a state of disequilibrium.

Because the disabled 6th and 7th grade Gagné subjects could not induce or retain a correct solution strategy as could the younger normal learners, it is suggested that normal and disabled children have different capacities to generalize from instruction. Only those disabled subjects who had been directly exposed to an intuitive conception of proportionality were able to retain the method taught or transfer it to other tasks. Thus, depending on the nature of the math task and the degree of transfer required, mildly disabled children may require instruction more carefully tailored to their developmental levels.

This study was not designed to isolate the effects of the instructional principles cited above which might have led to success or failure in the training. However, one may speculate as to what the effects may have been. It is felt that the simple examples given in the Case method enabled the one-to-many strategy to be more easily acquired, overlearned and retained than the algorithm with its higher cognitive load and less intuitive relationship to the tasks. Incorporation of incorrect strategies into the Case instruction seems to have had a less potent effect than overlearning the correct strategy, since almost all subjects reverted to the powerful additive strategy for some of the more difficult problems.

Solving the target problem throughout the training seemed to have a helpful effect on performance. For the standard and Gagné subjects it was clear that it would have been more effective to spend more time on complete problems so that they could better learn the algorithm.

It was felt too, that use of manipulatives rather than pictures enabled easier acquisition of the concepts. The necessity of using manipulatives may be considered a "developmental" phenomenon, and should therefore be an integral part of instructional planning for pre-formal operational children.

Problems to be addressed in the future might include confronting the limited transfer that all of the teaching strategies produced. It seems likely that there is a hierarchy of increasingly abstract proportionality situations which could be progressively presented (e.g. from discrete situations to mixtures) in a redesigned instruction. Only after the student has been exposed to the range of proportionality problems should the algorithm be taught. It may also be possible to experimentally isolate those factors responsible for the training effects noted, although for teaching purposes it may be sufficient to apply all of these in an instructional design. It would be useful to extend these methods to other math problems/concepts that appear developmentally related and which might necessitate instruction beyond the usual textbook methods (e.g. missing elements, place value, regrouping and fractions). Though time consuming, these instructional techniques may prove helpful in teaching those who have reached a limit in their cognitive development and for whom some facility with these concepts might prove beneficial, the mildly mentally retarded and adults who have completed schooling without having learned these concepts.

REFERENCES

Anderson, L.H. Developmental effects in learning hierarchy structure for problems involving proportional reasoning. Unpublished doctoral dissertation, University of California at Berkeley, 1979.

Case, R. Structures and strictures: Some functional limitations on the course of cognitive growth. Cognitive Psychology, 1974, 6, 544-573.

Case, R. Piaget and beyond: Toward a developmentally based theory and technology of instruction. In R. Glaser (Ed.), Advances in Instructional Psychology, Vol. 1 Hillsdale, New Jersey: Lawrence Erlbaum Associates, 1978.

Gagne, R.M. Conditions of learning (3rd. ed.). New York: Holt, Reinhart, and Winston, Inc., 1977.

Karplus, E., Karplus, R., & Wollman, W. Intellectual development beyond elementary school IV: Ratio - the influence of cognitive style. School Science and Mathematics, 1974, 74(6), 476-482.

Noelting, G. Stages and mechanism in the development of the concept of proportion in the child and adolescent. Paper read at Fifth International Interdisciplinary Seminar on Piagetian Theory and Its Implications for the Helping Professions, University of Southern California, Los Angeles, 1975.

DESIGNING TEACHING IN THE LIGHT OF RESEARCH ON UNDERSTANDING

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A considerable body of research now exists which plots conceptual 'maps' of particular areas of mathematics, identifying points of low and high difficulty, and, in some cases, describing the relevant naive conceptions and strategies. Less work has been done so far on the design of teaching based on this knowledge. Gold (1978) has recently completed an experiment comparing teaching methodologies ascribed to Gagne and to Case (1978) with a standard textbook treatment for the topic of proportion. In this paper I intend to discuss this experiment and also our own research investigating particular features of teaching method in the topics of directed number and algebra. Work is also in progress on choice of operation in decimal problems and proportion, but there is insufficient space to report it here. Broadly speaking, Gold has shown that three methods, designated 'standard text', 'Gagne' and 'Case' were in that order of increasing effectiveness. These methods contained a number of features, and it is of interest and importance to attempt to identify which features were responsible for the differences in effectiveness. I propose to consider these features in four groups.

I Analysis of the conceptual field and of the subjects' initial concepts and approaches to the chosen tasks

Gold's experiment drew on an analysis by Noelting of the orange juice proportion task, which identified some nine levels of problem, depending on the numbers involved (eg. ratios 2:1 or 7:5), each defined by the need for a strategy more sophisticated than the lower levels. Other aspects of the conceptual field which were included in Gold's material were

(a) the quantities might be 'symmetrically' related or not (water/juice and pieces/boxes of gum) and
(b) objects measured by different units (Mr Short and Mr Tall - in the test, but not in the training).

Not included were situations needing recognition of the non-relevance of proportionality, eg. in which additive comparisons were valid.

II Meaningful, concept-based teaching as against rote, symbol-based teaching

This includes

(a) sequencing from simpler to more complex versions of the whole target task, as against the accumulation and integration of subskills;
(b) dealing always with the conceptual task itself, rather than

with symbol-manipulations;
(c) the use of manipulative material to embody the conceptual task and to allow verification of solutions;
(d) the provocation of cognitive conflict by deliberately introducing expected misconceptions and leading pupils to verify their incorrectness;
(e) the offering in the teaching of a single key concept, as against a step-by-step procedure.

III (a) Individual training with individually appropriate feedback, as against class or group work, with the usual teacher/pupil interaction;
(b) mastery criteria for progress from one type of task to the next.

IV The sequencing of structural and contextual variations - ie. keeping the same context but varying the structure (eg. by interchanging what is given and what is to be found, or mixing problems of constructing a fourth value to complete a proportion with problems of comparing two given ratios); this contrasts with the practice in most standard texts of mixing problems from various contexts often with structural variations at the same time.

Of these features, group II constitutes the main difference between Gold's 'Case' and 'Gagne' treatments, and his results imply the superiority of the meaningful treatment, particularly in view of his observation that the 'Gagne' pupils who were successful on the delayed post-test were tending to adopt the 'Case' strategy spontaneously.

Individual, one to one teaching, with a mastery criterion, and control of the structural variations, were features of both of Gold's experimental treatments. They also were features of earlier successful experiments on the training of Piagetian conservations (Inhelder et al, 1974, Gelman, 1969, Sheppard, 1973, 1974). These show that the omission of feedback substantially decreases effectiveness (Gelman), and that observing the manipulations of the concrete material can be more effective than actually performing them (Gelman, Sheppard vs. Inhelder). An important practical question for research is how to retain the advantages of these features in the 1:30 situation of the classroom.

Thus Gold's experiment shows the effectiveness of a form of a meaningful teaching which focuses the pupil's attention on a key concept, using manipulative material to aid this, and which uses intrinsic checks to generate cognitive conflict and thus to correct misconceptions. (The particular features here do not appear to depend especially on Case's theory of working memory limitation).

Our own work so far has involved the conceptual analysis of the topics mentioned, the identification of pupils' initial concepts, and the development of teaching material which will incorporate most of the features described. The experiments will serve to validate our conceptual analyses and also to indicate the relative importance of some of the factors; but this is task-dependent, some principles being feasible and appropriate in one task but not in another.

DIRECTED NUMBER

In directed numbers we are developing teaching experiments based on the hypothesis that the deep-lying difficulties which many pupils experience are due to an inadequately connected conceptual structure; that is, the provision of rules for combining the numbers themselves is connected perhaps too weakly even with the number line and the co-ordinate plane, and hardly at all with other phenomena such as bank transactions and balances, journeys northwards and southwards on a motorway, fast and slow clocks, and so on. Another conceptual area which needs to be connected with this is the solution of algebraic equations, such as $12 - 3x = 5x - 84$. Marthe (1980), reporting to this group last year, gave relative difficulty levels for a set of problems concerning money transactions and journeys in two directions. We have used a closely similar set of problems, and have some figures for relative difficulty to compare with his. Our teaching, using these analyses, ahs been particularly based on the principles above, with the addition of making the pupils aware of the features of the problem which determine the levels of difficulty and of mkaing conscious translation of a problem from one context into another. Here we shall present mainly the prior analysis of pupils' approaches.

Pupils' concepts and approaches: motorway and money problems

The following problem will be used to illustrate the results:

> A car travelled 33 miles north along the M.1. It then made a second journey. If it had made only one journey straight from its starting point to its destination it would have made a journey of 16 miles north. How far and in which direction did it travel on the second journey?

This problem showed an 'atmosphere effect' - a prevalence of solutions which confused being in the north with travelling northwards. Even among the pupils who correctly subracted 16 from 33, many responded "17 miles north", (13/22 correct distance, 6 correct sign). This question seemed to be interpreted as if "overall the car travelled 33 miles north; one journey was 16 miles north; what was the second journey?" It is easy to see how this would arise from

scanning the question, focusing first on the numerical information, then scanning again to fit this into a comprehensible situation. Rewriting the question as a somewhat longer story had a large facilitating effect. Drawings produced in response to question 4 also suggested partial reading and sequential processing of the information; for example, some diagrams first showed a 27 mile distance, then marked a north place, then put Long Eaton near it.

Other difficulties appeared to be associated with information processing load - "start 11N, go 8N, arrive ?" was easy, while "start 11N, go ?, arrive 15S" was hard, and appeared to need the holding of two distances and directions in mind before a decision could be made about whether to add or subtract.

To summarise, the following factors all appear to affect the difficulty of these problems:

1. Familiarity of context and actual feasibility of the problem - in fact, problems in contexts differing like money and journeys appear to be handled in substantially different ways.
2. Problem structure : (a) whether state + transformation = final state, or two transformations (eg. journeys) combining to give a third (Marthe's STS and TTT). (b) position of unknown: like Grouws and Good (1976), we found 'unknown last' somewhat easier than 'unknown second', and 'unknown first' much the hardest; but this could be overlaid by other factors.
3. 'Atmosphere' effect; changes of direction; passing through zero.
4. Information processing load, including the number of reversals of order between problem wording and processing order.

Combining directed numbers

The strategies with which pupils approach problems given symbolically have also been studied. The first observation was that, in this case, the differences relating to the 'position of the unknown' appear not to exist. The table shows some results from two classes of 14 year olds. The figures are percentages of the pupils answering correctly.

Direct	Algebraic	Inversions	Large numbers
5 - 8	a = 5, d = 8; a - d	$\square$ - 8 = $^{-}3$	234 - 467
100	91	91	38
(-2) + $^{-}$(7)	b = -2, c = -7; b + c	$\square$ + (-6) = $^{-}9$	($^{-}$619) + ($^{-}$242)
93	86	93	66

$(^-10) - (^-3)$	$c = ^-7, b = ^-2;\ c - b$	$(^-9) - \square = ^-6$	
79	69	69	
$(^-2) - (^-7)$	$b = ^-2, c = ^-7;\ b - c$	$(^-2) - \square = 5$	$(^-453) - (^-785)$
44	47	56	43
$3 - (^-4)$	$d = 8, c = ^-7;\ d - c$	$\square - (^-5) = 9$	$423 - (^-572)$
71	50	43	50

With the addition questions there was a very close correspondence between the direct and inverse forms - unlike what we found with the verbally presented problems, and Gauws and Good found with simpler questions with 7-8 year olds. The algebraic forms are also surprisingly close. These throw into sharp relief the differences due to the signs of the numbers.

We have sought explanations for these facts in the pupils' strategies. We found the following rules to be used, generally with considerable confidence:

1. Subtraction is commutative. This was used only rarely for 5 - 8 (none in this sample), but commonly when any other distraction was present, eg. for $(^-2) - (^-7)$ or $234 - 467$.
2. If all numbers are negative, omit negative signs, operate, replace negative sign. This, with rule (1), produces the correct answer for $(^-10) - (^-3)$ but a wrong one for $(^-3) - (^-10)$ and probably accounts for the higher facility for the former.
3. For subtracting a negative, locate the first number on the number line and count down numerically, eg. $(^-10) - (^-3)$ is $^-10, ^-9, ^-8, \underline{^-7}$; and $3 - (^-4)$ is $3, 2, 1, 0, \underline{^-1}$.

ALGEBRA

Similar work on the solution of equations has included preliminary testing and interviewing and a small scale teaching experiment. The test is shown, together with the facilities in two classes in different schools with different approaches to algebra teaching.

		No. of pupils	
		28	25
1.	$3x + 7 = 28$	28	25
2.	$24 - 5x = 9$	17	18
3.	$29 = 14 - 5x$	3	2
4.	$1/4(3x + 5) = 14$	11	4
5.	$8x = 16 + 16x$	0	4
6.	$3x + 4y = 18$	1	7
	$3x + 6y = 26$		
7.	$17 + 6x = 2x + 9$	0	5
8.	$8y = 2y(4x - 1)$	4	0
9.	If $3x - 5 = 2y$, write TRUE, FALSE or CAN'T TELL beside each of the following:		
	$3x - 2y = 5$	17	10
	$2x - 3y = 5$	15	8

Interviews showed that the initial approach was usually quantitative rather than symbolic, that is by meanings rather than manipulative rules. This solved equation 1 by inspection. Equation 2 was often solved similarly, by considering what had to be taken from 24 to leave 9; this approach was usually successful in this case. The same appraoch with equation 3 led either to blockage, or to a recognition that x must be negative, and thence either to success by seeing that $5x = -15$, or to a switch to symbol-moving methods, often unsuccessful. (The inability to deal with negatives was the predominant factor in failure.) Equations 6 were dealt with in a similar intuitive way by some 30% of the second class. Where equation 4 was solved successfully it was generally not by multiplying both sides of the equation, but by multiplying out the bracket to give $3x/4 + 1\ 1/4 = 14$. When symbolic rules were attempted they took a number of untaught forms: some typical examples are given. Paul shows the switch from the quantitative to the symbolic approach, and in the latter he fails to take into account from which side the 16 has come.

Question 1:

Find the value of x in

$3x + 7 = 28$

Working:

$3x = 28 - 7$

$3x = 21$

$x = 7$

Explanation: since $3x + 7 = 28$ this means $3x$ is 7 smaller than 28 so $3x = 21$

Question 5:

Find the value of x in

$$8x = 16 + 16x$$

Working:

$16 = 16x - 8x$

$16 = 8x$

$2 = x$

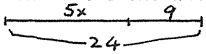
Explanation: I carried the $8x$ number over the = sign so the sign before it (e.g. + or −) changed, giving me $16 = 16x - 8x$ thus $16 = 8x$ so $x = 2$.

The mistakes thus consist of the following, which must be overcome if equation solving is to be meaningful:

1. avoidance of negatives;
2. not recognising the three views of (say) 8 - 5 as
   (a) two numbers combined by a binary operation,
   (b) a number 8 transformed by the function 'subtract 5',
   (c) two successive transformations, add 8, subtract 5;
3. giving priority to collecting like terms over the recognition of what changes of side they make (a deficiency in the concept of an equation).

Our teaching experiment consisted of a comparison between a 'rules' method - the provision of symbol-based, directly taught rules, but with explanations - to deal with these points, and a meanings method, in which all manipulations were worked out either from experiments with numbers or with concrete materials - coloured cubes for 1, 10, ⁻1, ⁻10 and bundles containing unknown numbers.

The success and confidence of pupils in dealing with equations such as $24 - 5x = 9$ by quantitative methods suggests that rather than teaching symbol-manipulative methods, it might be possible to extend the range of equations which can be dealt with meaningfully, by thinking of the quantities involved. We have tried training pupils to 'think which is the one which is the sum of the other two', to use line diagrams,



and to consider the truth of the Add-Subtract Law, A+B=C <=> A=C-B, B=C-A

REFERENCES

Case (1978), Gold (1978) - see papers in these Proceedings

Grouws, D. A. and Good, T. G.; Factors associated with 3rd and 4th grade childrens' performance in solving multiplication and division sentences. J. Res. Math. Educ., 155-171, 1976.

Marthe, P.: Additive Problems and Relative Numbers. Proceedings of IGPME, Warwick, 1980

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Reports on the research described here, on directed number, algebra and decimals (choice of operation) are available from Shell Centre for Mathematical Education, University of Nottingham, Nottingham, NG7 2RD, England.

THE DEVELOPMENT OF PROPORTIONAL REASONING IN FOUR DIFFERENT CONTEXTS

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Four different embodiments of proportional reasoning were studied, two involving problem-solving tasks — the Orange Juice and Sharing Cookies Experiments —, two bearing on numbers — Comparison of Fractions and Addition of Fractions. Twelve identical items were devised for each task, corresponding to stages which had already been described. This was presented to six grade levels: Elementary 6 and Secondary 1 to 5. Results were treated by an original type of scalogram analysis, where all correct results were kept. Satisfactory coefficients were obtained for the first three experiments, and stages were differentiated, while the last was an all-or-nothing task.

PROBLEM and INSTRUMENT

The development of proportional reasoning was studied at Grade 6 of the Elementary School and the five grades of the Secondary School, in four different contexts: the Orange Juice Experiment (Noelting 1980a), the Sharing Cookies Experiment (Noelting 1980b), and two new situations presented here for the first time: Comparison of numerical Fractions and Addition of Fractions. Corresponding items of the four experiments are given in figure 1.

A succession of 12 items was made up for the four Experiments, with identical numbers from one experiment to the other (Table 1). These had been chosen to correspond to stages which had earlier been identified.

SAMPLE

These items were presented in a group experiment to subjects at the six different grade levels mentionned (Table 2).

CORRECTION OF ITEMS

Period I and II items were corrected as passed or failed according to choice made

and correction of Period III items took into account explanations given. When only comparison of terms were given (e.g. "more orange juice, less water, more cookies, more people", etc.) correct answers were considered as accidental,due to centration. When any mention of an operation was found (division, multiplication) even if the explanation was incomplete, correct answers were accepted.

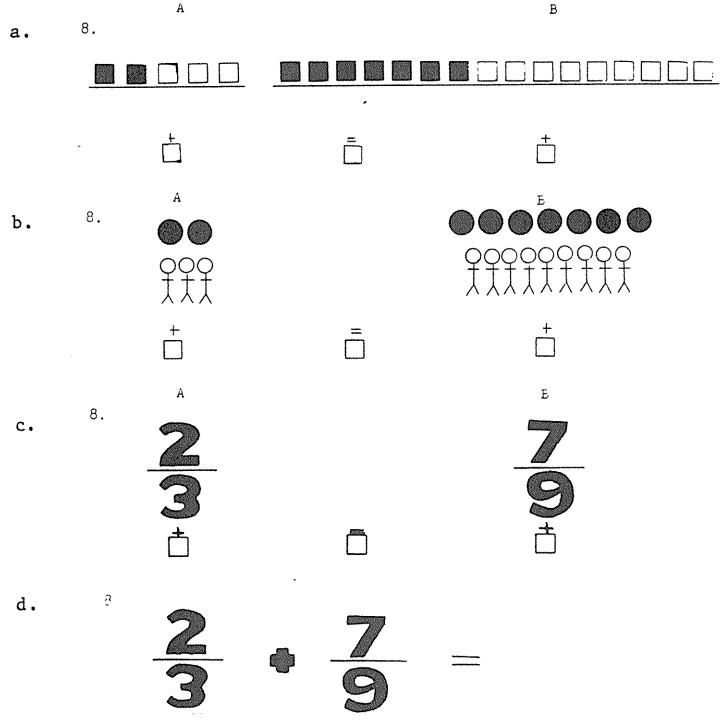

Fig. 1 - Item 8 of the four different experiments: Orange Juice (a), Sharing Cookies (b), Comparison of Fractions (c), Addition of Fractions (d).

TABLE I

IDENTICAL ITEMS OF ORANGE JUICE, SHARING COOKIES, COMPARING FRACTIONS, AND ADDITION OF FRACTIONS EXPERIMENTS, IN ORDER OF PRESENTATION (GROUP TESTING).

Item	Composition[a]	Correct answer[b]	Stage	Structure
1	2/5 vs. 4/5	B	I	Variation of either numerator or denominator.
2	1/2 vs. 1/3	A		
3	1/3 vs. 3/9	E	IIA	Equivalence of unit fraction[c]
4	2/4 vs. 3/6	E		
5	3/5 vs. 6/10	E	IIB	Equivalence of any fraction[c]
6	4/6 vs. 6/9	E		
7	3/1 vs. 5/2	A	IIIA	Non equivalence with multiple relations between two terms, either "within" or "between" ratios or fractions[d]
8	2/3 vs. 7/9	B		
9	6/2 vs. 8/3	A		
10	6/3 vs. 7/4	A		
11	3/5 vs. 5/8	B	IIIB	Non equivalence without multiple relations between terms.
12	8/5 vs. 5/3	B		

Notes - [a] For addition of fractions, the problem was set as an addition and the correct arithmetical answer was required.

[b] A, if left proportion is larger; B, if right proportion is larger; E, if both proportions are equivalent.

[c] Equivalences will be termed IIA1 and IIA2, also IIB1 and IIB2, when they have or have not, respectively, lowest terms.

[d] $IIIA_{int}$ when the multiple relation is "within" (e.g. 6/3 vs. 7/4); $IIIA_{ext}$ when the multiple relation is "between" ratios or fractions (e.g. 2/3 vs. 7/9).

TABLE 2

SAMPLE OF SUBJECTS FOR THE FOUR EXPERIMENTS

GRADES	N	Age[a]	
		Mean	Standard deviation
Elem.6	25	12;0	0;4
Sec. I	27	13;1	0;5
" II	25	14;0	0;4
" III	28	15;2	0;6
" IV	17	16;1	0;8
" V	27	17;2	0;7
TOTAL	149	14;6	1;10

Note - [a] Mean Age and Standard Deviation are given in years and months.

DEVELOPMENTAL SCALOGRAM ANALYSIS

A scalogram analysis was performed on results. However, a new technique was adopted. As all correct answers, after proper examination, were accepted, all failures at items statistically easier than the last item passed were considered as errors in the scalogram. This condition is more stringent than the usual Guttman procedure, but seems more suited to developmental studies, where a correct solution cannot be waved aside. It turns out also to be easier to accomplish and can be calculated by hand. For each experiment, a matrix is drawn up, with items in columns in decreasing order of success, and subjects in rows in increasing order of success. A dividing line is drawn, separating the last item passed for each subject, from items failed. This line in the total matrix need not be in stair-case order, but can admit overlaps. All errors at the left of the dividing line are tabulated. Then CR, or coefficient of reproducibility, MMR, or maximal marginal reproducibility, and PPR or coefficient of scalability, are calculated,as usual.

In Table 3a, 3b and 3c, the items are ordered in decreasing order of success for three experiments, and CR and PPR coefficients are given. These make up a so-called "perfect hierarchical scale", with CR $>.90$ and PPR $>.60$ for each experiment.

TABLE 3a

FREQUENCY OF SUCCESS AT INDIVIDUAL ITEMS OF ORANGE JUICE EXPERIMENT, WITH ITEMS ORDERED ACCORDING TO SUCCESS, CATEGORIZATION, AND RESULTS AT DEVELOPMENTAL SCALOGRAM

ITEM	COMPOSITION	FREQUENCY[a] OF SUCCESS	ERRORS AT[b] SCALOGRAM	STAGE	PERIOD
2	1/2 vs. 1/3	149	-	IB	I
1	2/5 vs. 4/5	147	2	IA	
4	2/4 vs. 3/6	118	13	IIA	II
3	1/3 vs. 3/9	115	13		
5	3/5 vs. 6/10	101	18	IIB	
6	4/6 vs. 6/9	71	34		
7	3/1 vs. 5/2	68	21		IIIA
10	6/3 vs. 7/4	67	18	$IIIA_{int}$	
9	6/2 vs. 8/3	64	15		
8	2/3 vs. 7/9	61	6	$IIIA_{ext}$	
12	8/5 vs. 5/3	32	4	IIIB	IIIB
11	3/5 vs. 5/8	31	-		
	CR[c]		0.92		
	PPR[d]		0.72		

Notes -

[a] Total numbers of subjects : N = 149

[b] These values correspond to failures only at items easier than last item passed.

[c] Coefficient of reproducibility at developmental scalogram.

[d] Coefficient of scalability at developmental scalogram.

TABLE 3b

FREQUENCY OF SUCCESS AT INDIVIDUAL ITEMS OF SHARING COOKIES EXPERIMENT, WITH ITEMS ORDERED ACCORDING TO SUCCESS, CATEGORIZATION, AND RESULTS AT DEVELOPMENTAL SCALOGRAM

ITEM	COMPOSITION	FREQUENCY[a] OF SUCCESS	ERRORS AT[b] SCALOGRAM	STAGE	PERIOD
2	1/2 vs. 1/3	149	-	IB	I
1	2/5 vs. 4/5	146	3	IA	
4	2/4 vs. 3/6	135	10	IIA	IIA
3	1/3 vs. 3/9	128	11		
10	6/3 vs. 7/4	112	23	$IIIA_{int}$	
9	6/2 vs. 8/3	110	25		
7	3/1 vs. 5/2	109	20		
5	3/5 vs. 6/10	103	18	IIB1	IIB and IIIA
8	2/3 vs. 7/9	71	24	$IIIA_{ext}$	
6	4/6 vs. 6/9	64	9	IIB2	
11	3/5 vs. 5/8	28	3	IIIB	IIIB
12	8/5 vs. 5/3	27	-		
	CR[c]	.92			
	PPR[d]	.62			

Notes - a, b, c, d (See Table 3a)

TABLE 3c

FREQUENCY OF SUCCESS AT INDIVIDUAL ITEMS OF COMPARISON OF FRACTIONS EXPERIMENT, WITH ITEMS ORDERED ACCORDING TO SUCCESS, CATEGORIZATION, AND RESULTS AT DEVELOPMENTAL SCALOGRAM

ITEM	COMPOSITION	FREQUENCY[a] OF SUCCESS	ERRORS AT[b] SCALOGRAM	STAGE	PERIOD
1	2/5 vs. 4/5	146	3	IA	I
2	1/2 vs. 1/3	139	8	IB	
4	2/4 vs. 3/6	132	14	IIA	IIA
3	1/3 vs. 3/9	127	13		
5	3/5 vs. 6/10	126	15	IIB1	
10	6/3 vs. 7/4	126	12		
9	6/2 vs. 8/3	125	12	$IIIA_{int}$	IIB and IIIA
7	3/1 vs. 5/2	118	17		
6	4/6 vs. 6/9	111	11	IIB2	
8	2/3 vs. 7/9	103	5	$IIIA_{ext}$	
11	3/5 vs. 5/8	86	2	IIIB	IIIB
12	8/5 vs. 5/3	80	-		
	CR[c]	.94			
	PPR[d]	.70			

Notes - a, b, c, d (See Table 3a)

CATEGORIZATION

The scale is then cut up in categories. It is found that, whereas in ratio, IIB items are all passed before IIIA items meet with success, in fraction, this is not the case. The difference between ratio and fraction is explained by the greater importance of "between" relations in ratio and of "within" relations in fraction..

Moreover, although the four embodiments of rational number presented here show a great similarity in pattern construction, the correlation between subjects is low. This confirms, behind certain specific differences, a result already obtained by Steffe and Parr (1968).

CHRONOLOGICAL DIFFERENTIATION

The grade distributions for each category of the various experiments are given in Tables 4a, 4b, 4c and 4d. The distributions of two adjacent categories were compared with the Kolmogorov-Smirnov test (Siegel 1956). It was found necessary to calculate two different probabilities: p_1 when $IIIA_{int}$ and $IIIA_{ext}$ were separated, p_2 when they were grouped. A probability of $p < .07$ was accepted as significant, as the test is not powerful, with values above considered non significant (n.s.).

In the Orange Juice Experiment bearing on ratio (Table 4a), stages II (A + B), IIIA (int + ext) and IIIB were significantly differentiated, without any differentiation between $IIIA_{int}$ and $IIIA_{ext}$.

In the Sharing Cookies Experiment bearing on fractions (Table 4b), stages $IIIA_{int}$ and $IIIA_{ext}$ were differentiated, and IIIB from the two when grouped.

In the Comparison of Numerical Fractions Experiment (Table 4c), only stage IIIB was differentiated from the test.

In the Addition of Fractions (Table 4d), this was also the case but at a still higher level of probability.

TABLE 4a

DEVELOPMENTAL STAGES IN ORANGE JUICE EXPERIMENT: STAGE/GRADE MATRIX

GRADES	N	STAGES				
		I	II	$IIIA_{int}$	$IIIA_{ext}$	IIIB
Elem 6	25	9	8	4	4	0
Sec. I	27	4	15	3	4	1
" II	25	0	5	4	6	10
" III	28	3	8	5	6	6
" IV	17	0	4	1	5	7
" V	27	1	3	5	7	11
TOTAL	149	17	43	22	32	35
p_1 [a]		-	<.06	n.s.	n.s.	<.07
p_2 [b]		-	-	<.05		<.07

Notes - [a] p_1 = Levels of significance of the difference in the distribution of grades for two adjacent stages, when $IIIA_{int}$ and $IIIA_{ext}$ are separated (Kolmogorov-Smirnov).

[b] p_2 = Levels of significance of the difference in the distribution of grades for two adjacent stages, when $IIIA_{int}$ and $IIIA_{ext}$ are grouped (Kolmogorov-Smirnov).

TABLE 4b

DEVELOPMENTAL STAGES IN SHARING COOKIES EXPERIMENT: STAGE/GRADE MATRIX

GRADES	N	STAGES				
		I	II	$IIIA_{int}$	$IIIA_{ext}$	IIIB
Elem 6	25	1	10	14	0	0
Sec. I	27	2	7	8	10	0
" II	25	0	1	5	7	12
" III	28	0	1	11	13	3
" IV	17	0	0	6	6	5
" V	27	1	1	5	9	11
TOTAL	149	4	20	49	45	31
p_1 [a]		-	-	<.02	<.02	n.s.
p_2 [b]		-	-	<0.001		<.06

Notes - [a,b] (See Table 1a)

TABLE 4c

DEVELOPMENTAL STAGES IN COMPARING NUMERICAL FRACTIONS : STAGE/GRADE MATRIX

GRADES	N	STAGES				
		I	II	IIIA$_{int}$	IIIA$_{ext}$	IIIB
Elem 6	25	4	5	9	7	0
Sec. I	27	0	2	8	2	15
" II	25	0	0	1	2	22
" III	28	0	3	3	3	19
" IV	17	0	0	4	0	13
" V	27	0	0	5	3	19
TOTAL	149	4	10	30	17	88
p_1 [a]		-	n.s.	n.s.	n.s.	<.01
p_2 [b]				n.s.		<.001

Notes - [a,b] (See Table 1a)

TABLE 4d

DEVELOPMENTAL STAGES IN ADDITION OF NUMERICAL FRACTIONS : STAGE/GRADE MATRIX

GRADES	N	STAGES				
		I	II	IIIA$_{int}$	IIIA$_{ext}$	IIIB
Elem 6	25	1	2	0	3	19
Sec. I	27	0	0	0	0	27
" II	25	0	1	0	0	24
" III	28	0	0	0	2	26
" IV	17	0	1	0	0	16
" V	27	0	0	0	0	27
TOTAL	149	1	4	0	5	139
p_1 [a]		-	-	-	-	<.001
p_2 [b]		-	-	-	-	<.001

Notes - [a,b] (See Table 1a)

REFERENCES

NOELTING, G., (1980a) The development of proportional reasoning and the Ratio concept. *PART 1: Determination of Stages.* Educational Studies in Mathematics, Vol II, pp. 217-253.

PART 11: Problem-Structure at Successive Stages, Problem-Solving strategies and the Mechanism of Adaptive Restructuring. Educational Studies in Mathematics. (in press)

NOELTING, G., (1980b) *The Qualitative and Quantitative aspects in the Development of Proportional Reasoning,* in Friedman, M., Das, J.P., and O'Connor, M., (eds) Intelligence and Learning, Plenum.

SIEGEL, S., (1956) *Nonparametric Statistics for the behavioral Sciences,* New-York, McGraw-Hill Book Company.

STEFFE, L.P., PARR, R.B., (March 1968) *The development of the concepts of ratio and fraction in the fourth, fifth, and sixth years of the elementary school,* Technical Report no. 49. Research and Development Center for Cognitive Learning, Madison: University of Wisconsin.

EARLY ADOLESCENTS' STRUCTURE OF PROPORTIONAL REASONING

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This study was designed to clarify the use of proportional reasoning by subjects between twelve and fourteen years of age. The variables considered were task structure (integral vs. non-integral ratios) and task demand (comparison vs. adjustment of ratios). The presence of integral ratios, especially within a recipe, made items easier. Additive procedures were widely used instead of proportional reasoning on items that did not have integral ratios. Task demand had little effect. The findings lead to the conclusion that the consistent use of proportional reasoning is not a developmental outcome in this age range.

Research results have consistently pointed to the great difficulty presented by proportional reasoning to many early adolescent students even though the study of ratio and proportion is a part of school mathematics instruction from the fifth grade onward. Research conducted about fifteen years ago (Lovell and Butterworth, 1966; Lunzer and Pumfrey, 1966) showed that proportional reasoning, at that time viewed as part of Piaget's stage of formal operational thought, was achieved considerably later than less quantitative aspects of formal thought. Some recent investigations have followed one of two distinct approaches and substantiated the earlier findings. In one series of studies (Noelting, in press), subjects were asked to compare ratios represented by orange juice mixtures prepared from concentrate and water in various amounts. In a second series (Karplus, Karplus, Formisano and Paulsen, 1979; Karplus, Karplus and Wollman, 1974; Kurtz and Karplus, 1979), subjects were asked to find an unknown quantity by equating two ratios of lengths, distances, or volumes.

The present study was designed to clarify the relationship between Noelting's and Karplus's investigations of the development of proportional reasoning. The subjects were selected and the tasks designed to answer the following questions:

1. Given a context in which comparison of the ratios of two ordered pairs is appropriate, what forms of reasoning do early adolescent students actually apply, and how frequently?
2. What differences in reasoning are elicited by tasks that require comparison of ratios (i.e. the Noelting procedure) vs. tasks using the same data but requiring the calculation of an unknown numerical value (Karplus procedure)?
3. To what extent is a subject's reasoning influenced by the structure of data

actually employed in the task? By "structure" we refer to the occurrence of integral ratios.

4. How consistently does a subject use a particular form of reasoning?

THE LEMONADE PUZZLES

To relate our investigation to the work of Noelting (in press), we devised the Lemonade Puzzles. In these puzzles, John and Mary prepare lemonade concentrate by dissolving a certain number of spoonfuls of sugar in a certain number of spoonfuls of lemon juice. For instance, John might use four spoonfuls of sugar and six spoonfuls of lemon juice, while Mary uses ten and fifteen spoonfuls, respectively.

We chose this setting for two reasons: (1) spoonfuls are smaller units of measure than Noelting's glasses and can therefore be considered a continuous rather than discrete measure; continuity of quantity was further emphasized in the illustrations, which presented the ingredients in measuring cups calibrated in spoonfuls; (2) both sugar and lemon juice are materials with distinctive properties, while the water mixed with orange juice in Noelting's tasks forms a neutral background.

Eight types of tasks were presented to the subjects (Table 1). The actual amounts of sugar and lemon juice in each item were selected to form various integral and non-integral ratios. Unlike Noelting, however, we avoided all single unit quantities since these could be used to form unit ratios both within a recipe and by comparing quantities between the two recipes.

Table 1. Data[a] Used in the Lemonade Puzzle Tasks.

	John's sugar	John's lemon	Mary's sugar	Mary's lemon		John's sugar	John's lemon	Mary's sugar	Mary's lemon
Item WB[b]	2	6	8	24	Item WBX[b]	3	6	9	15
	2	8	6	24		3	9	6	15
Item W	3	12	5	20	Item WX	4	12	6	16
	3	6	7	14		6	12	8	15
Item B	3	7	6	14	Item BX	6	8	12	15
	3	5	12	20		4	6	12	16
Item N	4	6	10	15	Item NX	3	5	7	11
	4	10	6	15		3	7	5	11

(a) Two versions of data were used, each with 60 subjects. This report will always refer to the combined results.

(b) W: within recipe ratio integral N: no ratio integral
B: between recipe ratio integral X: unequal ratios

THE SUBJECTS

Our subjects were 60 sixth and 60 eighth graders from a middle class suburban school with an ethnically mixed population performing near the average of standardized mathematics achievement tests. Half of each group were boys and half girls, selected randomly from an enrollment of about 150 at each grade level.

PROCEDURE

The Lemonade Puzzles were administered in about fifteen minutes during a 35-minute individual interview session that also included other items. Two or four questions were asked.

1. (Comparison) Whose lemonade concentrate is sweeter, John's or Mary's, or are they equal?
2. (After the response to #1) How did you come up with this answer?
3. (Adjustment: after the response to #2, if response #1 indicated unequal taste) How much lemon juice would Mary need with her ___ spoonfuls of sugar to make her concentrate taste just like John's?
4. (After the response to #3) Please explain your answer.

SCORING OF DATA

For the purposes of this preliminary report, we classified the students' explanations in answer to Questions 2 and 4 into the following four categories:

Category I (incomplete, illogical) -- don't know, guess, inappropriate quantitative operations.

Category Q (qualitative) -- qualitative comparison of amounts, using "more" or "less" or equivalent terms.

Category A (additive) -- using the data to compute and compare differences or remainders.

Category R (ratio) -- using the data to compute and compare correct ratios, possibly with arithmetic errors.

We also identified the data comparisons (sugar/lemon vs. sugar/sugar) that were used. Other aspects of the students' responses are still under study and will be reported separately.

RESULTS AND DISCUSSION

When the comparison and adjustment explanations are examined, it is clear that the eight items fall into two groups of four -- the "equal" items, WB, W, B, and N; and the unequal items, WBX, WX, BX, and NX. To provide an informative overview of our results, we have averaged the frequencies of all student responses in each category for the four items in each group and present these in Table 2.

Table 2. Average Response Frequency in Each Category (percent)

Category	Average for "Equal" Items WB,W,B,N		Average for "Unequal" Items WBX,WX,BX,NX	
	Comparison	Adjustment	Comparison	Adjustment
I	6	17	7	18
Q	28	6	26	3
A	23	28	43	44
R	43	0	25	25
Total	100	51*	100	90*

*Less than 100% because students who answered "equal" to the comparison question for an item were not asked the adjustment question for that item.

Certain overall features of our observations can be recognized in Table 2. First, 43 percent of the subjects used proportional reasoning on the "equal" items, while only 25 percent used proportional reasoning on the "unequal" items. Additive answers increased on the "unequal" items. The latter items are therefore clearly shown to be more difficult.

Category I and Category Q responses had virtually identical frequencies for the two groups of questions. We conclude that subjects giving these answers did not, in fact, perceive any distinction between the two groups of items. About half of the substantial number of students using Category Q for their answer on the comparison changed to Category I for the adjustment explanation, which required a more quantitative analysis.

Consistency of responses. Very few students' responses fell into the same category on all eight items. To obtain an overview of their consistency, we defined three levels of category use: high (6-8 responses in the category), medium (3-5 responses), and low (0-2 responses). Furthermore, we combined a student's responses to the comparison and adjustment questions of an item by using the higher category if there was a difference. Table 3 gives the results.

Table 3. Consistency of Responses (number of students).

High R	24	Medium R + Medium A	29	Medium R + Low other	2
High A	28	Medium R + Medium Q	1	Medium A + Low other	4
High Q	15	Medium A + Medium Q	6	Medium Q + Low other	6
High I	1	Medium Q + Medium I	2	Medium I + Low other	2

Note that 81 students' responses were almost completely included in Categories R and A, with about one-third of this number using ratios consistently, one-third using addition, and one-third using both approaches about equally. It appears, therefore, that proportional and additive reasoning are not so firmly established but that 29 students switched back and forth, strongly influenced by the numerical values.

Information processing patterns. We have pointed out that a lemonade puzzle can be solved by relating the amounts of sugar and lemon juice within each recipe first, then comparing recipes; or by relating the amounts of sugar and lemon juice between recipes, then comparing these relations. We found that 60 percent of all answers to comparison questions made use of the within recipe approach and that a little less than 20 percent used the between recipe approach. The remaining responses made use of only a single comparison or the nature of the processing pattern could not be identified. Except for minor fluctuations, these results also describe the responses to any one item.

When one considers only those subjects who used proportional reasoning on six or more items, however, one finds a different result. On the B items, which had an integral "between" ratio, 75 percent used the "between" comparison. On all other items, including the items BX, 60 to 90 percent used the "within" comparison. In other words, many of these students responded to the simple aspect of items B but not to that of items Bx, another indication of the difficulty of an unequal ratio task.

Effect of structure on proportional reasoning. Students' success in proportional reasoning on the eight types of items is displayed in Table 4, where the columns are headed by the structural designations introduced in Table 1. The items with structure WB or W were solved by about 60 percent of the subjects, while the items with structure NX, which required finding of a common denominator, were solved by only 10 percent. Intermediate in difficulty were items with structures B, WBX, or WX, near 35 percent in the use of ratio, and the items with structures N or BX, at a level of about 20 percent.

Table 4. Frequencies of Proportional Reasoning According to Item Structure (percent)

Sample	WB	W	B	N	WBX	WX	BX	NX
Comparison	55	63	38	18	34	35	19	10
Adjustment	-	-	-	-	34	35	23	4

Our earlier presentation and a more detailed analysis (not included here) indicate that lower rates of proportional reasoning on an item are accompanied by higher rates of additive reasoning. The combined use of these two quantitative approaches that employ all the data is near 65 percent for all items. We therefore have a substantial group of students whose behavior is consistent with two hypotheses: (1) they examine the data to find within recipe relationships; (2) they use operations that avoid fractions but know that proportional reasoning is applicable to the lemonade puzzles.

The theoretical analysis of Case (1979), which rested on working memory demands, did not lead to a clear distinction between additive and proportional approaches on that basis. It is therefore consistent with our observation of unstable behavior by many students. Yet it does not shed any light on why within recipe relations are favored so greatly, or why fractions are avoided. Noelting's (in press) theoretical comments do not clarify these issues either. We believe that the greater salience (Linn, in press) of the "within" relationships and a "fraction avoidance" syndrome resulting from elementary mathematics instruction explain our observations.

CONCLUSIONS

The outcomes of this study confirm previous observations in several respects. Noelting's (in press) identification of easier and more difficult problems depending on the occurrence of integral ratios was extended by a more careful design of item structure.

The extensive use of additive procedures was also confirmed, especially on items with non-integral ratios and even on items with integral "between" ratios. The readiness with which almost 25 percent of the subjects applied proportional and additive reasoning alternatingly as the task structure changed suggests that avoidance of fractions is a more compelling influence than maintaining an appropriate mathematical relationship. In our judgment, the evidence suggests that consistent use of proportional reasoning is not a developmental outcome, but depends instead on overcoming task related obstacles. A suitable teaching program should therefore be highly effective, as was found by Kurtz and Karplus (1979).

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REFERENCES

Case, R. Intellectual development and instruction: A neo-Piagetian view. In A. E. Lawson (Ed.) The psychology of teaching for thinking and creativity. Columbus, Ohio: ERIC/SMEAC, 1979.

Karplus, R., Karplus, E., Formisano, M., and Paulsen, A.C. Proportional reasoning and control of variables in seven countries. In J. Lochhead and J. Clements (Eds.) Cognitive process instruction. Philadelphia, PA: Franklin Institute Press, 1979, 47-108.

Karplus, R., Karplus, E. F., and Wollman, W. Intellectual development beyond elementary school IV: Ratio, the influence of cognitive style. School Science and Mathematics, 1974, 76(6), 476-482.

Kurtz, B., and Karplus, R. Intellectual development beyond elementary school VII: Teaching for proportional reasoning. School Science and Mathematics, 1979, 79(5), 387-398.

Linn, M. When do adolescents reason? European Journal of Science Education, in press

Lovell, K., and Butterworth, I. B. Abilities underlying the understanding of proportionality. Mathematics Teaching, 1966, 37, 5-9.

Lunzer, E. A., and Pumfrey, P. D. Understanding proportionality. Mathematics Teaching, 1966, 34, 7-12.

Noelting, G. The development of proportional reasoning and the ratio concept. Educational Studies in Mathematics, in press.

COGNITIVE CORRELATES OF PROPORTIONAL REASONING IN EARLY ADOLESCENCE

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This study examined the cognitive correlates of proportional reasoning in early adolescents. Correlates of competence and performance in proportional reasoning were examined separately. The sole predictor of competence was processing capacity, while the predictors of performance were fluid intelligence and cognitive restructuring. After the relationship with the cognitive variables were removed, gender and grade effects were found only in the performance analysis. It was suggested that variation in competence was due mainly to differences in processing capacity, while variation in performance once competence is acquired is due to differences in representational skills and specific experience.

In order to understand proportional reasoning it is necessary to identify both the specific mathematical operations and the general cognitive factors involved in the development and employment of proportional reasoning. In such an investigation it is important to distinguish between the correlates of proportional reasoning competence, and the correlates of its use in a wide range of situations once the competence is present, i.e., proportional reasoning performance (cf. Flavell & Wohlwill, 1969).

An examination of the literature on proportional reasoning suggests a number of possible cognitive correlates. According to Inhelder and Piaget (1956) proportional reasoning requires the presence of the formal reasoning structure. Siegler (1978) has suggested that encoding (representational) skills are essential for proportional reasoning. While Siegler did not suggest what general cognitive dimensions these encoding skills may be related to, an examination of the research on human abilities suggests a number of dimensions, such as: crystallized intelligence (having the appropriate representational units in the repertoire), fluid intelligence (being able to discover the appropriate encodings), divergent thinking (being able to generate many alternative encodings), and cognitive restructuring (being able to abandon one encoding in order to construct a more appropriate one). Pascual-Leone (de Ribaupierre & Pascual-Leone, 1979; Furman, 1980) and Case (1979) suggest, with some empirical evidence, that a certain information processing capacity (working memory, termed M capacity in Pascual-Leone's system) is necessary to engage in successful proportional reasoning. Pascual-Leone's analysis also suggests that field dependency and cognitive restructuring moderate the

the acquisition and manifestation of proportional reasoning. It is the purpose of the present study to examine the extent to which these potential correlates are actually related to proportional reasoning per se and also to clarify the cognitive dimensions that underly the essential encoding process in proportional reasoning.

Method

Of the 120 subjects completing the lemonade task who are described in Karplus, Pulos, and Stage (this volume) there were 87 who also completed the set of seven group administered tasks that test cognitive correlates. This sample of 87 was almost equally divided between males and females, sixth and eighth graders. The students were tested on the group tasks sometime after the individual interviews in two sessions during regular class periods in groups of approximately 30 students each.

The cognitive correlates tasks, all of which have been used previously with similar populations and found to have satisfactory reliability and validity, were: 1) a measure of process (M) capacity -- the Figural Intersection Task, FIT (Pascual-Leone & Burtis, 1974); 2) a measure of fluid intelligence -- a wide range series completion test; 3) a measure of crystallized intelligence -- a wide range vocabulary test; 4) a measure of cognitive restructuring -- the FASP embedded figures test (Pulos & Linn, 1979); 5) a measure of field dependency -- the Water Level Task, WLT (Pascual-Leone, 1974); 6) a measure of "formal reasoning" -- conservation of volume; and 7) a measure of divergent thinking, an alternative uses test (Wallach & Kogan, 1965).

Stepwise multiple regression analyses were employed to examine the relationship between proportional reasoning and the cognitive tests. After all cognitive variables were entered, grade and gender were then allowed to enter.

Two types of analyses were conducted in order to examine the cognitive correlates of competence and performance in proportional reasoning separately. In the competence analyses, data from all subjects were used and the dependent variable was a dichotomous score based upon whether or not the subject ever employed proportional reasoning in the lemonade puzzle. The performance analysis was conducted only on those subjects who manifested proportional reasoning on at least one item in the lemonade puzzle. The dependent variable in this analysis was the number of problems on which proportional reasoning was used.

Results and Discussion

The distribution of the number of proportional reasoning responses is presented in Table 1. This distribution does not appear to significantly differ from the distribution for the entire sample described in the preceeding paper.

Table 1. Distribution of the Numbers of Proportional Reasoning Responses (%)

Grade	Number of Proportional Responses								
	0	1	2	3	4	5	6	7	8
6 (N=44)	32	11	16	14	14	7	2	0	4
8 (N=43)	14	5	19	7	7	12	12	12	12

Competence Analysis. The stepwise regression for the competence analysis (Table 2) indicated that the only test significantly increasing the prediction was the FIT. This result suggests that of the cognitive factors investigated in the current study, the principle determinent of competence in proportional reasoning is processing (M) capacity.

Table 2. Stepwise Regression for the Competence Analysis

Variable[1]	R	r	R^2 change	F to enter	p
FIT	.54	.54	.29	25.75	.001
WLT	.56	.41	.03	2.80	ns
Volume	.58	.41	.02	1.88	ns
FASP	.59	.26	.01	.82	ns
Series	.61	.09	.02	1.87	ns
Vocabulary	.61	.09	.00	.40	ns
Uses	.61	-.11	.00	.01	ns
Grade Level	.63	.17	.03	2.95	ns
Gender	.63	-.20	.00	.20	ns

[1]Variables listed in the order of entry

In order to clarify the relationship between the FIT, M capacity, and the competence score, subjects were classified into discrete groups based upon their estimated M capacity. The percent of subjects engaging in proportional reasoning was: 24% for the group with an M capacity of four (the M-4 group), 88% for the M-5 group, 96% for the M-6 group, and 90% for the M-7 group. It appeared that a major discontinuity occurred only between the M-4 and M-5 groups. This observation is supported by a one-way ANOVA over the four groups, which yielded a significant overall main effect for M capacity ($F=18.99$, df 3, 82; p .001; eta =.41). The only significant difference between successive

groups (beyond the .01 level according to the Scheffe test) was between the M-4 and M-5 groups. It is interesting to note that task analyses and empirical results have found that either an M capacity of four (Case, 1980) or six (Furman, 1980; de Ripaupierre & Pascual-Leone, 1979) is necessary for the solution of proportional reasoning tasks, depending upon the task and strategy employed by the subjects. It would appear that the lemonade puzzles require somewhat more processing capacity than the task analyzed by Case and somewhat less capacity than the tasks studied by Pascual-Leone and associates. In any event, the results of the current investigation suggest that the subject must be able to coordinate at least five schemes (pieces of information) in order to manifest proportional reasoning on one or more lemonade puzzles.

Performance Analysis. The performance analysis was conducted on the 66 subjects who exhibited proportional reasoning on at least one of the eight problems. The stepwise regression (Table 3) for the performance score found a significant increase in prediction for only two cognitive variables, series completion and the FASP, and for grade level and gender.

Table 3. Stepwise Regression for the Performance Analysis

Variable[1]	R	r	R^2 change	F to enter	P
Series	.42	.42	.18	9.85	.01
FASP	.53	.35	.10	6.21	.01
Volume	.55	.18	.02	1.02	ns
WLT	.55	.14	.01	.62	ns
Uses	.57	.00	.01	.75	ns
Vocabulary	.57	.20	.01	.57	ns
FIT	.58	.18	.00	.04	ns
Gender	.65	-.21	.09	5.63	.01
Grade Level	.71	.37	.09	6.61	.01

[1]Variables listed in the order of entry

The relationship between the two significant cognitive variables and the proportional reasoning performance score may be better understood by examining task analyses for these two cognitive measures. Analyses of series completion tasks (e.g. Horn, 1978; Simon, 1976) suggest that one of the main task components which may be related to proportional reasoning is facility in the encoding of relationships among elements in a problem. Such a facility would appear to be important in tasks like the lemonade puzzle where the proper encoding of the relationships among the elements can greatly reduce the difficulty of the item, e.g., on some problems, those subjects who can perceive and represent

the between integral ratio will be faced with a simpler problem than those subjects who do not or cannot represent the between ratio and instead focus on the non-integral within ratio.

Analyses of embedded figure tests (Pascual-Leone, 1974; Witkin & Goodenough, 1977) suggest that a key component in these tasks which may be related to the generality of proportional reasoning is the ability to overcome initial responses or perceptions in order to emit a less common response or to construct a less salient encoding of the problem. In terms of the proportional reasoning task it may be that subjects low in cognitive restructuring tend to make problems more difficult by encoding them in a habitual manner and not being able to restructure the problem appropriately, e.g., in the lemonade puzzle subjects may look at only the within ratio when solving the problem rather than restructuring the encoding of the problem in terms of a between ratio when such a restructuring will lead to an easier solution.

It is interesting to note that the mere presence of a large number of verbal representational units in the subject's repertoire, as measured by the vocabulary test, or the ability to give a large number of possible encodings, as assessed by the alternative uses test, do not appear to be related to proportional reasoning, as least in the sample examined in this study.

In other words, it is postulated that fluid intelligence and cognitive restructuring are important in generalizing the application of proportional reasoning because they facilitate the most appropriate encoding of the problem. Fluid intelligence provides the basic competence to the easiest encoding, while cognitive restructuring allows the subject to overcome an initial encoding or a salient encoding in order to construct the most appropriate encoding.

The finding that the cognitive variables did not fully remove the relationship between grade and gender with the performance score suggest that other factors affect the proportional reasoning performance. While these hidden variables cannot be ascertained from the current study, some suggestions as to their nature may be offered. It seems possible that the grade effect may be due to specific mathematical instruction which occurs between the sixth and eighth grades. The gender effect may be due to attitudinal factors which may affect females' orientation toward mathematics. Both of these hypotheses suggest that the grade and gender effects are due to specific experience and/or

attitudes rather than to any general cognitive factors (see Stage, Karplus & Pulos, this volume).

Conclusions

The results of the present study suggest that competence in proportional reasoning is largely related to information processing capacity, while the generality of the usage of proportional reasoning is related to encoding skills. These conclusions would appear to support both Pascual-Leone's model of proportional reasoning and Siegler's model. In view of the lack of relationship between proportional reasoning and the conservation of volume task, the results do not seem to support the hypothesis that a formal reasoning structure, as defined by Inhelder and Piaget, is a necessary prerequisite for proportional reasoning. These results also support educational procedures based upon teaching proportional reasoning with emphases on minimizing information processing demands (e.g., Case, 1980) and on encoding skills (Kurtz & Karplus, 1979).

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References

Case, R. Intellectual development and instruction: A neo-Piagetian view. In A. E. Lawson (Ed.) The psychology of teaching for thinking and creativity. Columbus, Ohio: ERIC/SMEAC, 1980.

deRibaupierre, A., and Pascual-Leone, J. Formal operations and M power. In D. Kuhn (Ed.) Intellectual development beyond childhood. San Francisco: Jossey-Bass, 1979.

Flavell, J., and Wohlwill, J. Formal and functional aspects of cognitive development. In D. Elkind & J. Flavell (Eds.) Studies in cognitive development. New York: Oxford University Press, 1969.

Furman, I. M demand of problem solving strategies in Piaget's balance-scale task. Paper presented at the annual meeting of the American Psychological Association, 1980.

Horn, J. Human ability systems. In P. Baltes (Ed.) Life-span development and behavior. New York: Academic Press, 1978.

Inhelder, B., and Piaget, J. The growth of logical thinking from childhood to adolescence. New York: Basic Books, 1958.

Karplus, R., Pulos, S., and Stage, E. K. Early adolescents' structure of proportional reasoning. In R. Karplus (Ed.) Proceedings of the Fourth International Conference for the Psychology of Mathematics Education. Berkeley, CA: Lawrence Hall of Science, 1980.

Kurtz, B., and Karplus, R. Intellectual development beyond elementary school VII: Teaching for proportional reasoning. School Science and Mathematics, 1979, 79(5), 387-398.

Pascual-Leone, J. A neo-Piagetian process-structural model of Witkin's psychological differentiation. Paper given at the International Association for Cross-Cultural Psychology, Kingston, 1974.

Pascual-Leone, J., and Burtis, J. The FIT: Figural Intersection Test. A group measure of M capacity. Unpublished manuscript, York University, 1974.

Pulos, S., and Linn, M. C. The Find a Shape Puzzle (FASP). Berkeley, CA: Lawrence Hall of Science (Adolescent Reasoning Project), 1979.

Siegler, R. S. The origins of scientific reasoning. In R. S. Siegler (Ed.) Children's thinking. Hillsdale, NJ: LEA, 1978.

Simon, H. Identifying basic abilities underlying intelligent performance on complex tasks. In L. B. Resnick (Ed.) The nature of intelligence. Hillsdale, NJ: LEA, 1976.

Wallach, M., and Kogan, N. Modes of thinking in young children. New York: Holt, Rinehart & Winston, 1965.

Witkin, H., and Goodenough, D. Field dependence revisited (ETS Research Bulletin 77-16). Princeton, NJ: Educational Testing Service, 1977.

SOCIAL CONTEXT OF EARLY ADOLESCENTS' PROPORTIONAL REASONING

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This study examined the social context in which early adolescents' performance on proportional reasoning tasks took place. Attitudes toward mathematics were ascertained from survey and interview responses. Analysis focused on grade and gender differences in attitudes toward mathematics in order to account for the performance differences observed in this population on those groupings. The generally positive attitude toward mathematics was composed of different elements for different students and patterns of attitudes paralleled patterns of performance.

Pulos, Stage, and Karplus (this volume) distinguished between competence and performance in proportional reasoning. Following Flavell and Wohlwill (1969), competence was inferred from the demonstration of proportional reasoning at any appropriate time. Performance was designated as the number of times (of eight possible) that a subject appropriately applied proportional reasoning to the task. Regression analyses using seven cognitive tests showed that processing capacity alone accounted for a significant proportion of the variance in competence. Cognitive restructuring and fluid ability were related to performance.

Grade and gender were entered into regressions after the cognitive test scores. These variables did not account for significant proportions of the variance in competence. Each accounted for 9% of the variance in performance. Pulos et al. speculated that there were experiential and attitudinal factors that probably accounted for these contributions. The contribution of grade (following removal of cognitive factors) is likely to be related to the mathematics instruction received in 6th, 7th, and 8th grades, which is almost saturated with the manipulation of fractions. The contribution of gender to the regression is probably more complex, given the interrelationships among attitude, aspirations, and achievement.

For many years it was accepted that sex differences in mathematical ability, favoring males, emerge at adolescence (Maccoby and Jacklin, 1974). More recently, counter evidence has been provided by Fennema and Sherman (1978), who investigated the achievement of high school students in light of their mathematics backgrounds and found that males and females with similar course enrollment had similar achievement. They further found that superior mathematics achievement by males was only found in conjunction with less favorable attitudes by and towards females.

This study was designed to examine the performance of the subjects by grade and by gender, to investigate their attitudes towards mathematics and school, and to relate their attitudes to their performance.

METHOD

Of the 120 subjects completing the lemonade task described by Karplus et al (this volume) there were 82 students who also completed the Math Attitude Survey and 97 who completed the Math Attitude Interview. Since the Math Attitude Survey was administered in classroom groups of approximately 30 students each, classmates of the subjects in the other studies increased the sample size to 223 for some statistical analyses requiring larger sample sizes.

The Math Attitude Survey consists of 24 statements about mathematics which are rated on a 4 point Likert scale. It is an adaptation of a survey used to evaluate programs at the Lawrence Hall of Science that was based on the Fennema-Sherman Mathematics Attitudes Scales (Fennema and Sherman, 1976). It has acceptable reliability, with an alpha of .75. It was administered in in conjunction with the cognitive tests described in Pulos et al (this volume).

The Math Attitude Interview was developed for this study on the basis of classroom observations, informal conversations with students, and pilot interviews. Its purpose was to clarify and contextualize the responses to the Math Attitude Survey. It was administered by trained examiners as the first part of an interviewing session. Subjects were interviewed by someone other than the person who had administered the lemonade task.

RESULTS AND DISCUSSION

The distribution of performance on the lemonade puzzle is presented in Table 1.

Table 1. Frequencies of Proportional Reasoning According to Item Structure (percent)

Sample	WB	W	B	N	WBX	WX	BX	NX
All subjects	55	63	38	18	34	35	19	10
Grade 6	45	55	22	3	22	27	3	3
Grade 8	65	72	55	33	47	43	35	17
Male	63	70	53	27	42	37	32	17
Female	47	57	23	10	27	33	7	3

For all subjects, within recipe comparisons (W) are more likely to lead to use of proportional reasoning than are between recipe comparisons (B); integral ratios are more likely to lead to use of proportional reasoning than are non-integral ratios (X). On all structures, grade 8 students use proportional

soning more often than do females. As problems become more difficult, however, the decrement in performance increases. In both cases, the advantage of the more proficient group increases from 3:2 to 3:1. It appears that the increased demand of the between recipe comparisons and the non-integral ratios has more influence on the younger and the female students than on the older and the male students.

The responses on the Math Attitude Survey are presented in Table 2. This sample of early adolescents report generally favorable attitudes towards mathematics, recognizing that it is and will continue to be important and useful to them. They agree that "everyone can learn math," that "they will need math to do well in school, and that "knowing math helps (them) in many ways."

Table 2. Math Attitude Survey and Responses

Item #		Mean	Std. Dev.	n = 82
		(1 = never to 4 = always true)		
1.	I like to do math puzzles.	2.55	.86	
2.	No matter how hard I study, I will get low grades in math	1.56	.83	
3.	I like to have a definite rule to follow in math.	2.41	1.02	
4.	Outside of school, I don't use math.	1.77	.91	
5.	I like to guess an answer to see if I'm close.	1.89	.77	
6.	I think math games are interesting.	2.80	.96	
7.	Math is easy for me.	2.74	.83	
8.	I get nervous studying for a math test	1.96	.99	
9.	Most jobs require some math.	2.93	.84	
10.	I think math is boring.	2.07	.98	
11.	Drawing a picture helps me understand a problem.	2.01	.84	
12.	I do as little work as possible in math.	1.85	1.02	
13.	Knowing math helps me in many ways.	3.22	.90	
14.	Math is a lot of rules for numbers.	3.00	.89	
15.	Math is my favorite subject.	2.63	1.21	
16.	I wish I were smarter in math.	3.05	1.08	
17.	I think I could handle more advanced math classes.	2.44	1.06	
18.	I don't expect to use math when I get out of school.	1.54	.97	
19.	You don't need to have a good memory to succeed in math.	1.68	.91	
20.	I don't understand people who say they like math.	1.73	1.07	
21.	Playing games is a waste of class time in math.	1.83	1.03	
22.	I think everyone can learn math.	3.43	.93	
23.	Once I start working on a math puzzle, I find it hard to stop.	2.59	.97	

They disagree with the statements that "I don't expect to use math when I get out of school," "No matter how hard I study I will get low grades in math," and "You don't need to have a good memory to succeed in math."

Several analyses were performed to explore the gender-related differences in attitude. The results are presented in Table 3. Six items had significant sex differences for the Chi squares, none particularly illuminative of the performance difference given the lack of a difference on similarly worded items. For example, males think that math games are interesting significantly more often than females do, but the distributions of responses are similar concerning math puzzles.

Table 3. Analysis of Gender Differences on Math Attitude Survey

Item #	X^2 by sex	r with performance	r with competence	communalities males	communalities females
1.	-	.30($p \leq .001$)	.29($p \leq .001$)	.58	.67
2.	-	-	-	.28	.30
3.	p=.03 males	-	-	.38	.31
4.	-	-	-.22($p \leq .05$)	.56	.20
5.	-	.24($p \leq .05$)	.22($p \leq .05$)	.65	.62
6.	p=.03 males	-	-	.91	.73
7.	p=.02 males	-	-	.77	.60
8.	p=.01 females	-	-	.17	.50
9.		-	-	.56	.35
10.	-	-	-	.62	.48
11.	-	-	-	.47	.92
12.	p=.001 males	-	-	.32	.49
13.	-	-	-	.66	.42
14.	-	-	-	.45	.28
15.	-	-	-	.73	.64
16.	-	-.24($p \leq .05$)	-	.66	.59
17.	-	-	-	.30	.79
18.	-	-	-	.45	.53
19.	-	-	-	.50	.63
20.	-	-	-	.50	.74
21.	-	-.19($p \leq .05$)	-	.16	.24
22.	-	-	-	.34	.45
23.	-	-	.20($p \leq .05$)	.58	.52
24.	p=.01 males	-	-	.34	.43

It is interesting to note correlations of items with competence and performance that suggest an attitudinal contribution to performance. This is confirmed by the correlation of a total attitude score (the sum of the positive responses and the reflected negative responses) with performance, r=.22, p=.03; and not with competence, r=.11, p=.17.

Factor analyses for the total group, for males, and for females, revealed that some of the same items had substantially different communalities for each sex. The communality of a variable is the total variance of a variable that is accounted for by the combination of all common factors, i.e., the amount of variance that is shared by at least one other variable in the set (Kim, 1975). The communalities show that, for most of the variables on which there is a significant difference by sex, there is some difference in the amount of shared variance by sex. That may indicate that the item is not measuring the same thing in males and females. For example, on item 8, "I get nervous studying for a math test," females' greater agreement might measure agreement while males' tendence to deny the statement might be indicative of an unwillingness to admit anxiety rather than the lack of anxiety. The communalities suggest that there are several aspects of a general positive attitude toward math for one group that are operating differently for the other group, and vice versa.

Additional evidence of attitudes toward math and school in general was obtained in the Math Attitudes Interview, summarized in Table 4. Students were asked to describe their most recent school day. After this description, they were asked what the best and worst parts of the day had been. The majority of students cited a non-academic activitiy or event as the best part of the day and something associated with school as the worst part. More than a third of the students responded to the more focused "What is your favorite part of school?" with lunch, seeing friends, or a non-academic class (physical education, art, drafting). Many students indicated a particular subject as their least favorite part of school. Mathematics classes were cited as favorite more often than as least favorite. Overall, Math was chosen more often as "the subject you do best in" than any other subject, but females, sixth graders, and subjects using additive reasoning predominantly (on 6 or more of the 8 tasks) were more likely to say "English" than math. Although these proportions are not significantly different from each other at a 95% level of confidence, they add to the general picture from the global questions' answers, in which males, eighth graders, and subjects using proportional reasoning predominantly (6 or more times out of 8) gave a similar pattern of responses. A larger sample size might confirm that this group is more sophisticated and somewhat more cynical about school.

Table 4. Math Attitude Interview Responses

	all students n=97	males n=48	females n=49	6th n=51	8th n=46	high additive n=24	high proportion n=19
Best part of the day yesterday:							
lunch/friends	27%	24	31	26	28	30	15
getting out of school	26	32	20	18	35	16	40
other non-academic	22	23	20	29	13	20	15
subject/class	13	8	18	13	13	24	5
nothing	8	8	8	8	9	12	15
something special	4	6	2	6	2	0	5
Worst part of the day yesterday:							
subject/class	34	24	43	30	40	44	35
getting up/coming to school	23	29	16	15	31	16	20
something bad	19	23	14	18	20	8	25
other academic	10	10	10	20	0	16	5
other non-academic	7	6	8	8	6	8	0
nothing	7	6	8	10	4	4	10
Favorite part of school:							
non academic classes	23	23	23	4	42	8	55
lunch/friends	20	21	19	22	18	24	10
math	17	23	10	28	4	12	0
English	14	6	21	16	9	16	10
two or more subjs. w/math	10	11	10	10	11	12	10
two or more subjs. not math	8	4	13	8	9	16	0
history, social studies	6	8	4	6	7	4	5
Least favorite part of school:							
history, social studies	33	35	31	40	25	32	40
other school activities	19	12	25	14	22	12	5
English	17	21	13	23	11	8	15
math	12	9	16	20	5	28	0
foreign language	9	11	7	0	18	8	20
two or more subjs. not math	6	4	7	2	9	4	10
two or more subjs. w/math	4	6	2	0	9	0	0
Best subject:							
math	32	42	22	33	30	28	35
English	29	21	36	44	13	40	10
two or more with math	13	15	12	11	15	4	25
non academic	9	6	12	0	19	8	5
two or more not math	8	8	8	6	11	4	0
history, social studies	6	6	6	6	6	8	10
foreign language	2	2	2	0	4	0	10
Worst subject:							
history, social studies	27	27	27	29	24	20	25
English	26	32	19	39	12	40	15
math	21	12	29	22	19	20	5
two or more not math	11	10	12	5	17	5	20
foreign language	7	10	5	0	15	0	20
non academic	5	2	7	0	9	5	0
two or more with math	4	7	0	5	2	10	0

CONCLUSIONS

Examination of the performance of early adolescents on the lemonade puzzle shows that younger and female students have additional difficulty with structural changes that make tasks more difficult for all students. Responses to the Math Attitude Survey indicate that the general positive attitude toward mathematics is composed of different elements for males and females. Responses on the Math Attitude Interview further illustrate that younger, female, and additive reasoning students give similar responses to questions about school and math, while older, male, and proportional reasoning students tend to give a different set of responses. Patterns of attitudes toward math parallel patterns of performance in proportional reasoning. Bringing performance up to levels of competence may depend on greater recognition of these social context factors.

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REFERENCES

Fennema, E., and Sherman, J. A. Fennema-Sherman mathematics attitude scales. Journal Supplement Abstract Service, 1976, MS. 1225.

Fennema, E. Sex-related differences in mathematics achievement and related factors: A further study. Journal for Research in Mathematics Education, 1978, 188-203.

Flavell, J., and Wohlwill, J. Formal and functional aspects of cognitive development. In D. Elkind and J. Flavell (Eds.) Studies in cognitive development. New York: Oxford University Press, 1969.

Karplus, R., Pulos, S., and Stage, E. K. Early adolescents' structure of proportional reasoning. In R. Karplus (Ed.) Proceedings of the Fourth International Conference for the Psychology of Mathematics Education. Berkeley, CA: Lawrence Hall of Science, 1980.

Kim, J.-O. Factor analysis. In N. H. Nie, et. al. (Eds.) Statistical package for the social sciences (2nd Ed.). New York: McGraw-Hill, 1975, 468-514.

Maccoby, E. E., and Jacklin, C. N. The psychology of sex differences. Stanford, CA: Stanford University Press, 1974.

Pulos, S., Stage, E. K., and Karplus, R. Cognitive correlates of proportional reasoning in early adolescence. In R. Karplus, (Ed.) Proceedings of the Fourth International Conference for the Psychology of Mathematics Education. Berkeley, CA: Lawrence Hall of Science, 1980.

METAPHOR AND ANALOGY IN MATHEMATICS

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Abstract: A construction is given for identifying examples of structural metaphors in mathematics and some examples are discussed.

The claim that mathematics is, inter alia, a language is often made, but serious investigation of this assertion has been limited. The major focus of research in the area of language and mathematics has been on readability of mathematics texts or difficulty of verbal arithmetic problems (Nesher, 1976; Pimm, to appear; Rothery, 1980.) The sense of approach I wish to explore is to subject certain figures of speech, in particular analogy and metaphor, to close scrutiny, (figures which make natural language so powerful), and to see whether or not we can identify comparable processes at work within mathematics itself. One's initial inclination is to view analogy, and metaphor in particular, as uncertain methods of working and therefore unsuited to mathematics. I hope to show by means of examples culled from arithmetic and algebra (and in my talk, the calculus) that this is far from the case and in fact these processes are as central to the expression of mathematical meaning as they are in natural language.

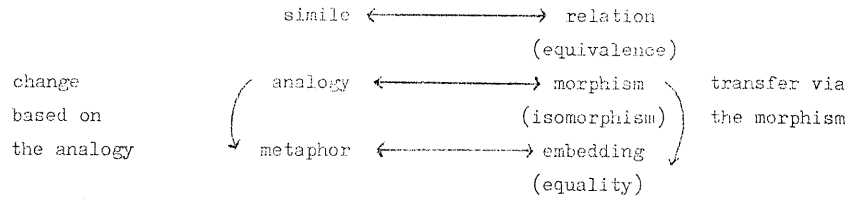
Before turning to examples, I wish to discuss a particular view of analogy and how metaphor relates to it. Metaphor is a particularly troublesome phenomenon for philosophers, linguists and literary critics alike, and no common terminology is agreed on by all writers (Black, 1962; Ortony, 1978. I by no means wish to claim that the construction identified in this paper characterises all cases of metaphor in mathematics; merely that it permits the identification of one class of metaphoric statements which I shall call structural metaphors. Some splendid work has been done on importing extra-mathematical metaphors or the dynamic metaphors implicit in the language of the calculus (Kaput, 1979).

The view of metaphor I wish to employ is one based on analogy (characterised and criticised by Black (op cit) as the 'comparison' view of metaphor).

Let us return to the Greek term analogia. Szabo (1978) informs us that, "It is less well known that this same word was not originally a grammatical or linguistic term, but a mathematical one ... . The Greek grammarians of Hellenistic times undoubtedly borrowed their term analogia from the language of mathematics. So in the last analysis, we are indebted to Greek mathematics for our word 'analogy' (pp. 145-6). The mathematical meaning is that of proportion, currently interpreted as equality of ratios, although recently David Fowler (1979, 1980) has shown how ill-defined the notation of ratio is in the Elements and that the celebrated Eudoxan book V is developing a theory of proportion rather than ratio.

Perelman* phrased the resultant extension of meaning of analogy as follows, "Analogy is analogical to a proportion", thus providing a reverse instance of what I am attempting to do in this paper. Thus an analogy links a relationship A:B to another C:D. A proportion is a symmetrical relation whereas the use of an analogy presumes a unidirectional use under the assumption that we know more about one relationship than the other. Hence by constructing this link we can thereby illuminate or evaluate the first relationship better. (Two-way flow is possible: for example, the development of computers and knowledge of the brain: both currently feed off analogies and metaphors from the other).

A metaphor is seen as a condensed analogy and we speak of "the C of B" or "A is a C". In order to understand a metaphor we must identify the latent analogy and thus our interpretation of it is dependent on this reconstruction (as often more than one is possible). Below is a table summarising my suggestion for links between some figures of speech in natural language and certain mathematical notions. The bracketed terms denote the strongest possible instance.



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* Part of this paragraph is based on a talk entitled, "Analogy and Metaphor in Science, Poetry and Philosophy," given by Chaim Perelman at the U. of Wisconsin - Madison on Nov. 8th, 1978. When asked, Dr. Perelman expressed the view that there was no metaphor in mathematics.

Thus if I wish to indicate examples of structural metaphors in mathematics, the above makes it clear that I am after statements of equality based on identification via a structure-preserving map.

The question, not surprisingly, is one of meaning. As René Thom has so clearly stated, "Any mathematician endowed with a modicum of intellectual honesty will recognise then in each of his proofs he is capable of giving a meaning to the symbols he uses" (1972, p.201). Why is a structural metaphor a problem for mathematics education? The central issue is that a morphism may preserve structure but it doesn't preserve meaning.

Consider the statement $2 - 3 = -1$. It involves an unexamined metaphor which leads us to presume identity of certain elements (and operations), but there are necessary conflicts as the relationship is not identity. The problems of taking a metaphor literally (as well as the resultant confusion), is known but this metaphor has all the more power to confuse as its existence is hidden by the symbolism. It is customary from primary mathematics onwards to identify the whole numbers with a subset of the integers using the homomorphism: $\mathbb{N} \to \mathbb{Z}$ taking n to $+n$, $\mathbb{N}$ is identified with $i(\mathbb{N})$. The homomorphism also identifies the operations and here is a second problem. Discussions of metaphor usually include the distinction between literal and metaphorical meaning. Let us look at the archetypal situations with $\mathbb{N}$ and operations on elements of $\mathbb{N}$ which provide their meaning. Elements of $\mathbb{N}$ arise as possible answers to the question "how many", addition from conflating two sets of distinct objects, subtraction from 'take away', multiplication reflects repeated addition and division, a model for equal sharing.

What are the equivalent well-springs of meaning for the integers and operations upon them? Under the above identification, the positive integers become the whole numbers by omission (and what an omission: for me, one of the most important distinctions is between signed and unsigned quantities and the two distinct types of numbers used to measure them). Because of the structural metaphor, we first require an extended concept of number itself, one including the possibility of direction (-1 makes no sense as an answer to "how many"). More importantly, is it a straightforward identification of two already extant systems, or do the operations on $\mathbb{Z}$ somehow emerge from those on $\mathbb{N}$?

Consider the case of multiplication. For me, there is no real-world situation to which we can refer. One force of the identification of $+2$ with 2 is

to permit the literal meaning of x in N to be used metaphorically in $\mathbb{Z}$. Thus +2 x +3 i.e. +3 + +3 (and having associated + with movement to the right - directed - on a number line, we obtain +6). +2 x -3 is acceptable also without too much stretching (assuming addition works on the whole of $\mathbb{Z}$). -3 x +2 we can approach via commutativity, though a justification for this is flimsy due to the absence of any situational meaning guiding our operations. The reasons become formal and reflect our desire for logical systems. (Not that this is inappropriate per se, but just consider the age range at which such operations are customarily taught). What can we do with -2 x -3? Our metaphor cannot help us here and this is, I believe, one of the causes of problems of multiplying negative numbers.

Why is this a problem? If the metaphoric quality of certain conceptual extensions in mathematics are not made clear to children, then specific meanings and observations (whether intuitive or consciously formulated) about the original setting will be carried over to the new setting where they are often inappropriate. The identification which is at the basis of the metaphor guarantees only that certain structural properties are preserved in the extended system, not the meaning. Thus, for example, that multiplying two numbers together results in a number larger than each of the original pair is a correct concommitant of viewing multiplication as repeated addition. This no longer holds with fractions. Why do we call 2/3 x 4/5 multiplication anyway? What is the root meaning of this operation? The extension of concepts in this fashion can result in the destruction of meaning presuming no distinction is made between the literal and metaphorical use, because the confusion arising from seeing such (often unanalysed) truths fail contaminates not only the extended system but also the one from which it grew.

The focal point of the metaphor is the use of equality. There seem to be at least two major, different uses of the symbol "=" in mathematics. The first is a fairly low-level, naming use, for example $f(x) = x^2$, where one side has meaning and the other is to be used as a label. Definitions can be of this sort; for instance, consider the statement (claim?) that $a^{-2} = 1/a^2$. (The move from N to $\mathbb{Z}$ as exponents is another instance of metaphor). Thus whenever we see a^{-2} for example, we transfer the meaning of the other side, giving force to Polya's injunction, "go back to the definitions" (Polya, 1948, p.89).

The second use is more worthy of the name theorem and arises from some kind

of identification of meanings and this is where both sides of the equality have meaning in their own right. For instance in the case of $\int_a^b f'(x)dx = f(b) - f(a)$, the two meanings are being equated under certain circumstances (although this, to a Greek, would be abhorrent, as how could an area equal a length?) Is $2 + 3 = 5$ to be classified in the second sense? How about $2 - 3 = -1$? In the former it is a dynamic equality where the left-hand side has been transformed by some equality and the result identified with the right-hand side. Thus $2 + 9$ is not an acceptable answer for me to the question "What is $4 + 7$?", because it still contains a verb. In addition to examples with elementary operations (for example, leaving a fraction as $7/4$ has an incomplete air about it - and what sense can be made of $7 \div 4 = 7/4$?), L. Copes (1976) has pointed out that "$\sqrt{\ }$" and "$\int . dx$" function in a similar way for many students. Eliminating verbs makes the answer static and hence, permanent, while their continued presence is indicative of a tension yet to be resolved. Consider in this light Wittgenstein's observation that mathematics always identifies processes with results. This provides a partial explanation students often have on seeing the definition $\log(x) = \int_1^x 1/t \ dt$. It is a naming equality yet they already have a meaning for $\log(x)$.

Examples of metaphor in mathematics often go under the name extension of meaning and are often accompanied by great distress. One example is the introduction of negative numbers, another that of complex numbers. Calling these objects numbers is a metaphoric extension broadening the notion. The extension of exponent notation firstly to negative numbers, then to rationals then to reals and even complexes provides another example ($e^{i\pi} = -1$ is perhaps the most famous metaphoric claim in mathematics). Another instance can be seen in Klein's Erlanger Programme (1872) where he says, "To make use of a modern form of expression, which to be sure is ordinarily used only with reference to a particular group, the group of all linear transformations, the problem might be stated as follows: Given a manifoldness and a group of transformations of the same; to develop the theory of invariants relating to that group." The novelty gets lost with time and with it the metaphoric content of the original insight of useful extension, it becoming a commonplace remark, the literal meaning. (cf. "Shakespeare is full of clichés", a dead metahpor being a cliché.)

Metaphors deny distinctions between things: problems often arise from taking structural metaphors too literally. Because unexamined metaphors lead us to assume the identity of unidentical things, conflicts can arise which can

only be resolved by understanding the metaphor (which requires its recognotion as such) which means reconstructing the analogy on which it is based. Teachers will often cease to use terms metaphorically when their concept is an expanded one, but this will not mirror the situation in most of their children's minds. The unexplained extension of concepts can too often result in the destruction rather than the expansion of meaning.

Bibliography

Black, M. Models and Metaphors Ithaca: Cornell U.P. 1962

Copes, L. Is there life after mathematics? A.M.M. 1980

Fowler, D. Ratio in Early Greek Mathematics 1979

Fowler, D. Book II of Euclids Elements and a pre-Eudoxan Theory of Ratio 1980

Kaput, J. Mathematics and Epistemology in Cognitive Process Instruction (eds. J. Lockhead and J. Clement). Franklin Institute Press 1979

Klein, F. Das Erlanger. Programm (1872) trans M.W. Haskell. Bulletin of A.M.S. July 1893, 3, 215 - 249

Nesher, P. Three Determinants of Difficulty in Verbal Arithmetic Problems. Educational Studies in Mathematics 1976, 7, 369 - 388

Ortony, A. Metaphor and Thought Cambridge C.U.P. 1978

Pimm, D. Review of research on linguistic factors related to addition and subtraction (to appear)

Polya, G. How to Solve It. Princeton; Princeton U.P. 1948

Rothery, A. (ed) Children Reading Maths A working paper of the Language and Reading in Mathematics Group

Szako, A. The Beginnings of Greek Mathematics. Holland, D. Reidel 1978

Thom, R. Modern Mathematics, does it exist? in Developments in Modern Mathematics Cambridge C.U.P. 1972

THE INTERPRETATION OF THE EQUAL SIGN: SYMBOL FOR AN EQUIVALENCE RELATION VS. AN OPERATOR SYMBOL

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This paper looks at recent research dealing with uses of the equal sign and underlying notions of equivalence or non-equivalence among preschoolers (their intuitive notions of equality), elementary and secondary school children, and college students. The idea that the equal sign is a "do something signal" [1] (an operator symbol) persists throughout elementary school and even into junior high school. High schoolers' use of the equal sign in algebraic equations as a symbol for equivalence may be concealing a fairly tenuous grasp of the underlying relationship between the equal sign and the notion of equivalence, as indicated by some of the "shortcut" errors they make when solving equations.

INTRODUCTION

On the topic of equivalence, Gattegno (1974) has stated:

> We can see that identity is a very restrictive kind of relationship concerned with actual sameness, that equality points at an attribute which does not change, and that equivalence is concerned with a wider relationship where one agrees that for certain purposes it is possible to replace one item by another. Equivalence being the most comprehensive relationship it will also be the most flexible, and therefore the most useful. (p. 83)

However, the symbol which is used to show equivalence, the equal sign, is not always interpreted in terms of equivalence by the learner. In fact, as will be seen, an equivalence interpretation of the equal sign does not seem to come easily or quickly to many students.

PRESCHOOL CONCEPTS OF EQUALITY

Once preschoolers begin to tag objects with number words in a systematic and consistent fashion, they seem, at some time after that, to be able to determine the numerical equality of two sets. Gelman (Gelman & Gallistel, 1978) has found evidence that the preschool child's (3- to 5-years-old) judgment of whether two sets are numerically equal ordinarily rests on whether they yield the same cardinal numeral when counted (see also, Siegel, 1978). In other words, to deter-

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[1] This expression was first coined by Behr, Erlwanger, and Nichols in their 1976 PMDC Technical Report (S. Erlwanger, personal communication, June 1980).

mine whether two different and not necessarily homogeneous sets, A and B, are numerically equal, the preschooler's normal principle is "Count them and see". His/her statement that "they're both the same" would correspond symbolically to "card. (A) = card. (B)". This can be considered a comparative notion of equality -- based on the child's ability to count two different sets and to compare their numerosities.

Preschoolers' acquisition of the ability to count two distinct sets is generally followed by the acquisition of the ability to put together two distinct sets and to count the number of elements in their "union" (Brush, 1978; Fuson, 1979; Gelman & Gallistel, 1978). This counting of the number of elements in the combined set leads to an operator notion of equality which emphasizes the result of the arithmetic operation. This corresponds symbolically to "card. (A) + card. (B) = card. (A ∪ B)" where A and B are disjoint sets.

Thus we can distinguish two intuitive meanings of equality among preschoolers: one involving a comparison between two sets where the child counts the elements of the two different sets and, on the basis of the same cardinality, establishes their equality; the other involving the addition of two sets where the child (s/he may or may not count the items in the individual sets first) combines the two sets and then counts the elements in the resulting set. As will be seen, it is the latter, the operator notion of equality, which is appealed to when the equal sign is introduced in school.

THE USE OF THE EQUAL SIGN IN ELEMENTARY SCHOOL

By the time most children enter school, they have sound intuitions about addition and subtraction (Ginsburg, 1977). It is these intuitions which are first symbolized in their school arithmetic. Many children learn fairly quickly to read and write the elementary written symbolism of simple arithmetic, but do not necessarily understand it the same way we do. Whereas we can look upon equality sentences as equivalence relations (involving the reflexive, symmetric, and transitive properties), primary school children, Ginsburg points out, interpret + and = in terms of actions to be performed. One of his second grade subjects typically said that in $3 + 4 = \square$, "the equal sign means what it adds up to". Others read "3 + 5 = 8" as "3 and 5 make 8". Many first and second graders, when asked how to read $\square = 3 + 4$, would answer, "Blank equals 3 plus 4", but then add: "It's backwards! Do you read backwards?" and then change it to $4 + 3 = \square$. As can be seen, one consequence of the child's interpretation of equalities in terms

of actions is that he finds it difficult to read arithmetic sentences that do not reflect the order of his calculations. Ginsburg also found that many children cannot read sentences that express relationships like $3 = 3$.

It can be argued that these notions reflect the kind of instruction that these children have received. One might then assume that later exposure to equality sentences involving the commutative and associative properties might broaden the elementary school child's notion of the equal sign. However, this does not seem to be the case. Behr, Erlwanger, and Nichols (1976) did a study with children in grades 1 to 6 which investigated their view of equality sentences. When Behr et al. asked sixth graders about the meaning of "3 = 3", a typical response was: "This could mean $6 - 3 = 3$ or $7 - 4 = 3$". Behr and his colleagues emphasized that there was no evidence to suggest that children changed in their thinking about equality as they progressed to upper grades; in fact, even sixth graders seemed to view the equal sign as a "do something signal".

Another type of sentence which Behr presented to his subjects involved equalities such as $4 + 5 = 3 + 6$. A common response to this was: "After '=' should be your answer. It's the end, not another problem", followed by the transformation of the initial equality into two sentences -- "4 + 5 = 9" and "3 + 6 = 9" -- and the comparison of the results of those statements. This kind of reaction brings into question the psychopedagogical validity of the "name for a number" approach, that is, that "$4 + 5 = 3 + 6$" because both $4 + 5$ and $3 + 6$ are other names for 9. The "name for a number" idea, advocated approximately twenty years ago by the School Mathematics Study Group (SMSG) and adopted by most current elementary school mathematics text books, assumes that the young learner can assimilate an equivalence view of the equality symbol; for, indeed, "is another name for" is an equivalence relationship defined on ordered pairs of numbers $\left[(a,b)\ \mathcal{R}\ (c,d) \text{ iff } a + b = c + d\right]$. However, as both Denmark(1976) and Collis (1974) have shown, these assumptions are not warranted.

Denmark designed a teaching experiment to teach the concept of equality as an equivalence relation to a group of first grade students (before they encountered the + and = signs in school). This study showed that students were able (by means of activities with a balance and the corresponding written equations) to acquire some flexibility in accepting the use of the equal sign in a variety of sentence structures (e.g., $3 = 3$, $3 + 2 = 4 + 1$, $5 = 4 + 1$); however, the equal sign was still viewed primarily as an operator, not as a relational

symbol. Furthermore, the data do not support the conjecture that if students (first graders) are provided with appropriate instructional experiences in which they encounter the use of the equal sign in a variety of sentence forms, they will acquire a conceptualization of equality as a relation between two names for the same number.

Collis has also provided evidence of children's inability to deal with the "two names for the same number" approach. Based on his observations of children's mathematical behavior, he has distinguished various levels of capability with respect to their "Acceptance of Lack of Closure", that is, their ability to hold unevaluated operations in suspension. Collis has found that primary school children who are between the ages of approximately 6 years and 10 years require that two elements connected by an operation be actually replaced by a third element. Quite clearly, according to Collis, they cannot handle equations such as $4 + 5 = 3 + 6$. The child needs literally to be able "to see" a unique result before the operations on numbers mean anything to him, that is, $4 + 5 = 3 + 6$ must be written $4 + 5 = 9$. Later (between approximately 10 years and 13 years), the learner may regard the outcome as unique, but may not need to make the actual replacement to guarantee this. There is less reliance on "seeing" the uniqueness in the results of his operations, though he is still bound by empirical evidence. After about age 13, Collis points out, the learner is willing to infer beyond physical models and to use specific cases for forming adequate generalizations.

THE USE OF THE EQUAL SIGN IN HIGH SCHOOL AND COLLEGE

For many 13-year-olds, however, this is a transition period -- transition between requiring the answer after the equal sign and accepting the equal sign as a symbol for equivalence. In a teaching experiment which investigated the construction of meaning for non-trivial algebraic equations (Herscovics & Kieran, 1980; Kieran, 1980), subjects (12- to 14-years-old) were first asked what the equal sign meant to them, followed by the request for an example showing the use of the equal sign. It is telling that most of them described the equal sign in terms of the answer and limited themselves to examples involving an operation on the left side and the result on the right. The ensuing teaching session focussed on extending subjects' use of the equal sign to include multiple operations on both sides (e.g., $4 \times 3 + 5 = 20 - 3$). Despite initial insistance from one subject on writing $4 + 3 = 6 + 1$ as $4 + 3 = 7$, and from another on inserting the "answer" between both sides (i.e., $5 \times 3 = 15 = 10 + 5$),

subjects seemed in general to be quite comfortable with equality statements containing multiple operations on both sides. They justified them in terms of both sides being equal because they had the same value. The comparisons that subjects were eventually able to make between left and right sides of the equal sign suggest that the equality symbol was being seen at this stage more as a relational symbol than as a "do something signal". The right side, by this time, did not have to contain the answer, but rather could be some expression that had the same value as the left side. This same interpretation carried over into their conceptualization of the meaning of simple algebraic equations. For example, equations such as $2x + 3 = 4x + 1$ were described by subjects: "If you know what number x is, then 2 times that number plus 3 has the same value as 4 times that number plus 1".

However, the ability to consider an algebraic equation as an expression of equivalence because both sides have the same value does not seem to be sufficient for an adequate conceptualization of the equation-solving process. For not only does equation solving involve a grasp of the notion that right and left sides of the equation are equivalent expressions, but also that each equation can be replaced by an equivalent equation (i.e., one having the same solution set). Unfortunately, very little research has addressed itself to the question of how this concept is acquired by high schoolers. That both these notions of equivalence may be quite fuzzy for many students is reflected in the procedures they use to solve equations. Byers and Herscovics (1977) have provided examples from the written work of some students which yield evidence that the equal sign is not being used consistently as a symbol for equivalence:

Solve for x :

$$\begin{aligned} 2x + 3 &= 5 + x \\ 2x + 3 - 3 &= 5 + x - 3 \\ 2x &= 5 + x - x - 3 \\ 2x - x &= 5 - 3 \\ x &= 2 \end{aligned}$$

And :

$$\begin{aligned} x + 3 &= 7 \\ &= 7 - 3 \\ &= 4 \end{aligned}$$

It may be argued that these students possess an underlying equivalence notion of the equal sign and that they are just using shortcuts in their procedural work. However, it can also be contended that their written work reflects a basic lack of awareness of the equivalence role of the equal sign. These "shortcut" errors suggest that many high school students are still interpreting the equal sign as an operator symbol -- as a signal to do something in order

to get the "answer". Even college students taking calculus, when asked to find the derivative of a function, frequently seem to be using the equal sign merely as a link between steps:

$$\begin{aligned} f(x) &= \sqrt{x^2 + 1} \\ &= (x^2 + 1)^{\frac{1}{2}} \\ &= \tfrac{1}{2}(x^2 + 1)^{-\frac{1}{2}} D_x(x^2 + 1) \\ &= \tfrac{1}{2}(x^2 + 1)^{-\frac{1}{2}} (2x) \\ &= x(x^2 + 1)^{-\frac{1}{2}} \\ &= \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

(Clement, 1980, p.7)

Whether these errors are due to taking shortcuts or to the application of the operator principle or both is a question which needs to be investigated further. Future studies ought to be undertaken which investigate the notions of equivalence held by high school and college students and also the nature of instructional activities which would be most effective in teaching the concept of equivalence.

SUMMARY

Mathematics makes no distinction between the left and right sides of expressions such as $3 + 2 = 5$, $3 + 5 = 7 + 1$, or even $3x + 3 = 4x + 2$. They are all considered examples of equivalence relations. However, as has been pointed out in this paper, the concept of equivalence is an elusive one not only for elementary school students but for high schoolers as well. That the equal sign is a "do something signal" is a thread which seems to run through the interpretation of equality sentences throughout elementary school, high school, and even college. Early elementary school children, despite efforts to teach them otherwise, view the equal sign as a symbol which separates a problem and its answer. This thinking remains as children get older and advance to the upper elementary grades. Junior high school students are able to relax some of the constraints placed earlier on the interpretation of the equal sign to make room for arithmetic equalities (with multiple operations on both sides) where both sides "yield the same value". It is not clear, however, whether or even how these beginnings toward interpreting the equal sign in terms of an equivalence relation develop into an awareness of the notion of equivalent equations in high school algebra and college calculus. The procedures used by students to solve equations and to find the derivative of a function would seem to indicate that high school and college students are also interpreting

the equal sign in terms of an operator symbol, albeit at a more sophisticated level, rather than as a symbol for an equivalence relation.

REFERENCES

Behr, M., Erlwanger, S., & Nichols, E. How children view equality sentences. PMDC Technical Report No. 3. Florida State University, 1976. (ERIC Document Reproduction Service No. ED144802).

Brush, L.R. Preschool children's knowledge of addition and subtraction. Journal for Research in Mathematics Education, January 1978, 9 (1), 44-54.

Byers, V., & Herscovics, N. Understanding school mathematics. Mathematics Teaching, December 1977, 81, 24-27.

Clement, J. Algebra word problem solutions: Analysis of a common misconception. Paper presented at annual meeting of American Educational Research Association, Boston, April 1980.

Collis, K.F. Cognitive development and mathematics learning. Paper presented at Psychology of Mathematics Education Workshop, Centre for Science Education, Chelsea College, London, June 1974.

Denmark, T., Barco, E., & Voran, J. Final report: A teaching experiment on equality. PMDC Technical Report No. 6. Florida State University, 1976. (ERIC Document Reproduction Service No. ED144805).

Fuson, K.C. Counting solution procedures in addition and subtraction. Paper presented at Wingspread Conference on the Initial Learning of Addition and Subtraction Skills, Racine, Wisconsin, November 1979.

Gattegno, C. The common sense of teaching mathematics. New York: Educational Solutions, 1974.

Gelman, R., & Gallistel, C.R. The child's understanding of number. Cambridge, Mass.: Harvard University Press, 1978.

Ginsburg, H. Children's arithmetic. New York: Van Nostrand, 1977.

Herscovics, N., & Kieran, C. Constructing meaning for the concept of equation. The Mathematics Teacher, 1980.

Kieran, C. Constructing meaning for non-trivial equations. Paper presented at annual meeting of American Educational Research Association, Boston, April 1980.

Siegel, L.S. The relationship of language and thought in the preoperational child: A reconsideration of nonverbal alternatives to Piagetian tasks. In L.S. Siegel & C.J. Brainerd (Eds.), Alternatives to Piaget. New York: Academic Press, 1978.

MATHEMATICAL INTUITION, WITH SPECIAL REFERENCE TO LIMITING PROCESSES

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In this paper intuition is conceived as the utilization of an individual's current cognitive structure in a novel manner. The cognitive structure is a product of previous experience and mental activity. This develops a complex mental image of the concept which may contain potential conflicts. Limiting processes contribute to the intuition of infinity yielding quite different intuitions from the formal theory of cardinal numbers.

§1 INTUITION AND COGNITIVE STRUCTURE

Intuition is a *cognitive* activity and its description should be in cognitive terms. Hebb (1972) or Young (1978) describe how the brain acts through a mechanism of excitation and inhibition of neurons, causing activities of cell assemblies in the brain. Sensory information, for instance from the eyes, is conveyed both to the area of the brain which gives rise to the sensation of sight and also, by a network of diverging paths, to other parts of the brain, setting up complex neuronal excitations. The stimulation of neurons passes messages to others, which excite or inhibit them, making them more or less sensitive to subsequent stimulation. In this way global mental activity is made up of an amalgam of local processes which have been selectively excited. Faced with a novel situation, the brain can only make sense of it be reacting with an amalgam of mental activity from its current structure. This mental activity may be anything but logical and it is such activity, preformed relatively quickly without analysis, which I term intuition. One may wish to qualify this by excluding logical or reflective processes, as did Bruner 1966, Hadamard 1945, Poincare 1913, Skemp 1971,1979, in different ways; one may wish to include characteristic properties (Fischbein 1978). For this paper I shall only concentrate on what I believe to be the central property of intuition: the global amalgam of local mental processes using existing cognitive structure, as stimulated by a novel situation.

§2 CONCEPT IMAGE AND CONFLICTS

Vinner (1975) defined an individual's *mental picture* of a concept to be the set of all visual representations (including symbols) that are associated with the concept. He defined the individual's *concept image* to consist of the mental picture together with all the properties associated mentally with the concept in the mind of the individual (Vinner 1980). These ideas have been extended (Tall & Vinner 1980), the concept image being regarded as the cognitive structure consisting of the mental picture together with all properties and processes associated mentally with the concept. Thus the concept image consists of all the mental structure, conscious or unconscious, that shapes the individual's notion of that concept. This concept image need not be totally coherent. Depending on the context, different parts of the concept image may be activated. At any given time the portion of the concept image that is activated is called the *evoked concept image*. At different times different concept images may be evoked and if these are

communicated to an observer then the latter may note that there are conflicts in what the individual says or does. However, the individual may not be at all disturbed by the conflicts. Such potential conflicts only become *cognitive conflicts* when they are evoked simultaneously. Fischbein (1978, p.155) notes:

> *Two opposite interpretations may coexist for a long time without the subject being aware of the contradiction.*

On the other hand a cognitive conflict may be partly subconscious, in which case the individual may feel a sense of unease and confusion, yet not know the reason for it.

Quite distinct from the complex structure of the concept image is the *concept definition* which is the form of words used to describe the concept. Such a concept definition may be *formal* and given to the individual as part of a mathematical theory, or it may be *personal,* invented by the individual to (partially) describe his concept image. A personal concept definition is part of the concept image, the formal concept definition may or may not be.

The formal concept definition is part of a formal mathematical theory. In this way it has implications which may be deduced within the theory. The term *potential conflict factor* is used to denote any part of the concept image or any implication of the concept definition which may conflict with another part or another implication.

A potential conflict need not be apparent to the individual concerned. To create actual *cognitive* conflict it is necessary for that individual to evoke two mutually conflicting factors simultaneously. There is therefore the distinct possibility of a potential conflict between part of the concept image an an implication of the concept definition which never gives rise to cognitive conflict, simply because the formal definition and its consequences never become part of the individual's concept image. On the other hand the experiences encountered by the individual may develop a concept image full of potential conflict which only becomes congnitive conflict at a later stage. An individual's concept of infinity develops in this way through an amalgam of different experiences and cognitive processes. The intuition of infinity arises from a complex concept image with a variety of potential conflict factors.

§3 MEASURING INFINITY

In a recent paper (Tall 1980c) I demonstrated that when measuring properties are extrapolated to the infinite case, a self consistent mathematical theory arises which includes infinitesimals and infinite elements. This theory can be in the form of non-standard analysis (Robinson 1966, Keisler 1976 a,b) or a simpler theory (Tall 1980 a,b). A full understanding of such ideas is not necessary at this stage but it is vital to note that there exist perfectly acceptable formal interpretations of infinity quite different from cardinal infinity. The existence of such a theory has two vital implications. First, different finite experiences (measuring as opposed to counting) can lead to different notions of infinity, giving a concept image containing potential conflicts. Second the use of the cardinal paradigm (or any other single framework) for interpreting intuitions of infinity may seriously distort the evidence.

Hence the need to view the notion of infinity as a complex concept image amalgamating a variety of cognitive experiences instead of a developing formal notion measured against the particular touchstone of infinite cardinals.

§4 ACTUAL INFINITY OF SETS

Fischbein (1978 p.159) asserts:

> *"the natural concept of infinity is the concept of potential infinity, the only one which Aristotle admitted as meaningful and which was predominant until Cantor".*

Young children make a great leap when they understand that the process of counting is potentially infinite but its actual infinity cannot be realised. However, the processes of mathematical instruction can lead us to a blurring of the distinction between the actual and the potential infinity of sets. In a questionnaire, which I administered, 42 final year university mathematics students were asked if the natural numbers, $\mathbb{N}$, existed as a coherent mathematical idea. All 42 responded in the affirmative. The next day I expressed my interest that all had given a positive response and asked if they were aware of the difference between potential and actual infinity. We had a five minute discussion on the topic and then I requested that those who felt it necessary to distinguish between potential and actual infinity of $\mathbb{N}$ to write "yes" on a piece of paper, otherwise "no". There were 42 "no's". Modern set theory, with the all important symbol $\mathbb{N}$ for the natural numbers, has retrained our primary intuitions of infinity to accept actual infinities of sets as a secondary intuition.

But this is by no means the end of the story. Although students may acknowledge the actual infinity of *sets*, other experiences reinforce the *potential* infinity of *processes*.

§5 POTENTIAL INFINITY OF DYNAMIC PROCESSES

Many different kinds of limiting process are met by students (chords approaching tangents, iterative processes for computing numbers, $\lim_{x\to\infty} f(x)$, $\lim_{x\to a} f(x) = \infty$, $\lim_{n\to\infty} s_n$, $\sum_{n=0}^{\infty} a_n$, and so on). In England such concepts are usually introduced informally first, with a formal definition coming later, if at all. (The School Mathematics Project gives formal definitions of limits of functions and sequences only at the very end of the course.)

This means that a concept image is built up long before any formal concept definition is given. With such subtle concepts this is a sensible approach, it may be the only practical approach, but it does mean that certain implicit properties which are not part of the concept definition become part of the concept image.

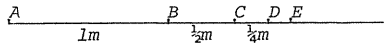
In an investigation (Schwarzenberger and Tall 1978) it was noted that a large number of good mathematicians entering university assumed that if $s_n \to s$, then s_n was never equal to s. As one put it

> *"$s_n \to s$ means s_n gets close to s and n gets large, but does not actually reach s until infinity..... Infinity is an imaginary concept invented by mathematicians, useful in describing limits etc."*

Here the concept image includes a fact ($s_n \neq s$) which is not part of the formal concept definition. This fact often survives after long exposure to formal mathematics. In a tutorial which I held with four final year B.Ed. honours students (shortly to become mathematics teachers), all four insisted that s_n can never equal s.

In a questionnaire (Fischbein et al. 1979 p.68) administered to younger pupils (grades 8 and 9) two questions were:

(i) *Given a segment AB = 1m. Let us add to AB a segment BC = ½m. Let us continue in the same way adding segments of ¼m, ⅛ m, etc. Will this process of adding segments come to an end?*



(ii) *Let us consider question (i). What will be the sum of the segments AB+BC+DE+ ... etc.?*

Out of 107 students, 84% thought the process in (i) would never end, 14% thought it would. In (ii) only 6% thought the sum of the segments would be 2, 17% thought it would be less than 2, and 15% thought it would be infinity. Even before the limiting process had been discussed the concept image intuitively alights on the infinite nature of the *process* rather than the numerical limit.

In a poster (Tall 1980d) presented at ICME IV I reported a questionnaire on limiting processes for mathematics students arriving at university. These were good students (with A or B grades in mathematics A-level). They were asked:

Did you use the notation $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ *at school? If your answer is 'yes', explain (if you can) what the following are:*

δx, δy, $\frac{dy}{dx}$, dx, dy, δ, d.

These are not regular mathematics questions, but they give insight into the individual's concept image. I was particularly interested in their interpretation of the meaning of dy which I classified to include 22 out of a total of 70 giving infinitesimal-type responses. Sixteen of these specifically mentioned "infinitesimal", "infinitely small" or "smallest possible" quantities (the term "smallest" almost certainly used in a superlative, rather than mathematical, sense). The other six responded that dy was the 'limit of δy' which such comments as

"dy is the limit as δy tends to zero"

or *"The only possible meaning I can attach to this is* $dx = \lim_{\delta x \to 0} \delta x = 0$".

The latter responses once more underline the fact that many students tend to visualise a limit as a dynamic *process* rather than a numerical quantity. Other topics in the questionnaire show that the *majority* of students regard the limit as a dynamic process.

§6 INTUITIVE ORDERS OF INFINITY

Though Fischbein asserts that 'potential infinity' is the only natural concept of infinity, limiting processes lead to a natural secondary intuition of various orders of infinity. This arises by considering the relative rates at which processes increase in size. For instance as n tends to infinity, so do n^5 and 2^n, but 2^n tends to infinity "faster" than n^5.

With a class of 20 average university students (grads C and D at A-level mathematics) we computed various limits on hand calculators. An interesting example was $s_n = n^5/(1.1)^n$ which increases to begin with, getting very large as n increases, then after n = 80 or so its behaviour changes and for much

larger n it tends to zero. On a written assignment students were asked to compute various limits, including $n^2/(n^2 + 1)$ and $n^5/(1.1)^n$.

The students had been taught to write the first as

$$\frac{n^2}{n^2+1} = \frac{1}{1+1/n^2} \to \frac{1}{1+0} = 1.$$

One student responded

$$\frac{n^2}{n^2+1} \to \frac{\infty}{\infty} = 1.$$

I interviewed her and pointed out that a similar "computation" would give

$$\frac{n^5}{(1.1)^n} \to \frac{\infty}{\infty} = 1 .$$

She firmly responded "no it wouldn't because in this case the denominator is a *bigger* infinity, and the result is zero". This response was close to extrapolating properties which lead formally to a kind of non-standard analysis, with different infinities as in §3, though in the current mathematical paradigm of cardinal number it would likely be rejected.

Historically it is interesting to note that Leibniz had an intuition of various orders of infinity, suggested by the kinds of functions for which he computed derivatives. He said that as the earth is infinite with respect to a ball, so the distance of the fixed stars is doubly infinite with respect to the ball (Boyer 1949,p.212). Meanwhile Leibniz rejected actual cardinal infinity to avoid the apparently conflicting properties enunciated by Galileo who had shown that infinite sets could be put in one-one correspondence with proper subsets. How times change! Now we accept the cardinal infinity of Cantor but reject the measuring infinity of Leibniz, despite the modern non-standard theory of Robinson. Yet if we look at the intuitions of children we still find notions which extrapolate to non-standard analysis rather than cardinal infinity. Fischbein et al 1979 p.75 gave an example where the sum of

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots$$

was stated to be

$$s = 2 - \frac{1}{\infty} \text{ , because there is no end to the sum of segments.}$$

Here the potential infinity of the limiting process leads to a concept consonant with non-standard analysis. It conflicts with cardinal theory where infinities cannot be divided.

In the same paper (p.73) a pupil justified a one-one correspondence between a line and a line segment as follows:

> *"The line has no magnitude and therefore it is possible to establish a correspondence between a big point of the line and a small point of the line segment".*

Here the experience of practical drawing with points having various sizes leads to an extrapolation with indefinitely small points of different dimensions. Again it is closer to a measuring infinity rather than a cardinal one.

How we distort the intuition of the young by viewing them through the spectacles of a cardinal paradigm! And in saying that I realise how a "measuring infinity paradigm" is another form of distortion, though it has certain intuitive properties. (Non-standard analysis has an axiom which says that any "elementary" property remains true in the extended theory; therefore extrapolating "elementary" properties leads to true intuitions.) Following Lakatos 1978, who speaks on behalf of incomplete theories in history, one should make a plea that the intuitions of students should be respected whether they fit with an accepted formal theory or not. Who knows, a later mathematician may canonize such an intuition with a new formal theory!

§7 CONCLUSION

The concept of infinity varies from one individual to another and need not be globally coherent. As Fischbein has observed, the individual need not be aware of conflicting factors unless they are evoked simultaneously in a context seeking overall consistency.

I contend that these factors occurring in the concept of infinity are a typical case of intuition in particularly extenuating circumstances. They reveal intuition as the amalgam of local mental processes using current cognitive structure which in turn is a product of ongoing mental activity and previous experience.

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References

Boyer C.B. 1949 The History of Calculus and its Conseptual Development reprinted Dover 1959

Bruner J.S. 1966 "Towards a Disciplined Intuition" in Relevance of Education 98-113 Penguin 1974

Fischbein E. 1978 Intuition and Mathematical Education Proc.2nd Int.Conf. for the Psych.of Math.Educ. 197-202.

Fischbein E. Tirosh D. & U. Malamed 1979 Is it possible to measure the intuitive acceptance of a mathematical statement? Proc.3rd.Int.Conf. for the Psych.of Maths.Educ. 67-79

Hadamard J. 1945 The Psychology of Invention in the Mathematical Field reprinted Dover 1954

Hebb D.O. 1972 Textbook of Psychology 3rd edition Saunders

Keisler H.J. 1976a Elementary Calculus Prindle, Weber & Schmidt

Keisler H.J. 1976b Foundations of Infinitesimal Calculus Prindle, Weber & Schmidt

Lakatos I. 1978 Cauchy and Continuum Mathematical Intelligencer 1, 151-161

Poincare 1913 "Mathematical Creation" in Foundations of Science New York: The Science Press

Robinson A. 1966 Non-Standard Analysis North Holland

Schwarzenberger R.L.E.S. & Tall D.O. 1978 Conflicts in the Learning of real numbers and limits Mathematics Teaching 82

Skemp R.R. 1971 The Psychology of Learning Mathematics Penguin

Skemp R.R. 1979 Goals of Learning and Qualities of Understanding Proc. 3rd.Int.Conf. for the Psych.of Maths.Educ. 197-202

Tall D.O. 1979 The Calculus of Leibniz - an Alternative Modern Approach Mathematical Intelligencer 2 54-55

Tall D.O. 1980a Looking at Graphs through infinitesimal microscopes, windows and telescopes Mathematical Gazette(to appear)

Tall D.O. 1980b Infinitesimals constructed algebraically and interpreted geometrically Mathematical Education for Teachers (to appear)

Tall D.O. 1980c The notion of infinite measuring number and its relevance in the intuition of infinity Educational Studies in Mathematics (to appear).

Tall D.O. 1980d Intuitive infinitesimals in the calculus (Poster presented to ICME IV)

Tall D.O. & Vinner S. 1980 Concept image and concept definition in mathematics with particular reference to limits and continuity (paper presented by first author to conference of Swedish Mathematicians, UHA, St.Andrew's, Scotland. March 1980)

Vinner S. 1975 The Naive Platonic approach as a teaching strategy in arithmetic Educational Studies in Mathematics (339-350)

Vinner S. 1980 Concept image, concept definition and the notion of function (in preparation)

Young J.Z. 1978 Programs of the Brain OUP

CONCEPT IMAGES AND COMMON COGNITIVE PATHS IN THE DEVELOPMENT OF SOME SIMPLE GEOMETRICAL CONCEPTS

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Some notions are going to be defined by means of which a model for acquiring certain type of concepts, having strong visual components, will be constructed. The notion of significantly common cognitive path will be introduced. It will be used, together with the model, to analyse some data related to the development of simple geometrical concepts; however, the main purpose of the paper is just to introduce the notion and the model which seem to have certain potential to explain phenomanae in the processes of concept formation.

§ 1. Mental pictures and Mental Images of Concepts

Let C denote a concept and let P denote a certain person. The P's mental picture of C is the set of all pictures that have ever been associated with C in P's mind. Besides the mental picture of a concept there might be a set of properties associated with the concept (in the mind of our person P). For instance, somebody might think (incorrectly) that an altitude in a triangle should always fall inside the triangle. He might think (correctly) that a triangle has 180°. This set of properties together with the mental picture will be called by us the concept image (there is a more detailed discussion of the notions in Vinner, 1975 and 1980).

§ 2. Concept Definitions and their Role in Forming the Concept Images

By "Concept Definition" we mean here a verbal definition that accurately explains the concept in a non circular way. For some of our concepts we also have concept definition in addition to the concept image. For many other concepts we do not. For instance, we do not have a definition for "house", "orange" etc., although we have very clear concept images for them. They were acquired when we were children probably by means of "ostensive definitions". It is true that some concepts as these above could be introduced to us by means of verbal definitions. The word "forest" could be introduced to us by saying: "many many trees together are forest". We were supposed then to visualize many trees together and thus to form a concept image. However we claim that (1) in order to handle concepts one needs a concept image and not a concept definition. (2) Concept definitions (in case the concept was introduced by means of a definition) will remain inactive or even will be forgotten. In thinking, almost always the concept image will be evoked.

2. In a geometry textbook (Geometry with Coordinates, SMSG, part I, pp. 143-144) an angle is considered as "the union of two concurrent rays". A straight angle is "the union of two opposite rays". There is only one drawing of a straight angle in the book, of course, a horizontal one. Again, will it be a surprise if a student will identify only straight angles which are horizontal?

3. In another book by SMSG (Mathematics for Junior High School, Vol. 2, part I, pp. 194-195) right triangles are discussed. In all of them one side of the right angle is horizontal (there are ten right triangles altogether). It seems unnecessary to ask our question once again.

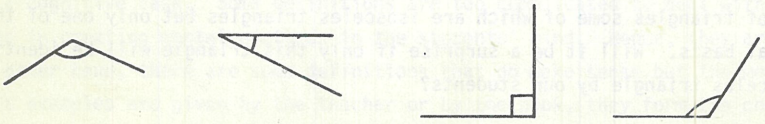
Thus, revealing the concept images of our students become very important for teaching. Not only that it might give us better understanding of our students (knowing what caused them to act as they acted) but also it might suggest some improvements to our teaching which formed such wrong concept images.

§ 5. The Sample, the Questionnaire and the Results

We tested 550 students in grades 7, 8 and 9. Their schools were officially declared as culturally deprived (namely, over 60% of the students are culturally deprived). The students were supposed to learn several geometrical concepts in their elementary schools, concepts as ray, angle, triangle etc. In their textbooks, these concepts were introduced by means of examples and formal definitions ("two concurrent rays are an angle" and so forth). The students did not have any official geometry lessons before the test but they met the above geometrical concepts in their mathematical activities from time to time.

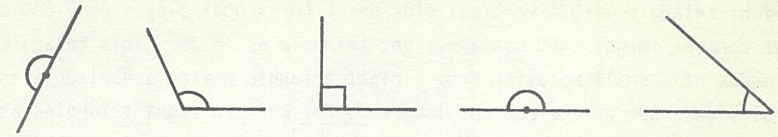
Out of the four questions in the test (which are concerned with this paper) three had an identification task and one had a construction task. Each question item will be denoted by a capital letter after which, in parentheses, will be written the percentage of right answers.

QUESTION 1: CIRCLE EVERY OBTUSE ANGLE.



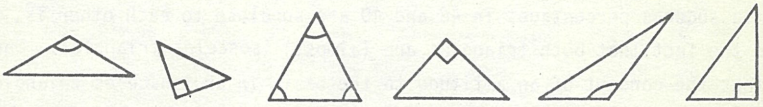
A (62.7) B (82.4) C (87.4) D (76.7)

QUESTION 2: CIRCLE EVERY STRAIGHT ANGLE.



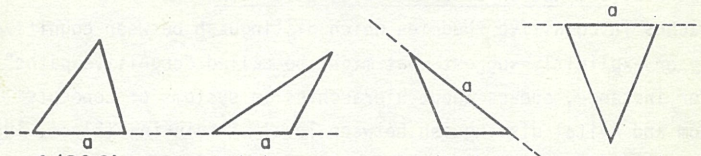
A (79.6) B (96.5) C (96.7) D (92) E (95.4)

QUESTION 3: CIRCLE EVERY RIGHT TRIANGLE.



A (86.4) B(67.6) C (76) D(41.3) E (92.2) F (75.6)

QUESTION 4: DRAW THE ALTITUDE TO THE SIDE DENOTED BY a IN EACH OF THE FOLLOWING TRIANGLES:



A (22.9) B (8) C (38) D (41.6)

The following can be said about the concept images:

1. The Obtuse Angle. Obtuse angles which have a horizontal ray are identified more easily than others. Some concept images contain only obtuse angles with horizontal rays. As we know teachers and textbooks, we can say that in both there is a tendency to draw obtuse angles with a horizontal ray. Probably the "gravitational factor" has also some role in forming the above concept image. An angle is "stable" ("can stand") only if it has one horizontal ray (the other one being ascending rather than descending). This might cause people to draw angles with a horizontal ray. As a result of this a concept image that contains only obtuse angles with a horizontal ray can be formed.

2. The Straight Angle. The concept image of many students for this notion contains only horizontal straight angles. Everything that was said about the obtuse angle can be said about the straight angle as well.

3. The Right Triangle. 76% of the students have a concept image for this notion that contains a right triangle with a vertical side and a horizontal side. Only 68% of the students have concept images that contain right triangle as in 3B. This triangle is obtained by means of a small rotation from a right triangle having a horizontal side and a vertical side. 59% of the concept images do not contain right triangles as in 3D. The gravitational factor plays a major role also here.

4. The Altitude in a Triangle. Note first that less than half of the students succeeded even in the easy items of the question. The failure in 4B can be a result of a (sometimes implicit) common belief that an altitude should always fall inside the triangle. The fact that the success percentages in 4C and 4D are so close to each other is, perhaps, due to the fact that both triangles are (almost) isosceles triangles. Our guess here is that the concept of an altitude to the basis in an isosceles triangle took the place of a general concept of altitude. Note that here the "gravitational factor" is not dominant anymore.

§ 6. The Notion of Common Cognitive Path

There are some approaches in cognitive theories which distinguish between cognitive levels and implicitly or explicitly suggest what might be called "cognitive paths" in learning. Gagné, for instance, speaks about hierarchies in systems of concepts (Gagné , 1970). Bloom and Avital distinguish between level of thinking (Bloom, 1956, Avital, 1968). Piaget points at stages in the intellectual development. In this paper we deal with simple isolated concepts that have several aspects or components. These aspects are not necessarily acquired simultaneously. Let us denote by A, B, C, D, four different aspects (components) that partially form a certain concept (or concept image). Assume that a certain learner acquired these aspects in the order they were written above. Then A$\rightarrowtail$B$\rightarrowtail$C$\rightarrowtail$D is a cognitive path of the above concept for the above learner. Now assume that there is a group of people such that A$\rightarrowtail$B$\rightarrowtail$C$\rightarrowtail$D is a cognitive path for everybody in the group. Then we will say that it is a common cognitive path for this group. In such a group we will not find people who know D without knowing also A, B and C or people who know C without knowing also A and B and so on. Suppose we had a question that examines aspects A, B, C and D of our concept and we administered it to a group of people (not necessarily the same one as above). Denote by a, b, c, d respectively the subgroups of people that answered correctly the items that test aspects A, B, C, D. Suppose, finally, that it was found that $a \supsetneq b \supsetneq c \supsetneq d$. We may claim then that A$\rightarrowtail$B$\rightarrowtail$C$\rightarrowtail$D is a common cognitive path for this group (in the sense that nobody in the group can know D without knowing also A,

B, C and so on). If x is a set denote by m(x) the number of elements in x. It is easy to see that $a \supsetneq b \supsetneq c \supsetneq d$ is equivalent to

(1) $m(a) > m(b) > m(c) > m(d)$

(2) $\frac{m(a \cap b \cap c \cap d)}{m(d)} = \frac{m(a \cap b \cap c)}{m(c)} = \frac{m(a \cap b)}{m(b)} = 1.$

(It has some similarity to Guttman scales, see for instance SPSS, 1975). Certainly it is not realistic to expect (1) and (2). It seems that in a more realistic situation it will be reasonable to expect in (2) a ratio smaller than 1 but quite close to it, say 0.8. However, a simple analysis (that will be given elsewhere) shows that this is not the case. As a result of this analysis we suggest the following definition.

Definition: Let A, B, a, b be as above and assume $m(a) > m(b)$. Denote by N the total number of people in our group. Define now two numbers by the following:

$r(a, b) = \frac{m(b) \cdot m(a)}{N}$, $\Delta(a, b) = m(a \cap b)$ (It is easy to see that r(a, b) is approximately the number of members of b who are also in a, provided b was randomly chosen.) Assume $\Delta(a, b) > r(a, b)$. We will say that $\Delta(a, b)$ is α- significantly greater than r(a, b) if the probability of χ^2 comparing r(a, b) (as an expected value) and $\Delta(a, b)$ (as an observed value) in b is less than α. Finally, we will say that $A \rightarrow B$ is an α-significantly common path, if $\Delta(a, b)$ is α - significantly greater than r(a, b). If $\Delta(a, b)$ is α-significantly smaller than r(a, b) we will say that $A \not\rightarrow B$ is definitely not a common cognitive path. In all other cases we will say that there is no indication on a significantly common cognitive path.

Thus we have defined our notion for two step path but it is obvious how this definition can be generalized for any number of steps. For instance, if we have aspects A, B, C and we have $m(a) > m(b) > m(c)$ then in addition to r(a, b), $\Delta(a, b)$ that we should compare according to the above definition we should also compare r(a , b, c) and $\Delta(a, b, c)$.

We will apply now the above notion to the questions of our test. In questions 1 - 3 we had an identification task. Such a task generally include two kinds of items. Those which should be identified as the concept and those which should be identified as the non-concept. Call identification of items of the first kind by C-identification (concept identification) and identification of items of the second time by NC-identification). Thus in questions 1-3 we will look for two different common cognitive paths, one for the C-identification and the other for the NC-identification. In question 4 we will look only for one cognitive path. The results are the following:

Question 1: D→A is an α-significantly common cognitive path, α = 0.005, for the C-identification. There is no indication on significantly common path for NC-identification.

Question 2: D→A is an α-significantly common cognitive path for C-identification, α = 0.005. There is no indication on significantly common cognitive path for NC-identification.

Question 3: F→B→D is an α-significantly common cognitive path for C-identification, α = 0.005. E→A→C is an α-significantly common cognitive path for NC-identification, α = 0.005.

Question 4: D→C→A→B is an α-significantly common cognitive path for the construction task, α = 0.005.

References

Austin, J.L. and Howson, A.G., Language and Mathematical Education, Educational Studies in Mathematics, 1979, 10, 161-197.

Avital, S.M. and Shettleworth, S.J., Objectives for Mathematics Learning, Ontario Institute of Studies in Education, Bull. No. 3, 1968.

Bloom, B.S. (Ed.) The Taxonomy of Educational Objectives, Handbook I, Cognitive Domain, McKay Company, 1956.

Gagné, R.M., The Conditions of Learning, 2nd edition, Holt, Rinehart and Winston, 1970.

Piaget, J. and Inhelder, B., The Growth of Logical Thinking from Childhood to Adolescence, Basic Books, 1958.

Vinner, S., The Naive Platonic Approach as a Teaching Strategy in Arithmetic, Educational Studies in Mathematics, 1975, 6, 339-350.

Vinner, S., Concept Definition, Concept Image and the Notion of Function; to appear in Educational Studies in Mathematics.

SPSS (Statistical Package for the Social Sciences), N.H. Nie et al., second edition, McGraw-Hill, 1975, pp. 528-533.

The effects of previous knowledge on learning geometry

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As part of a general research program[1] about the utilisation of cognitive models for teaching and learning processes in school a research group at the University of Osnabrück investigates changes of states of cognitive development during the mathematics' instruction of 5th grade students (about 10 years old). For two instructional units, "axial reflection" and "nondecimal number systems", teaching programmes have been developed. Additionally, diagnostic tests were constructed which aim specifically at previous knowledge and the content dependent mechanisms of cognitive processing. The influence of these tested learning-prerequisites on the actual processes of learning and thinking were studied by individual observation of each student during his work on the teaching programme. The investigation aimes at a comparison of content specific and content independent properties of thinking processes in both areas "axial reflection" and "nondecimal number systems" to clarify the relation of more area specific and more general modes of thinking. This report is restricted to the area of axial reflection and exemplifies our procedures and gives some first results.

1. Theoretical foundations

The visual objects of school-geometry develop the human ability of thinking in a specific way: geometrical relationships are perceived by direct visual contact and thinking about these relationships can proceed within this direct visual contact. Many teachers assume that the pupil perhaps produces spontaneously semantic interpretations while his cognitive system is concretely interlocked with the object of thinking. But adverse to the assumption of many teachers exatly this the pupil is unable to do. But only later, after having attributed meaning to some relationships (e.g. to the position of two lines as "vertical") he will be able to make further explorations. Concerning the area of axial reflection, the objects of thinking are lines and points in the plane, and just with these the 10 years old pupil has scarcely accumulated every-day experience. But in spite of this he relativly quickly achieves to organize the perceptual field into parts, unities and subunities. (Koffka, 1935).

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In the twenties and thirties of this century gestalt psychologists have been involved with questions like these. Examples are the gestalt-laws of Wertheimer (1923) or the treatment of geometrical-optical illusions by Rausch (1966) containing important hints about the perceptual mechanisms which may be effective in connection with the relationships of axial reflections. Recently Palmer gave examples for the structural description of two-dimensional forms starting from the cognitive theory of propositional representation (Palmer 1976). He assumes points as elementary perceptual units which combine into higher order units. The position of a line e.g. must then be described by diverse parameters: position of the midpoint, direction from one endpoint to the other, and by the distance of both endpoints. To code special cases of straight lines all parameters of the line must be specified. In concern with our investigations it is interesting to know which parameters catch attention of the pupil while he is contemplating the entire abstract axial symmetric figure, and what is difficult for him. For many of the ten years old pupils this first interaction with abstract lines necessitates a complete re-organisation of stored knowledge as described by Norman and Rumelhart (1976). This means learning by creating a structure which is identical with the new formation of a schema (Norman and Rumelhart 1976).

The notion of schema mentioned above has been alredy used by Selz, Piaget and Bartlett for the description of thinking processes and has again won importance in connection with cognitive theories recently. We hypothesize that the student activates existing schemata in the course of his constructive mental interaction with the axial-reflection-figure or constructs new ones. As part ot the cognitive apparatur schemata can be conceived as stored plans for actions which also can controll the excention of these actions (Neisser, 1976). For a more precise description the action-control theory of W. Hacker which describes the construction an utilisation of action plans can be of a certain importance. W. Schnotz (1979) developed on the base of that theory an action-theoretic model of cognitive learning which implies the following assumptions:

- Thinking is an inner action with copies or images (ideas, concepts,

memory complexes) which is based on action-programmes and procedes hierarchically organized.

- The thinking of different individuals differs in
  a) the disponability of images and the proper action-programmes.
  b) the complexity of the operative image system (OIS). (OIS is a construct that serves as representation function and simultaneously as a kind of medium of the inner actions):
  c) the possibility to act on higher levels of codification, i.e. the higher the learning level of an individual is the fewer are the inner actions necessary to find a solution.

We tried to adjust our abservational methods to the described theoretical structures. The teaching-programmes were constructed in a way to make interences about these cognitive machanisms possible which take part in the whole learning process. The pupil will be observed in respect to the kind of action-programmes he chooses and the mode of their utilisation during the work on the different tasks exposed to him.

2. Construction of teaching-programmes

The construction of teaching programmes was guided by the following criteria:

1. Consideration of the age level in choosing curricular contents out of the theme complex "axial reflection".
2. Goal of the curricular unit should be to unloose the reflection from the global pattern "original figure - symmetry axis - image figure". The pupil should learn to pay attention to details and to the relationships between the parts of the global pattern. In this way the pupil should be enabled visualise in a different way forms which, in their total appearance he has perceived only intuitively as he was hardly able to describe them. The regularities of the perceived pictures must be verbalized by the pupil. By this means he learns the formulation of the mathematically essential properties of the axial reflection. In a next step these shall be used as constructive design for the production of new reflections.
3. To prevent, the cognitive development during the acquaintance with axial reflection from becoming, biased to certain constellations, a comprehensive notion of reflection has been developed by variation of irrelevant properties of the tasks. For instance the position of the axis was varied or the form of the original figure.

4. In consequence of procedural considerations also the instructional structure of the learning sequence had to be varied in order that individual behaviour of the pupils can be observed. These variations concern
   - the type of questioning: The pupil will be introduced to the problem by a very open instruction without further reference. The pupil utters whether we has understood the problem, whether he sees a possible solution and how he intends to obtain it. In case the student does not signify the important aspects of the problem to solve, the programme tests whether he knows these aspects and whether they are of proper significance for him. The amount of information thereby transmitted to the student is minimal and offers only graduated hints toward a solution of the problem.
   - The abstractness of presentation and of the demanded mental action: E.g. sentences, concepts and relations.
   - The linkage of mathematical terms to certain relationships.

3. The construction of the pretest

The combination of tasks for testing the previous knowledge include the following demensions:

1. The previous knowledge concerning physical reflection.
2. The previous knowledge about paper-folds and fold-axis.
3. The comprehension of symmetry-properties.
   At the description of symmetrical pictures the students' insight into symmetrical properties may become obvious. Moreover they understand and describe non symmetrical properties in symmetrical pictures.
4. The knowledge of geometrical concepts.
   In order to realize a students idea about a certain concept he has to work on that concept in the "previous-knowledge-tasks" in several stages.

I. First the objective is the "isolated concept", which the pupil has to describe or to represent by a drawing.

II. The concept will be discussed with the pupil. If necessary, the teacher gives certain hints. So it can be confirmed in which context the pupil conceives that concept, especially to what extent there has been given a mathematical or a specific everday description.

III. The students have to describe the relations which explain a

given concept by means of a diagram (e.g. "vertical").

IV. The pupils have to deal with concepts actively. (Examples: a) draw a parallel line b) draw a line which has a distance of 2.5 cm to a given one.)

5. The modalities in the description of visually exposed geometrical relations.

6. The significance of geometrical properties as seen by the pupils (presented figures) and the apprehension of geometrical laws. By means of both last mentioned dimensions inferences should be possible to what parts of a geometrical form a pupil pays attention and which verbalisations he chooses to express his observations.

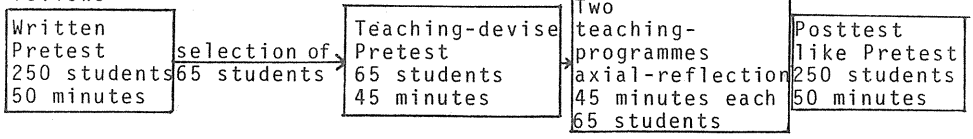
7. Technical skills. Here the different drawing skills ment which enable the pupil to construct mirror-images.

. Employment of teaching devices

Part of the previous-knowledge-tasks as well as the teaching programmes have been produced for the use on the electronic teaching device STA TM 150. By these standardized curricular units and questions a better comparability of the individually differing answers and procedures of the students has been achieved. The device exposes the stepwise instructions simultaneously audible (headset) and visible (slides, projector), controlled by a programme which branches in accord to the graduation of questions. The devise is easy to work on by the student. The below sitting teacher chooses branchings in suitable dependence with the pupils' responses. The questions are answered by the pupil partly in written and partly in oral form. Moreover the student explains his procedures while working on the problems and all of his utterances are sound-recorded.

. Design of the investigation

The study has been presented in June 1980 and was organized as follows



The evaluation of the study is presently carried out on qualitative and quantitative terms. The main results shall be interpreted in two directions:

a) given a certain previous knowledge, can it be shown to have

facilitated or aggravated definable parts of the learning.

b) Conversly it should be possible to understand observed facilitations and aggravations in the learning process as consequence of certain previous knowledge.

Finaly the possibility should be explored to design a hierarchy of curricular contents in accordance with the necessary cognitive presuppositions of specific modes of geometrical thinking. A pilot study from February 1980 conveys some results which contain parts of structures of the kind just mentioned. Some of these results will be given below.

6. Some examples

1. Effects of insufficient technical skills of the pupils working with the "geometric triangle"[1)]

The use of "geometric-triangle" makes easy geometrical constructions and drawings which are of course more precise but also more laboriously carried out by ruler and compasses; the "geometric-triangle" may serve for: measuring extents, halving straight lines, drawing of parallels in a defined distance, drawing of verticals, drawing and measuring of angles. At this age-level it is also used for the construction of mirror images. But to be sure, many teachers neclect thereby, that also the use of the "geometric-triangle" has to be learned and needs exercise. We detected the following difficulties:

I. It seems hard for the pupils to coordinate simultaneously required actions with the"geometric triangle".

As illustration from the pretest: A parallel to a given line shall be drawn through a given point. To achieve this the pupil moves the "geometric-triangle" up to this point and then pays attention to the parallel position of the edge of the "geometric-triangle". He thereby looses the point and finally draws the line past the point.

The same pupils produce the following defect in the teaching programme: The connective between original- and picture-point lies vertical to the axis. In constructions the picture-point the middle-line of the "geometric-triangle" must be on the axis, then the distance of the wanted picture-point will be marked. But while measuring the distance the pupil does not pay attention to the vertical position of the "geometric-triangle".

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1) From the german word "Geo-Dreieck"

II. Effects of unprecise knowledge of a geometrical concept.
The use of unprecise geometrical concepts leads to specific faults in solving tasks which require the correct application of these concepts.
Illustration from pretest: For a certain pupil the mirror axis constitutes always a part of the axial-symmetric figure and is consequently an extent and no straight line. The same pupil makes the following fault in the teaching programme. They have drawn a triangle and for two of its edges the corresponding picture points. Required is the mirror-image of the third edge.

> At first the pupil has to construct the mirror-axis by use of the given picture points. But here the pupil draws only this part of the axis (the extent) which is "limited" by two points of the figure. He says that he had only a quarter of the mirror axis and did not know how to construct the whole mirror-axis in order to draw the missing picture point.

III. Apprehension of geometrical relationships.
In the teaching-programme there will be scrutinized wether the student recognizes laws according to which mapping is to be executed. So the pupil has to know that he can find those laws by contemplating the total picture, that he can e.g. imagine anxiliary lines between the original figure and the picture figure and investigates these in respect to their position.
Illustration from the pretest: Most pupils do only pay attention to the important relations in geometrical configurations after having been adequatly instructed. Therby they regard at first only the extension measures of straight lines. The relative position of straight lines (e.g. vertical) is scarcely regarded.
Fault in the teaching programme: Most pupils realize that the connective between original point and picture point is halved by the axis but only few apprehend that the connective extents vertical to the axis.

IV. Effect of a special introduction to axial reflection.
A special introduction into the mentioned curricular unit has detectable influence upon the procedures at working with

"mirror-tasks". In this introduction the pupils gathered their experiences in respect to axial-symmetry exclusivly by placing little plates into axial-symmetric figures.
The following mistakes and peculiarities have been observed later at the construction of mirror images:

a) the contours of geometrical figures are reproduced correctly, but all lines within the figure are forgotten or wrongly drawn
b) reproducing open graphs, additional lines are drawn in order to create closed figures.
c) figures which are not positioned adjacent to or on the axis will be connected with the mirror axis and then reflected.

In case of an introduction with a real mirror or folding of paper also specific modes of thinking of the students result which in turn create special difficulties for the solution of axial-symmetrical tasks.

1. Hacker, W.: Allgemeine Arbeits- und Ingenieurpsychologie; Berlin 1973
2. Koffka, K.: Principles of Gestalt Psychology, New York,1935
3. Neisser, M.: Cognition and reality. Reading/Engl.: Freeman, 1976
4. Palmer, S.E.: Visuelle Wahrnehmung und Wissen: Notizen zu einem Modell der sensorisch-kognitiven Interaktion In: Norman, Rumelhart D.: Strukturen des Wissens, Wege der Kognitionsforschung, Klett-Cotta 1978
5. Rausch, E.: Probleme der Metrik (Geometrisch-optische Täuschungen) In: Handbuch d. Psych., 1. Bd., 1. Hälfte, Göttingen 1966a
6. Rumelhart, D.n. Norman D. Accretion, tuning und restructuring: three modes of learning
7. Schnotz, W.: Lerndiagnose als Handlungsanalyse, Weinheim 1979
8. Wertheimer,M.: Untersuchungen zur Lehre von der Gestalt II. In: Psychol. Forsch. 4, 1923

THE SENSE OF MATHEMATICAL EXPLANATION FOR 11-YEAR-OLD PUPILS (WITH RESPECT TO A COMBINATORIAL PROBLEM)

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In this paper we describe the organisation and we give the main results of an experimental situation devised to observe the elaboration - and to study the sense - of pupils' explanations with respect to the solution of a combinatorial problem. We examine the relation between the elaboration of an explanation and the problem-solving process, and the relation between proof and explanation.
We observed 11-year-old and 15-year-old pupils, for the moment only the first group has been fully analysed and the results are reported here.

Introduction

To get children to be able to build and to write proofs, and to be aware of the necessity to produce them, is certainly one of the main problems of mathematical education.

Usually to reach their end, teachers use empiricist and impressionistic methods : they insist on the necessity to prove some propositions, they show proofs to their pupils and ask them to build some ; they judge their proofs, correct them, and accept or refuse them. From these examples and sanctions, pupils have to build up and idea of what a proof is.

It is well known that this aim is not reached, particularly at the end of compulsory schooling (16 years of age). To solve this problem, which is not only concerned with the notion of proof but also with the problem-solving process, we think that it is necessary to answer two questions :

1. what does "to prove" realy mean for pupils at the different compulsory school levels ?
2. what are the didactical conditions which get the pupil to be conscious of the necessity to build proofs, and then to master that activity ?

Our research is centred on the first question. We devised experimental observations with pupils 11 and 15 years old (in this paper we only report on the first) to observe :

- How pupils accept the result of their problem-solving process as a solution of the problem.

- The existence of a proving process, for example critical analysis of solving-process, use of counter-examples...
- The elaboration of a formulation of their solution: Is it a proof-explanation ?

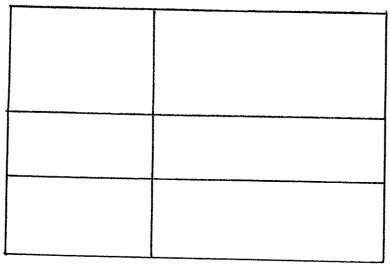
The observation apparatus

The didactical process as we briefly described it above implies a didactical contract in the classroom in which the teacher receives the pupil's proof explanation as an object (which he will accept or refuse) and not as a tool for proof. In such a situation it could be that pupils believe that what they have to do is to show their comprehension of some mathematical point, or to describe in a standard form the process they use to solve the problem.
So, for our experimental observation we try to break the ordinary didactical contract, organizing relations and communications between pupils to favour the appearance of proving processes. We set it up as follows :
Eight pupils divide themselves into two teams of four pupils. In each team pupils forme two pairs. During the observation each pair is in a separate room, those of the same team being able to communicate by writing (there is no communication between teams). The same problem is proposed to each team, the target being to solve it giving the best explanation. The winner is to be indicated after a debate with the children at the end of the experiment.
In this situation :

- Pairs of the same team have to make proposals or to criticize the solutions proposed by their partner pairs. As the elaboration of an explanation is a common project, and because each pair has no authority over the other, it would seem that the propositions would not be accepted or refused without discussions which would be the starting point for proving process.
- The affective stake of the situation (the interest in being the winner) increases the evaluation of the risk coming from the uncertainty of the result. It can make pupils aware of the necessity to prove.

We choose a problem wich is not too difficult so that each pupil can start the resolution. Also we want to privilege the procedural aspects. It is possible in a domain where pupils have not too much difficulties with the concepts involved. Finally we do not want classical school mathematical problems for which pupils can refer to standard discourse which would limit their liberty of analysis. So the problem is the following:

How many rectangles are there in this figure ?



Each pair was recorded on a tape recorder and observed ; for each team a person ensured the communication between pairs and recorded the exchanges. (The analysis and records of these observations are reported in detail in two Research Reports - one for 11-year-olds, one for 15-year-olds - available at the Laboratoire IMAG, Grenoble).

The identification of the proving-process and its analysis are conducted with respect to the categories of the Lakatos description of the logic of mathematical discovery. The existence of a proving-process does not mean that pupils are conscious of the necessity to give a proof. We identify it in their attempt to show in an explanation the necessity of their result relating to the data.

The main results of the observation of the 11-year-olds pupils

* Search for the solution, the proving process

  After an attempt to solve the problem empirically most of the pupils use an enumeration procedure underlaying which is an a priori classification of the set of solutions. This indicates the use of a classification criterium which in general remains implicit. One pair who formulates it, in fact uses it to designate a class of solutions and does not extend it to designate the others or to search other solutions.

  All pupils at some time or other, raise the problem of exhaustiveness : are there any more rectangles left ? But this doubt with respect to the result they have already acquired does not incite them to examine their procedure.

  In fact it seems that they substitute to the problem we propose, the new task : count as many rectangle as possible in the figure. So to reduce their anxiety it is sufficient to check that they use correctly their procedure and to verify that they do not enumerate solutions twice. So the result obtained is the best at that moment.

So we do not really find genuine proving processes, nevertheless we observe in two cases behaviours which are foundamental with respect to the Lakatos process :

- In searching for new solutions, pupils often return to the procedure they have already used. They use it again many times but it is viewed globally and not as an articulation of elementary operations. Some remarks show that they see the systematic features of the procedure, but they are not sufficiently able to formulate it to get moving a proving-process. In some cases pupils return to the objects to eliminate some of them (the square) or to find new ones. In this last case they do not link the properties of the objects to the procedure. This return takes place in an empirical quest.
- Rejection of squares is the occasion for a debate amongst pupils. Mathematical arguments are developped to accept squares as solutions. But decisions are not taken on an agreement on relations between squares and rectangles, but under authority arguments.

* The elaboration of explanation

First of all to explain is to formulate a text. So to fulfil instructions pupils try to express themselves in a good language and to do their best in presentation.

To explain the result means to show how it has been obtained. Implicitly children aim at describing the procedure they have used. But even if that target is explicitly formulated, they just describe the set of rectangles they have enumerated showing the classification associated with their procedure. The shift we observe between what pupils intend and what they really do, emphasizes the precariousness of studies dealing with pupils' explanations without taking into account their conditions of elaboration. This shift could be associated with the pupil's incapability to grasp his procedure as an articulation of elemantary operations.

There is during the formulation period opportunity to get moving a critical analysis as such as that we find in the logical process described by Lakatos. Here it is not a matter of improving a proof but the description of a set of objects. So it is necessary to build a code, so that there is absence of ambiguity, and possibility of decoding. We observe such a process once. We report it below, this is not an exact translation of the French text (wich is already avalaible in the report we have cited) but it conveys the athmosphere and sense of the discussion.

Bertrand	*: And this here, what can we call it ?*	They look for designation for : 
Eric	*: What d'y mean ?*	
Bertrand	*: This, how cou' we describe it ?*	
Eric	*: 4 rectangles together*	One is proposed, it is criticized by a counter-example which is rejected
Bertrand	*: But it could be :* 	
Eric	*: But I said together !*	
Bertrand	*: But what about :* 	an other counter-example is proposed...
Eric	*: That's not a rectangle !*	
Bertrand	*: That's just what I mean*	
Eric	*: So, 4 rectangles together forming a rectangle.*	then the designation is changed.

But such a process demands that pupils be not too self-centred, but we observe it is not the case for most of them.
The elaboration of an explanation shows their feeble decentring, i.e. to take their own activity as a subject of thinking. Indeed they formulate their consciousness of the necessity to be understood by their partners ; but does not lead to an analysis of their formulation, with the object of discovering possible decoding problems. Pupils put their faith in the explanation they formulate, with respect to their own feelings of clearness : what is for them sufficiently obvious must be necessarily so for others. From this self-centred feature proceeds a refusal of self-criticism which leads to a massive use of authority arguments or to blocked situations in which each one sticks to his guns. We remark that the fact that pupils think little or much of their partners reinforces these behaviours.

REFLECTIVE INTELLIGENCE AND MATHEMATICS PERFORMANCE

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Data from three studies conducted in Lebanon and involving 12 to 14 years old students are analyzed to test the hypothesis that mathematical performance increases with higher reflective intelligence, as measured by Skemp Test, irrespective of the type and organization of mathematical content, teaching method, and the cognitive complexity of the mathematical task.

Intelligence in the Piagetian sense is an extension and a perfection of adaptive processes. Piaget identifies two types of intelligence: Sensori-motor and reflective. Sensori-motor intelligence is the permanent state of equilibruim towards which all successive adaptations of sensori-motor mature tend. These include perceptual activity, the formation of habits and pre-verbal activity. The full development of sensori-motor intelligence is normally achieved before appearance of language ($1\frac{1}{2}$ - 2 years). On the other hand, reflective intelligence is, according to Piaget, the permanent state of equilibrium towards which all successive adaptations of cognitive nature tend. These include representative thought, intuitive thought, and concrete operational thought. Reflective thought normally exists from the age of eleven to twelve years. Piaget (Piaget, 1947/1967) defines reflective intelligence as consisting of "reflecting (in the true sense of the word) on these operations and, therefore, operating on operations or on their results and consequently affecting a second degree grouping of operations" (p. 148). Skemp (1961) modified Piaget definition of reflective intelligence as "the functioning of a second order system which:

1. can perceive and act on the concepts and operations of sensori-motor system,
2. can act on them in ways which take account of these

relationships and of other information from memory and from the external environment,

3. can perceive relationships between these concepts and operations." (p. 49)

Using this definition, Skemp (1961) constructed a four-part instrument to measure reflective intelligence: Concept Formation, Reflective Activity on Concepts, Operation Formation, and Reflective Activity on Operations. Concept Formation consists of items each of which has three examples of a concept, three non-examples, and three instances to be judged as to their belongingness to the concept. A similar pattern holds for Operation Formation except that the operation (after discovering it) has to be applied to three given instances. Reflective Activity on Concepts is the logical multiplication of two attributes of two known concepts. Reflective Activity on Operations consists of the operation of combining two known operations, reversing a known operation or combining and reversing two known operations. Skemp (1961) investigated the hypothesis whether the presence of reflective intelligence is a necessary, but not sufficient, condition for mathematics achievement. Skemp correlated reflective intelligence measures with the mathematics achievement of 10th-and 11th- grade students in England. The correlations for the 11th-grade between a mathematical criterion and parts two, three, and four of Skemp Test were 0.58, 0.42 and 0.72. The corresponding correlations for the 10th-grade were 0.56, 0.48 and 0.73. Skemp concluded that the data support the hypothesis: The two highest correlations being those between the two tests of reflective activities and mathematics, although an unexpected relatively high correlation between the use of operation and mathematics was observed.

Harrison (1967) studied the predictive power of Skemp Test using mathematics achievement as a criterion. He investigated whether the addition of measures of reflective intelligence to measures of general intelligence improves the prediction of mathematics performance. A battery of tests including general intelligence tests and Skemp Test was administered to 125 8th-grade students. He

concluded that measures of Operation Formation and Reflective Activity with Operations significantly improve the prediction of mathematics performance.

The present paper provides data from three studies conducted in Lebanon to support the hypothesis that for students of age twelve to fourteen years, mathematics performance increases with higher reflective intelligence irrespective of the nature and organization of mathematical content, teaching method, and the cognitive complexity of the mathematical task.

Study 1

The purpose of Study 1 (Jurdak, 1973) was to compare the effectiveness of two teaching methods having two distinct levels of emphasis on mathematical structure in organizing and presenting the same mathematical content. In addition the study investigated the relationship of reflective intelligence to performance in a second-order mathematical structure whose learning depends on an already learned first-order system.

As two models of a second-order system, the integral powers of 2 and 3 were chosen as a suitable mathematical content for 8th-graders in Lebanon. Hypothetically, the learning of these two models depends on the learning of a first-order system i.e. integers with addition. Two organizations of this piece of mathematical content were constructed in such a way that one organization tended to emphasize the structural properties of the models in developing operations and algorithms (E),whereas, the second attempted a direct approach with no explicit emphasis on structural properties (C).

A sample of 228 8th-grade students were selected and assigned to the two treatments. The sample was given two parts of Skemp Test: Operation Formation and Reflective Activity with Operations. Using the sum of the two scores, the sample was divided equally into three parts: High (H), middle (M), and low (L).

Achievement measures were taken at the conclusion of the experiment (two weeks) using specially-constructed tests. The tests were classified into four cognitive levels (computation, comprehension, application, and analysis) using the taxonomy developed by the National Longitudinal Study of Mathematical Abilities (NLSMA) (Romberg & Wilson, 1970).

Table 1 gives the means and standard deviations of the post test scores for each reflective intelligence level and each cognitive level.

Table 1

Means and Standard Deviations of Posttest Scores (Study 1)

		Cognitive Level							
Group	n	Computation		Comprehension		Application		Analysis	
		$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
Experimental									
H	38	12.7	2.0	6.8	1.3	5.5	2.2	4.1	2.7
M	38	9.5	3.4	5.2	1.7	4.0	1.9	1.9	1.8
L	38	7.8	2.7	4.6	1.5	2.8	1.6	1.1	1.5
Control									
H	38	11.4	2.4	6.1	1.5	6.4	1.8	4.3	2.4
M	38	7.7	2.8	4.4	1.5	5.2	1.6	2.0	1.6
L	38	7.6	2.6	3.7	1.8	3.4	2.0	0.9	1.5

Note. Maximum scores for computation, comprehension, application and analysis are 15, 8, 5, 5.

It was found that the two orthogonal contrasts comparing (H) to (M) and the average of (M) and (L) to (H) were significant ($P < .05$) and favoring the higher level for each treatment group and each cognitive level. Treatment effect was significant only for computation and comprehension favoring the E group. Interaction between treatment and reflective intelligence was not significant.

Study 2

The purpose of Study 2 (Ibrahim, 1979) was to investigate whether providing students with experience in playing a game whose rules and strategies reflect a certain mathematical structure facilitates the learning of that structure. Reflective intelligence was used as a moderator variable.

Two games devised by Kaufmann and Steiner (1969) were used. The games whose rules and strategies reveal the mathematical structures of an operational system, semi-group, and group were played by the experimental group (E) prior to instruction in these structures. The control group(C) played, prior to instruction in the three structures, games whose rules did not reflect those of the structures.

A sample of 250 9th-grade students in Beirut area were given two parts of Skemp Test: Operation Formation and Reflective Activity with Operations. Thirty four students whose scores clustered around the mean score were selected to form the middle group (M). Thirty four students one standard deviation below and above the mean were selected to form the low (L) and high (H) reflective intelligence group.

Evaluation of performance was done using specially constructed tests which were classified into recognition and recall, algorithmic thinking, and open search according to the taxanomy developed by Avital and Shettleworth (1968).

Table 2 gives the means and standard deviations for each of reflective intelligence group and each cognitive level.

For each treatment and each cognitive level the two orthogonal contrasts comparing (H) to (M), and the average of (M) and (L) to (H) were significant ($p < .05$) and favored the higher group. Treatment and interaction between treatment and reflective intelligence were not significant.

Table 2

Means and Standard Deviations of Posttest Scores (Study 2)

Group	n	Cognitive Level: Recall $\bar{x}$	Recall s	Algorithmic Thinking $\bar{x}$	Algorithmic Thinking s	Open Search $\bar{x}$	Open Search s
Experimental							
H	17	25.1	3.7	23.6	3.7	17.7	6.5
M	17	18.1	4.0	15.5	3.8	9.5	3.4
L	17	13.5	4.3	11.1	3.3	5.0	3.1
Control							
H	17	24.6	3.8	24.9	3.2	14.2	5.4
M	17	16.6	4.7	16.7	4.5	8.4	4.0
L	17	12.2	4.6	13.5	1.8	4.3	3.5

Note: Maximum score for recall, algorithmic thinking, and open search are 33, 32, and 35 respectively.

Study 3

Study 3, a study conducted only recently, had as its purpose the comparison of the effectiveness of two organizations of a part of the geometry curriculum in the intermediate school. The two organizations had the same terminal objectives, namely, knowledge and application of the parallelogram and its properties. However, one approach was based on congruence axioms and the second on transformations axioms.

A sample of 181 9th-grade students were given two parts of Skemp Test: Operation Formation and Reflective Activity with Operations. Based on the score of Skemp test, the sample was divided into three levels: Middle (M), clustered around the mean; (L), one standard deviation above the mean. Twenty subjects were selected from each level and assigned randomly and equally to the two treatments.

At the conclusion of the experiment which lasted for 18 fifty-minutes periods, the students were evaluated by using criterion-referenced tests. The latter were classified as knowledge and comprehension,

and application. Table 3 shows the means and standard deviations for Study 3.

Table 3

Means and Standard Deviations of Posttest Scores (Study 3)

Group	n	Cognitive Level: Knowledge and Comprehension $\bar{x}$	Knowledge and Comprehension s	Application $\bar{x}$	Application s
Transformation					
H	10	22.0	2.7	25.2	5.2
M	10	19.0	3.9	17.6	8.5
L	10	18.8	2.9	15.9	6.8
Congruence					
H	10	22.1	2.7	24.5	4.0
M	10	20.5	2.0	20.4	7.7
L	10	20.7	2.3	19.6	7.2

Note: Maximum score for knowledge and comprehension is 26 and for application 34.

Again the contrasts comparing (H) to (M) as well as the contrast comparing the average of (M) and (L) to (H) were found significant ($p < .05$) and always favoring the higher level for each treatment and each cognitive level. Treatment and interaction effects were not significant.

Discussion

The three studies provide support to the hypothesis of the present paper. Better mathematical performance was associated with higher reflective intelligence which varied in terms of the type and organization of mathematical content, the method of presentation, the nature of the evaluation taxanomy, and the level of the cognitive complexity of the criterion task. The absence of interaction between reflective intelligence and method of instruction and/or organization of content seems to point to an underlying ability which operates in a rather dominant way as far as mathematics performance is concerned.

The question arises whether it is possible, if at all, to develop, or

accelerate the development of, the ability of reflective intelligence. Would it be possible to find ways of structuring the learning environment in order to promote the development of reflective intelligence? Is the heuristic method of teaching with its emphasis on building the learner's awareness of the relationships among the elements of the problem one such method?

Reflective intelligence has a number of advantages over measures of general intelligence. First, reflective intelligence seems as good predictor, and even a better predictor,of mathematics performance than measures of general intelligence. Second, by virtue of its definition and theoretical basis, reflective intelligence seems more readily analyzable into operational methods than general intelligence in as much as the theories of mental factors have not been able to analyze the group factor of mathematical ability. In the words of Peel (1971) such "attempts to analyze this in greater detail reveal number, verbal, spatial, thinking elements intricately mixed up with material content."

REFERENCES

Avital, S.M., & Shettleworth, S.J. Objectives for mathematics learning: some ideas for the teacher. Ontario Institute for Studies in Education, Bulletin No. 3. Toronto: the Institute, 1968.

Harrison, D.B. Reflective intelligence and mathematics learning. Unpublished doctoral dissertation, University of Alberta, 1967.

Ibrahim, M.S. The effect of selected structured games and of reflective intelligence on the acquisition of three mathematical structures, by third intermediate students. Unpublished M.A. thesis, American University of Beirut, 1979.

Kaufman, B., & Steiner, G.H. Checker games in operational systems as media for an inductive approach in teaching algebra. Educational Studies in Mathematics 1968-69, 1, 445-478.

Jurdak, M.E. The effects of emphasizing mathematical structural properties in teaching and of reflective intelligence on four selected criteria (Technical Report No. 275). Madison: Wisconsin Research and Development Center for Cognitive Learning, 1973.

Piaget, J.L. /The psychology of intelligence/ (M. Piercy, trans.). London: Routledge & Kegan, 1967 (originally published, 1947).

Peel, E.A. Psychological and Educational Research bearing on mathematics teaching. In teaching school mathematics (W. Servais and T. Varga, Eds.). Paris: UNESCO, 1971.

Romberg, T.A. & Wilson, J.W. The Development of Tests. NLSMA Report No. 7. Stanford: Stanford University, 1969.

Skemp, R.R. Reflective intelligence and mathematics. British Journal of Educational Psychology. 1961, 31, 45-54.

QUELQUES THEMES HEURISTIQUES

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This paper deals with some essential features concerning heuristic training as practiced at the University Louis Pasteur. This kind of teaching is related with some heuristic themes. Those are types of general research behaviour, commonly observed during solving processes ; they do not depend on the mathematical content of each submitted problem.

Qu'est-ce qu'un problème ? Ce mot désigne, dans la littérature des activités très diverses proposées à des élèves. Dans [3] [3bis], on distingue sept types d'exercices visant des objectifs différents, provoquant des comportements spécifiques tant chez les étudiants que chez les enseignants. Ils nécessitent chacun des pédagogies appropriées.

Dans cet exposé, nous n'évoquons qu'une seule de ces significations. Ici un problème (ou exercice de recherche) est une question pour laquelle l'élève ne dispose pas a priori de stratégie ou d'algorithme tout prêt. Il faudra donc chercher longtemps, souvent plus d'une heure, quelquefois des semaines. Un élève sérieux et travailleur n'est pas assuré de pouvoir résoudre tous les problèmes que l'on lui proposera. Un échec n'est pas significatif, contrairement à une réussite.

Il s'établit alors un contrat didactique spécifique. L'élève et le maître savent qu'on n'attend pas de réponse rapide : tout ce qu'on peut exiger est une recherche active. Il est exclu qu'un problème puisse être sanctionné par une note, puisque l'échec n'est pas significatif. Tout au plus, la réussite pourra donner lieu à un renforcement.

La pratique des séances heuristiques est peu compatible avec les contraintes scolaires traditionnelles : il est pourtant essentiel de leur réserver une place importante dans l'enseignement. Une séance par mois constitue déjà une formule acceptable.

Peut-on enseigner l'heuristique ? Si l'on adopte cette description, il devient clair que l'heuristique, art de résoudre des problèmes, ne peut être enseignée comme la dactylographie, en promettant le succès assuré pour tout élève qui s'applique. On ne peut préparer un élève à affronter des situations déroutantes (par définition) sans s'exposer à ce qu'il soit quelquefois dérouté. Tout ce qu'on peut faire, c'est d'entraîner l'étudiant à résoudre des problèmes variés et à tirer les leçons de chaque tentative. Cette préparation sera jugée efficace, si l'étudiant parvient à résoudre plus souvent et plus vite

les problèmes qui lui seront soumis. L'entraînement consistera donc, après une séance de recherche d'une heure ou deux, à examiner, analyser, critiquer les démarches du chercheur. On mettra notamment en évidence tout ce qui n'est pas directement lié au contenu mathématique du problème posé, mais qui relève plutôt du processus général.

C'est ainsi que l'on peut mettre en évidence les conseils des "tables heuristiques" de Polya [5]. On remarquera que les mêmes conseils valent pour des contenus mathématiques divers, et que ce ne sont pas des instructions algorithmiques. Par exemple, si l'on demande : "Connais-tu un problème analogue à celui qui t'est posé ?" il n'existe pas de procédé automatique pour obtenir la réponse. L'analogie cherchée est à inventer ; et le résultat sera d'autant plus efficace que cette invention ne sera pas immédiate. De même si l'on conseille d'essayer de dessiner une représentation graphique, il arrive fréquemment que le mode de représentation suggestif soit la clé du problème difficile à trouver.

Un peu d'heuristique générale. L'observation clinique des processus heuristiques [2], révèle que le déroulement de la recherche se décompose souvent en cycles. Chaque cycle est caractérisé par le domaine de recherche, ensemble des objets manipulés par le chercheur, au cours du cycle. Celui-ci comporte plusieurs phases, dont certaines ont déjà été décrites par John DEWEY [1], et il se termine par un micro-eureka qui déclenche une modification profonde du domaine de recherche. Le dernier cycle aboutit à l'eureka final, lorsque le sujet est convaincu d'avoir résolu la question.

En plus des considérations générales qui viennent d'être présentées, l'heuristique étudie quelques thèmes, qui sont des démarches communes à beaucoup de résolutions de problèmes, mais qui dépendent cependant de l'énoncé proposé.

On se propose ici d'énumérer quelques thèmes heuristiques, en signalant simplement les plus connus, et en développant quelque peu la présentation des autres.

Quelques thèmes

A) Elargissement et restriction du domaine de recherche

Il arrive fréquemment que les données d'un problème dépendent de trop de paramètres : l'inconnue est alors à chercher dans un domaine de recherche beaucoup trop vaste. Dès que le chercheur en prend conscience il pourra essayer de restreindre le champ.

Un des procédés consiste à fixer quelques-unes des variables, et à commencer par traiter le problème ainsi modifié ; puis, dans un second temps à examiner comment la solution évolue lorsqu'on libère les données provisoirement fixées. (Méthode de la variation des constantes).

Au lieu de fixer les variables, il est parfois plus ingénieux

d'inventer des contraintes qui lient convenablement les données. Le choix de ces relations est souvent l'élément décisif de la solution : il est à inventer.

Dans d'autres circonstances, au contraire, le domaine de recherche est trop rigide : il est difficile de jouer avec la situation proposée. On cherchera alors à briser cette rigidité, en traitant un problème plus général obtenu en introduisant des paramètres auxiliaires. C'est ainsi que dans de nombreux problèmes dont les données et les inconnues sont discrètes, on autorise provisoirement les paramètres à devenir réels, pour que le chercheur puisse bénéficier des ressources de l'analyse infinitésimale.

Une tactique analogue consiste à dynamiser certains problèmes trop statiques. Par exemple, si l'énoncé demande de démontrer que trois droites AA', BB', CC' sont concourantes, on pourra reformuler la question de façon que les points C et C' deviennent variables, et l'on prouvera que la droite (variable) CC' , pivote autour du point de rencontre de AA' et BB' .

Ces thèmes heuristiques, illustrés par des exemples très nombreux et variés donnent lieu à un entraînement satisfaisant, surtout si l'on arrive à convaincre les étudiants que ces changements de points de vue sont efficaces dans toutes sortes de problèmes.

B) Grignotage d'un problème.

Cette idée s'applique aux problèmes où le nombre des inconnues est apparemment très grand. Le cas typique est celui des problèmes de mots croisés (autant d'inconnues que de cases à remplir). Une tactique efficace consistera à rechercher systématiquement les points faibles. Il s'agit, au sens strict, de quelques inconnues que l'on peut déterminer immédiatement Par exemple, dans un "jigsaw puzzle", où les inconnues sont les positions relatives des pièces, les points les plus faibles sont évidemment les coins, puis les bords du rectangle à reconstituer.

Dans un sens plus large, un point faible peut être constitué par un groupe d'inconnues, que l'on ne peut pas déterminer au début, mais dont l'étude affaiblit d'autres inconnues.

Dès que l'on a trouvé la valeur de quelques inconnues "faibles", le domaine de recherche diminue, et on constate que dans ce problème restreint de nouveaux points faibles sont apparus.

Voici deux exemples typiques :

Problème 1

3	6	2	0	0	4	4
6	5	5	1	5	2	3
6	1	1	5	0	6	3
2	2	2	0	0	1	0
2	1	1	4	3	5	5
4	3	6	4	4	2	2
4	5	0	5	3	3	4
1	6	3	0	1	6	6

Problème 2

	a	b	c	d	e	f
a						
b						
c						
d					b	
e				c	a	b
f						a

Dans le problème 1, on a placé les 49 pièces d'un jeu de dominos, dans une boîte 7 x 7 , et on a essayé de prendre une photo : malheureusement les lignes de séparation, entre les rectangles 2 x 1 qui constituent les pièces de dominos ne sont pas apparues sur la photographie. Il s'agit de reconstituer ces traits manquants !

Remarques : S'il n'existait qu'un couple 66 de cases contiguës, ce serait un point très faible du problème : on pourrait immédiatement l'entourer. Or, ce couple apparaît ici deux fois (cf ↑). On ne peut donc savoir celui de ces deux couples qu'il faudra entourer.

Le coin en haut et à gauche est un point faible au sens large. Il ne permet pas de déterminer immédiatement où se trouve le domino 36 . Mais, en raisonnant sur ce coin, on peut tirer des conclusions concernant les autres couples 36 disséminés sur la figure, ce qui permet dès le début de placer certains traits de séparation. Et alors le couple 03 devient un nouveau point faible.

Le problème 2 (cf [4]) demande de compléter la table de Pythagore d'un groupe à 6 éléments. Le premier sous-problème "faible" consistera à déterminer l'élément neutre du groupe, et les données prouvent immédiatement que ce n'est ni d, ni e , ni f . Un autre argument montre aussitôt que ce n'est ni c , ni b . Ces résultats permettent alors de remplir 11 cases : on a commencé à grignoter le problème en passant de 31 inconnues, à 20 seulement !

Une particularité psychologique des problèmes de grignotage est qu'à la fin de chaque cycle, le chercheur est certain d'avoir progressé.

Ce sont donc des problèmes qui sont "motivants", même pour ceux qui ne parviennent pas à la solution définitive. De plus, beaucoup de problèmes de grignotage mal construits s'effondrent dès que l'on a déterminé beaucoup d'inconnues.

C) Reconnaissance de formes.

Il y a des personnes qui, contemplant l'expression $\frac{\text{Arctg } x}{x} + \frac{\text{Log } x}{1+x^2}$, s'aperçoivent rapidement que c'est la dérivée du produit $\text{Arctg } x \cdot \text{Log } x$. Il est difficile d'enseigner à avoir de telles idées, faciles pour certains, "introuvables" pour d'autres. Cependant il est indispensable d'entraîner des individus à "reconnaître des formes" (selon la terminologie de la Gestalttheorie). Le cas le plus facile est celui, où tous les indices de reconnaissance figurent sous les yeux du chercheur. Ainsi l'idée peut apparaître par diverses voies (dans l'exemple cité, en examinant l'un ou l'autre terme). Dans un second temps, il faudra confronter le chercheur avec la Vénus de Milo : c'est une situation, où il convient d'abord de restituer les éléments manquants. Par exemple, en contemplant $\frac{\text{Arctg } x}{x}$ on reconnaît une partie de la dérivée précédente, et on essaie de la reconstituer. Dans d'autres cas, il conviendra d'apporter des modifications pour reconnaître des formes : par exemple, $\frac{-b+\sqrt{b^2-4ac}}{2c}$ n'est pas l'expression habituelle d'une formule connue.

On peut cependant parvenir à y reconnaître une racine de l'équation

$$cx^2 + bx + a = 0 .$$

Il apparaît qu'une pratique systématique et prolongée d'activités comportant des reconnaissances de forme, apporte des progrès dont l'importance toutefois ne saurait être assurée.

C'est ainsi que l'on développe l'intuition. Parmi les nombreuses significations que Westcott [6] donne à ce mot, nous retiendrons celle-ci :
"L'intuition est l'aptitude à tirer des conclusions à partir d'informations partielles". Ainsi dans le croquis ci-dessous, des tracés rudimentaires permettent de reconnaître un jockey.



Bien entendu, l'intuition est une prise de risque. Il convient de vérifier son bien-fondé.

En géométrie, l'élève doit être entraîné à reconnaître certaines situations à partir de quelques indices, et à reconstituer des éléments manquants.

Ces exemples ne constituent qu'une faible partie des thèmes heuristiques, autour desquels s'organise l'enseignement dispensé, sur ce sujet à l'Université Louis Pasteur, aux étudiants, aux professeurs en exercice, et par l'intermédiaire de ces derniers à des élèves plus jeunes.

Citons d'autres thèmes heuristiques : le transfert, la rectification de tir, l'analyse et la synthèse, l'utilisation des transformations et des invariances, etc...

BIBLIOGRAPHIE

[1] J. DEWEY — Comment nous pensons. Flammarion Paris (Traduction Decroly).

[2] G. GLAESER — In La problématique et l'enseignement de la mathématique. C.I.E.A.E.M. 1976 (W. Vanhamme éditeur).

[3] IREM de Strasbourg — Le livre du problème - fascicule I - Cedic - Paris.

[3bis] — Didaktik mathematischer probleme und aufgaben (Trad. du précédent) Vieweg. Braunschweig 1980.

[4] IREM de Strasbourg — Pour une mathématique vivante en seconde. Publication de l'A.P.M.E.P.

[5] POLYA — Comment poser et résoudre un problème (How to solve it). Dunod - Paris.

[6] WESCOTT — Psychology of intuition Holt. Rinehart and Winston New-York 1968.

METACOGNITION: THE MISSING LINK IN PROBLEM SOLVING?

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This paper describes the instructional component of an NSF-supported project investigating the problem-solving processes of fifth and sixth grade students. The instruction has been based on the work of Polya and Krutetskii, and on some work done by psychologists interested in metacognition. A brief review of metacognitive research is presented, followed by a description of the instructional program used in fifth and sixth grade classes during the past year.

This paper describes one component of an NSF-supported project, Problem Solving Processes of Upper Elementary and Junior High School Mathematics Students, currently being conducted at San Diego State University under the direction of the second author. The purposes of the project are to analyze the processes used by students in solving mathematical problems and to develop effective instructional techniques for the teaching of problem solving. This paper will focus on the instructional component of the project; a companion paper - "Problem-Solving Processes of Ten and Eleven Year Old Children" - will describe another component of the project.

A distinctive feature of the project is that the instructional component has been designed to integrate themes drawn from the works of Polya (1957) and of Krutetskii (1976) with themes drawn from developmental cognitive psychology, such as metacognition and metamemory. Since it is assumed that the reader is familiar with the works of Polya and Krutetskii, only the metacognitive literature is reviewed herein. Following an overview of this research, the current instructional work of the project will be briefly described.

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METACOGNITION

The term "metacognition" is used to describe one's knowledge concerning one's own cognitive processes and products. Two related themes tend to dominate current research: (1) with development, individuals adopt an active, self-directive role in certain areas of cognition, and (2) individuals develop the ability to monitor and evaluate their own cognitive processes (Markman, 1977).

Probably the most actively studied area of metacognition is called "meta-memory" - one's awareness of information storage and retrieval. One of the most consistent findings of developmental memory research is that older children remember more than younger children. Corresponding to this memory improvement is a gradual emergence of self-interrogation and regulation in a wide variety of situations (e.g., Flavell & Wellman, 1977). Brown and DeLoache (1979) suggested that the growth of self-regulation and sophisticated rehearsal strategies is responsible for the more successful memory performance of older children.

Although several possible metacognitive explanations for the superior memory performance of older children (Brown & DeLoache, 1978), Chi (1978) has suggested that performance is not necessarily related to age but rather to knowledge of stimuli. Her work clearly demonstrates that children can outperform adults on memory tasks for which the children have greater knowledge; for example, young chess players' memories for chess positions were superior to non-chess-playing adults' memories. Furthermore, she suggests that, in general, the amount of knowledge that a person possesses about a specific content area can determine to a large extent how well the person can perform in memory and metamemory tasks.

Further support for Chi's hypothesis may be found in the work of the Russian psychologist Istomina (1948/1975), who found that young childrens' memory for meaningful (to the child) stimuli was better than for standard experimental stimuli; furthermore, the children utilized explicit regulatory behaviors in remembering the meaningful stimuli. Similarly, Wellman, Ritter, and Flavell (1975) reported uncharacteristic planful behavior on the part of very young children who were asked to remember the location of a toy. Thus, knowledge of the stimuli seems to be intimately related to the utilization of regulatory strategies for memory.

Although the picture is reasonably clear for elicited memory activities, far less is known about spontaneous activities. A number of studies of children generating combinations and permutations have shown that the likelihood of keeping external records of progress increases with age and is positively related to success in solving such problems (Siegler & Liebert, 1975). Similarly, Brown and DeLoache (1978) have reported that older children (fifth grade and older) spontaneously use study strategies, such as underlining text and taking notes, while reading, whereas younger children do not exhibit these behaviors. A plausible metacognitive explanation for all these spontaneous behaviors is that those who utilize them do so because they believe that such regulatory activities will enhance the future retrieval of the information and that the information is likely to be useful in the future.

Baron's (1976) review of developmental studies of cognitive strategies has suggested certain important metacognitive or control processes. Among the most salient central cognitive strategies identified are the imposition of subjective organization on information in the task, the setting of realistic limits on the dimensions of the task, and the use of multiple solutions to check results. These strategies appear to play an important role in the processes of planning and carrying out a problem solution.

In an interesting study of planfulness and metacognition, Pellegrino and Schadler (1974) found that the likelihood of a student inventing a procedure for comparing the sizes of two parallelograms, on the basis of experience with comparing the sizes of two rectangles, was greatly improved if the student was asked, "What do you think I'm going to ask you to do?" and given an opportunity to propose several possible tasks, instead of simply being asked, "Which one is bigger?" . In our experience with children, we have observed that, when given data and asked to propose a question, children rarely pose a question that they cannot answer, and they usually produce the answer quite readily. The metacognitive relationship between problem posing, planning, and problem solving would appear to be an important one.

If Chi's hypothesis is correct, the readily observable metacognitive differences between adults and children might also be observed when one contrasts the behavior of novice and expert problem solvers in a task domain; however, there is evidence to the contrary. For example, Dansereau, Long, McDonald,

and Actkinson (1975) found that even good students know very little about techniques they might use to remember better the information they are studying. Furthermore, Simon and Simon's (1978) expert physics problem solver showed little evidence of metacognitive analyses in his problem-solving behavior; in fact, the novice problem solver's protocols had many more instances of metacognitive analyses. It may be that the metacognitive behaviors of experts become so integrated into problem-solving routines that they become difficult to observe, or it may be that the metacognitive analyses are only needed at certain points in the development of expertise. Once a sufficient level of expertise has been generated, the metacognitive aspects of thinking assume a secondary role, with technique and execution assuming primacy.

Despite the fact that metacognitive behaviors may sometimes be difficult to detect, one can find clear indications of metacognitive differences between very capable and less capable persons in a variety of areas. For example, Reese's (1977) review of studies on paired-associate learning suggested that compound imagery elaboration facilitated long-term retention. Weinstein (1978) has proposed the general efficacy of elaboration as a learning strategy and has suggested that elaboration skills might be specifically addressed in instruction.

METACOGNITION AND MATHEMATICAL ACTIVITY

According to Chi's hypothesis, high ability students would be expected to exhibit qualitative differences in mnemonic strategies and expertise when compared with low ability students. Krutetskii (1976) has reported that high ability students tended to remember for long periods of time the mathematical structure of problems they have solved and general reasoning schemes for proofs they have given, whereas low ability students were characterized by a poor memory for such material. It is not clear from Krutetskii's report whether or not subjects were specifically told to remember information from their mathematics work; however, since this instruction was not mentioned in Krutetskii's report, it seems reasonable to interpret his results as evidence of spontaneous mnemonic planfulness on the part of high ability students.

Silver (in press; 1979) has found that high ability students tend to recall important structural aspects of a problem, whereas poor students tend to remember, if anything, only the details of the problem's statement. Since the subjects in my research were presented with a solution to the problem and instructed to "remember all that you can about the problem and its solution," the results indicate that high ability students, when preparing for elicited retrieval, engaged in a careful study of the problem's structure and ignored the problem's context.

The inability of the low ability students to remember the structural aspects of the mathematics they study may be symptomatic of a more general metacognitive deficit; namely, the failure to notice structure. Krutetskii has noted that high ability students are characterized by the tendency to apprehend rapidly the structure of a mathematical problem. From the research cited, it may be inferred that apprehension of structure is an important, possibly metacognitive in its planful aspects, component of mathematical understanding.

In general, if one believes that there is an underlying structure to mathematics, then one is more likely to believe that the problems one solves or the theorems one proves today are likely to be useful tomorrow. This, in turn, is likely to influence memory regulation. In particular, one is likely to use different strategies for remembering what one learns than one would use to remember a random array of discrete facts. For example, one is more likely to organize incoming information into "chunks" on the basis of one's experience and knowledge of what is <u>really</u> important in mathematics, i.e., problem structure and generalized arguments. It is clear that one's belief in underlying structure is likely to be associated with the set of behaviors described by Krutetskii.

It is interesting that Krutetskii's description of mathematical abilities does not contain any explicit reference to metacognitive processes. A commonly held, yet rarely examined, belief is that competence in mathematics, especially mathematical problem solving, is correlated with "awareness" of one's own mental processes. In support of this claim, one often finds Gurova (1969) cited. She reported success both in observing a link between problem solving ability and awareness of one's mental processes and in instructing students to be more aware of their problem-solving processes.

It seems fairly clear that the first of Gurova's successes is not generalizable. A sizable number of protocols, collected in a number of different areas, indicate that one cannot distinguish between good and poor problem solvers on the basis of metacognitive utterances. For example, Simon and Simon (1978) reported that their "novice" physics problem solver gave many more metacognitive utterances than their "expert" did. Furthermore, Silver (in press) found that very few students, even capable ones, gave evidence of any awareness of having been helped to solve a new problem by the discussion of a structurally related problem, despite the fact that most of the successful solutions used the method from the discussion -- a method that was not used by anyone in solving the original problem!

A plausible explanation for these findings is that metacognitive processes may become generally automatic and hence inaccessible during verbalization either during or after problem solving. This explanation is consistent with Ericsson and Simon's (1980) extensive analysis of retrospection, introspection, and concurrent verbalizaiton.

Although awareness of mental processes is not always readily apparent in the protocols of capable subjects, it does appear to be a necessary condition for attaining expertise in a task domain. For example, Polya's (1957) heuristic questions are likely to be quite useful to a person who is learning to solve mathematical problems. As the heuristics become internalized, one would expect to find evidence of their usage in subject's protocols. After a person attains a reasonably high level of expertise in solving mathematical problems, the heuristics may be less observable, but their role in the development of that person's problem-solving ability is likely to have been significant. Thus, awareness of mental processes is likely to play a critical role in the attainment of expertise in a task domain.

THE CURRENT WORK

The first year of this longitudinal study of students' problem-solving processes has been completed. During the past year, the project has been actively involved with a fifth grade class (29 students) and with a sixth grade class (36 students). Within each class there was a considerable range in mathematical abilities; nevertheless, the typical student in these

classes is above average according to national norms. Next year, the instructional component will involve the same students from the fifth grade as they become sixth graders and students from a new fifth grade class.

Our reading of the metacognitive literature has led us to design instruction with a primary emphasis on process rather than product. Moreover, unlike many of the published reports of instructional studies aimed at increased utilization of specific heuristic suggestions, our work with specific processes has been embedded in a general metacognitive framework. Thus, for example, considerable class time is spent discussing and thinking about processes.

The two principle foci for the instructional activity are increasing students' awareness of (a) the act of solving a mathematical problem as a multi-faceted complex of processes and (b) the fact that many problems permit multiple methods of solution. The focus on process has led us to include a number of heuristic strategies in our instruction. For example, specific attention is given to (a) drawing a diagram to aid in understanding a problem, (b) making a chart or table to aid in detecting patterns, (c) searching for patterns in problem data, (d) identifying irrelevant (and, consequently, relevant) information in a problem statement, (e) judging the sufficiency of problem data, (f) proposing a simpler problem that may be easier to solve, and (g) thinking of previously solved related problems that may suggest useful processes. Furthermore, instruction has placed a heavy emphasis on planning -- "thinking before doing" -- as an important component of successful problem solving. In some class activities, students are specifically required to develop an entire plan for a problem's solution before they are allowed to begin the actual calculations.

One feature of our focus on process was the creation of a student-generated list of useful problem-solving processes. Midway through the instructional period, the list was created and posted in the classroom. The list then served as a useful referent for students in solving problems and for the teacher in discussing problem solving, and the list was subsequently modified as the year progressed.

The second focus has led us to include several class sessions in which a problem is presented, students work independently or in small groups to solve it, and the various methods used by individuals are identified in a summary discussion. As in all project work, the emphasis in these discussions are the processes used by different successful solvers.

Another feature, related to the second focus, is the problem of the week (POW). A POW was a non-routine mathematical problem for which the students were given up to one week to solve. After solutions were turned in and reviewed, time was spent discussing various processes used in students' solutions. In this year's instruction 13 POWS were used. Next year it is anticipated that the problem of the week will become a regular weekly feature of the instructional program.

Another of the instructional techniques used to focus both on processes and on multiple solution paths was the focused discussion. After a problem has been worked on by the students, a general discussion of useful and useless approaches, was undertaken. This discussion allowed for the gradual refinement of one or more "good" methods to solve the problem. The format allowed for the active involvement of students and gave the teacher ample opportunity to focus on structure and process. According to DeCorte (1980), the discussion method was pioneered in the Netherlands by Kohnstamm (1952) and has been used successfully in that country. The authors would appreciate being advised of other similar work that is being done elsewhere in the world.

REFERENCES

Baron, J. Intelligence and general strategies. In G. Underwood (Ed.), Strategies of information processing. New York: Academic Press, 1978.

Brown, A. L., & DeLoache, J. S. Skills, plans, and self-regulation. In R. S. Siegler (Ed.), Children's thinking: What develops? Hillsdale, N. J.: Lawrence Erlbaum, 1978.

Chi, M. T. H. Knowledge structures and memory development. In R. S. Siegler (Ed.), Children's thinking: What develops? Hillsdale, N. J.: Lawrence Erlbaum, 1978.

Dansereau, D., Long, G. L., McDonald, B. A. & Actkinson, T. R. Learning strategy inventory development and assessment. (AFHRL, Technical Report 75-40). Lowrey Air Force Base, Colo.: Air Force Human Resources Lab, 1975.

DeCorte, E. Processes of problem solving: Comparison of an American and a European view. Instructional Science, 1980, 9, 1-13.

Ericsson, K. A., & Simon, H. A. Verbal reports as data. Psychological Review, 1980, 87 (3), 215-251.

Flavell, J. H. & Wellman, H. M. Metamemory. In R. V. Kail & J. W. Hagen (Eds.), Perspectives on the development of memory and cognition. Hillsdale, N. J.: Lawrence Erlbaum, 1977.

Gurova, L. L. School children's awareness of their own mental operations in solving arithmetic problems. In J. Kilpatrick & I. Wirszup (Eds.,), Problem solving in arithmetic and algebra. Stanford: SMSG, 1969.

Istomina, Z. M. The development of voluntary memory in preschool-age children. Soviet Psychology, 1975, 13 (4), 5-64 (originally published, 1948).

Kohnstamm, Ph. Keur uit het Didactisch Werk Prof. Dr. Ph. Kohnstamm. (A selection from the work of Ph. Kohnstamm in educational psychology.) Groningen: Wolters, 1952.

Krutetskii, V. A. The psychology of mathematical abilities in schoolchildren, J. Kilpatrick & I. Wirszup (Eds.). Chicago: University of Chicago Press, 1976.

Pellegrino, J. W., & Schadler, M. Maximizing performance in a problem-solving task. Unpublished manuscript, University of Pittsburgh, LRDC, 1974.

Polya, G. How to solve it (2nd ed.). Garden City, N.Y.: Doubleday, 1957.

Reese, H. W. Imagery and associative memory. In R. V. Kail & J. W. Hagen (Eds.), Perspectives on the development of memory and cognition. Hillsdale, N.J.: Lawrence Erlbaum, 1977.

Siegler, R. S., & Liebert, R. M. Acquisition of formal scientific reasoning by 10- and 13-year-olds: Designing a factorial experiment. Developmental Psychology, 1975, 11, 401-402.

Silver, E. A. Recall of mathematical problem information: Solving related problems. Journal for Research in Mathematics Education, in press.

Silver, E. A. Student perceptions of relatedness among mathematical verbal problems. Journal for Research in Mathematics Education, 1979, 10 (3), 195-210.

Simon, D. P., & Simon, H. A. Individual differences in solving physics problems. In R. S. Siegler (Ed.), Children's thinking: What develops? Hillsdale, N. J.: Lawrence Erlbaum, 1978.

Weinstein, C. Elaboration skills as a learning strategy. In H. F. O'Neil (Ed.), Learning strategies. New York: Academic Press, 1978.

Wellman, H. M., Ritter, K., & Flavell, J. H. Deliberate memory behavior in the delayed reactions of very young children. Developmental Psychology, 1975, 11, 780-787.

PROBLEM SOLVING PROCESSES OF TEN AND ELEVEN YEAR OLDS

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This paper describes one aspect of an NSF funded project on the longitudinal development of problem-solving processes of upper elementary and junior high school students. The project is studying the processes used by ten and eleven year olds as they solved or attempted to solve problems in interview settings. This paper describes the solutions generated for one problem. The problem was geometric in nature, had rich numerical patterns underlying its structure, and was related to a problem the students had been presented in class. Examination of students' solutions revealed some interesting tendencies. For example, high ability students and/or highly field independent students exhibited much more systematic planful behavior. The paper concludes with a general discussion of observations to date and educational and research implications.

Mathematical problem solving, always considered an integral component of the school mathematics curriculum, is receiving increased attention from both curriculum developers and researchers. The focus of this attention is primarily on the process aspect of problem solving. Curriculum specialists have begun to include problem-solving processes as part of the standard curriculum and to look to researchers for answers to such questions as: How do students learn problem-solving processes?, How much emphasis should be placed on process?, When and where in the curriculum should it be included?, and What are the long-range implications of this emphasis? This paper will focus on the problem solving processes of ten and eleven year olds. It describes one aspect of an NSF supported longitudinal study (currently being conducted at SDSU under the direction of the first author) designed to trace selected classes of students for up to three years in classroom settings. A companion paper at IGPME, "Metacognition: The Missing Link in Problem Solving?" (by Silver, Branca and Adams) will describe another aspect of the study.

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Mathematics educators have for a long time recognized the development of the ability to solve problems as one of the most important goals of mathematics instruction. The recent "Agenda for Action" of the National Council of Teachers of Mathematics recommends that problem solving be the focus of school mathematics in the 1980's. Although there is widespread agreement as to the importance of problem solving in the mathematics curriculum, there is little agreement as to the components of an effective method of developing problem solving ability in children (Lester, 1975). Instead, educators and mathematicians alike are indicating that current approaches to the teaching of problem solving are inadequate. The report of the Snowmass Conference on the K-12 Mathematics Curriculum (Mathematics Education Development Center, 1973) expressed such a viewpoint when it called for extensive modification of current programs to include systematic instruction in problem solving.

RESEARCH DIRECTIONS

There has been a tremendous amount of research in the field of mathematics education that is classified under the heading of problem solving. Several bibliographies and reviews have been undertaken in an effort to consolidate the research and generalize their results. Riedesel (1969) listed 83 outstanding articles and research reports from the past fifty years. Gorman (1967) identified 293 studies on word problems between 1925 and 1965, while Suydam (1967) found 84 studies of problem solving undertaken from 1900 to 1965. Most of these studies were experimental studies in which an objective measure of problem-solving ability was obtained and was subsequently related to conditions of the study and/or student background variables, thus focusing upon the product aspect of problem solving. These studies are marked by extreme variations in populations and samples, in research designs and methodologies, and in problem-solving instruments and tasks used. In general, the research permits little, if any, consistent generalization.

A comprehensive review of the literature dealing with mathematics problem solving (Kilpatrick, 1969) pointed out the apparent lack of direction in the research and the absence of any systematic investigation of problem solving. Kilpatrick states that "few studies build on previous research: few studies have an explicit theoretical rationale." Furthermore, these studies have contributed little to the development of effective instructional techniques for the teaching of problem-solving skills.

During the past few years, research on problem solving has experienced some dramatic changes in focus and direction. Influenced by studies outside the content area of mathematics, a growing number of mathematics educators have begun concentrating on the process aspect of problem solving rather than on the product aspect. The development and validation of criteria for assessing performance on information gathering tasks based upon the logical structure of the problem by Rimoldi et al. (1964), the information processing approaches to the study of problem solving including the systematic analysis of protocols by Hunt (1968), and Paige and Simon (1966), and the emerging theory of state space representation of problems described by Newell and Simon (1972) have led mathematics educators to begin to ask whether similar investigations can be made in the area of mathematics problem solving. The recently translated (1969-1976) studies of Soviet researchers on the learning and teaching of mathematics have provided methodological influences for American mathematics educators to more easily undertake these investigations.

A catalyst for research in the area of problem solving was recently provided by a workshop on mathematical problem solving sponsored by the Georgia Center for the Study of Learning and Teaching of Mathematics (Hatfield, 1978). This workshop brought together a number of researchers in the area of problem solving and emphasized the process aspect of problem solving. A number of papers discussing both empirical findings and methodological concerns were discussed at the workshop and a growing body of knowledge is emerging. (eg. Kantowski, 1977; Kilpatrick, 1978; Schoenfeld, 1979; Webb, 1979.)

CURRENT WORK

As part of an NSF funded project on the longitudinal development of problem-solving processes of upper elementary and junior high school students, 24 ten and eleven year olds were interviewed as they solved or attempted to solve non-routine mathematical problems. The children were selected from the fifth and sixth grades of an elementary school on the basis of the results of two mathematics inventory tests. The tests were given at the beginning of the academic year (September and November) before the project began. If a student's scores on both tests were in the upper third of the class, then the student was considered to be in the "high" group. If a student's scores on both tests were in the lower third of the class, then the student was in the "low" group. All

students that met these requirements were interviewed. In the sixth grade, there were 6 students in the high group and 5 students in the low group. In the fifth grade, there were 4 students in each group. Additional students having scores in the middle third of the class were selected so that a total of 12 students from each class were interviewed. Each student was interviewed individually. Students were asked to think aloud and the interviews, lasting from 30 to 60 minutes, were taped.

Because of time and space limitations, general implications will be discussed after a description of the performances of students on only one problem. Additional results will be made available during the paper presentation.

The first problem (Figure 1) presented to the students was chosen because of the following attributes: (A) the spatial presentation mode with its possible interactions with field dependence-independence, (B) the rich numerical patterns underlying its structure, and (C) the processes involved in solving it were related to a problem presented to the whole class about two weeks earlier called "Strictly Squaresville" from NUMBER SENSE AND ARITHMETIC SKILLS published and distributed by Creative Publications. In "Strictly Squaresville" students had to count the numbers of squares in 1x1, 2x2, 3x3, 4x4, and 5x5 square grids, record the information in a table, and look for a pattern. In the class discussion, the systematic recording of information was emphasized.

Q. How many triangles can you find in this figure?

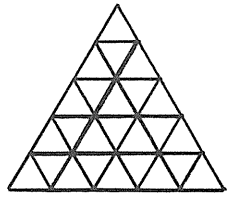

Fig. 1 The Triangle Problem

The triangle problem was initially conceived of as a counting problem in which attention to different sizes and, possibly, orientation are important. Figure 2 indicates how the 48 triangles present can be broken down by size and orientation. The notation (a,b) indicates a triangles of this type that pointed "up" and b such triangles that pointed "down." The designations "up" and "down" refer to the orientation of the triangle within the original figure.

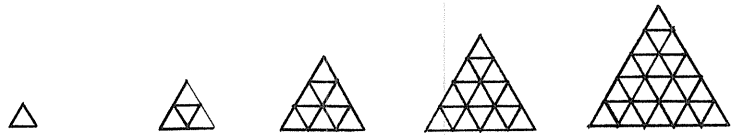

Fig. 2 Breakdown of Triangles by Size and Orientation

In the interview, students were asked to read the problem and solve it. If they did not solve the problem correctly, they were given a series of hints as needed. Only one student was able to get the correct number of triangles without hints. The first hint was to ask them if the problem reminded them of any other problem they had worked before and to suggest the "Squaresville" problem if they were still unable to work it. The next hint, if needed, was to ask the student to look for other sized triangles and finally to look for other orientations of the triangles. Students were then asked how many triangles they thought were in a 10-10-10 triangle. A diagram of a 10-10-10 equilateral triangular grid was shown to them.

Student performance on the problem varied considerably. Differences were observed in both how students perceived the problem and how systematic they were in obtaining its solution. Some students found it difficult to find triangles other than the small ones. Others found different-sized triangles only after they were asked to look for them. Most of these students were not systematic, exhibiting almost random search patterns and losing track as they counted. The more successful students not only saw or were aware of other-sized triangles, but attempted to count them in a systematic way. In general, students did not do much writing or recording of information and they found it

difficult to talk about the different-sized triangles, starting out by calling them small, large, and middle-sized. When encouraged to describe them, they usually talked in terms of the number of small triangles in them, and not in terms of the length of the sides of the triangles.

As might be expected, the problem reminded only some students of the "Squaresville" problem. Some students saw it as being like the Group Embedded Figures Test (Witkin, Oltman, Raskin, & Karp, 1971), a test given to the students before the project began as a measure of field dependence-independence [FDI]. When asked about the 10-10-10 triangle, many students responded that it would be twice as many, especially if the interviewer had in some way called attention to the length of the sides of the triangles. Some students thought in terms of a comparison of area and responded that a 10-10-10 triangle would have 4 times as many triangles since it was made up of 4 of the 5-5-5 triangles. This does not take into account the new sizes in the 10-10-10 triangle not found in the original problem, but it does appear to be a primitive form of the heuristic advice, "Think of a related problem." or "Can you use the result of a previously solved problem?"

Three different approaches that were used by students in solving the problem are discussed below:

1. Concentrating on numerical patterns. For example, one student observed that each row had two fewer triangles than the one below it and added 9, 7, 5, 3 and 1 to get the 25 small triangles instead of counting. He did not, however, attempt to use this technique in finding any other-sized triangles.
2. Focusing on one particular size at a time. For example, one student counted all the triangles of a particular size first by counting the triangles in the corners, and then by moving around the triangle and counting along the sides. This approach generally proved to be difficult because of the tendency to count some of the triangles more than once.
3. Focusing on one particular horizontal line as a locus for bases. For example, one student indicated that she found the triangles by focusing on the bottom part of the triangle putting a dot on the middle point either mentally or with her pencil.

Of course, most students exhibited a tendency to use more than one approach and varied in their consideration of the orientation issue. For example, one of the most successful solvers counted the large triangle first, followed by the

small triangles, then systematically counted the triangles of the other sizes. In counting the other triangles, she first counted the "up" triangles and then the "down" triangles; however, she made no specific mention of orientation when counting the small triangles.

If one compares the behavior of high ability students with that of low ability students one notes some interesting differences. For example, high ability students were more likely to keep a written record of their work and to exhibit planful thinking, whereas low ability students rarely did either. Furthermore, low ability students tended to trace every triangle they counted, whereas high ability students appeared to operate without this form of geometric closure.

If one compares the behavior of high and low field independent students, one detects an important interaction between this trait and certain problem-solving behaviors. Students with low FDI scores tended to exhibit much less planful behavior and systematic counting than their high FDI counterparts. Furthermore, the low FDI students seemed generally less able to benefit from finding a particular exemplar of a certain triangle or from hints that were supposed to lead to generalizations from particular geometric examples.

GENERAL DISCUSSION

In addition to the triangle problem, the ten and eleven year olds solved or attempted to solve a variety of other problems. Included in interviews or classroom discussions were problems from logic, number theory, geometry, and algebra. The processes observed raise a number of questions and concerns regarding educational and research practices.

Overall, most students did little planning or thinking ahead. There was little recording of either information or preliminary results in solving problems. Students tended to either give up if they could not see how to solve a problem right away or else jump right into some computational procedures and continue until a result was achieved without regard to what preliminary results indicated. Students, for the most part, were unable to verbalize their awareness of problem-solving processes. Few students were systematic in what they did and few kept written track of what they were doing as they progressed. Students differed greatly in how they perceived problems and what they attended

to. For some students, diagrams proved very helpful whereas others saw little value in constructing them. When asked if they had seen the problem before or one like it, some students could cite examples of related problems. However, there was little evidence that students remembered specific processes used in solving other problems that applied and the notion of related problems came primarily as an afterthought.

In terms of educational implications, programs need to be developed that make students aware of processes and structure and give students experiences that emphasize planning, step-by-step or systematic procedures, recording of information, organization of information, diagramming, and labeling. These programs should take into account the individual differences of students and emphasize the variety of problem solving processes for a single problem. Furthermore, the interaction with field independence is perhaps worthy of further attention.

In terms of research implications, there is much to be done in the area of evaluation of problem-solving skills and processes. How does one get at the thought processes of problem solvers? Thinking aloud does afford an opportunity to understand better what is occurring but in many cases it is apparent there is much more going on than what is stated. In analyzing the processes of ten and eleven year olds one is struck by the wide variability of processes. Some of this variability is a function of the specific problem while some depends on general individual differences. Descriptive measures in each of these areas are needed.

REFERENCES

Gorman, C. J. A critical analysis of research on written problems in elementary school mathematics (Doctoral dissertation, University of Pittsburgh, 1967). Dissertation Abstracts, 1968, 28 (12), 4818A-19A.

Hatfield, L., "Research on Mathematical Problem Solving: An Overview.: In L. Hatfield (Ed.) Mathematical Problem Solving. ERIC-SMEAC, Columbus, Ohio, 1978.

Hunt, E. Computer simulation: Artificial intelligence studies and their relevance to psychology. Annual Review of Psychology, 1966, 19.

Kantowski, M. G. Processes involved in mathematical problem solving. Journal for Research in Mathematics Education, 1977, 8 (3), 163-180.

Kilpatrick, J. Variables and methodologies in research on problem solving. In L. Hatfield (Edl). Mathematical Problem Solving, ERIC 1978.

Kilpatrick, J. Problem solving and creative behavior in mathematics. In J. W. Wilson & L. R. Carry (Eds.), Reviews of recent research in mathematics education. Studies in Mathematics, Volume 19. Stanford: School Mathematics Study Group, 1969.

Kilpatrick, J. and Wirszup, I. Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. 1-14. Stanford, School Mathematics Study Group, 1969-75.

Lester, F. K., Jr. Mathematical problem solving in the elementary school: Some educational and psychological considerations. Paper presented at the Research Workshop on Problem Solving in Mathematics Education at the Center for the Study of Learning and Teaching Mathematics, The University of Georgia, May 1975.

Mathematics Education Development Center. Report of the Snowmass Conference on the K-12 mathematics curriculum. Washington, D. C.: C. S. Department of Health, Education, and Welfare, 1973.

National Council of Teachers of Mathematics. An Agenda for Action: Recommendations for School Mathematics of the 1980s. Reston, VA: The National Council of Teachers of Mathematics, 1980.

Newell, A., & Simon, H. A. Human problem solving. Englewood Cliffs, N. J.: Prentice-Hall, 1972.

Number sense and Arithmetic Skills, Mathematics Resource Project. Palo Alto, CA: Creative Publications, 1977, University of Oregon, p. 206.

Paige, J. M., & Simon, H. A. Cognitive processes in solving algebra word problems. Problem solving: Research, method and theory. New York: John Wiley, 1966.

Riedesel, C. A. Verbal problem solving: Suggestions for improving instruction. Arithmetic Teacher, 1964, 11.

Rimoldi, H. J. A., et al. Problem solving in high school and college students. Cooperative Research Project No. 2199. Chicago: Loyola Psychometric Laboratory, Loyola University, 1964.

Schoenfeld, H. H. Explicit heuristic training as a variable in problem solving performance. Journal for Research in Mathematics Education, 1979 10, 173-187.

Suydam, M. N. The status of research on elementary school mathematics. Arithmetic Teacher, 1967, 14.

Webb, N. Processes, conceptual knowledge and mathematical problem solving ability. Journal for Research in Mathematics Education, 1979, 10, 83-93.

Witkin, H. A., Oltman, P. K., Raskin, E., & Karp, S. A. A manual for the Embedded Figures Test. Palo Alto, Calif.: Consulting Psychologist Press, 1971.

UPSTAGING PIAGET: THE CHILD AS DIALECTICAL CONSTRUCTIVIST

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It is argued that the emphasis given to the stage aspect of Piaget's theory of cognitive development is inappropriate for educational purposes. The idea of the child as a dialectical constructivist is proposed as being both more useful to teachers and a more accurate fundamental image for the theory as a whole. This view is illustrated by excerpts from two interviews with children.

When the history of twentieth-century thought comes to be written it seems likely that Jean Piaget will be seen as one of the towering intellectual figures of our era. Over a period of some sixty years he has synthesized ideas from biology, logic, philosophy, psychology and mathematics to forge the discipline of genetic epistemology. His own output has been enormous and he has collaborated with a number of leading scholars to produce a theory of intellectual development which is elaborate, ambitious, sweeping, difficult and controversial.

Of the numerous professional groups which have attempted to interpret and apply Piaget's ideas, educators have perhaps had the most frustrating time. Despite the apparent richness of the research materials and the provocative nature of the findings about the mechanics of child thought the task of teasing out the educational implications of Piaget's general theory has proven to be a difficult one (Duckworth, 1979). Central to the problem is the fact that Piaget has never been particularly interested in pedagogical issues. It may well be, as Gruber and Voneche contend (1977), that "there is no Piagetian educational dogma...if there were...it would be necessary to destroy it" (p.694). Nevertheless, since at present no fully-fledged alternative theory exists, it is the case that, explicitly or implicitly, teachers and curriculum developers frequently base curriculum rationalizations on their understanding of Piaget.

To a great extent in educational circles, particularly in North America, Piaget's theory has been reduced to the idea of developmental stages. For some educators it would appear that the postulated invariant progression from sensorimotor to formal thought <u>is</u> the Piagetian thesis. There are two reasons why, for educational purposes, this emphasis on stages is unfortunate. The first is that this

is one of the parts of the overall theory which now appears most vulnerable. Critics such as Brainerd (1978) might be expected to be suspicious, but the nature of recent research findings has been such that even supporters like Boden (1979) conclude that "the stage-concept has become increasingly dubious" (p. 153). And from within the Genevan citadel itself we find Vonèche (1979) saying "there seems to be no evidence from the long term studies of any acceleration of concept formation at a particular point making a definite stage transition. Is this a serious blow to Piaget's theory? Certainly not...the concept of stage is not so central to Piaget's theory" (p. 776).

Independent of the validity of the stages hypothesis, however, there is a second major problem. Indeed, it may well be that further work in the area will show that stages of intellectual development, perhaps in some modified form, do exist. (Mathematics educators have played and one hopes that they will continue to play a major role in clarifying some of these issues. The value of cooperation in this area between theoreticians and practitioners is considerable (Kuhn, 1979).) This would not alter the fact that the preoccupation with stages has blinded some educators to rather more fundamental aspects of the theory. To use the terminology of the theory somewhat loosely, we have tried to assimilate Piaget's ideas into an existing structure when an accommodation or major shift in our views has been necessary. By focusing on the the hierarchical nature of stages and seeing them in a tightly structured and static way one can maintain a positivistic view of the nature of knowledge and a neobehavioristic vision of the learning process (Kamii & DeVries, 1978).

If the picture of a child leaping a set of fixed intellectual hurdles distorts the basic Piagetian position, what fundamental image will capture it? It seems that there are two characteristics of the theory which are fundamental. The first is that the child actively constructs his own reality. It is not the case that some pre-existent reality is inserted into the child's mind (Richards & von Glaserfeld, 1980). The second is that this construction process is dialectical. The critical analysis of existing structures yields new structures which incorporate the old ones. The new synthesis is the product of an equilibration between thesis and anthesis.

The image of the child as dialectical constructivist is compatible with the research results of psychologists like Donaldson (1978) and Smedslund (1977) who has written "when I meet a small child I always take for granted that, within his limited sphere of activity and given his own premises, he is logical,

and my problem is to understand what his expressions mean, and hence to grasp his existential situation" (p. 4). In addition it is consistent with some of the more general epistemological views of thinkers such as Popper (1969), Lakatos (1976) and Kelly (1963). The picture of the 'child as scientist' (Karmiloff-Smith & Inhelder, 1975; Phillips, 1978; Sinclair, n.d.) is a positive one and one which teachers should find easy to understand. The use of clinical interview methods with mathematics pupils by Ginsburg (1977) and others has revealed the ingenuity of children when they attempt to make sense out of certain concepts and processes.

Given the constraints of space we can conclude this paper with only the briefest of examples of children acting as dialectical constructivists in the area of mathematics. First we mention the case of a three-year old child trying to understand something of the workings of a digital clock (Higginson, 1980). Contrary to Elkind's (1979) view it is not the case that "the digital clock takes all of the logical complexity out of telling time" (p. 9). In the spontaneous and natural setting of a child-initiated conversation with a parent in the family home (the sorts of conditions which are now seen as being crucially important for the study of cognition (Scarr, 1979)) many of Gelman's (1979) contentions about the intellectual abilities of preschool children are illustrated. From the statements of the child it seemed that the two main areas of difficulty and hence of theorizing, were numeral recognition and number sequencing. For instance, when the clock showed 7:57 we had the comment, "Two sevens, seventy-seven". When the clock turned to 7:58 an immediate question was "When is it going to be eighty-eight?". At 8:02 we had the question "Is that a two or a five?".

The equivalence of some symbols under certain geometric transformations caused problems. Sixes and nines were tricky as were twos and fives. The arbitrary boundary between letters and numerals was cheerfully ignored. Despite the difficulties the prevailing mood was one of intense interest, excitement and strong curiousity. The social context was warm and supportive and the self-image of the child was very positive. The number of rules used by the child was small and it was fairly easy to follow the thought patterns.

There was considerable contrast therefore, with our second example which is taken from some work in progress on children's cognitive and metacognitive ideas about mathematics (Burnett et al, 1980). Ann was a twelve-year old grade seven

pupil who talked with a researcher about fractions and decimals. Ann was failing grade seven mathematics at the time, as she claimed were over half of her classmates. By her own admission she did not understand decimals very well. There seemed little doubt of this once the interview began since it was established quite quickly that she was unable to name any number larger than 4.17 and she contended that 2/3 + 2/3 + 2/3 + 2/3 = 8.3 . And yet it would be a major, albeit understandable, error to conclude that this child lacked mathematical ability. For a detailed examination of her statements suggested the existence of a logical system which was, in its own way, imaginative, coherent and original. Unfortunately for Ann her system gave quite different answers in many cases from those expected by her teacher.

Central to her idiosyncratic view was a set of interconnected ideas which were so woven into the fabric of her mathematical thought that it was difficult to see how she would ever be able to eradicate their influence completely. The first 'axiom' of her system was that "zero is nothing". Next came the idea that a blank is a placeholder which could be any number. When the decimal point is seen as a placeholder some of Ann's unorthodox ideas about number begin to become somewhat more clear. In some miasmic sense 1.2, 12, ½, 1/2, 1-2 and 102 all took on a fuzzy resemblance and could be substituted for one another almost at will. One could not give a number larger than 4.17 because the 'decimal placeholder' part of the number could have taken on any value you wish. 2/3 + 2/3 + 2/3 + 2/3 = 8/3 which in turn is equivalent to 8.3 . Like Erlwanger's Benny (1973) Ann had a highly developed sense of mathematical tolerance. 1/3 + 2/3 = 3/3 when done in fractions and 3/3 in her system was equivalent to 3.3 . If, however, one "changes first" one gets that the sum of 1.3 and 2.3 is 3.6 . To Ann it seemed eminently reasonable that if you "do it in fractions" you get one answer and if you "do it in decimals" you get another.

Given the divergence of her rules from those expected of her Ann had to carry out some very complex rationalizations to arrive at 'the right answer' in the cases where she knew in advance what these should be. Here we had the ironical situation of a child who was seen as being mathematically inept carrying out calculations which would appear to have been much more difficult than the ones expected of her. Consider, for instance, her response to a textbook exercise where pupils were asked to fill in the six empty slots in a three by three magic square. Three of the corner slots were filled by 2.6 , 3.7 and 5.0 and the children were told that the sum of any column, row or diagonal was 11.4 .

Ann's initial difficulties stemmed from the fact that she had never before seen a magic square and felt, with some justification, that one might have diagonals with one, two or three elements. With some coaching she concluded that one needed to put 5.1 between 2.6 and 3.7 . She then added the 3.7 to the 5.0 to get 8.7 and began to think how she might get to 11.4. After a little thought she said with some confidence " Yeah, you could take three from that (the 7 of 8.7) and add it on to that (the 8 of 8.7) and that would make it... no hold it-that came out to- eight point seven- so you could take three from the sevenand add it on to that and that would make eleven point four." (Her observation is equivalent to noting that 2.7 = 3.0 -0.3 .)

Ann's case raises some general questions of considerable importance. How typical is she? How did she get into this situation? What could be done to help her? How can teachers cope with problems like this in the classroom? These are not easy questions to answer. What is certain, however, is that we will make very little progress on any of them unless we recognize the constructive and dialectical nature of childrens' thought processes.

REFERENCES

Boden, M. Piaget. London: Fontana, 1979.

Brainerd, C. J. The stage question in cognitive-development theory. The Behavioral and Brain Sciences. 1978, 2, 173 - 213.

Burnett, D. J., Higginson, W. C., Hills, G. L., Osser, H., & Schiralli, M. G. Mathematical knowledge: A study of cognition and metacognition. Research project, Queen's University at Kingston. In progress, 1980.

Donaldson, M. Children's minds. London: Fontana, 1978.

Duckworth, E. Either we're too early and they can't learn it or we're too late and they know it already: The dilemma of "Applying Piaget". Harvard Educational Review. 1979, 49, 297 - 312.

Elkind, D. Piaget and developmental psychology. In F. B. Murray (Ed.) The impact of Piagetian theory on education, philosophy, psychiatry and psychology. Baltimore: University Park Press, 1979.

Erlwanger, S. H. Benny's conception of rules and answers in IPI mathematics. Journal of Children's Mathematical Behavior. 1973, 1, 7 - 26.

Gelman, R. Preschool thought. American Psychologist, 1979, 34, 900 - 905.

Ginsburg, H. Children's arithmetic: The learning process. New York: Van Nostrand, 1977.

Gruber, H. E. & Voneche, J. J. The essential Piaget: An interpretive reference and guide. New York: Basic, 1977.

Higginson, W. C. Berry undecided: A digital dialogue with a three-year old. Mathematics Teaching. June 1980. In press.

Karmiloff-Smith, A. & Inhelder, B. If you want to get ahead, get a theory. Cognition, 1975, 3, 195 - 212.

Kamii, C. & DeVries, R. Physical knowledge in preschool education: Implications of Piaget's theory. London: Prentice Hall International, 1978.

Kelly, G. A theory of personality: The psychology of personal constructs. New York: Norton, 1963.

Kuhn, D. The application ofPiaget's theory of cognitive development to education. Harvard Educational Review, 1979, 34, 340 - 360.

Lakatos, I. Proofs and Refutations. Cambridge: Cambridge University Press, 1976.

Popper, K. R. Conjectures and refutations: The growth of scientific knowledge. London: Routledge and Kegan Paul, 1969.

Phillips, D. C. The Piagetian child and the scientist: Problems of assimilation and accommodation. Educational Theory, 1978, 28, 3 - 15.

Richards, J. & von Glaserfeld, E. Jean Piaget, psychologist of epistemology: A discussion of Rotman's Jean Piaget: Psychologist of the real. Journal for Research in Mathematics Education,1980, 11, 29 - 36.

Scarr, S. Psychology and children: Current research and practice: Introduction to the special issue. American Psychologist, 1979, 39, 809 - 811.

Sinclair, H. The child as scientist: An interview. Edwardsville, Illinois: The Teachers' Centre Project, SIU. No date.

Smedslund, J. Piaget's psychology in practice. British Journal of Educational Psychology, 1977, 47, 1 - 6.

Vonèche, J. J. Conclusions In S. Modgil & C. Modgil (Eds.) Toward a theory of psychological thought. London: NFER, 1980.

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TRANSLATION PROCESSES IN MATHEMATICS EDUCATION

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In addition to the usual (but too often overlooked) geometrical figures which in many respects are abstract, "the repertoire" of modes of representation is extensive: Venn diagrams, tree diagrams, Cayley tables, arrow graphs, flow diagrams, cartesian graphs... as well as letters, numerals in various positions, brackets,... (Bell, A.W., 1976). Even from a pedagogical point of view, most text-books today make use of a wide (if not wild) variety of diagrams and pictures meant to promote understanding. Everyone certainly agrees that the use of symbolism in mathematics thinking is fundamental.

However, particular uses of symbolism appear generally to be overlooked, namely: the translation processes. By a translation process, we mean the psychological processes involved in going from one mode of representation to another, for example: from an equation to a graph.

In recent years, Lesh (1979) points their importance in real problem solving and Burton, L. (1979) in an inventory of problem-solving skills mentions the notion of translation in her list. Burkhardt, H.* (1977) stresses their crucial role in mathematical modelling. Bell, M. (1979) identifies this fondamental component of mathematical competence. Bessot, A. and Richard, F. (1979) conducted a research on tree diagrams and graphs. It is not easy to repertoriate the literature dealing with translation processes since quite often the idea is tackled indirectly. For instance, researchers concerned with the notions of models, intuitive models (for example, see Fischbein, 1977) implicitly touch upon this question.

We personaly got interested in this topic in a recent research project dealing with the "Interpretation of cartesian graphs representing situations" (Janvier, C. 1978).

We have selected seven points on which we shall briefly report our findings in this paper. During our talk, we shall illustrate most points with examples and we invite the participants to share their experience.

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* To whom we extend our thanks for the idea.

1. The question of giving names

If we limit the modes of representation of variables to four, namely: verbal description, table, graph and formula (equation), we can represent translation processes in the context of handling variables by a 4X4 table.

TRANSLATION PROCESSES

From \ To	Situations, Verbal Description	Tables	Graphs	Formulae
Situations Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading Off		Curve Fitting
Formulae	Parameter Recognition	Computing	Sketching	

The reader most probably feels a little uneasy about how a few cells were filled in, or about the names given to a few processes. He is kindly invited to fill those cells from scratch and discover the source of his hesitation. In fact, the quasi-impossibility of defining (at least uniquely) a few processes arises from the ill-defined context in which a particular translation is achieved. Despite this limitation, this array is very useful in that it gives a total picture of the question and helps us to single out and illustrate a variety of its aspects.

The reader has noted that we did not fill the diagonal. In fact, such processes exist which we could call transposition.

2. Direct or indirect?

Let us simply look at a sub-table for the time being (setting aside the verbal description, sometimes of situation).

TRANSLATION PROCESSES

From \ To	Situations, Verbal Description	Tables	Graphs	Formulae
Situations Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading Off		Curve Fitting
Formulae	Parameter Recognition	Computing	Sketching	

As you notice, a few arrows are now added to account for alternative ways to achieve translations. More precisely, these arrows indicate that the translation "table → formula" is often carried out as "table → graph → formula" and "formula → graph" as "formula → table → graph". Actually indirect processes are substantially different than direct ones. A study of several mathematics programs has showed that they exclusively develop the indirect version of many processes of the tables. Could we do otherwise? Would it be worthwhile?

3. The source and target paradigm

A translation involves two modes of representation. For the modes equation and graph, for instance, we have the translations: "graph → equation" and "equation → graph". To achieve directly (and correctly) a given translation, one has to transform the source "target-wise" or, in other words, to look at it from a "target point of view" and derive the results. For instance a graph has to be examined equation-wise (with respect to its equational features). This simple comment suggests teaching strategies as well as points to the importance of the inverse process (which we prefer to call complementary) which is symmetrically located with respect to the diagonal. In fact, our research suggested that processes were best developed in (symmetric) pairs, in our case: "graph → verbal description" (interpretation) and "verbal description → graph" (sketching). This principle is already applied by those teaching methods of foreign language which stress "listening" and "talking" in language laboratory.

4. The role of language

We have just stated that the source has to be examined "target-wise" but our research results equally show that in the process language or words play a central role. Actually, we noted that verbal tags were given to the relevant elements and the execution of the process was carried out through an efficient handling of those verbal tags. In a way, source ans target were verbally simplified. Even in a "graphical context", the speech appeared to be essential.

5. Application to the teaching of music

Clearly, this type of analysis can be applied to other areas of mathematics (proportion, arithmetic operations...) and other domains of education. We shall take very easy examples of a process involved in music, namely some related to a single note. We shall distinguish four modes of representation of a single note: the sound (vibration in the air), the act of playing it (position of finger(s), body...), the name of the note (ex.: A or La) and its written symbol on the staff. We can consider then all the questions posed so far on the following table.

TRANSLATION PROCESSES

From \ To	Vibrating note	Played note (action)	Name of note	Symbol on staff
Vibrating note		Play by ear	Tell name of a note	
Played note (action)				
Name of note	Sing			
Symbol on staff	Sing		Read music	

a) You see that we have not given names to all cells!

b) Play by ear: "vibrating note (heard) → played note (action)" is a direct process. For many, the process is often impossible or sometimes achieved

indirectly: "vibrating note → name of a note → played note".

c) Traditional teaching method stresses the process "name of note → played note" while the Suzuki method is based on the process "vibrating (heard) note → played note (action)".

d) Look at how inverse processes are interrelated!

6. More complex thinking processes in mathematics education

This section states opinions rather than research observations.

We believe that the study of generalisation, abstraction, proof, symbolisation... could be made more meaningful if they were considered as made out of basic translation processes.

The study of intuition should equally gain from using the "translation process" framework. We believe with Fischbein that intuition are based on schemas, verbal descriptions... which are implicitly used in action. We think that the development of intuition is often related to the development of processes which are subsumed and later called into play.

7. Translation processes and curriculum designs

Using the idea of translation processes involving variables, we carried out in our Ph.D. dissertation * (Janvier, 1978) a comparison between two mathematics programs which proved most interesting as for the improvements it suggested. Translation processes for other topics could be used fruitfully.

CODA (only to be discussed next year)

Lately, we have been studying the question from different points of view. To answer the question: "Is there anything behind all those modes of representation?", we are inquiring about answers semiology (science of meaning) could give. Also, since each mode has its intrinsic level of complexity, we wonder how the theory of information and the idea of entropy could help out.

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* Subsidised copies can be obtained from the author for $10.00.

REFERENCES

Bell, A.W. The Learning of General Mathematical Strategies. Ph.D. dissertation. Shell Centre for Math. Education, University of Nottingham, 1976.

Bell, M. What Most People Need From Mathematics: Teaching Mathematics so as to be useful. Unpublished one page paper, 1979.

Bessot, A. and Richard, F. Commande des variables dans une situation didactique pour provoquer l'élargissement de procédures en vue d'étudier le rôle du schéma. Thèse collective de 3e cycle, Université de Bordeaux I, 1979.

Burkhardt, H. Seven sevens are fifty? Mathematics for the real world. Inaugural Lecture. Shell Centre for Math. Education Publication, University of Nottingham, 1977.

Burton, L. The classification of problem-solving skills and procedures - presentation of an inventory. Proceedings of the 3rd International Conference of IGPME, Warwick, 1979.

Fischbein, E. Image and Concept in Learning Mathematics. Educational Studies in Mathematics, 1977, 8, 153-165.

Janvier, C. The interpretation of complex cartesian graph representing situations - studies and teaching experiment, Ph.D. Dissertation, University of Nottingham, 1978.

Lesh, R. Some Trends in Research and the Acquisition and the Use of Space and Geometry Concepts. Critical Reviews in Mathematics Education, Materialen and Studien, Band 9. Institut für Didactik der Mathematik der Universität Bielefeld, 1979.

A CODING SYSTEM FOR INVESTIGATING SELF-REGULATED PROCESSES IN PROBLEM SOLVING

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The purpose of this project was to develop a method of coding mathematical problem solving processes used by middle school children and to generate a bank of problems. The problems and coding instrument are intended for use in an investigation into the relationship between visual spatial skills and mathematical problem solving processes and the role that self-regulation plays in this relationship.

The most important goal of mathematics instruction, which is the development of problem solving abilities (NCTM, 1980), is not being met in American children (NAEP, 1979). The fact that there is actually evidence of a decline in problem solving abilities makes the situation worse. It is not enough any more to be told that our children are not able to solve problems; we need to know about the skills and processes lacking as well as those present and how they are used.

Researching the mental processes required in problem solving is a complicated task since these processes are so difficult to identify. Good techniques and research models are needed so that these problem solving processes can be better understood.

The teaching experiment, a new form of natural experiment developed in the Soviet Union (Menchinskaya, 1968), has come to this country as a promising type of research which can be used to filter out the processes used in problem solving. This method, in contrast with quantitative research, focuses on process more than product in problem solving. A disadvantage of this kind of methodology, though, is the lack of well-developed techniques for recording and evaluating behaviors being studied in the experiment (Kantowski, 1977).

This investigator's future research plans require the development of a coding instrument sensitive to middle school problem solving behaviors related to problem variables, heuristics, visual spatial skills, and the problem solver's own regulation of these skills (Moyer and Schultz, Note 1). This paper

presents this instrument and the procedure used to develop it. It is most akin to Kantowski's (Note 2) and requires coders who are trained to discriminate problem solving behaviors and record them sequentially on a coding form. This will be done during a one-on-one session with coder and student.

Problem solving is defined as the process of combining existing ideas to do a task for which there is no readily accessible algorithm to arrive at a solution. Krutetskii (1976) says that problem solving ability is a function of two variables -- the problem solver and the problem. Therefore different heuristics and different problem types are delineated.

Since it is possible that a problem solver might have all the necessary knowledge for the solution of a problem, but not be able to relate the important features of the problem to this knowledge, the use of heuristics is important to avoid fixation in problem solving (Ivanitsyna, 1970). Offering heuristic hints to the problem solver provides the opportunity to stimulate the fixated problem solver. Krutetskii recommends to the researcher the use of hints in testing sessions so that the suddenness of insight might be observed. For these reasons, hints are included in the instrument presented.

Visual spatial skills, also a focus of the mentioned planned research, include perceiving spatial patterns and manipulating objects in imagination. Self-regulation refers to the problem solver's tendency to approach problems through a personalized pattern of choosing symbolic, pictorial, and manipulative strategies.

PROCEDURE

The development of this coding instrument occurred in three phases. During phase 1 the following was drafted: (1) a bank of 30 problems with corresponding solutions and hints, (2) the coding instrument, and (3) a pretest (not discussed further in this paper) for screening children whose content knowledge is not enough to interact with the problems to offer any meaningful data. Then 9 middle school children were individually observed as they attempted to solve a subset of 10 problems from the bank of 30 so that each problem was attempted by a total of 3 children. During phase 2 the coders compared their results of phase 1 for reliability. Then the problems, corresponding hints, and coding instrument were revised. Finally, in phase 3

the problem hints and coding instruments were reviewed for validity by 2 classroom teachers and 2 university professors of mathematics education. After making further minor changes, 3 subjects were observed solving a subset of 10 problems (so that each of the 30 problems was solved once). After these observations, which were made simultaneously by 2 newly trained coders, results were compared for inter-coder realiability. The coding partners were required to come to a minimum of 85% accuracy of each other.

On the coding instrument shown in Figure 1, behaviors are listed as both Behaviors Observed and Hints Requested. Each list corresponds to 6 heuristics

Date _____ Subject M F Reading Level _____ Problem _____

Coder ____ School _____ Pretest Score _____ Problem Attempted ___

_____, _____, _____, _____, _____, _____, _____, _____, _____,

_____, _____, _____, _____, _____, _____, _____, _____, _____,

_____, _____, _____, _____, _____, _____, _____, _____, _____

Behaviors Observed

1	O-P	Tells Plan(s)
	O-O	Tells Operation(s)
	O-A	Tells Algorithm Rule
2	O-SP	Describes Similar Problem
3	O-LB	Looks Back or Reviews
4	O-EP	Explains Problem
5	O-DP	Draws Picture
	O-US	Uses Sticks
	O-UC	Uses Chips
	O-UM	Uses Money
	O-UP	Uses Paper (cuts or folds)
	O-UF	Uses Fingers
6	O-GT	Guess-then-Test (Trial/Error)

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	O-C	Uses Calculator
	O-CS	Checks Solution
	O-R	Reads or Rereads Problem
	X-R	Reasoning Error
	X-C	Computation Error

****

	C	Answers Correctly
	X	Answers Incorrectly
	G	Gives Up

Hints Requested

1	H-P	Plan
	H-O	Operation
	H-A	Algorithm Rule
2	H-SP	Simpler (similar) Problem
3	H-LB	Looking Back or Review
4	H-EP	Explanation of Problem
5	H-DP	Drawn Picture
	H-DS	Demonstration w/Sticks
	H-DC	Demonstration w/Chips
	H-DM	Demonstration w/Money
	H-DP	Demonstration w/Paper
6	H-GT	Guess-then-Test

Intervention

I-D	Request for demonstration of solution (Show me.)
I-C	Clarification (What do you mean?)

Comments:

Figure 1. Coding Instrument

(Polya, 1957): (1) using a plan, (2) recalling a similar problem, (3) looking back, (4) restating the problem, (5) drawing a figure, and (6) working backwards. If a problem solver tells the coder what plan is being followed (subjects will be instructed to think out loud), then that behavior is recorded as an observed behavior; whereas if the problem solver requests a plan, then it is recorded as a hint requested. The same holds for the remaining hints.

During phase 1 of the development of the coding instruments, it was found that the "use of a plan" and "drawing a figure" heuristics were too general for the middle school student, so they were subdivided accordingly. The "use of a plan" now refers to a general plan, a plan for using a certain operation (e.g., addition, subtracting, counting, etc.), or algorithm rule (e.g., determining where to place decimal point). "Drawing a figure" now includes the use of a variety of representational models such as sticks, chips, etc.

In addition to the above modifications two other heuristics were renamed to help the coder more easily recognize how to record certain behaviors, also to put them into terms that would be used by the problem solver when requesting a hint. "Restating a problem" is renamed "explaining the problem" and "working backwards" is renamed "guess-then-test".

The coding instrument also allows for the recording of other behaviors such as using the calculator, checking a solution, and reading the problem. It also identifies different kinds of errors, accuracy of the final solution, and giving up on a problem. Should the coder need further explanation or a demonstration from the problem solver, there are two codes for that as well. All coded behaviors, hints, and interventions are to be listed from left to right on the blanks provided using the acronyms.

The problems developed vary like Krutetskii's according to: I amount of information given (too much or too little), II having flexibility of thinking (different solution processes or seeming barriers), III spatial content (present or not present), and V number of steps needed for reaching solution (one step or more than one step). There are 6 of each type, and 3 of each

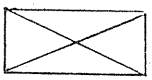
subtype noted in parentheses above.

Figure 2 shows one sample problem displayed with solution and hints, etc. just as the coder would see it. This problem requires flexibility of thinking because of the seeming barriers to solution. The hints which correspond to each of the

Type II Flexibility of Thinking B. Seeming Barriers to Solutions

Problem:

1. How many different triangles are there in the figure?



Solution:

Trace and count the triangles in the figure.

8 triangles

Hints:

1. Plan. Trace all the different kinds of triangles you see in the figure.

   Operation. Count

   Rule. None

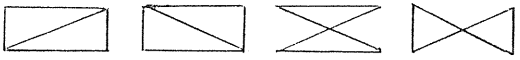
2. Similar problem. How many triangles are in this figure?



3. Looking back or Review. Review what the student has done to this point.

4. Explaining the Problem. There are many different triangles included in the figure. Trace the ones you see and count the total.

5. Picture. (Show one at a time)



   Demonstration. None

6. Guess-then-test. None

Figure 2. Sample problem, solution, and heuristic hints.

6 heuristics discussed earlier are listed under the solution for the coder's convenience. If there is no hint that is appropriate for a particular heuristic, "none" is written. The subject only sees the written problem with illustration (if there is one) on a separate display.

References Notes

1. Kantowski, M.G. Successful problem solving in geometry and selected process variables. NIE (G78-0225) Basic Skills Group Project Abstract. (1978-1981)

2. Moyer, J.C. and Schultz, K.A. Interrelationships between visual spatial skills and mathematical problem solving processes. Research proposal submitted to NIE, Teaching and Learning Program, January 1980.

References

Ivanitsyna, E.N. Achieving skill in solving geometry problems. In J. Kilpatrick and I. Wirszup (Eds.), Problem solving in geometry. Soviet studies in the psychology of learning and teaching mathematics, Vol. 4. Stanford School Mathematics Study Group, 1970.

Kantowski, M.G. Processes involved in mathematical problem solving. Journal for research in mathematics education. 1977, 8, 163-180.

Krutetskii, V.A. The psychology of mathematical abilities in school children. Translated by Teller, J. Kilpatrick, J. and Wirszup, I., (eds.) University of Chicago Press, 1976.

Menchinskaya, N.A. Fifty years of Soviet instructional psychology. In J. Kilpatrick and I. Wirszup (Eds.), Soviet studies in the psychology of learning and teaching mathematics, (Vol. I). Stanford, Calif.: School Mathematics Study Group, 1969.

National Assessment of Education Progress. Changes in mathematical achievement, 1973-1978. Denver: NAEP, 1980.

NCTM. An agenda for action. Reston, Virginia: NCTM, 1980

Polya, G. How to solve it, Second Edition. New York: Doubleday and Co. Inc., 1957.

PRESCHOOLERS' ABILITIES TO RECOGNIZE COUNTING ERRORS

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This study investigated the knowledge underlying children's counting performance through an examination of their ability to recognize counting errors and to distinguish these errors from unconventional but correct counting. Three-, four-, and five-year-olds' recognition of omitted word, extra word, skipped object, and doubly counted object errors were examined. Each of these errors could occur in the beginning, middle, or end of the row. In addition, each type and location of error was examined in a context of short (3 or 4 item) or long (9 or 10 item) rows. Four types of unconventional but correct counting as well as standard correct counting were also presented. Results indicated that (1) children in all age groups were very good at recognizing standard correct counting as correct; (2) in general, children were poorest at recognizing unconventional but correct counting; and (3) ability to detect true counting errors increased with age, with the three-year-olds only consistently identifying skipped object errors, while the five-year-olds identified all errors more than 90% of the time. Error recognition was significantly better on small than on large sets, as was detection of errors made at the end of a count compared to those in the beginning or middle. A positive relationship was found between a child's ability to detect counting errors and accuracy of his own counting. Overall, the results seem to suggest that the knowledge underlying very young children's counting may be better characterized as rules about the standard procedure than as general principles.



Traditionally, the behavior of pre-schoolers has been characterized in terms of the skills and knowledge that they lack when compared to older children. Recently, however, researchers have started to focus on the capabilities pre-schoolers do have (Gelman & Gallistel, 1978; Siegler, 1980). Counting is an ideal context for investigating young children's abilities and how they develop. Even young children are not totally incapable; in fact, most of them are surprisingly good counters at least within some contexts, such as counting small sets of objects (Gelman & Gallistel, 1978; Fuson & Mierkiewicz, 1980). Yet, there are marked differences between the counting skills of younger and older children, so developmental changes in capabilities are observable. In addition, counting itself is an important skill for investigation because it is a prequisite for many arithmetic concepts as well as being part of everyday life.

A major question is what kind of underlying knowledge generates children's counting performance. One claim is that this performance stems from children's knowledge of a set of counting principles (Gelman & Gallistel, 1978). However, this was inferred primarily from tasks children performed correctly. Thus, it is difficult to know if they understood the underlying principles or just knew how to

execute the procedure correctly. In addition, performance that did not conform to these principles was attributed to difficulties in executing the procedure rather than a lack of knowledge of the principles. The validity of this claim could not be easily assessed. Data from spontaneous performance alone is not adequate for distinguishing whether counting is based on a knowledge of principles or simply on knowledge about the procedure.

A different perspective on this issue is provided by examining children's recognition of counting mistakes as errors and to distinguish these mistakes from unconventional but still correct counting. The purpose of this study is to examine young children's ability to make these distinctions.

Counting consists of two subskills: saying the number words in the correct order and establishing a one-to-one correspondence between the words and the objects to be counted. One of Gelman's principles is related to each of these parts . The stable order principle stipulates that the number words must be given in a stable (repeatable) order. The one-one principle specifies that one and only one number word is to be associated with each object in the set being counted. These principles differ, however, in their relationship to their respective counting subskills. Knowledge of the one-one principle should be sufficient to elicit correct correspondence during counting; however, saying the correct string of number words requires specific knowledge (the words) in addition to the stable order principle. Due to the hypothesized sufficiency of an understanding of the one-one principle to produce correct word-object correspondence in counting, this aspect of counting was examined in the present study.

The one-one principle can be enacted in different ways depending on the characteristics of the counting situation, such as whether the objects to be counted are movable or fixed. The focus of this study was limited to children's understanding of the one-one principle in a single context-- counting rows of non-movable objects. Fixed objects were used because they are involved in so many counting situations encountered by young children (e.g., counting pictures in a book , counting children in a group). Rows were selected over some other arrangement, such as a random configuration, to minimize the effects of other factors such as memory for objects already counted. (In a row, pointing to an object usually serves to separate the counted from uncounted items and thus avoids the memory problem.)

Note that in counting fixed objects the one-to-one correspondence between number words and objects is usually mediated by pointing. Thus, two correspondences are actually involved--word-point and point-object--which together result in a one-to-one corespondence between number words and objects ($W \rightarrow P \rightarrow O$).

This analysis suggests two main sources for counting errors: violations of the word-point or point-object correspondences. Each of these correspondences can be violated by having either too many

or too few words, points, or objects. This produces four principle types of errors: omitted word (an object pointed to without a word), extra word (an object with two words and one point), skipped object (an object for which no word or point was given), and doubly counted object (assigning two successive word-point pairs to one object). Three of these errors, omitted word, skipped object, and doubly counted object, were found to be the most common correspondence errors made by three, four and five-year olds in counting rows of non-movable objects (Fuson & Mierkiewicz, 1980).

In line with Gelman and Gallistel's (1978) observations, location of error (beginning, middle, or end of a row) was varied for each of the error types above .

Note that all of these errors could involve multiple violations (e.g., triple pointing to a block, skiping several adjacent blocks, etc.). In this study, we restricted our attention to single violations.

In addition to the error types above, four types of unconventional counting that adhered to the one-one principle were included. In three of these, the order in which the objects were counted was unconventional: (1) the objects were counted in the opposite direction from that of the child; (2) the objects were counted by starting in the middle, counting to the end, and then going back to the other end of the row and counting the uncounted objects (e.g. o o o o o o o o o o); and (3) counting alternate objects in one direction, then reversing directions and counting the uncounted ones (e.g. o o o o o o o o o o). (The rows had items alternating in color, so this amounted to counting all the red chips, then all the green chips to minimize memory problems.) The fourth "error" consisted of double pointing to an object while saying only one word--a violation of both the word-point and point-object correspondences, but not a counting error because the one-to-one word-object correspondence is maintained.

In addition to type of error and location, set size was also included as a task variable. An understanding of a general priniciple implies that one should be able to apply it in a range of appropriate situations; children understanding the one-one principle should be able to recognize violations of it in counting sets of any size. On the other hand, children may only be able to recognize counting errors within sets they could actually count, suggesting that knowledge less general than a priniciple may underly their counting performance. Consequently, children were asked to detect errors in counting both small (3 and 4) and large (9 and 10) rows of items. (Row length was limited to 10 to minimize attentional difficulties found during pilot testing with sets as large as 20.)

Overall, there were 72 count trials, with each type of error or pseudo-error presented in each of the four possible set sizes. Each trial contained only one error, if any. Forty-eight were true error trials containing omitted word, extra word, skipped object, and doubly counted object errors in either the beginning, middle, or end of the row. Sixteen were pseudo-error trials in which the counting was done in a non-standard order, or a dual correspondence error was committed. Eight trials of standard

correct counting were also included.

The trials were randomly ordered with the restriction that each quarter of the set of trials contained one of each type of error and pseudo-error and two correct count trials. A set order of presentation was used for half of the subjects while the reverse order was used for the rest.

METHOD

Subjects

Thirty three- , four- , and five-year-olds from a predominantly middle class pre-school participated in the study. Age groups were balanced by sex and order of presentation of the count trials.

Materials

Rows of 3, 4, 9, and 10 plastic chips spaced approximately an inch apart on cardboard strips were used. Chips within each row alternated in color (red or green) to maximize the children's ability to distinguish adjacent chips.

Procedure

Each child was brought individually from his or her classroom to a vacant room within the school and given the following instructions: "This is Scruffy (the puppet). Scruffy is just learning how to count. Today, I'm going to ask you to help Scruffy learn to count better. First, I'd like you to show Scruffy how to count. Please count this row of chips very carefully." A row of 10 chips was then presented and the child's counting performance recorded. "Very good. Now, Scruffy is going to count for you. Scruffy is just learning how to count. Sometimes when he counts he makes mistakes. He knows his numbers, but sometimes when he counts the chips, he does it wrong. Scruffy is going to do some counting for you. I want you to watch him very carefully to see if he makes a mistake. Let's try one for practice. Watch carefully." Scruffy then counted the row of 10 chips, skipping both three adjacent middle chips and the last one. (These errors were used because pilot testing indicated that a large number of children recognized one or the other of these as a mistake.) The regular set of trials was then presented. After the puppet counted each row, the child was asked "Did Scruffy make a mistake?" If the child responded affirmatively, s/he was then asked "What did Scruffy do wrong?"

RESULTS

The data were scored in terms of the number of misconceptions: undetected true errors, unconventional counting identified as incorrect, and correct conventional counting said to be incorrect. Although some undetected true errors were probably attributable to poor attention,

especially those by the older children, all such errors cannot be blamed on inattentiveness. Often younger subjects would correctly imitate the puppet's error, claiming that it was not a mistake, and thus indicating a gap in their knowledge about counting.

Of major interest were possible developmental changes in children's abilities to detect counting errors and to distinguish them from unconventional but correct and standard correct counting; the effect of set size on this ability; children's recognition of different types of errors and the related effects of location of the error and set size; and the relationship between the type of true errors a child identified and the type of errors he made in his own counting.

Children's abilities to identify true errors, pseudo-errors and correct counting were examined through a 3(Age) X 3(Type of error) X 2(Set size) repeated measures analysis of variance on the percent of misconceptions (trials with incorrect responses). This analysis indicated that: (1) Children in all age groups were very good at identifying standard correct counting as such, with even the three-year-olds making errors on only 5% of these trials. Set size had no effect on their ability to recognize correct trials. (2) Overall, children were poorest at recognizing unconventional but correct counting, making judgement errors on 36% of the trials, compared to such errors on only 21% of the true error trials and 3% of the correct trials. In addition, recognition that unconventional counting was correct did not improve with age, with three- , four- , and five-year-olds making errors on 25%, 41% and 41% of those trials, respectively. The superior performance of the three-year-olds probably reflects a tendancy to say the puppet counted correctly rather than a better understanding of counting. (3) Identification of true errors did improve markedly with age; the three-year-olds performed at chance level by failing to detect almost half of the counting errors (42%), while the the fours and fives missed only 15% and 6% of those errors respectively. (4) In general, performance was better on small than on large sets (16% vs 23% misconceptions). Although performance on identifying correct trials was comparable for both set sizes, more true errors and pseudo-errors were correctly identified on small sets than on large ones. This difference appears to be at least partially attributable to children's tendancy to use the cardinality of the set to make error judgements about the short rows (e.g., "The puppet counted to four, but there's only three chips.") but not about the longer ones.

Performance on the true error trials was compared across age groups according to type of true error, set size, and location of error. A 3(Age) X 4(Type of error) X 2(Set size) repeated measures analysis of variance revealed significant main effects of all three variables and significant type of error by age, type of error by set size, and type of error by age by set size interactions. For all age groups, children were best at identifying skipped objects as errors, with even the three-year-olds detecting these errors on 83% of the trials. Four- and five-year-olds did so on 93% and 98% of their trials respectively. The main effect of set size again indicated that error identification was better on small than large sets. The significant set size by type of error interaction was due to almost equivalent

detection of skipped object and extra word errors on both small and large sets (94% and 89% of skipped object errors identified on small and large sets, respectively; 74% and 69% of extra word errors, respectively), compared to large differences in recognition of omitted word (86% and 69% for small and large sets) and doubly counted object errors (81% and 72% respectively). The age by type of error set size interaction seems mainly attributable to a large improvement in four-year-olds' detection of omitted word and skipped object errors on small sets, with only one-tenth the improvement on recognition of these errors on large sets. A similar increase occurs for five-year-olds in recognition of doubly counted object errors on small sets but not on large ones.

Location of error indicated a significant main effect with 69% of the errors occurring at the end of a count recognized compared to 54% of the middle and 59% of the beginning errors. There were no significant interactions of location with other variables.

Overall, our subjects made few mistakes in counting the ten item row; in fact, none of the four- or five-year-olds made any errors at all. Consequently, our comparison of the types of errors a child can identify with the types of counting errors s/he makes is limited to the performance of our three-year-old subjects. Four three-year-olds made no mistakes in their counting. On the average, these perfect counters failed to identify 5.5 of a possible 48 errors. In contrast, the two children making only a single mistake in their own counting each failed to detect an average of 22.5 errors and the remaining four subjects who made multiple counting errors failed to recognize a mean of 30 errors. In general, then, it seems that the children who make the most counting errors were poorest at detecting them. However, due to the small number of counting errors made by our subjects, our data is too limited to establish a more direct relationship between recognition of an error and making the error while counting.

CONCLUSIONS

Together, these results, especially the effects of set size and children's difficulties in dealing with unconventional but correct counting, raise questions about the generality of their underlying knowledge about counting. Instead, their understanding seems to develop in a limited, piecemeal fashion. Our data suggest that one of the earliest criteria for correct counting, used even by three-year-olds, is that it is wrong to totally skip an object; that is, to not point to it or to not say a word for it. The "only once" aspect of the one-to-one correspondence does not appear as a criteria for correct counting until later. Even then, older children's difficulties with correct but unconventional counting seem to suggest that the knowledge underlying their counting performance may be better characterized as rules for executing the standard procedure rather than general principles.

REFERENCES

Fuson, K.C. & Mierkiewicz, D.B. A detailed analysis of the act of counting. Paper presented at the Annual Meeting of the American Educational Research Assoication, Boston, April, 1980.

Gelman, R. & Gallistel, C.R. *The child's understanding of number.* Cambridge, Mass.: Harvard University Press, 1978.

Siegler, R.S. Information processing approaches to development. To appear in P. Mussen (Ed.), *Manual of child psychology (history, theories, and methods volume* (W. Kessen (Ed.)). New York: Wiley, in press.

THE COUNTING WORD SEQUENCE AS A REPRESENTATIONAL TOOL

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Six types of changes are outlined for the sequence of counting words that lead it from early use in the counting of objects to later use as objects which themselves get counted in several kinds of arithmetic solution procedures. These six changes are an increase in the speed of production of the sequence, the derivation of relations on the counting words, the acquisition of sequence meanings for the words, the acquisition of ordinal and cardinal meanings for particular words, the acquisition of symbolic and written forms of the words, and the linking of sequence to cardinal or ordinal meanings through the use of the sequence in counting in cardinal and in ordinal contexts.

The first use of the counting word sequence as a problem solving tool is in the counting of objects. Later the counting words themselves become the objects that are counted, thereby adding some more sophisticated counting procedures to the mathematical problem solving repertoire. This paper outlines six types of changes in the counting word sequence that lead from this early object use to the later representational use. Some of these later uses are then briefly described.

The acquisition of the standard sequence of counting words by middle-class American children begins by age two (Gelman and Gallistel, 1978) and extends into the decades for five-year-olds (Fuson and Mierkiewicz, Note 1, reported mean lengths of correct counting sequences averaged across two samples to be 43.6 for 5- to 5½-year-olds and 83.9 for 5½- to 6-year-olds). During the period of acquisition, the form of the counting sequence produced by children seems to be the following: a first portion that is some beginning part of the standard sequence (e.g., "one, two, three, four"), a second portion that is not the standard sequence but which is produced with some consistency by a given child (e.g., "six, eight, nine, eleven"), and a final portion that has no consistent pattern over repeated productions (e.g., "sixteen, fourteen, twelve, twenty, six, fifteen, eleventeen, nine"). The stable portion varies from child to child and usually consists of words from the standard sequence with some omissions. The final seemingly random portion consists of counting words only. For some children it contains only words not in the first two portions and for other children it contains words

that were in the first two portions. More details of the nature of these early counting sequences are reported in Fuson and Richards (Note 2) and Fuson and Mierkiewicz (Note 1).

When a child has acquired a portion of the standard counting word sequence, further changes occur within this sequence that lead eventually to the objectification of the counting words, i.e., the words become the things which are counted in some kinds of problem solution procedures. In the terms of Steffe, von Glasersfeld, and Richards (Note 3) the words are verbal unit items which are counted. Six of the changes that occur between age three and eight and which contribute to this objectification are listed and discussed below.

1. THE OPERATIONAL EFFICIENCY OF THE PRODUCTION OF THE SEQUENCE INCREASES.

Case, Kurland, and Daneman (Note 4) reported that the mean speed with which shapes on cards were counted increased linearly with age from 840 milliseconds per object at age four to 510 milliseconds at age six to 210 milliseconds for adults (numbers approximated from their Figure 3). According to Case et al., this increase in operational efficiency resulted in an increase in available M-space, the central reserve of attention which activates schemes to execute operations and which stores products of such schemes. The increase in M-space which results from an increase in the operational efficiency (the speed) with which the sequence of counting words is produced means that the faster counter can count and also "do something else" while counting. This "something else" may be, for example, remembering the size of the sets already counted, noticing the words being produced and observing relations between them, or noticing and counting (by some keeping-track mechanism) the words being produced. Thus a freeing of processing capacity for some activity concurrent with the production of the counting word sequence can enable the words being produced to become the focus of attention and the objects of thought. Case et al. presented evidence that the previously established developmental increase in M-space results from a maturation-based increase in the speed with which schemes can be activated from central storage. However, they also reported that massive practice in counting objects (a mean of 835 trials) increased the operational efficiency and the M-space of four-year-old subjects by about a year. Therefore both maturation and practice may contribute to an increase in the speed of counting objects, and therefore to the ability to reflect upon the counting words being produced.

2. RELATIONS ON THE COUNTING WORDS ARE DERIVED AND THEN LATER STORED AS SEPARATE FACTS.

The acquisition of a portion of the standard sequence of counting words is followed by a decomposition of the sequence as partial productions of the sequence become capable of being emitted and as relations on the words come to be derived from the sequence (Fuson and Richards, Note 2; Fuson and Richards, Note 5). Partial productions of the counting sequence involve first the ability to start at an arbitrary point within the sequence instead of always starting at the beginning of the sequence and then later the ability to stop at an arbitrary preassigned word. These partial productions enable relations on the counting words to be derived more easily than when the whole sequence must be repeated in order to establish some relation between two counting words. The relations that are derived on the counting words are "the immediate successor" or "comes immediately after" ("seven comes immediately after six"), "successor" or "comes after" (when counting, "nine comes after five"), and "between" ("eleven comes between nine and twelve"). The first two relations have inverses: "the immediate predecessor" (or "comes immediately before") and "predecessor" (or "comes before"). An associative chain theory of serial learning (learning a sequence) asserts that the sequence is learned by the establishment of associative links between successive words in the sequence, i.e., that "the immediate successor" relation is the means by which the sequence is initially acquired. Such a theory has not proven adequate to account for serial learning in the laboratory nor does it fit the counting word sequence. If the counting sequence is initially a chain of associated words, any single word ought to be able to elicit its immediate successor. However, for three- and four-year-olds, a preceding portion of two or three words of the sequence elicits significantly and considerably more (85% vs. 45%) immediately following words than does a single word (Fuson and Hall, Note 6). For example, "five, six, seven" elicits "eight" significantly more often than does "seven". Thus the counting word sequence seemingly is first learned as a whole "string" of connected words, and the words then become separated and related in various ways. The "comes after" relation may be established in one of at least three different ways: through the successive application of immediate successor relations ("seven comes immediately after six" and "six comes immediately after five" so "seven comes after five"); through an empirical test of whether one says one number while counting to another ("seven comes after five because I say five while counting to seven"), or through an intuitive comparison of initial counting segments for two words ("seven comes after five because 'one,

two, ..., seven' is longer than 'one, two, ..., five'"). These possibilities are discussed further in Fuson and Hall (Note 7). The final relation "between" may be derived independently or it may result from a combination of "comes after" and "comes before" ("eight is between five and nine because it comes after five and before nine").

3. THE SEQUENCE RELATIONS PRODUCE SEQUENCE MEANINGS THAT BECOME ASSOCIATED WITH THE COUNTING WORDS.

4. ORDINAL AND CARDINAL MEANINGS ARISING OUTSIDE OF THE PRODUCTION OF THE COUNTING SEQUENCE BECOME ASSOCIATED WITH PARTICULAR WORDS.

5. SYMBOLIC AND WRITTEN FORMS OF THE WORDS BECOME LEARNED.

Changes 3, 4, and 5 all refer to meanings or stimuli which come to be associated with the words in the counting sequence. As relations on the counting words come to be derived and then stored as facts, meanings associated with these relations (e.g., six comes before seven) begin to be associated with the counting word. These counting word meanings are those which derive only from the "alphabetical order" of the counting words and involve meanings relating counting words to other counting words. Cardinal word meanings describe "how many" entities are in some specified set. Ordinal word meanings refer to one particular object in relation to a whole ordered set of objects. From the second or third year of life, children hear the number words used by adults in cardinal ("There are five people in your family.") or ordinal ("Big Bird is fourth in line.") situations, and these cardinal and ordinal meanings gradually become associated with the words. Because the words used in counting and the words used to describe the size of a set are identical, these counting and cardinal meanings, though arising in different contexts, become attached to the same words. Most of the ordinal words are the counting/cardinal words with a suffix "-th" added, so these ordinal word meanings may become associated with the counting/cardinal words to some extent. Numerals also begin to be learned from, for example, Sesame Street, cards, numbered blocks, or older siblings playing school, and these are associated with the counting/cardinal words ("5 is five"). In school, printed words also become associated with the now very familiar auditory words.

6. CARDINAL AND ORDINAL MEANINGS ARE LINKED TO COUNTING MEANINGS BY THE USE OF THE COUNTING WORD SEQUENCE IN COUNTING OBJECTS FOR CARDINAL AND FOR ORDINAL PURPOSES.

Many of these different meanings arise in quite different contexts and are likely to remain for a while context-bound and not well linked together. A given word in a cardinal context will elicit only cardinal meanings and so

forth. However, the use of the counting word sequence to count objects for cardinal and for ordinal purposes begins to link cardinal and ordinal meanings to counting meanings by juxtaposing these meanings in time. The first meaning elicited is the counting meaning and then this must be translated to the cardinal or ordinal word meaning: "One, two, three, four, five (counting word). Ah. There are five (cardinal meaning) plates on the table." or "One, two, three, four, five, six (counting word). Oh. She is sixth (ordinal word) in the race." Failure to connect the counting meaning to the cardinal meaning after counting a set of objects results in a lack of the "cardinality rule" (Schaeffer, Eggleston, and Scott, 1974) or the "cardinality principle" (Gelman and Gallistel, 1978); the child not making this connection does not respond to a question of "how many" with the last word said in counting but simply recounts the set again.

A connection between counting and cardinal meanings is necessary for the counting-on procedure to be used in addition situations (Fuson, Note 8). In this procedure, if the child knows the cardinal number of two sets ("There are eight here and five there. How many in all?"), he makes a complex association of several meanings summarized by the utterance of the single word "eight" and continues counting from that word for as many words as are in the second set: "eight, nine, ten, eleven, twelve, thirteen." (See Fuson, Note 8, for more details concerning this procedure). This is obviously a much more efficient procedure than the counting-all procedure in which all of the elements must be counted. Counting-back x words and counting back to x are analogous procedures often used for subtraction situations (Carpenter, Hiebert, and Moser, Note 9; Moser, Note 10).

SUMMARY

With all six of these changes, the sequence of counting words has moved from a somewhat laboriously produced memorized chant of nonsense syllables to a flexibly and easily produced sequence of words, each with a complex of meanings attached. This sequence of words can now become the object of thought, and the words can become things which are themselves counted. The counting word sequence has now become a representational tool which can be used for many arithmetic problems. This tool is used by counting up or down the sequence. In different problems, the starting word or the ending word or the number of words to be counted up or down may or may not be known. The counting steps may be in ones, or threes, or fives, or any number. The operation may be addition, subtraction, multiplication, or division. The sequence production may use only one size of counting step or more than one

(e.g., 36 + 24: "thirty-six, forty-six, fifty-six, fifty-seven, fifty-eight, fifty-nine, sixty"). The solution procedure may be primarily direct counting up or down the sequence or it may place some known arithmetic fact within the sequence and operate upon it there. Research tracing the development of the counting word sequence from its initial acquisition to its later representational use in arithmetic is just beginning. Details of the ways in which the six changes outlined in this paper effect the later representational use is not at present well understood. Further research may clarify the processes involved and also contribute to instructional ways of facilitating this growth in children.

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Reference Notes

1. Fuson, K. C., & Mierkiewicz, D. A detailed analysis of the act of counting. Paper presented at the Annual Meeting of the American Educational Research Association, Boston, April 1980.

2. Fuson, K. C., & Richards J. Children's construction of the counting numbers: From a spew to a bidirectional chain. Paper presented at the Annual Meeting of the American Educational Research Association, Boston, April 1980.

3. Steffe, L. P., von Glasersfeld, E., & Richards J. Children's counting types. Manuscript submitted for review. 1980.

4. Case, R., Kurland, M., & Daneman, M. Operational efficiency and the growth of M-space. Paper presented at the Biennial Meeting of the Society for Research in Child Development. San Francisco, March 1979.

5. Fuson, K. C., & Richards, J. The acquisition and elaboration of the sequence of counting words. In preparation.

6. Fuson, K. C., & Hall, J. W. The representation of young children's counting words: Evidence for complex chaining. In preparation.

7. Fuson, K. C., & Hall, J. W. The acquisition, properties, and functions, of the counting word sequence. In H. Ginsburg (Ed.), The Development of Mathematical Thinking. New York: Academic Press, expected 1981.

8. Fuson, K. C. An analysis of the counting-on solution procedure in addition. In T. Romberg, T. Carpenter, & J. Moser (Eds.), Addition and Subtraction: A Developmental Perspective. Expected 1981.

9. Carpenter, T., Hiebert, J., & Moser, J. The effect of problem structure on first-grader's initial solution processes for simple addition and subtraction problems. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, 1979.

10. Moser, J. A longitudinal study of the effect of number size and presence of manipulative materials on children's processes in solving addition and subtraction verbal problems. Paper presented at the Annual Meeting of the American Educational Research Association, Boston, April 1980.

References

Gelman, R., & Gallistel, C. R. The child's understanding of number. Cambridge, Ma: Harvard University Press, 1978.

Schaeffer, G., Eggleston, V. H., & Scott, J. L. Number development in young children. Cognitive Psychology, 1974, 6, 357-379.

COUNTING AND THE CONSTRUCTIVIST PROGRAM

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In this paper I explore some connections between the Constructivist program in the philosophy of mathematics, and the development of mathematical abilities in children. I use a language, created by the constructivist mathematician Alexander Yessenin-Volpin, in order to describe the act of counting, and the development of the ability to produce natural number series. This provides a basis for distinguishing numerical from pre-numerical behavior.

"'Can you do Addition?' the White Queen asked. 'What's one and one and one and one and one and one and one and one and one?'
'I don't know,' said Alice. 'I lost count.'
'She can't do Addition,' the Red Queen interrupted."
Lewis Carroll (1896), p. 220

John Stuart Mill (1891) asserted that the basis for numerical understanding -- indeed all knowledge -- is experience. Observed facts regarding collections of objects, say that a group of three objects ○○○ can be separated in two parts ○○ and ○ , provide meaning for numbers and numerical formulae. Gottlob Frege (1894, p.9) attacks Mill with the following rhetorical question :

> "But what in the world can be the observed fact, or the physical fact (to use another of Mill's expressions), which is asserted in the definition of the number 777,864?"

What indeed!

Rather than taking this as a rhetorical question, used to cast doubt on the experiential basis for three, I take it to point to an important distinction which I make by speaking of the number three and the number name 777,864. The former's sense is determined by its appearances in various natural number series. The latter is a properly determined application of number names in the numeration system. In this paper I focus on the development of the numbers to which we can count -- the counting numbers. These must be distinguished from such as 777,864. While there are some kinds of experiences with 777,864 similar to those with 3

(adding 5, taking away 2), the former has clearly a different character. It is not directly grounded in counting, or perceptual, experiences and, furthermore, what are trivial (or "mental") operations with three (doubling, squaring, dividing, and so on) become "paper-and-pencil" operations with 777,864.

The counting numbers must also be distinguished from the results of calculations with number names. For example, 10^{12}, though well formed, yields a number name with which we clearly have no counting experience. These calculations themselves are grounded in operations with counting, insofar as they have numerical significance.

In this paper I begin by indicating a language, developed by the constructivist mathematician Alexander Yessenin-Volpin (see especially 1970; but also 1980), for speaking about events and processes generally, but, in particular, about natural number series. I then adopt this language in order to give a description of children's counting actions which I consider numerical. The ability to produce a natural number series is seen as crucial for distinguishing numerical from non-numerical behavior.

I EVENTS AND PROCESSES

Mathematics, from the Constructivist perspective, is taken to be an activity -- a series of actions. It is done by persons. Each of these actions may be taken as an event in a discrete process. A discrete process results from applying a method in some prescribed manner. Such a process which terminates is a construction (cf. Hennix, 1979, p.5). In this paper I am only concerned with terminating processes, i.e. with constructions.

Events within a discrete process are connected by a tactic of attention, i.e. a way of following the connections. Events occur, or arrive, in time. An event which arrives in a particular process may, or may not, be identified with an event which arrives in a process at a different time, or in a different place. Such an identification is made by following a tactic of collation. That is, the intention to reproduce a process does not by itself guarantee that the events will be identified. Such an identification would be a separate, third, process itself comprised of events of identification, i.e. events which identify events in the former processes.

An accomplished construction, C, can serve as the basis for a new construction, called its load (C_{load}). That is, the new construction is made "on the basis" of the accomplished construction. Each event, c, of the basis bears a part of the load, $load_c$. With the occurence of each event in the basis (the old construction) there is a prescribed manner for following a method which results in an event of the load (the new construction). Each $load_c$ is a feasible construction just in case such a construction has been accomplished for all events preceding c in the basis. The new construction, C_{load}, is feasible and consists in the reproduction of C with $load_c$ following each step c as it occurs.

II NATURAL NUMBER SERIES

The above section indicates a way to talk, generally, about events and processes. This can be seen to apply, in particular,to natural number series. A natural number series is a discrete process. It has a system of generating rules for constructing the events of the process -- the numbers. Moreover there must be an evident tactic of attention which connects these events, i.e. it must be able to be seen as a discrete process (Volpin, 1970, p.26). Numbers, from this perspective, are events. They occur in natural number series, and are connected by following the rules for generating the series (see Geiser, 1979; Hennix,1979). Numbers which occur in natural number series at different times, or in different places, may, or may not, be identified. Thus, we begin by rejecting the uniqueness of the natural number series -- one of the primary hidden assumptions of classical mathematics. However, in some cases we may be able to identify one series with all or part of another through a process of events of identification -- following tactics of collation. This would depend on the generating rules for the natural number series in question. The particular natural number series we consider is the series of counting numbers, i.e. of those to which it is possible to count (as opposed, for example, to a series containing 777,864). Any finite length of this series terminates, and, therefore, is a construction.

III COUNTING

Counting serves as a basis for constructing feasible natural number series. Yet counting itself need not produce such a number series, and need not be considered as numerical behavior. Counting, we have argued elsewhere, is the production of a sequence of number words such that each number word is accompanied by the production of a unit item (Steffe, Thompson and Richards,

1980; Steffe, von Glasersfeld and Richards, 1980).

A number word sequence is the uttering of number words taken as a single linguistic unit, i.e. as events within a discrete process (cf. Fuson and Richards, 1980). Number word sequences may be produced as a distinct activity, or as a part of a counting process. In the former case the number words are not used to indicate, i.e. to point to, anything, and there is no counting. Nothing is being counted. In counting, the number word indicates a unit item. There are three major stages of counting depending on the nature of the unit item (i.e. on what the child seems to generate and be aware of, while counting): 1.) Counting with perceptual unit items -- perceptual, i.e. visual, auditory, tactile,items are the "material" for unit construction. 2.) Counting with motoric unit items -- motoric, i.e. proprioceptive, items are the "material" for unit construction. 3.) Counting with abstract unit items -- unit items themselves are the "material" for unit construction.

It is my contention here, that numerical behavior -- the production of a natural number series -- only arises in the last stage of counting: counting with abstract unit items.

IV NUMERICAL BEHAVIOR

In this paper I investigate the development of natural number series in children. The ability to produce natural number series distinguishes numerical from pre-numerical behavior. There are three behavioral manifestations of this new constructive process:

1.) Conservation. The child conserves the results of counting. This is not only Piagetian number conservation, i.e. a maintenance of number across physical transformations, but it is marked by the disappearance of recounting -- a set of objects once counted remains counted. In Volpin's terms, the child has identified events from distinct processes by following tactics of collation. In other (Fregean) terms one might say that number is becoming an objective idea,but this phraseology is unfortunate because it suggests number as object rather than event.

2.) Integration. The child is able to integrate. An integration is taking a potential, or manifested, sequence of counting actions as a unit (cf. Steffe, Thompson and Richards, 1980, p.5). In the following protocol Donna performs an integration during the solution of a task:

> Interviewer: (Places a card with 12 checkers glued to it, seven of which are covered, in front of Donna.) There are

seven checkers here (points to cover). How many are there altogether?

Donna: Seven under here (points to cover)?

Interviewer: Yes.

Donna: Eight, nine, ten, eleven, twelve (points at the individual checkers in synchrony with utterances). Twelve.

Once she was assured that there were seven checkers covered, she considered them as having been counted and picked up the counting sequence with "eight" and the first visible checker. Donna constructed "seven" by integration. She was able to begin by uttering "eight" because she took the potential counting actions leading to eight as a single unit, and did not feel the need to reconstruct seven through actual counting actions. This analysis of the protocol (taken from Steffe, Thompson and Richards, 1980, p.5) fits well with Volpin's language. Donna took a potential construction - the terminating discrete process of counting to seven - as a single event. This event served as the initial event of the above counting process. The continuation of the count, from this event, was a single process which depended on the presence of the checkers (counting with perceptual unit items). Donna's ability to integrate was quite advanced, but she required only the more primitive perceptual counting to complete the task.

3.) Double-counting. The child is able to "double count", i.e. to produce two counting sequences. For example, to solve "Five plus what makes eleven" a child might respond "Six is one, seven is two, eight is three...". In the above terms the first sequence serves as the basis for the second. The basis may depend on the presence of physical objects, or motor movements, in which case the child is counting perceptual unit items or motor unit items, respectively. The second sequence, however, is based on the availability of units from the first sequence and the child is counting the appearance of unit items in the first sequence.

V ABSTRACT UNIT ITEMS

All of these actions, although in particular the third, indicate that the child is able to count with abstract unit items. A counter with abstract unit items is aware of the fact that "the numerical structure designated by a number-word is a composite of so many individual units" (Steffe, von Glasersfeld and Richards, 1980, p.13). And again, the counter with abstract unit items has "acquired the capability of going from a number-word to the conceptual structure which that word designates, i.e. to the internal construct that

constitutes the particular numerosity" (ibid., p.13). The conceptual structure which constitutes a particular numerosity depends on actions of identification and integration. Events (in particular the production of units from different items) from disparate counting processes are identified. This act of identification -- of ignoring differences -- strips elements of their qualities (Piaget, 1970, p.37; but see especially von Glasersfeld, 1980, pp.2 and 12). There must also be an integration -- an act of taking these actions as a unit.

It is my contention in this paper that the significant step marking the appearance of counting with abstract unit items is the ability to produce natural number series. The above behavioral indicators can be seen as related actions which depend on the availability of natural number series. Taking the last first:

3.) Double counting. The second process in double counting is clearly a natural number series. It is constructed from a counting process, which itself is a construction. The child is counting unit items, and is not tied to sensori-motor material for counting. The material for the units -- what is created by the counter while counting -- is the pattern of the basis counting process, what von Glasersfeld (1979; 1980) calls the "attentional pattern". This is the pattern of attending to the action of creating unit items in the base construction. The nature of the events in the natural number series is determined by the load on each of the events in the basis construction. This is the process of going from the first construction to the second construction. In the first construction an item is taken as a unit and indicated by a number-word, say "six". In the second construction the "sixth" unit from the first construction is itself taken as the material for a unit item which is indicated by the number-word "one".

2.) Integration. Double counting itself requires an integration at each step (in each load$_C$). In order to operate on patterns, i.e. in order to take the unit items from the basis process as material for counting, the child must create a pattern from these units. The child must integrate the counting actions from the first series in order to continue the count for the second series. Let us return to our example above. The child has said, "Six is one." When the child takes a new item as a unit in the basis construction and indicates it by "seven", then the pattern of two unit items (indicated by "six" and "seven") must be created, or else the child would say "one" again. That is the child would say, "Six is one, seven is one..." rather than, "Six is one, seven is two,...". What is indicated in the second process, in the

natural number series, is a single unit, whose internal structure constitutes the particular numerosity.

1.) Conservation. This seems to be an important prerequisite, rather than a sufficient condition, for natural number series. Nevertheless, by making identifications across separate counting episodes the child lays the foundation for creating a pattern in common. On the basis of tactics of collation a child identifies "eight" in one counting process with "eight" in another. But what occurs when a child counts eight apples and eight houses, and sees this as the same eight? From our perspective, the child has followed tactics of collation, which have led her to disregard the items being counted (apples and houses). This allows the child to attend to the action of taking something as a unit, considering the unit qua unit. This is the foundation for the natural number series.

VI CONCLUSIONS

Numbers occur as events in natural number series. Counting and calculation may very well occur without a natural number series being produced. I take this activity to be non-numerical. Thus, a child who cannot integrate may be taught an algorithm for counting-on. For this child, the product of this procedure will be a number-name. The procedure itself will be rigid, and not readily adapted to new situations. In contrast, Donna, in the above protocol, faced a situation which was new to her, yet because integration was available as an operation her sophisticated solution resulted.

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REFERENCES

Carroll, L. Through the looking-glass. New York: Signet, 1960 (1896).

Frege, G. Die Grundlagen der Arithmetik. Breslau, 1884. Translated by J.L. Austin, The foundations of arithmetic. Oxford: Basil Blackwell, 1959.

Fuson, K. and Richards, J. Children's construction of the counting numbers: types of number-word sequences. Paper delivered at AERA, Boston, 1979.

Geiser, J. Unpublished notes. 1979.

Hennix, C. Computability on different natural number series. Paper delivered Mid-Hudson Mathematics Forum, 1979.

Mill, J.S. A system of logic, 8th edition, New York; Harper and Brothers, 1891.

Piaget, J. Genetic epistemology. New York: Columbia University Press, 1970.

Steffe, L., Thompson, P. and Richards, J. Children's counting in arithmetial problem solving. In T.A. Romberg, T. Carpenter and J. Moser Proceedings of the Wingspread Conference on Initial Learning of Addition and Subtraction, Racine, Wisconsin, 1979.

Steffe, L., von Glasersfeld, E., and Richards, J. Children's counting types. 1980 (submitted).

von Glasersfeld, E. The conception and perception of number. Paper presented at the Symposium on Models of Mathematical Cognitive Development, Athens, Georgia, 1979.

von Glasersfeld, E. An attentional model for the conceptual construction of units and number. 1980 (submitted).

Yessenin-Volpin, A. The ultra-intuitionistic criticism and the antitraditional program for foundations of mathematics. Proceedings of the Summer Conference on Intuitionism and Proof Theory, Buffalo, New York. Amsterdam : North-Holland, 1970.

Yessenin-Volpin, A. On the main problem in the foundations of mathematics. Proceedings of the Boston Colloquium for the Philosophy of Science, 1974, 1980.

THE INTERNAL REPRESENTATION OF OPEN SENTENCES IN ARITHMETIC

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The paper presents a theoretical model which will account for observed differences in children's solving eight open sentences forms of addition and subtraction problems (including two spatial forms). The major hypothesis in this study is that reaction time on each of the open sentences reflects processes involved in performing the given task, and is proportional to the 'distance' of its surface structure from the underlying additive relation. The 'distance' is expressed in terms of both the sequential order (a kind of syntactic component) and the interpretation of the given numbers and signs (a kind of semantic component). Some empirical results which support the major hypothesis are presented, but they are only tentative at present.

What kind of operation is presented by the open sentence a + □ = c ? On the surface it is an addition operation since it contains the symbol '+'. From the logical point of view it is, however, a kind of subtraction operation since the sum and one of the addends are given and the second addend is missing, which is characteristic of subtraction. When a child is presented with such a sentence a conflict may arise by virtue of the fact that he is simulataneously exposed to two different processes. The one is information processing of a sequential string given in an arithmetic symbolism, and the other is performing the logical analysis which will lead him to find the missing addend and not the sum of the two numbers presented to him despite the presence of the plus sign.

Since it is concerned here with rapid mental processes and I would like to suggest a theoretical model which will account for ditinguishable overt behaviors, I have employed Sternberg's methodology for inferring internal processes from measurements of reaction-time (RT). Studying 'mental arithmetic' by means of RT measurements was the focus of several studies conducted recently by cognitive psychologists (Ashcraft and Battaglia, 1978; Ashcraft and Stazik, 1979; Banks, Fujii and Kayra-Stuart, 1976; Groen and Parkman, 1972; Parkman and Groen, 1971). Unlike the above mentioned studies, which concentrated on one operation and comparison of numbers (in most

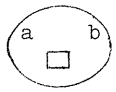
cases the calculated sum and a stated number given in the stimulus), this study is aimed at finding parameters which distinguish among various open sentences as will be detailed below.

THE PROPOSED THEORETICAL HYPOTHESIS

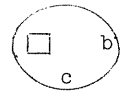
The various open sentences of addition and subtraction are taken here as commands for operations (or actions) which all relate to one underlying additive relation. The major hypothesis in this study is that RT on each form of the open sentence reflects the processes involved in performing the action called upon. Moreover, the specific RT is proportional to the 'distance' of its surface-structure from the underlying additive relation (its deep structure). The specific forms of open sentences in this study are:

Canonic Forms	Non Canonic Forms	
I. a + b = □	III. □ + b = c	V. a + □ = c
II. a - b = □	IV. □ - b = c	VI. a - □ = c

Spatial Forms

VII. 

(Finding the sum)

VIII. 

(Finding an addend)

These Forms can be classified according to their Logical Task into two classes:
1) Finding the sum: Forms I, IV and VII
2) Finding an addend: Forms II, III, V, VI and VIII

The open sentences (I - VI) in contrast to the spatial tasks (VII - VIII) can be characterized by two main properties: a) They are sequentially ordered and must be read from left to right, and b) they include the symbols '+' and '-' with their different semantic interpretation. The major hypothesis stated above will now be more specifically stated in terms of cognitive processing, and further examined by means for reaction time measurements.

SEQUENTIALITY OF THE INFORMATION AND THE POSITION HYPOTHESIS

As has already been noticed (Grouws, 1974; Lindvall and Ibarra, 1980;

Nesher, 197. ; Weaver, 1971, 1973) the position of the missing number is the core of the nature of the task. Traditionally open sentences with '+' and '-' were given first only in their canonical form. Non canonical open sentences were delayed until Algebra was studied. Now, such sentences are introduced in the primary grades. The early introduction of non-canonical sentences was probably inspired by two sources: the Piagetian notion of reversability and the mathematical notion of inverse operations which aimed to emphasize the logical structure of the additive relation. The introduction of non-canonical open sentences in the absence of algebraic tools left open the question of how children solve such open sentences. As data on the solution of non-canonical sentences has accumulated (Carpenter et-al,1979;Lindvall et-al,1980; Suppes et-al,1967;Vergnaud,1979)it became apparent that the non-canonical open sentences should be further classified according to the position of the unknown (1st, 2nd and 3rd position). I suggest then the following hypothesis:

The Position Hypothesis: The task demanded by the open sentences I - VI is an action performed sequentially in time, with respect to an already acquired underlying structure, on the basis of the available surface structure information at each point in time.

From the operational point of view, The Position Hypothesis claims that the time required for the performance of an open sentence will vary according to the position of the unknown, in the following manner:

1st position	$\square * b = c$	The largest RT
2nd position	$a * \square = c$	
3rd position	$a * b = \square$	The smallest RT

THE SEMANTIC INTERPRETATION OF THE OPEN SENTENCE AND THE SIGN HYPOTHESIS

When the sign '+' vs. '-' is written in an arithmetic sentence it affects the understanding of the whole string of numbers. Reading from left to right it is understood that in the presence of the '+' sign, the first number in the string is one of the addends of the additive triple of numbers, while if the '- sign

appears in the sentence, the first number should be read as the sum of the given additive triple. This interpretation is embedded in the meaning of the signs '+' and '-'. Since these signs are first learned in school with natural numbers, they also receive as one of their semantic interpretations the notion of 'increasing' for '+' and 'decreasing' for '-'. There is also a growing knowledge of the semantic feature concerning commutativity for the '+' sign vs. non-commutativity for the '-' sign.

The knowledge of the semantic features of the '+' and '-' signs is especially significant if an open sentence is given in its non-canonic form. In such cases it is crucial that the semantic features of the operation sign be considered in the complete context of the entire sentence and not be evaluated as an isolated sign (like a lexical item) with its interpretation.

An example will illustrate this best; consider the following open sentence: VI. $\square - b = c$. The child usually, when reading such a sentence makes errors when he attends to the semantic feature of 'decreasing' for '-' and immediately substracts b-c or c-b (depending upon which of the two numbers, b or c is larger). This is a common error for such an open sentence in spite of the fact that it contradics another semantic feature of the '-' sign, namely, that a sentence containing the '-' sign starts from the left with the sum (i.e. the 'largest' number in the case of natural numbers). I suggest, then, the following hypothesis:

The Operation Sign Hypothesis: The signs '+' and '-' are both derived from the same underlying structure (deep structure) among triples of numbers with primacy for the '+' sign.

The primacy of the '+' sign might in part be attributed to the history of learning, but might also be attributed to the difference in its semantic features (+increasing; +commutativity). These features of the '+' sign enable partial violation of the strict order (of the addends) and therefore impose in its canonic form less restrictions for the child than the '-' sign.

It is hypothesized therefore that forms with the '-' sign will demand larger RTs for their processing than forms with the '+' sign, controlling all other variables.

A Note About the Actual Numbers Involved in the Open Sentences: Studies about the internal representation of numbers (Moyer and Landauer, 1967, 1973; Restle, 1970; Shepard, Kilpatric and Cunningham, 1975) suggest that appropriate internal representation for different digits smaller than 9 would correspond in an analog manner to their magnitude. Such an approach also agrees with Bank's (1975) and Ashcraft's (1978) proposal to consider the numbers in the framework of long term semantic memory having semantic features such as (+large) or (+small). Thus, the two specific numbers provided in any open sentence should be taken into consideration as influencing the encoding of the given task.

The variables: Logical Task; Position of the missing number; and the operation sign were examined in our study. The variable related to the actual numbers in the stimuli was controlled.

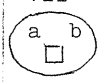
THE EXPERIMENT AND RESULTS

The tasks in the experiment consisted of the eight forms mentioned on p.2, given to second graders at the beginning of the school-year. These students were exposed in their first grade to an experimental program which presents them with the spatial forms (VII and VIII) for the logical tasks involved in the underlying additive relation among triples of numbers, before learning the use of the conventional signs '+' and '-'. There were six triples of numbers for each form arranged in a Latin Square to avoid mechanical answers. The position of the '=' sign was also controlled. The Ss were examined individually (n=22) and RT measurements were obtained. (details of the procedure are omitted here).

Table 1 will present mean RTs for each form of open sentences; the mean is over the six triples of numbers and the 22 Ss.

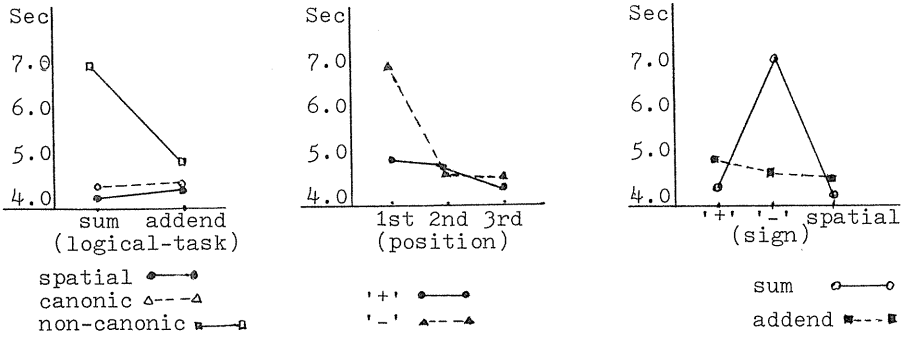
TABLE 1

Mean RTs for each form of open sentences

The Form	I a+b=□	II a-b=□	III □+b=c	IV □-b=c	V a+□=c	VI a-□=c	VII 	VIII 
RT in seconds	4.17	4.35	4.77	6.97	4.56*)	4.55*)	4.10	4.40

*) RTs obtained from different Ss.

The following graphs will provide some of the major findings:



A two-way analysis of variance (Logical Task X Position);(Position X Sign) and (Logical Task X sign) was performed on RTs to the eight forms. All of them were found to be significant (at the level of at least .05) with strong effects of interaction. The variable Position has the strongest effect and the Logical variable - the least strong effect.

It was found that the spatial form VII, is possibly the closest to the underlying stored structure and posed the least constrains; the canonic forms are next in 'distance'; and the non-canonic forms are the most difficult (the largest 'distance'). In each of the above cases the use of the '-' sign makes it more difficult, except for the form VI which needs additional comment: In form VI (a-□=c) if the sequential processing is violated, it will still yield a correct answer and therefore the mean RT probably represents a mixture of correct and erroneous procedures as will also be noted later from the regression lines. We have no explanation for the fact that form II took less time than form VIII.

We have also employed a simple regression analysis for all the eight forms in the lines suggested by Groen and Parkman (1972) for simple addition (Form I). The idea was to find, in an exploratory manner, to what extent one can assume similar processes for the various forms. Data below will present the regression equation for each form as a function of its best predictor (out of four predictors), and the proportion of explained variance by this predictor.

Table 2

The Open Sentence Form	The Regression Equation with Best Predictor	r^2
I. a+b= □	RT = 2.45 + .22(sum)	.62
VII. (a □ b)	RT = 2.31 + .23(sum)	.73
III. □ +b=c	RT = 3.00 + .22(sum)	.56
II. a-b=□	RT = 3.27 + .39(min)	.87
V. a+□=c	RT = 2.93 + .57(min)	.76
IV. □-b=c	RT = 1.57 + 2.13(min)	.75
VIII. (□ c b)	RT = 2.50 + .47(c-b)	.73
VI. a-□=c	no signinficant predictor	

I regard the above results as tentative for two reasons: only six triples of numbers were employed, and the results of forms V and VI were obtained from a similar, but not the same Ss. I also think that we should look more closely for individual differences. Bearing the above comments in mind, it seems that the two parameters in the regression lines (the intercept and the slope) along with the best predictor, point out that different processes involved in the various forms of open sentences. I must postpone stronger conclusions which emerged from this exploratory study until more data from an ongoing study will be available.

BIBLIOGRAPHY

Ashcraft, M. H. & Battaglia, J. Cognitive arithmetic: Evidence for retrieval and decision processes in mental addition. Journal of Experimental Psychology: Human Learning,1978,4, 5, 527-538.

Ashcraft, M. H. & Stazyk, E. H. Mental arithmetic: An information processing tutorial,Focus on Learning Problems in Mathematics, 1979, 1, 1.

Banks, W.P.,Fujii, M. & Kayra-Stuart, F. Semantic congruity effects in comparative judgement of magnitudes of digits.Journal of Experimental Psychology:Human Perception and Performance,1976,2,435-447.

Carpenter, P.T & Moser, J.M. The development of addition and subtraction problem solving skills. A paper presented at the Wingspread Conference, November, 1979.

Groen, G.J. & Parkman, J.M. A chronometric analysis of simple addition. Psychological Review, 1972, 79, 4, 329-343.

Grouws, D.A. Solution methods used in solving addition and subtraction open sentences, Arithmetic Teacher, 1974, 21, 255-261.

Lindvall,C.M. & Ibarra, C.G. CT procedures used by primary grade pupils in solving open addition and subtraction sentences. Journal for Research in Mathematics Education, January 1980, 51-62.

Moyer, R.S. & Landauer, T.K. Time required for judgements of numerical inequality, Nature, 1967, 215, 1519-1520.

Moyer, R.S. & Landauer, T.K. Determinants of reaction time for digit inequality judgements. The Psychonomic Society Bulletin, 1973, 1,3, 167-168.

Nesher, P. & Katriel,T. Two cognitive modes in arithmetic word-problems, Paper presented at the second annual meeting of the International Group for the Psychology of Mathematics Education. Osnabruck, West Germany, 1978.

Parkman, J.M. & Groen, G.L. Temporal aspects of simple addition and comparison, Journal of Experimental Psychology,1971, 89, 2, 335-342.

Restle,F. Speed of adding and Comparing Numbers. Journal of Experimental Psychology, 1970, 83, 2, 274-278.

Suppes,P. & Groen G.L. Some counting models for first grade performance data on simple addition facts. in Research in Mathematics Education, NCTM, 1967.

Shepard,R.N. ,Kilpatric, D.W. & Cunningham,J.P. The internal representation of Numbers. Cognitive Psychology, 1975, 7, 82-138.

Vergnaud, G. A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. A paper presented at the Wingspread Conference, Wisconsin, 1979

Weaver, J.F. Some factors associated with pupils' performance levels on simple open addition and subtraction sentences. The Arithmetic Teacher, 1971, 18, 513-519.

Weaver, J.F. The symmetric property of the equality relation and young children's ability to solve open addition and subtraction sentences. Journal for Research in Mathematics Education, 1973, 4, 45-56.

GEOMETRY STRUCTURES AND PROCESSING STRATEGIES IN YOUNG CHILDREN

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This research was designed to determine whether strategies (more specifically, algorithms and heuristics) are generated spontaneously by children in response to problems presented by geometric transformations (slides, flips and rotations). The data, derived from children 4 to 11 years of age, were in the form of videotaped records. An analysis of these data reveal various combinations of algorithmically-based procedures associated with successful problem solutions and differently deployed strategies associated with unsuccessful solutions.

Theories of two kinds have been proposed to explain the nature of cognitive development and the ability to reason in particular contexts such as mathematics. One class of theories ordinarily conceives of development as a stagewise progression of cognitive operations and functions. In this view mathematical and similar knowledge is acquired by the development and operation of cognitive structures, in which each structural level is superceded by and integrated into a more complex organization. Two examples are the developmental theories of Piaget and Gal'perin, although each differs from the other in its characterization of developmental stages and in the assigned origin of cognitive structures.

The second class of theories emphasizes the processes and functions inherent in simple and complex problem solving, as well as in a broad range of cognitive behaviors. In this group are theories of information processing, strategy development, hypothesis testing, problem solving, and algo-heuristic learning. These theories, in the main consist of process-oriented formulations designed to detail the procedures utilized in cognitive activity, in contrast to the stage-developmental theories, which attempt to define the cognitive structures underlying cognitive processes and activities.

Despite the fact that the two classes of theory, structural and functional, are put in opposition to one another in the debates of theoretical adequacy, they do not necessarily conflict; on the contrary, they may be conceived as complementary. This complementarity has been noted before and various efforts have been made to integrate the two kinds of formulation. One approach is to expand Piagetian theory, internally so to speak, by emphasizing the functional as well as structural aspects of that theory. This is exemplified in the work of Piaget's immediate colleagues (Inhelder, Sinclair & Bovet, 1975). Another approach appears in the effort to coordinate aspects of Piaget's research (either the experiments or the theory) with information processing theory. Klahr, for example, incorporates Piagetian experimental data into an information processing formulation (Klahr & Wallace, 1976), whereas Pascual-Leone (1976) weds information processing and Piagetian theory into a single conception of cognitive development and processing. An attempt at integrating hypothesis theory, developmental theory and information processing theory appears in the theoretical proposals of Gholson and Beilin (1978).

Theoretical accounts of mathematical reasoning, particularly in respect to geometry, show the same dichotomy as the more general formulations of cognitive functioning. The dominant theoretical model to date derives from Piaget's stage-theoretic conception of geometrical reasoning and the child's developing conception of space. Piaget's views have influenced a considerable number of mathematics education and mathematics researchers. Evidence of this influence is found to be in most contemporary accounts of research in geometry, and in other aspects of mathematical reasoning as well (e.g., Martin, 1976; Lesh, 1978). The appeal of Piaget derives, at least in part, from his adoption of mathematical theory, particularly the Klein Erlanger program and the Bourbaki group formulations, as models or descriptions of cognitive functioning (see Martin, in Osborne, 1976). Secondly, he reduces conceptions of space to specific mathematical (i.e., geometric) structures. Third, his

emphasis on the development of invariance structures from transformation data has particular interest to geometers (see Weinzweig for a description and critique of this view, in Lesh 1978).

Piaget's structuralist theory is not the only stage theory that interests mathematicians and mathematics educators. Recent attention has been drawn to the work of Russian educators and researchers who have based their recent curriculum developments in geometry on the work of the Dutch educators, van Hiele and van Hiele-Geldof. The van Hiele theory is based on levels of thought development in geometry with emphasis on the effects of learning defined by the content and methods of instruction (see Wirszup, in Martin, 1976, for a description and analysis of this position).

Although the alternative class of theories, the process theories, is widely represented in general cognitive theory, it is more limited in its application to mathematical reasoning, in particular to geometry, although this state of affairs is rapidly changing (see in particular, Greeno, 1977).

Process-oriented theories or models share a number of properties despite their differences. Collectively they add a description of the processes involved in problem solution and reasoning that is generally neglected in structuralist accounts of cognitive development. From the orientation of the structuralists, however, these accounts neglect the constraints on learning provided by the logico-mathematical or other rule structures that are constructed in development.

The general thesis guiding our present research is that stage-developmental theories presently available yield limited information concerning the processes basic to given types of mathematical problem-solving and knowledge acquisition. Thus, instruction designed in accord with the properties of structural developmental theories alone are not likely to ensure the acquisition of cognitive capacities necessary for mathematical problem solving. At the same

time, however, developing such capacities independently of cognitively organized operations appears unlikely as well.

More specifically the research is designed to test whether solution strategies which include algorithms and heuristics, are generated "spontaneously" by children in dealing with geometry tasks, prior to their acquiring such strategies through formal (school) instruction. Strategy, as we use the term, refers to a consistent approach to concept attainment or problem solution, it includes both algorithms and heuristics. Algorithms are procedures with a finite number of steps. They are (a) constrained by specificity, i.e., all actions of the user are unambiguously determined by instructions (rules), (b) they are applicable only to a set of problems belonging to a particular class, and (c) an algorithm is directed to a sought-after goal, and if properly applied is always achieved. Heuristics, on the other hand, are solution procedures with an indeterminate number of steps, the application of which do not guarantee a solution. Their applications tend to be more novel and flexible and are generally associated with more creative aspects of problem solving.

A test of the integration of stage and functional (strategy) theory was sought in an assessment of the young child's knowledge of geometric motions (flips, slides and rotations) in the context of four types of plane geometry tasks, anticipation tasks that ostensibly required transformational imagery, equivalence of figures tasks constructed so as to elicit measurement strategies, matching parts tasks that assessed the child's ability to establish correspondences between parts of figures, and a combination of motions task that tapped the child's ability to combine knowledge of individual geometric transformations. The subjects were between the ages of 4 and 11 years and the data presented here are based on videotapes of 16 subjects across this age span.

A variety of consistently used strategies manifested themselves in the childrens' responses to the aforementioned tasks. This was particularly evident in those tasks involving the measurement of congruent and non congruent geometric figures (triangles) that differed by a transformation (slide, flip or rotation).

Examples of these strategies are:

Visual Matching in which eye scans and head turning are used to compare two or more geometric figures.

(Tactual) edge Matching (of one or two sides) - with (a) slides and rotations but no flips of the figures, and (2) the same with flips.

Progressive edge Matching in which each side of the figure (e.g., triangle) is successively matched before an equivalence judgment is made.

Superposing (with rotation and/or flip).

Each of the foregoing strategies may be said to represent a specific algorithmic procedure for attaining a measurement solution. What we observe in addition are heuristics based on varying combinations of these algorithmitically-based procedures.

One pattern (heuristic) associated with successful congruence measurement in the 6-year-old is Approximation (or estimation) Matching followed by Fine tuning, in which Visual matches lead into Progressive edge Matching and/or Superposing.

Unsuccessful measurement, for example, in a child of the same age (6 yrs.), shows Approximation matches based on Visual matching alone or followed by Edge matching with one side of the figure.

These selected observations are in accord with our proposed thesis that successful mathematical problems solving and reasoning is associated with solution strategies that are deployed differently from those associated with unsuccessful performance, and the lack of presumed cognitive structures.

REFERENCES

Gholson, B. & Beilin, H. A developmental model of human learning. In H. W. Reese & L. P. Lipsett (Eds.) Advances in child development and behavior. Vol. 13. New York: Academic Press, 1978.

Greeno, J. G. Natures of problem solving abilities. In W. K. Estes (Ed.) Handbook of learning and cognitive processes. Hillsdale, N. J.: Erlbaum, 1977.

Inhelder, B., Sinclair, H. & Bovet, M. Learning and the development of cognition. Cambridge: Harvard University Press, 1974.

Klahr, D. & Wallace, J. G. Cognitive development: An information processing view. Hillsdale, N. J.: Erlbaum, 1976.

Lesh, P. A. (Ed.) Recent research concerning the development of spatial and geometric concepts. Columbus, Ohio: ERIC/SMEAC, 1978.

Martin, J. L. (Ed.) Space and geometry: Papers from a research workshop. Columbus, Ohio: ERIC/SMEAC, 1976.

Osborne, A. R. (Ed.) Models for learning mathematics: Papers from a research workshop. Columbus, Ohio: ERIC/SMEAC, 1976.

Pascual-Leone, J. Metasubjective problems of constructive cognition: Forms of knowing and their psychological mechanism. Canadian Psychological Review, 1976, 17, 100-125.

SOME STEPS IN THE UNDERSTANDING AND THE USE OF SCALES AND AXIS BY 10-13 YEAR-OLD STUDENTS

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This study describes protocols produced by 10-13 year-old students faced with the task of setting data on an axis. Over 50 different sorts of protocols were obtained. A classification of these protocols enables us to analyze different operations of thinking that are necessary to meet the task. Representing a data by a point on an axis involves a "detour" through the representation of a magnitude by a segment, which raises several subsequent problems among which the inclusion principle and the mapping of a segment onto a point. A hierarchical model of steps summarizes the findings.

Before I give details about our work on the understanding of scales, I need to say a few words about the theoretical framework and the methodology. This is essential to a psychological approach of didactics.

A conceptual field is a field of preconcepts and concepts connected with one another, the understanding of which by children develops over a very long period of time, through different classes of situations. By this I mean that:

1. it is misleading to study separately the acquisition of concepts which are interconnected.
2. it usually takes many years for children to acquire these concepts.
3. concept formation usually takes place by facing new situations and solving new problems.

Let me give the example of "multiplicative structures".

- It would be absurd to make separate studies for the concepts of multiplication, division, proportion, fraction, ratio, rational number, linear function, dimensional analysis, etc. which are all associated with one another.
- The acquisition process of preconcepts and concepts in this field covers a wide range of ages, from 8 to 16 or 17 at least (and even from 3 or 4 for some preconcepts).
- Experiments with problem-solving situations involving the comparison of ratios and the use of different properties of linearity, are essential for all sorts of studies: didactical , evaluative and analytical.

Hierarchies (partially ordered scales) of complexity happen to be important aims and findings of research on the formation of a conceptual field. But different kinds of hierarchies should be distinguished:

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. hierarchy between classes of problems: the solution requires the use of different relationships and of different properties of these relationships.

. hierarchy between procedures (or strategies) : the same class of problems can generally be solved by different procedures using different properties of the relationships involved.

. hierarchy between symbolic representations: the solution may require the use of a symbolic representation of the problem. Different representations exist, that symbolize different aspects of the relationships.

Such an analysis of complexity has been made (and is still being made by our group) to study the acquisition of two main conceptual fields: additive structures on the one hand, multiplicative structures on the other hand (see references)

This paper is devoted to an incidental question concerning symbolic representations, the use of scales and axis: more precisely the task of setting and reading numerical data on an axis. This task is not usually isolated but rather a part of more complex situations. The analysis reported here should lead to other experiments on this question which is essential for mathematical education (and for education).

EXPERIMENT

Having been struck by the difficulties met by 10-13-year-old students in using an interval scale to set dates of birth on a slip of paper representing the flow of time (although ordering dates is an easy task for them), we have organized, during the last school year, an experiment scheduled in the following way. The purpose of this schedule will appear later in this paper.

1. PRETEST

Each task consists of setting 7 data on a line drawn on a slip of paper (50 centimeters long, 10 centimeters wide). Four different sorts of data are used in four different tasks:

I. weights of babies; ranging from 0,800 (premature) to 4,400 kg; no data between 2 and 3; scale 1 centimeter to 100 grammes.

II. ages of young children; ranging from 8 months to 4 years 4 months; no data between 2 and 3; scale 1 centimeter to 1 month.

III. dates of birth; ranging from 15-7-67 to 31-5-70; no date in 1969; scale 1 centimeter to 1 month.

IV. performances in javelin throwing, ranging from 67,75 meters to 70,60 meters; no data between 69 and 70; scale 1 centimeter to 10 centimeters.

In order to use the decimal notation which is suitable for weights and lengths but not for ages and dates of birth, the data were written with different units, for instance: 3 kg and 450 grammes, 2 years and 7 months, 13 may 1970, 67 meters and 90 centimeters.

In addition to the distinction between time (ages, dates) and other magnitudes (weights, lengths) there is an important difference between tasks I and II on the one hand, and tasks III and IV on the other hand. In tasks I and II, the scale is compatible with the setting of zero and all data on the available space. In tasks III and IV, either there is no natural zero (dates of birth), or the scale is not compatible with the setting of zero and the data on the slip of paper (javelin throwing performances).

Each student has been faced with two tasks, on two different weeks: either I and III, or II and IV.

2. DIDACTIC SEQUENCE

Several tasks and lessons are planned during five weeks (one session a week) in order to help students to overtake the main difficulties observed at the pretest. This part will be explained verbally during the IGPME session.

3. POST-TEST

Identical to the pretest; each student is faced with the same two tasks as the ones he had at the pretest on two different weeks. The post-test takes place about 5 months after the pretest.

Most of the research-work has been devoted to the analysis and classification of the students' protocols at the pretest. To characterize these protocols we have used different criteria, referring to the different operations of thinking needed to meet the tasks successfully:

- ordering
- representing magnitudes as segments of a line
- considering the same segment as the common part of different segments (principle of inclusion)
- mapping a segment to one of its extremities
- and others.

This approach may look a bit strange to those who have not studied this task. We were also surprised by our findings when we first examined the pretest protocols. The 150 students (aged from 10 to 13 years: last grade of primary school, two first grades of secondary school) produced as many as fifty different kinds

For instance some students do not make any division of the line and others do. But among those who do so there are at least three different attitudes:

- either the division is used as a referenced graduation with regular marks: every month or every 100 grammes is marked, or at least every year or every kilogramme.
- either the division is not referenced (no regular mark) and is only used for counting.
- or even the division is not used at all in the subsequent actions of the student.

Among those who write numbers along the graduation, one finds a few children that read twice the same graduation, once for kilogrammes and once for hundreds of grammes, for instance; and a few children also use two independant graduations for two different systems of unities (protocols will be presented during the session).

A very frequent category of protocols consists of different segments representing different data "end-to-end". For instance three successive weights A, B and C are represented on the same line by three segments.



This can be done either when there is no division of the line, or when the division is used only as a counting tool. One can easily understand why a referenced graduation, as defined above, is uncompatible with an "end-to-end" procedure.

This procedure entails different sorts of interesting things, mainly because the line is not long enough to represent all the data "end-to-end" with the given scale:

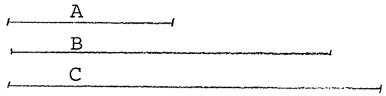
- the child may then continue on a new line, and then another line, and so on.
- he may also change the scale.
- he may also use two different scales: the one given for hundreds of grammes or for months, and another one for kilogrammes or for years (1 centimeter for 1 kilogramme for instance).
- he may also omit the first part of the data and keep only the "decimal" part of them (cutting off data).

This end-to-end procedure is very interesting because it is typical of children who have difficulties with the inclusion principle.

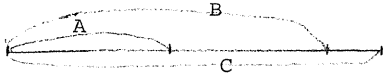
If data A is smaller than data B, the segment between O and A should represent both data A and a part of data B.

$$A < B \implies OA \subset OB$$

This is so difficult for children that some of them draw different lines for the different data, although they do succeed in starting from the same origin.



Other students set the different data on the same line, but find it necessary to explain that each data is represented by a segment starting from O.



In all these cases, children do not usually use a referenced graduation but only a division for counting or no division at all. This can be done because the students have a graduated rule at their disposal.

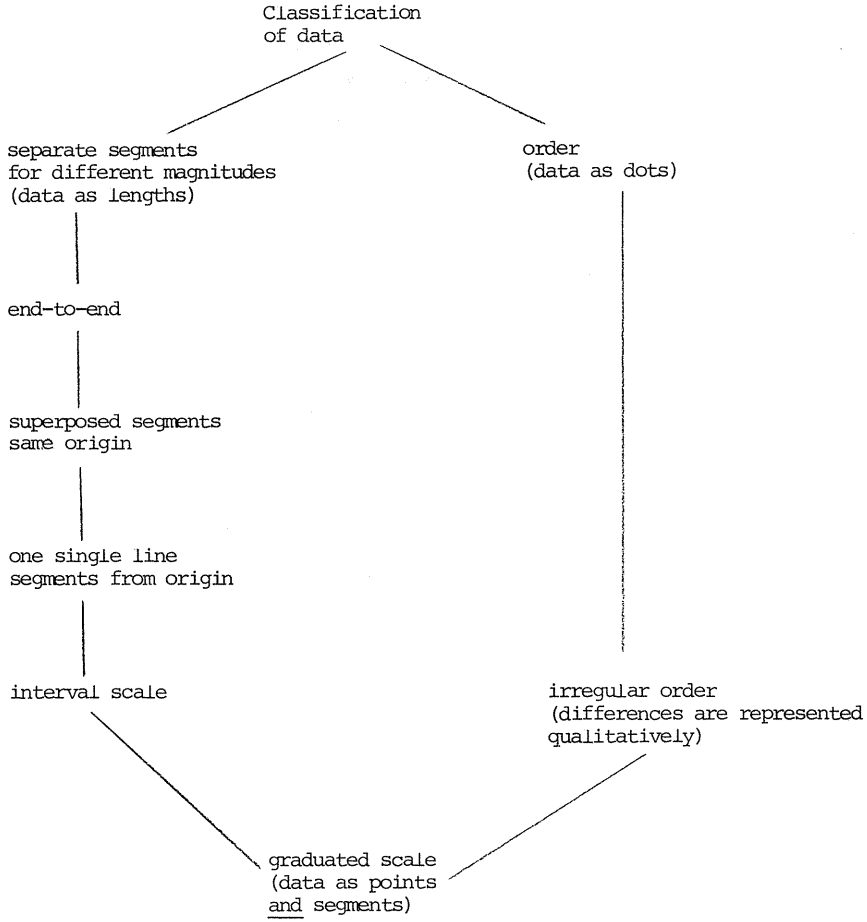
It is easy to imagine the difficulties met by these children when no natural origin exists or when it is not possible to represent it on the slip of paper. Then comes a new distinction between children who are able to use differences on an interval scale, and those who are not able to do so, and feel it necessary to count from the origin.

Of course, children who use a referenced graduation and read it in a univoque fashion, do not meet these problems. But how have they reached that referenced graduation? Is it just a natural way of setting data regularly, each data being represented by a dot, or does it require a new operation of thinking, namely the mapping of the segment onto its final extremity; this mapping then consists of identifying a particular point with the interval between the origin and this point.

One does find protocols that represent a mere ordering of the data; and ordering does not require more than thinking of data as dots. Is it plausible that this "dot-representation" of data is transferred directly into the dot-representation of a regularly-referenced graduation?

The answer given by this experiment is NO. And moreover it shows how difficult it is for students to reach the concept of a referenced graduation, especially in the case where the origin cannot be represented on the line: each data must then be interpreted as the extremity of a segment, a part of which is outside the slip of paper; moreover the different segments are included in one another, outside the slip.

The hierarchy can be summarized in the following diagram :



REFERENCES

VERGNAUD, G., ROUCHIER, A., RICCO, G., MARTHE, P., METREGISTE, R., GIACOBBE, J. Acquisition des structures multiplicatives dans le premier cycle du second degré. Publication de l'IREM d'Orléans et du Centre d'Etude des Processus Cognitifs et du Langage. IREM d'Orléans: Domaine Universitaire - 45045 Orléans Cedex.

VERGNAUD, G., BENHADJ, J., DUSSOUET, A. et al. La coordination de l'enseignement des mathématiques entre le Cours Moyen 2ème année et la classe de sixième. Recherches Pédagogiques, n°102, INRP 29 rue d'Ulm, 75005 Paris.

VERGNAUD, G. The Acquisition of Arithmetical Concepts. Educational Studies in Mathematics, 10, 1979, 263-274.

VERGNAUD, G. A Classification of cognitive Tasks and Operations of Thought involved in Addition and Subtraction Problems. To appear in: "The Initial Learning of Addition and Subtraction Skills", Seminar held in Wingspread, 26-29 November 1979.

A LEARNING HIERARCHY FOR RATIOS AND FRACTIONS

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A certain discrepancy was obtained when three experiments bearing on proportional reasoning — two involving concrete material, one with numerical fractions — were compared to addition of fractions, with similar items. The first three experiments presented a quasi-normal distribution, while the last one was an all-or-nothing curve. The first one was interpreted as evidence of a constructive process of development different from the last one. Various aspects related to the development of fractions and ratios have been treated, and a unifying structure for the nature of rationals and operations on the rationals is presented.

THE GAP BETWEEN SCHOOL LEARNING AND COGNITIVE DEVELOPMENT

A comparison of pupils performance at the Secondary level at four different tasks involving ratios and fractions and also numerical symbols, (Noelting and Gagné, 1980) shows that there is no direct correspondance between possessing algorithms bearing on rational numbers and problem-solving competence in the same field.

This is illustrated in Figure 1, where two tasks involving proportional reasoning — the Orange Juice Experiment bearing on ratio and the Sharing Cookies Experiment bearing on division and fractions — are compared to two tasks bearing on numerical symbols — Comparison of Fractions and Addition of Fractions. All four tasks consisted of twelve items involving identical numbers. They were submitted to six school levels: Elementary 6 and Secondary 1 to 5.

The distribution of subjects in % among the stages for the Addition of Fractions Experiment will be found. Here we have an all-or-nothing distribution. Practically all subjects (94%) are at Stage IIIB, with a sprinkling at other stages. These subjects are all capable of adding fractions when problems are presented in the familiar context.

The situation begins to change when the problem is presented as a comparison in magnitude of two fractions. The curve for the Comparison of Fractions Experiment will also be found. The percentage of subjects who succeed at all problems and are therefore at stage IIIB, now drops to 60%. The rest of the distribution is quasi-normal, with a peak at stage $IIIA_{int}$ with 20% of subjects.

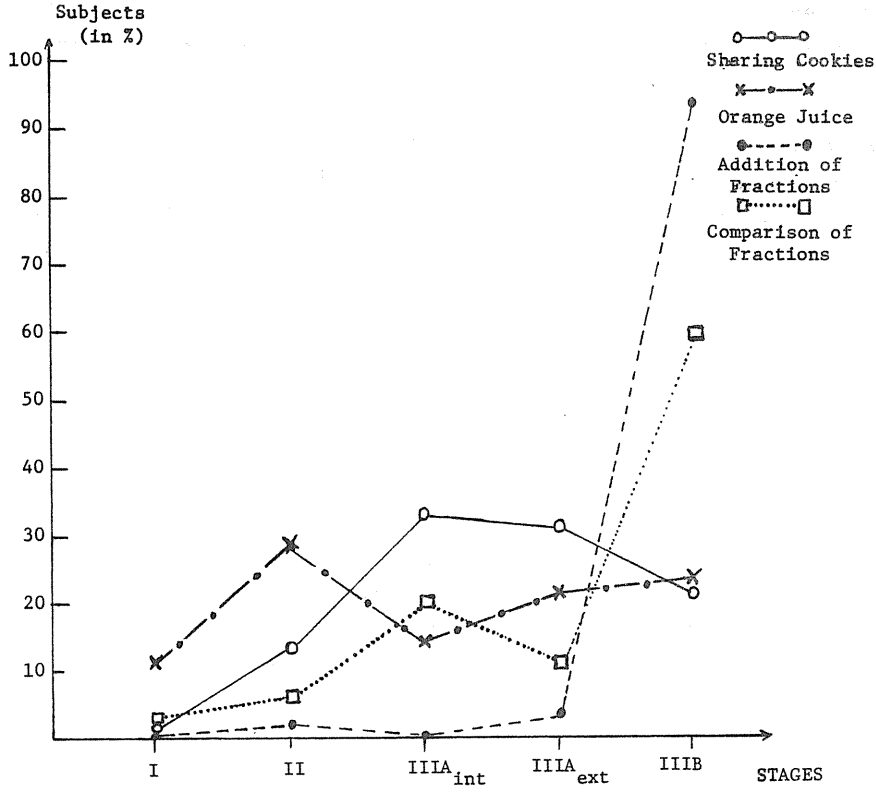


Figure 1: Distribution of subjects among the stages for Sharing Cookies, Orange Juice, Addition of Fractions and Comparison of Fractions Experiments.

Let us now consider the corresponding curves for the two problem-solving experiments. Instead of an all-or-nothing situation note the quasi-normal curve for the distribution of subjects among stages for the Sharing Cookies Experiment, with a peak at 33% for stage $IIIA_{int}$. The Orange Juice Experiment is similar. It shows a dip for $IIIA_{int}$ as it bears on ratio, which requires an external strategy. Otherwise, the curve is quasi-normal too, with a peak at stage II (A + B), with 29% of subjects. These two curves show that some homogeneous process of hierarchical construction is at play in these two problem-solving situations where a subject has to rely on his own strategies for answering. Fewer subjects are found at end stages, and more at middle stages. The distribution is quasi-normal. No transfer effect of schooling is found in problems

which seem to be the exact transcript in concrete situations, of the symbolic numerical problems given. Algorithms are forgotten and subjects cope on their own.

These results seem to show that cognitive mechanisms involved in the construction of algorithms are largely ignored at school. Teaching seems to be a process introduced from without and simply applied to adolescent mind, without any interest as to how it works or what are its needs. Knowledge acquired in this fashion simply breaks down when any real problem arises.

This paper will touch on some points which have come out from our recent research on rational numbers.

THE HIERARCHICAL DEVELOPMENT OF THE CONCEPTS OF RATIO AND FRACTION

A concept such as ratio or fraction is constructed through a series of qualitative changes which imply a change in *structure*. Structure is defined as the set of relations between the four components of two ratios or fractions. When they are composed, they can be compared in magnitude or operated upon (e.g. added). Complexity in structure makes a rational number problem easier or more difficult. These qualitative changes determine stages.

It has been shown that each level is a particular part of the following. Thus development occurs through a process of differentiation of simple strategy through reversal or complementation, to give two sub-strategies which are integrated in the more complex structure which follows. The processes involved have been described for ratio as a process of adaptative restructuration of a dialectical or interactive nature (Noelting, 1980).

THE TWO LEVELS OF COGNITIVE ACTIVITY

Each stage is defined in algebraic terms, and the particular strategy applied to solve problems of the stage are expressed algebraically.

However, each problem or item is expressed arithmetically as a particular embodiment of the general algebraic structure. Thus the strategy applied at stage IB (unit fractions):

$$a = c,\ b > d, \text{ thus } (c,d) > (a,b)$$

corresponds to items: (1,5) vs. (1,2), (1,3) vs. (1,1), (1,5) vs. (1,3), etc.

Cognitive activity takes place at two levels: i) the operative level of the subject, who examines data, organizes it, takes the necessary steps to solve the problem; this is a general type of activity and is described algebraically; ii) the factual level of the data, which are the particular facts implied in the actual problem, which are here transcribed arithmetically. Cognitive activity is always an interplay between the general and the particular. Strategies and schemes are general, data are always particular. No possibility of generalization would exist if this distinction was not the case at the outset. Cognitive activity inside a stage is made up of the constant interplay between generalization and particularization. Particular data is processed by a general strategy. Various strategies can be chosen in front of a problem. The suitable strategy must first be found. This first step is a qualitative assessment of the situation and takes time. This first moment thus implies assessment and choice. Then the strategy must be applied with care. This second moment requires precision. These two levels of processing data: qualitative (qualis = which) and quantitative (quantum = how much) are always present. School training has for too long induced students, into rushing, into quantification, without any evaluation of the qualitative nature of the problem involved. A clear distinction about the two levels of cognitive activity can remedy this situation.

PROPER AND IMPROPER FRACTIONS

At all levels, problems presenting proper or improper fractions were found. These problems were of same difficulty when their structure was the same. Working out improper fractions led to curious situations. The expression 3/1 left some adolescent puzzled. Others treated it as 3 halves. In school, there is a tendency to immediately transform an improper fraction into a whole plus a fraction. This is unnecessary.

With the introduction of ratios, the coexistence of proper and improper fractions will further be generalized.

THE TWO TYPES OF RELATIONS INTERVENING IN RATIONALS

At all levels of rational number construction, two types of relations were found in rationals: relations binding terms "within" the ratio of fraction; relations "between" corresponding terms of two ratios or fractions.

Freudenthal (1978) had already insisted on that duality, although he had chosen the set of terms of the same kind as criterion, and had given this distinction as follows:

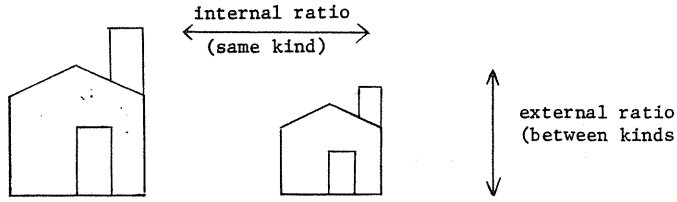


Figure 2: The two types of ratio involved in proportion.

Our distinction uses as criterion the relation itself, and we call "internal" or "within", the relation between terms "within" the mathematical entity (called ratio or fraction), and "external" or "between",the relation "between" two equivalent expressions of this entity, as illustrated in Figure 3.

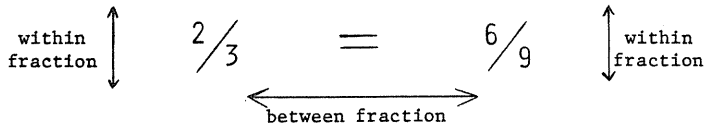


Figure 3: The two types of relations in an equivalence class of a ratio or a fraction.

Corresponding strategies are termed "internal" when they involve a division and "external" when they involve a co-multiplication.

Thus item 7 (3/1 vs. 5/2) is solved by an internal strategy, 5/2 being transformed into $2\frac{1}{2}/1$, giving 3/1 vs. $2\frac{1}{2}/1$, with comparison of numerators. This "unit-factor" method has been stressed by Heron and Wheatly (1978).

Item 8 (2/3 vs. 7/9) is solved by an external or co-multiplicative strategy, 2/3 being transformed into its equivalent 6/9, giving 6/9 vs. 7/9, with comparison of numerators. We have shown elsewhere that the first type of strategy leads to the percentage algorithm, the second to the Common Denominator algorithm. We had called the first type of item $IIIA_{int}$, the second type $IIIA_{ext}$,

and data had shown that they formed distinct stages in specific instances. The problem has to be investigated further.

The parallelism of the distinction made by Freudenthal and ourselves, (with opposite terms based on set or relation) goes to showing that this distinction is fondamental in the theory of rationals.

LOGICAL AND ALGEBRAIC OPERATIONS IN RATIONALS

The distinction just made can explain a difference between two types of operations performed on rationals: i) Composition, quantitative in nature: $6/9 + 7/9 = 13/9$, ii) Equivalence, qualitative in nature: $2/3 \times (3/3) = 6/9$. Transformations involved in equivalence should be considered as operations just as are arithmetic operations. We have termed them co-multiplications. An other qualitative or logical transformation is the extraction of units: $13/9 = 1\ 4/9$.

EQUIVALENCES OF UNIT FRACTIONS AND ANY FRACTIONS

When fractions are taught, the concept of unit fractions is introduced, before passing to any fractions. The same sharp distinction should be maintained when equivalences of fractions or ratios are introduced. The student should be made to play at length with the one-many equivalence (stage IIA). Concrete material can help: 1 person/3 cookies, 2 persons/6 cookies, etc.; 1 glass of orange juice/2 glasses of water, 2 glasses of orange juice/4 glasses of water, etc.

Later, the many-many relation, characteristic of stage IIB, can be introduced: 2/3, 4/6, 6/9, showing the difference between the internal relation "within" each ratio and the external relation leading to equivalence.

PREPARING THE COMMON DENOMINATOR AND PERCENTAGE ALGORITHMS AT STAGE IIIA

Algorithms are just a condensed rule for problem-solving strategies. They must first be constructed by the pre-adolescent before a cryptic mathematical expression is set down for further abstract symbolic treatments. This construction proceeds step-wise. The comparison in strategies used to solve problems of IIIA and IIIB levels will illustrate the point. We will first consider external "strategies" leading to the common denominator algorithm through co-multiplication.

TABLE 1

Preparation of the common denominator algorithm (IIIB level) by single combination of equivalence and addition (IIIA level)

COMPONENTS	STAGE	
	IIIA	IIIB
Problem	$\frac{2}{3} + \frac{7}{9} = \square$	$\frac{3}{5} + \frac{5}{8} = \square$
1st algebraic[a] operation: LCM of denominators	$3 \vee 9 = 9$	$5 \vee 8 = 40$
1st logical[b] operation: equivalence of first fraction	$\frac{2}{3} \times \frac{(3)}{(3)} = \frac{6}{9}$	$\frac{3}{5} \times \frac{(8)}{(8)} = \frac{24}{40}$
2nd logical operation: equivalence of second fraction	———	$\frac{5}{8} \times \frac{(5)}{(5)} = \frac{25}{40}$
2nd algebraic operation: addition of numerators	$\frac{6}{9} + \frac{7}{9} = \frac{13}{9}$	$\frac{24}{40} + \frac{25}{40} = \frac{49}{40}$

Notes – [a] We shall call algebraic those operations which compose two quantities to give another quantity of the same system.

[b] We shall call logical those operations which change a quantity into an equivalent.

In Table 1, the successive steps for solving item 8, of IIIA level, and item 11, of IIIB level, are given. Logical operations, where a quantity is changed into its equivalent, are carefully distinguished from algebraic operations, where two quantities are composed, to give another quantity of the same system. It will be easily seen how IIIA items are simple instances of IIIB algorithms. These IIIA items should be worked out at length and generalized to IIIB items, and finally axiomatized. In Table 2, the same is shown for strategies making up the unit factor method and leading to the algorithm of percentage (internal strategies based on double division).

TABLE 2

Preparation of the percentage algorithm (IIIB level)
by the unit-factor method (IIIA level)

COMPONENTS	STAGE	
	IIIA	IIIB
Problem	$\frac{6}{3}$ vs. $\frac{7}{4}$	$\frac{3}{5}$ vs. $\frac{5}{8}$
1st algebraic[a] operation: HCF of denominator	$3 \wedge 4 = 1$	$5 \wedge 8 = 1$
1st logical[b] operation: equivalence of first fraction	$\frac{6}{3} = \frac{2}{1} = 2$	$\frac{3}{5} = .6$ (= 60%)
2nd logical operation: equivalence of second fraction	$\frac{7}{4} = 1\frac{3}{4}$	$\frac{5}{8} = .625 = (62.5\%)$
2nd algebraic operation: difference	$2 = 1\frac{3}{4} + \frac{1}{4}$	$.625 = .6 + .025$

Notes - [a, b] See Table 1.

A UNIFYING STRUCTURE FOR THE NATURE OF RATIONALS AND OPERATIONS ON RATIONALS

These developmental studies lead to the interpretation of rationals and operations on rationals within the same framework.

Let us consider the interpretation of algebraic operations in terms of set theory. Addition is considered a union of disjoint sets (Fig. 4).

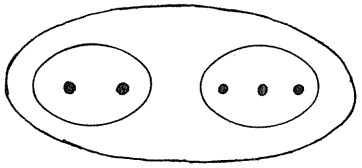


Figure 4 - Additions as union of disjoint sets.

Multiplication is considered a mapping of a set on another set or Cartesian product (Figure 5).

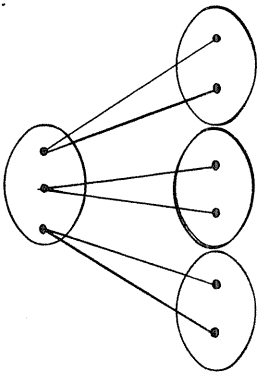


Figure 5 - Multiplication as a mapping of a set on another.

The same distinction was made when defining an equivalence class of rationals, where the binding force between terms defining the rational number was considered a relation, that between corresponding terms of two equivalent rationals was considered a set binding terms of the same "kind".

This leads to the following matrix where algebraic operations on rationals (multiplications and additions) are related to rational-internal and rational-external relations in the equivalence class of a rational. (See Table 3).

THE DEVELOPMENTAL HIERARCHY IN MATHEMATICAL STRUCTURES

A final table (Table 4) will give the hierarchy which developmental studies like this one leads to and which could be a guideline for constructive mathematics.

TABLE 3

The two types of operations: logical and algebraic, explaining the nature of rational number and the operations that can be performed upon them.

	DICHOTOMY II	
DICHOTOMY I	LOGICAL	ALGEBRAIC
RELATION SET	within rational number between rational number	multiplication addition

TABLE 4

Developmental hierarchies of mathematical structures

Period		Typical items	Mathematical structures and strategies
I Preoperational	1	2/5 vs. 4/5	Natural numbers and their multiplicative inverses
	2	1/2 vs. 1/3	Equality and order between /E
II	3	1/2 vs. 3/9	Algebra of sets and relations
	4	2/4 vs. 3/6	Equivalence of one-many relations
Concrete Operation	5	3/5 vs. 6/10	Equivalence of many-many relations
	6	4/6 vs. 6/9	(also Additive and Multiplicative Groups)
IIIA	7	3/1 vs. 5/2	Unit- Factors -- One-many relations
Early Formal	9	6/2 vs. 8/3	
Operation	10	6/3 vs. 7/9	
	8	2/3 vs. 7/9	Combination of conjunction and disjunction — Many-many relations
IIIB	11	3/5 vs. 5/8	Boulean algebra
Late Formal Operation	12	8/5 vs. 5/3	Common denominator and percentage algorithm

Note that multiplicative inverses appear contemporaneously with additive inverses. We should call $\mathbb{E}$, or Egyptian numbers, the set of natural numbers and their multiplicative inverses, i.e. natural numbers and unit fractions. The natural hierarchy, both historical and developmental, in number construction, seems to be:

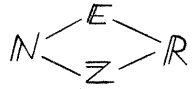


Figure 6 - The construction of rationals from natural numbers.

We should add that Algebra of sets and Algebra of relations correspond to so-called "Concrete operations" in developmental psychology and are contemporary with the additive and multiplicative groups. A transition is made up of IIIA level items, where one finds non equivalence based on unit factors, and the first combination of conjunction and disjunction in the addition of fractions with denominators multiple of one another.

At the final IIIB level, one finds the Algebra of propositions and Boulean algebra which are termed "Formal Operations" in developmental psychology.

A natural hierarchy has here been shown to exist between these mathematical structures.

REFERENCES

NOELTING, G., and GAGNE, L., (1980), *The development of proportional reasoning in four different contexts* (this volume).

HERRON, J.D. and WHEATLEY, G.H., *A Unit Factor Method for solving Proportion Problems*. The Mathematics Teacher. 1978, 71, 1, 18-21.

FREUDENTHAL, H., Weeding and Sowing, *Preface to a science of mathematical education*. Dordrecht, Holland. Boston: U.S.A., D. Reidel Publishing Company (Eds.), 1978.

NOELTING, G., (1980a) The development of proportional reasoning and the Ratio concept. *PART I: Determination of Stages*. Educational Studies in Mathematics, Vol. II, pp. 217-253.

PART II: Problem-Structure at Successive Stages, Problem-Solving strategies and the Mechanism of Adaptive Restructuring. Educational Studies in Mathematics. (in press)

THE HIERARCHIES IN MATHEMATICS UNDERSTANDING FOUND IN THE CSMS (MATHEMATICS) RESEARCH

K. Hart, Chelsea College, London University

The CSMS research team investigated secondary school children's understanding of eleven mathematical topics by presenting 10,000 of them with word problems in a pencil and paper format and also by interviewing 300 children using the same problems. A matching across topics was made by assigning items of approximately the same facility to the same "stage", most of the sample had completed two test papers and γ coefficients test/test were available. Four "stages" were imposed on the data, each described in terms of the mathematical demand and the methods used by children in the interviews.

The mathematics team of the project "Concepts in Secondary Mathematics and Science" based at Chelsea College carried out research into secondary school children's understanding of mathematics from 1974-1979. The team investigated eleven topics which appear in the British secondary school mathematics curriculum. Word problems were written which embodied key ideas of a particular topic, required a minimum of computational skills and which were (as far as possible) free of technical terms. An important phase of the research was the interviewing of about 30 children on the problems for each topic. From these interviews the methods the children used to solve the problems were found, particularly the incorrect methods which led to errors. After a number of revisions and trials the problems were written in the form of paper and pencil tests and given to a representative sample(on IQ) of the English(secondary)child population. About 10,000 children aged 11-16 were tested, the sample for each year group for each test being taken from six or more schools (both rural and urban) and approximating to the normal distribution of IQ scores.

After the test papers were marked, the items on each were formed into a hierarchy based on the following criteria:

1) Items were said to be in a group if of approximately the same facility and possessing an item/item homogeneity coefficient ($\emptyset$) above a stated value.

2) Children were judged as succeeding at a 'level' if they correctly solved two thirds of the items in that group

3) The groups were scalable in that children who succeeded at one

level had also succeeded on all easier groups of problems. The error rate (children succeeding on harder groups of items without success on all easier groups) was kept to a minimum - less than seven per cent. The scalability was tested by Guttman Scalogram Analysis.

Most children in the sample of 10,000 had taken two test papers so it was possible to quantify the connection between success on one topic and another by calculating the γ coefficient. The values of γ ranged from .85 to .59. An exact match between levels in any two topics was not possible in that the number of levels varied from topic to topic and for any one level in a particular topic hierarchy there was not necessarily a level in another of comparable facility. To provide some guidance to teachers the totality of hierarchies was divided into four 'stages' (note that stages refer to sub-divisions of a crossmatching of levels of hierarchies). The content of levels or items in each stage was the same for each year group although the position of the line drawn to delineate each stage was at a different facility since the items themselves were solved by different percentages of children in each age group, thus:-

Table 1. Limits of Four Stages (Hardest Items)

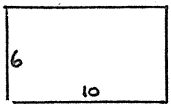
	13 yrs	14 yrs	15 yrs	
Stage 1	70	70	80	per cent
Stage 2	48	50	58	(facility)
Stage 3	18	22	30	
Stage 4	2	5	8	

The most productive way of describing the contents of the four stages seemed to be in terms of both mathematical content and the methods used by children to solve them (found from the interviews). It is acknowledged that some children solve problems using sophisticated methods but the most naive successful methods are used in the descriptions since if all children had available the same strategies leading to success then there could no longer be items classed as 'difficult'. The research did not test the efficiency of any particular teaching programme and the topic being tested may have been studied some considerable time before the test date. The methods the children used in the interviews were essentially what remained when the teaching had finished and what they felt content to apply to the problems posed them.

Stage One items to a large extent involved whole numbers in a familiar

context such as:-

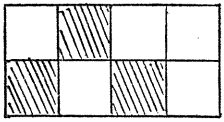
What is the area of this shape ?



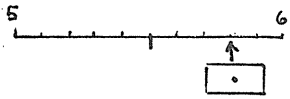
Write down any number between 4100 and 4200.

The conventions or symbols of fractions and decimals also occurred:-

What fraction is shaded?



This number is



When fractions had to be used (as distinct from being recognised) whole number rules could be applied i.e. the fractions could be regarded as a number of pieces of the same size and the problem was to count the total number of pieces, such as in 3/8 + 2/8 in problem form. It was noticeable in the interviews that when one might suppose subtraction was the relevant operation, the children tended to 'count on'. Thus asked if a man paid 3/5 of his wages in tax, what fraction was left, they might say "5/5 altogether, 3/5, 4/5, 5/5" and not set out the computation 1 - 3/5. The only equivalences appearing in this stage were those which involved doubling or a half e.g. 5/10 = ?/30 when instead of a comparison across the equal sign the child would say "5 is half of 10, what added to itself gives me 30". Similarly in the ratio problems, halving or doubling could be attempted successfully.

In Graphs, Stage 1 questions involved counting on block graphs or counting across squares to plot points with whole number coordinates. The use of counting was also evident on the Measurement items when in many cases the counting of square tiles or half tiles would suffice for finding the area of a figure. In the questions dealing with the reflection of a line, grids were usually provided so the subsequent finding of a mirror image was very close to finding the coordinates of a point.

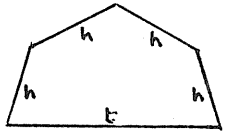
Success on Stage one items could be achieved by using:1) whole number concepts 2) the first conventions for labelling fractions (or decimal fractions) 3) counting. Stage Two items built on these aspects. The

The child succeeding at Stage two could give the decimal label 1.43 for the area of a square and parts of a square and write 2.74 for a point on a scale between 2.7 and 2.8.

It was an indication that he was confident within the numeration system although the items did not require him to do anything with the decimals. In Fractions there were more equivalences e.g. 4/12 = 1/?; these could be solved by a process of addition and did not necessarily require the use of a multiplicative method. Equivalences were also used but when either a diagram was provided or the fraction was a familiar one.

A significant appearance at Stage 2 was the use of the formula for the area of a rectangle (probably the first formula British children meet). The formula had to be used twice, or the area was given and one of the dimensions had to be found. At no point however was the child multiplying fractions or decimals to find the area; this type of problem was considerably harder.

The conventions of using letters and the language of generalised arithmetic first appeared at Stage two. The child did not need to appreciate that a letter represented a whole set of numbers, for stage two items he needed to use only one replacement value. He must have been aware of the conventions that 'a' belongs with 'a' and 'b' with 'b' however, e.g.

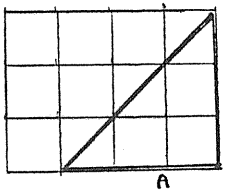


perimeter equals =

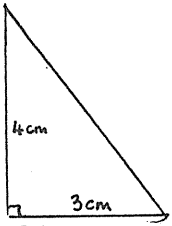
In graphs although all relevant information was given in time/distance graphs the child who succeeded was able to connect the story with the graph, interpret the questions (which were still part of the story) with their symbolic representation on the graph and interpret the graph to answer those questions.

In Measurement the basic ideas of volume occurred. The questions dealt with the conservation of volume when the object was divided into pieces; that volume could be decomposed and reassembled (as when building with cubes) and that the volume of a cuboid could be obtained by counting layers.

The dimensions of the cuboids were all whole numbers and it was likely that at the end of Stage 2 the volume formula was used. The area of a triangle could be found even though tiling was not shown, compare the facilities of these two questions:-



Area of A =



Area of A =

	Age 12	13	14	Age 12	13	14	
Success	77.5	86.9	91.4	31.4	38.7	47.5	per cent

In Geometry the mirror line was no longer parallel to the edge of the page (or coincident with the lines on a grid) and the centre of rotation, given initial and final positions of a flag, had to be recognised as such.

All these aspects of stage two items point to the child coming to terms with the conventions of mathematics. He was still not however using the operations of multiplication and division on other than whole numbers and even there he could survive by using some form of repeated addition.

Stage three contained two levels of the Ratio hierarchy, there was a large gap between being able to double and halve and the continued application of doubling and halving as for example in an enlargement 5:2. Since very few children on interview solved items of this type by multiplying by a fraction this was discounted as a common strategy on stage three items. Fraction paper items which required the multiplication of two fractions were commonly in stage four. Stage three problems involved a combination or comparison of fractions with different denominators and the avoidance of the view that two fractions were four whole numbers, pairs of which could be combined or formed a number pattern. For example a common error (13 per cent) in finding the replacement for Δ in $\frac{2}{7} = \frac{\square}{14} = \frac{10}{\Delta}$ (stage three) was to find a number pattern in the denominators and so give Δ =21.

Until this stage the child could rely on whole number models to interpret operations. It was commonly found that when children were asked to supply a story to illustrate whole number division they wrote about sharing sweets between friends (see Brown and Küchemann, 1976). It is a considerable

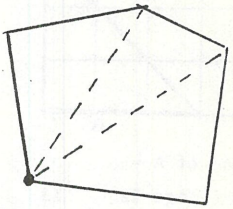
step to '60 ÷ 0.3' which occurred in Stage three.

A common feature of Algebra, Geometry and Vectors items in stage three was the absence of complete information in pictorial form. So that for example a Stage three Algebra item was:-

In a shape like this you can work out the number of diagonals by taking away 3 from the number of sides.

So, a shape with 5 sides has 2 diagonals;

a shape with k sides has ..... diagonals.



A stage three reflection item showed the lines of reflection on a grid but the mirror image (or the original point) came off the page. The child could no longer count to find a solution as he did on Stage one items.

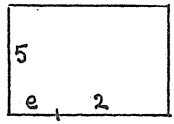
To succeed at stage four items the child had to be able to work with fractions and decimals and see their use. Two stage four decimal items were '24 divided by 20' and '16 divided by 20', to solve these the model of division based solely on sharing objects was no longer sufficient. In the first case, the 'four left over' caused problems; it was often written as .4, in the second case many children (44 per cent of 12 year olds) stated 'it is impossible'. The fractions questions required multiplication or division e.g.

A man is driving in France. He knows that 1Km is the same length as $\frac{5}{8}$ mile. His hotel is $\frac{2}{3}$ Km from the petrol station. What is this distance in miles? .....

In Graphs the outstanding features were the use of variables (x, y) to represent not just one pair of numbers but a whole range and to connect points on a grid with an algebraic expression such as y=2x+3. Similarly on the Algebra paper a single replacement as in "u=v+3, v=1, u=?" (Stage two) was no longer adequate, "which is the larger 2n or n+2? Explain" (Stage 4) required the child to try a range of values of n. The common error here was to choose 2n because it involved multiplication and if the child was still fixed within the set of whole numbers he naturally said 'multiplication always makes bigger' just as he believed 'division always makes smaller'. The syntax of Algebra had to be firmly established;

although using the area formula with whole numbers was a stage one item, the following was a stage four question:-

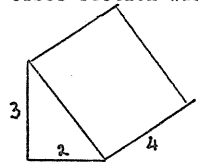
What is the area?



The problem was not what 'had to be done' but how to do it, when the 'e' and '2' had to be combined and we did not know the value of 'e'.

The Ratio questions required a ratio to be found and applied, it was no longer possible to use halves and doubling to build up to an answer (e.g. enlargement in the ratio 5:3 was a stage four item). Since the children interviewed overwhelmingly used correct additive strategies in the solution of ratio problems it was not surprising that on the harder questions they opted for an incorrect addition strategy in which they used a - b rather than a/b (see Karplus 1975, Piaget 1967).

Stage four Measurement items were mainly concerned with volume in which the figure was not cuboid or the dimensions were fractions. It was no longer possible to count layers or if one did, the cross section was a triangle so a stage two idea had to be applied, as in:-



Questions which required the child to generalise (e.g. a,b,d are positive whole numbers, a/b is less than a/d when ------) were in stage four. Questions which required the child to appreciate that there were an infinity of numbers were also in stage four:- How many different numbers could you write down which lie between 0.41 and 0.42?

The difference between the stages seems likely to be in terms of methods used as well as the complexity of the numbers involved.

References:

M.Brown and D.E. Küchemann."Is it an 'add' Miss?" Mathematics in School, 5,5, and following issue (part 2), 1976.

K.M. Hart. Secondary School Children's Understanding of Mathematics (Research Monograph) Chelsea College, 1980.

R. Karplus, E. Karplus, M.Formisano & A.C.Paulsen. Proportional Reasoning and Control of Variables in Seven Countries. Advancing Education Through Science Oriented Programs, Report ID-65, June 1975.

STUDIES BY THE MATHEMATICS WORK GROUP OF THE UNIVERSITY OF WISCONSIN R & D CENTER

This introduction presents a brief background to the following papers by Moser, Carpenter, and Romberg & Collis. The current research program at Wisconsin is investigating the development of addition and subtraction concepts and skills in young children. The major vehicle for this investigation is a three-year longitudinal study of children from first to third-grades. Begun in September 1978, data collection is scheduled for completion in May 1981. Data presented in the papers are taken from the longitudinal study or from satellite studies that use the same data gathering procedures.

In these studies a number of variables are under investigation, including types of verbal problems, the problem solving behaviors exhibited by the subjects, selected cognitive skills, performance on written arithmetic tasks, and the nature of teacher and pupil actions observed in the classes during instruction. Details are contained in the various publications from the Mathematics Work Group (e.g., Carpenter & Moser, 1979). In the papers that follow only children's performance on the problem solving tasks and the strategies they use are examined.

The problem solving interviews involve six basic types of verbal problems: 1) Joining (addition) 2) Separating (subtraction) 3) Part-part-whole, missing addend (subtraction) 4) Part-part-whole (addition) 5) Comparison (subtraction) 6) Joining, missing addend (subtraction). This classification of problem types is based on semantic characteristics of the problems (Carpenter & Moser, 1979). Overall, results indicate this semantic model is an excellent predictor of the strategies children use to solve given problems. For example, young children tend to solve the following Separating problem by constructing a set of 13 objects and removing 8 of them: Eric had 13 stamps. He gave 8 of them to Gary. How many stamps did Eric have left? At a more advanced stage children might count backwards from 13 rather than modeling with objects. However, for the following Joining, missing addend problem, involving the same numbers, children usually solve it by constructing a set of 8 objects and finding how many more would be required to bring the set to 13 or by counting from 8 to 13 and keep track of the number or counts. Eric has 8 stamps. How many more stamps does he need to put with them to have 13 stamps altogether?

At each separate administration of these six tasks, nouns and verbs within the problem statement are systematically changed. Within each problem, two of three numbers from the number triple $x + y = z$ $(x < y < z)$ are given (smaller number

problems are $5 \leq z \leq 9$, and larger number problems are $11 \leq z \leq 15$). Within each interview four levels of problems are given. The levels differ by use of smaller numbers vs. larger numbers and whether counters are present or absent. In the longitudinal study the interviews will be conducted nine times so that a picture of childrens' growth can be determined.

Pupil responses to each problem are coded for correct response, model, strategy, and error. The problem solving strategies have been divided into four categories. However, the specific strategies used within a category vary with problem types. The categories are: Inappropriate. These are responses, which if followed produce an incorrect answer. Included are use of wrong operation, repeating one of the given numbers of the problem, guessing, or no answer at all. Also included are uncodable responses and missing data from terminated interviews. Direct modeling. Here sets of objects such as cubes or fingers are used as direct representations of problem entities. Actions are then performed on the model sets. Included are counting all (addition), separating (subtraction), adding on (subtraction) and matching (subtraction). Use of counting sequences. A counting sequence is entered at some point other than "one" where the entry point corresponds to one of the given numbers in the problem. Depending upon the child's interpretation of the action or relationship presented in the problem, counting will proceed forward or backward. Four kinds of strategies are identified; Counting on, Counting up, Counting down from, and Counting down to. Use of mental operations. Children appear to translate the problem directly into abstract, mental mathematical representations. No visible counting or operations on physical models can be observed. Included here are recall of known basic facts and heuristics (a derivation of some fact from some other known fact).

Moser's paper examines the incidence of use of strategies that can be characterized as inventive - that is, relatively advanced as well as not likely to have been formally taught.

Carpenter's paper focuses specifically on heuristic strategies.

Romberg and Collis relate performance on the problem solving tasks to the children's cognitive abilities.

These are only a few of the interesting ideas and findings from the rich data set being gathered by the staff at the Wisconsin R & D Center on the development of addition and subtraction concepts and skills.

WHAT IS THE EVIDENCE THAT CHILDREN INVENT PROBLEM SOLVING STRATEGIES

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Results taken from the longitudinal study being conducted at the University of Wisconsin give strong indication of inventive behavior on the part of young children. A large incidence of counting on strategies for addition problems by beginning first-graders confirms results of earlier work by Groen and Resnick. Use of a counting up strategy for subtraction problems whose semantic structure ordinarily induces a subtractive strategy is also taken as evidence for invention. There is also a significant incidence of use of more sophisticated counting-sequence and mental operation strategies by children in a problem situation where they are kept from using their ordinarily preferred physical modeling strategies.

I would like to talk about the inventive behavior of young children. But, to talk about invention, we should first decide what is meant by the term. What I do not mean is invention in the purest sense which is to create something brand new for the very first time, as say Edison and the light bulb. Hadamard's (1945) idea of mathematical invention as being synonomous with discovery is a good approximation since he characterizes discovery as dealing with phenomena or laws that already exist. Yet, I'm not totally happy with "discovery" either because what I am thinking of is something different than Columbus discovering America. A closer description is Resnick's (1978) who calls invention a process where people acquire new mathematical knowledge by constructing for themselves new organizations of concepts and new procedures for performing mathematical operations. So, in essence it comes to youngsters figuring things out for themselves as opposed to simple application of formally taught facts and skills. Still, I choose to not adhere too closely to Resnick's description since I think she means to imply that children exhibit inventive behavior only where no instruction has taken place. I do not dispute this possibility, but the behavior I want to talk about may well have resulted from children learning something from others. And furthermore, I want to extend the definition a bit to look at highly provocative problem solving behavior which involves the use of perhaps-taught strategies in novel situations.

THE RESEARCH EVIDENCE

Addition Problems. The first set of data deals with problem situations similar to those of Groen and Resnick (1977), the results of which led to her formulation of invention. In their study, ten preschool children were taught how to solve simple addition problems of the form $m + n$ where each of the addends was between

1 and 5. The children were instructed to model each of the addends with blocks, combine the sets and then count the combined set. Invention was said to have occurred if a child eventually solved the problem mentally by counting on from the larger of the two addends. If the problem involved 2 + 5, the child would count, "*five*, six, seven." This is the so-called *min* method. The theoretical notion is that the child sets a counter in his or her mind at the *max* (m,n) and then increments that counter *min* (m,n) times. This method is the most efficient counting model for simple addition problems, where efficiency is defined as incrementing the least number of times. Half of their subjects were said to have exhibited inventive behavior.

Our evidence, based upon individual interviews of a much larger sample, corroborates these findings. We shall look at behavior of beginning first-graders in October by when no specific instruction in any addition or subtraction skills was given, and in January by which time the prescribed instruction dealt with simple measurement skills, counting, writing numerals, some geometry, and solving very rudimentary problem situations at the physical and pictorial level. The discussion is restricted to the two addition problems of the interview. One is the Joining problem (e.g., Wally had 3 pennies. His father gave him 6 more pennies. How many pennies did Wally have altogether?) and the other is the Part-Part-Whole problem (e.g., Anne has 4 green apples. She also has 7 red apples. How many apples does Anne have altogether?)

I want to consider three different strategies. The first is what we call Counting On From Larger (CL) which is identical to the *min* method, except that we allow fingers or objects as aids when keeping track of how many times the counter is incremented. The second strategy is Counting On From Smaller (CS) which can be described as the *max* method since the counter is set at *min* (m,n) and then incremented the *max* (m,n). While not as efficient as the *min* method, the *max* method of counting on is nevertheless more efficient than counting all and does represent a significant mental accomplishment on the part of the young child (cf. Fuson, in press). In all addition problems used in our interview, the smaller number is always read first in the problem. The third strategy is what we call the Heuristic strategy (cf. Carpenter, 1980) and involves a conscious mental manipulation of some already known mathematical information. Given that the children are in early first grade, all three of these strategies do represent, as Resnick characterizes, a construction of new organizations of concepts or procedures; that is to say, inventive behavior.

At the time of each interview, six basic problem types (2 addition and 4 subtraction) were presented under a maximum of four different conditions involving the crossing of presence or absence of manipulatives with smaller (between 6 and 9, incl.) or larger (between 11 and 16, incl.) sums. At the time of the October interview, 38 (out of a possible 144) different children used the CL strategy at least once, with some also using the CS strategy. Another 22 children used only the CS strategy. Altogether the CL or CS strategy was exhibited 148 times. By the second interview in January a total of 81 different youngsters, well over half the sample, used either the CL or the CS strategy. These two strategies appeared on 243 occasions. The heuristic strategy is perhaps an even more striking instance of inventive behavior. As Carpenter (1980) points out, there is a richer variety of mental operations possible. At the first interview, 25 different children used a heuristic, while by the time of the second interview that number had climbed to 44. I should point out that not all instances occurred during the addition problems. Some were observed during the subtraction problem situations, to which I would now like to turn my attention.

Subtraction Problems. While in no way intending to denigrate the behavior of youngsters on addition problems, we should keep in mind that the addition situation is a relatively unambiguous one. The task set by Groen and Resnick (1977) is clear; the results of our own work (Carpenter, Hiebert & Moser, in press) also show that the two addition problems discussed above pose no serious difficulty for first-graders. As a counter point, it is instructive to look at performance on subtraction problems. In fact, I want to concentrate on the two hardest ones, the Part-Part-Whole, missing addend (e.g., There are 13 children on the playground. Eight are boys and the rest are girls. How many girls are on the playground?) and the Comparison (e.g., Glen won 4 prizes at the fair. His sister Connie won 7 prizes. How many more prizes did Connie win than Glen?).

First we draw a parallel to the addition discussion given earlier. What sort of out-of-the-ordinary behavior was shown during the first and second interviews? To answer this question, consider the Part-Part-Whole, missing addend problem and the problem that always directly preceded it in the interview, the Separating problem which is the classic "take-away" type that appears in most American textbooks as the primary and initial example of subtraction. Both problems have the numbers presented in the order (sum, larger addend). The semantic content of the Separating problem strongly influences the choice of strategy - a subtractive one. Most subjects either use direct modeling by objects or fingers to first construct the larger set and then take away the required amount so as to be able to count

the remainder set, or they use a counting analogue, which we call the Counting Down From (DF) strategy which amounts to setting the internal counter at the larger number and then decrementing the required number of times, where reading the number on the counter gives the answer. While DF is more sophisticated than direct modeling, it still does not appear to represent very inventive behavior for this problem. Because of its setting in the interview situation, the Part-Part-Whole, missing addend problem tended to induce similar behavior.

What, then, might be an inventive behavior? Consider the counting on procedure, which we call Counting Up from Given (UG). Set the counter at the smaller number [the given larger addend] and then increment the counter until it reads the larger number [the sum]. Not only does this involve an easier process -- incrementing is easier than decrementing -- but it is also more efficient. The number of increments is less than the number of decrements in the DF strategy. It also represents a behavior similar to our CL strategy for addition in that a conscious decision is made to start operating not on the first number read in the problem, but the second. At the time of the first interview in October, 19 different children used the UG strategy for these two problems. By January, this number had increased to 27.

For the remaining portion of this paper, I want to look over a period of a year and a half. The longitudinal approach to investigating children's development permits a helicopterian view from which we can see the whole picture, not just of a single interview or of a single problem type. This view also permits consideration of possible conscious creative problem solving which I would like to take as another example of inventive behavior. An immediate impression from the view on high is that many children exhibit erratic patterns of attacking the six problem types, thereby allowing no real conclusions to be drawn. But, looking at the more "stable" subjects, some interesting results do emerge.

If a child goes along for a noticeable period of time using a predominant strategy (with some occasional variations permitted) which closely maps the semantic structure of a problem or his/her perception of that structure, but then somewhat dramatically changes to a different strategy, then I submit that invention or discovery of the applicability of that different strategy has taken place. This is particularly so if the newly applied strategy is more efficient than the earlier one. In looking at the results, I find that this sort of thing occurred on 33 separate occasions for the Part-Part-Whole, missing addend problem and on 32 separate occasions for the Comparison problem.

Finally, let's take a look at another problem situation that comes up within the interview situation. Four conditions are used in this order: smaller sums with manipulatives, smaller sums without manipulatives, larger sums with manipulatives, and the last, larger sums without manipulatives. In the first three conditions, direct modeling is always possible, either with the manipulatives, or in the case of the smaller sums, with fingers. When numbers get larger than 10, finger modeling gets difficult. The problem situation is this: What does the child who has been modeling for the first three conditions do when this strategy is essentially taken away from him/her? This question is especially salient for the Comparison problem solver who has been using a Matching (construction of two sets, putting them into one-to-one correspondence, and counting the excess) strategy. A child who then switches to a higher level counting sequence strategy or a heuristic is taken to have exhibited inventive behavior. I do not claim that the new strategy has been invented on the spot. It may, in fact, have been taught or learned elsewhere. But the fact that it was spontaneously used in a true problem situation is considered to be an example of what this paper is about. For the Part-Part-Whole, missing addend problem, this type of behavior occurred in 27 different instances and for the Comparison problem, in 24 different instances.

In short, the evidence from the Wisconsin studies suggests very strongly that children are inventive. Why are they inventive? Well, the old adage says, "Necessity is the mother of invention." Perhaps, if we as educators want to encourage inventive behavior, we should see to it that children are confronted with realistic and challenging problem situations.

References

Carpenter, T.P. Heuristic strategies used to solve addition and subtraction problems. Paper presented at the fourth annual meeting of the International Group for the Psychology of Mathematics Education, Berkeley, Calfironia, 1980.

Carpenter, T.P., Hiebert, J. & Moser, J.M. First-grade children's initial solution processes for simple addition and subtraction problems. Journal for Research in Mathematics Education, in press.

Fuson, K.C. An analysis of the counting-on solution procedure in addition. In T.A. Romberg, T.P. Carpenter and J.M. Moser (Eds.), Addition and subtraction: A developmental perspective, in press.

Groen, G.J. & Resnick, L.B. Can preschool children invent addition algorithms? Journal of Educational Psychology, 1977, 69, 645-652.

Hadamard, J. The psychology of invention in the mathematical field. Princeton, NJ: Princeton University Press, 1945.

Resnick, L.B. The role of invention in the development of mathematical competence. Paper presented at the Workshop on Children's Mathematical Learning, Learning Research and Development Center, Pittsburgh, PA, 1978.

HEURISTIC STRATEGIES USED TO SOLVE ADDITION AND SUBTRACTION PROBLEMS

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Heuristic strategies in which children use known addition and subtraction combinations to generate solutions to unknown combinations are described. Results of several studies that document the development of Heuristic strategies are discussed, and the significance of these strategies is examined.

Most recent research on children's acquisition of addition and subtraction concepts, including our own, has focused on the development of a variety of counting strategies that children use to solve different problems. We have also identified, however, another class of strategies that are not based primarily on counting. For lack of a better term, we have labeled these strategies "Heuristic strategies."

We have observed a number of different heuristic strategies. Their common feature is that children use known facts to generate the solution to problems involving unknown facts. For the most part, heuristic strategies are based on known combinations whose sum is 10 or on doubles, but other combinations are also used. When appropriate, some children will use the result of a previous problem to generate the result of a subsequent problem.

CHARACTERIZATION OF HEURISTIC STRATEGIES

We have identified several distinct classes of heuristic strategies. One we have labeled "Decomposition strategies" because they involve breaking up one of the terms in the sum or difference. Heuristic strategies involving combinations whose sum is 10 are frequently of this type. For example, one student explained this solution to an addition problem involving the sum 4 + 7 as follows:

"Seven and 3 is 10, so I put 1 more on there and got 11."

In other words, the 4 was represented as 3 + 1 because the child knew that 7 + 3 = 10 and could use this fact to generate the answer to 4 + 7. Note that this particular solution involved both the commutative and associative properties of addition.

A Decomposition strategy can also be used with doubles. One student explained her solution to a problem involving the sum 6 + 8 as follows

"6 + 6 = 12 and 2 more is 14."

In this case, the 8 was represented as 6 + 2 because the child knew the double 6 + 6. Although both examples decomposed one of the addends, they seem to involve different ways of initially analyzing the problem and may represent distinct strategies. In the second example, the larger addend was represented as a sum with the smaller addend as one of the terms. The first example involves an additional step. Before the child could decompose the 4, he first had to figure out how to represent the 4. The different steps in each example may be represented schematically.

$7 + 4 \rightarrow 7 + \square = 10 \rightarrow 4 = 3 + 1 \rightarrow 7 + 3 = 10 \rightarrow 10 + 1 = 11$

$6 + 8 \rightarrow 8 = 6 + 2 \rightarrow 6 + 6 = 12 \rightarrow 12 + 2 = 14$

A second major class of heuristic strategies may be labeled "Compensation strategies" because a change in one of the terms of the problem is offset by a compensating change in one of the other terms. For example, one student explained her solution to a problem involving the sum 4 + 9 as follows:

"Ten add on 4 is 14 and 1 less than 14 is 13."

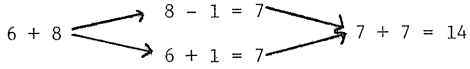
In other words, she used a fact in which one of the addends was one greater than the corresponding addend in the given problem. Consequently, she subtracted 1 from the answer the known fact yielded. This can be represented as follows:

$4 + 9 \rightarrow 9 = 10 - 1 \rightarrow 10 + 4 = 14 \rightarrow 14 - 1 = 13$

In this case, an increase of 1 in one of the addends is compensated for by decreasing the resultant sum by 1. An increase in one of the addends can also be compensated for by a decrease in the other addend. For example, one student explained the solution to a problem involving 6 + 8 as follows:

"I took one from the 8 and gave it to the 6. 7 + 7 = 14."

This can be represented as



Other variations of these basic strategies are possible. We have observed some that involved a combination of known facts and counting and some that involved known facts combined with some sort of physical representation of the problem with cubes or fingers. We have also observed a number of heuristic strategies involving facts other than doubles or 10 complements.

Heuristic strategies are more difficult to apply to subtraction problems than was the case for addition. A change in the subtrahend has the opposite effect of a corresponding change in the minuend. Therefore, one has to be much more careful to what kind of alteration has been made in the original problem. Nevertheless, a number of children used heuristic strategies to solve subtraction problems. In general, the strategies for subtraction fall into the same basic classes as the addition strategies. For example, to explain his solution for 9 - 6, one child reported the following decomposition strategy:

"Nine take away 4 is 5, take away 2 is 3."

9 - 6 ---> 6 = 4 + 2 ---> 9 - 4 = 5 ---> 5 - 2 = 3

Another child used the following compensation explanation for 11 - 3:

"Three from 10 is 7. It would have to be 1 more because it's 11."

11 - 3 ---> 11 = 10 + 1 ---> 10 - 3 = 7 ---> 7 + 1 = 8

Children also frequently explain their solution to subtraction problems in terms of related addition problems. To solve 14 - 8 one child reported:

"Seven and 7 is 14; 8 is 1 more than 7. So the answer is 6.

14 - 8 ---> 8 + □ = 14 ---> 7 + 7 = 14 ---> 7 + 1 = 8 ---> 7 - 1 = 6

Students are somewhat more prone to make logical errors in solving subtraction problems. For example, the following solution to 13 - 9 inappropriately compensated for a change in the subtrahend:

"Thirteen and 10 would give 3; so 13 and 9 give 2."

RESULTS

Selected results from the longitudinal study are summarized in Table 1. These results demonstrate that the use of Heuristic Strategies is not limited to a select group of superior students. By the end of first-grade, over half the students had used a heuristic strategy at least once; and by the middle of second-grade, over three-fourths of the students had done so. Children who used Heuristic Strategies, however, did not use them consistently. For the smaller numbers, only one child used a Heuristic strategy more than 3 times out of the 12 problems administered during any of the interviews. For the larger numbers, only one first-grade child used more than three Heuristic strategies. Four second-graders used more than three heruistic strategies at the September interview and 12 did so in January.

TABLE 1
SUMMARY OF FREQUENCY OF HEURISTIC STRATEGIES

		Grade 1			Grade 2	
		Sept.	Jan.	May	Sept.	Jan.
Number of children using a Heuristic	Sums less than 10	10	26	21	22	17
	Sums between 10 and 20	9	10	24	27	38
Number of Heuristics Used	Sums less than 10	19	43	29	36	28
	Sums between 10 and 20					
Cumulative* Total		15	39	59	70	79
Total N		94	101	102	99	96

*Number of children who used an Heuristic Strategy in this or any preceding interview.

In a related study that has just been completed, Glen Blume has found that Heuristic strategies are used as early as kindergarten. Twenty-two percent of the kindergarten children in his study used a Heuristic strategy at least once. He found, however, that there was a distinct shift in the types of strategies used. Over three-fourths of the kindergarten children in his study used a Compensation strategy, but over two-thirds of the first-graders used a decomposition strategy. Although we have not systematically classified Heuristic strategies in the longitudinal study, we have also boserved a distinct shift from Compensation to Decomposition strategies in older children.

DISCUSSION

It is tempting to attempt to fit Heuristic strategies into a stagelike model of the development of addition and subtraction skills. It seems logical that children would initially model addition and subtraction problems using physical objects. This stage would be followed by a stage in which children use more abstract counting strategies. Next would come heuristic strategies, which would ultimately lead to the recall of basic facts. Our data suggests, however, that things are not as simple as this. We have attempted to related the use of Heuristic strategies to the use of other strategies. So far, we have found no clear evidence of a relation between heuristic strategies and other strategies. Heuristic strategies are often found in combination with the most elementary counting and modeling strategies.

Heuristic strategies, however, potentially play a significant role in the development of addition and subtraction concepts and skills. Brownell (1928) strongly argues that students must be capable of the reasoning involved in Heuristic strategies in order to give meaning to memorized addition and subtraction combinations.

Understanding addition and subtraction requires that students have some basis for relating the magnitude of different combinations, that they know how different sums and differences can be composed and decomposed, that they understand the relation between addition and subtraction, and that they understand basic properties of addition like commutativity and associativity. These basic concepts are all involved in Heuristic strategies.

From an information processing perspective, one might argue that Heuristic strategies provide a logical basis for storing addition and subtraction combinations in long term memory. These strategies for generating addition and subtraction combinations are also more closely related to strategies that might be used to generate multiplication combinations, for which counting strategies are too cumbersome.

In order to determine whether we could increase children's use of Heuristic strategies, we should consider the results of a recent study of Japanese children by Giyoo Hatano. He found almost no incidence of the counting strategies that are so prominent in American research. Instead, Japanese children tended to rely on a Decomposition strategy. This study suggests that the strategies that children construct are related to instruction, and appropriate instruction could encourage the development of Heuristic strategies.

REFERENCES

Brownell, W.A. The development of children's number ideas in the primary grades. Supplementary Educational Monograph No. 35. Chicago: University of Chicago Press, 1928.

ıTIVE LEVEL AND PERFORMANCE ON ADDITION AND SUBTRACTION PROBLEMS

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An initial examination relating cognitive level and performance on the six types of addition and subtraction for young children suggests that the higher the cognitive level the better the performance. However, the increase is primarily due to reduction in use of inappropriate strategies and not in differential use of different appropriate strategies.

In this paper an intial look at problem solving performance data from a series of related, collaborative studies carried out by the authors in Tasmania in 1979 and 1980 is presented. We assumed that the evolution of children's performance on addition and subtraction tasks must be related to their cognitive abilities. The strategy adopted to examine this assumption involved several steps: First, we identified memory capacity (M-space) for a population of children of ages 4-8; (Romberg & Collis, 1980a). Four M-space tests were administered. Second, we identified cognitive processing capabilities for the same set of children (Romberg & Collis, 1980b). Fifteen different tests were given. From a factor analysis of those scores a quantitative factor, a qualitative correspondence factor and a logical reasoning factor were identified. Third, from that data we identified six groups of children with specific cognitive characteristics. A cluster analysis procedure was used to group the children. Group 1 children operate at M-space level 1, are capable of handling qualitative comparisons and transformations at a moderate level, and are incapable of dealing with quantitative tasks or logical reasoning. Group 2 children operate at M-space level 2, handle qualitative correspondence tasks and cannot handle quantitative and logical skills (but were considerably better than Group 1 on all tasks). Group 3 children also operate at M-space level 2, are high on qualitative correspondence, have developed the specific counting skills of counting-on and counting-back, are inadequate in their use of those counting skills on the transitive reasoning, and are inadequate on logical reasoning. Group 4 children operate at M-space level 3, are high on qualitative correspondence and all the quantitative tests, but are inadequate on the logical reasoning test. Groups 5 and 6 are at M-space levels 3 and 4. They reach the ceiling on the qualitative correspondence tests, have very high scores on all the quantitative tests, and also are high on logical reasoning. Because these later two groups were both small, only contained third graders and displayed no differences in cognitive processing scores, the data for these children have been combined.

Next, we measured (on three occasions over a three month period) the performance and the strategies used on the addition and subtraction problems for a sample of children at each grade level. The following data is from the first of those occasions. Table 1 presents a summary of the population's performance data (% correct) for the six groups of children with differing cognitive levels on the six item types. As one reads down each column the relationship between cognitive level and performance is obvious. With one trivial exception, the higher the cognitive level the better the performance on these tasks. There are important differences between item types. The addition tasks and separating

Table 1. Cognitive Level by Item Type; Percent Correct

Item Type (2 or 4 items taken by each subject)

	n	Joining(+)	Separating(-)	PPW,missing addend(-)	PPW(+)	Comparison(-)	Joining missing addend(-)	Total
Cognitive level 1	5	50	60	30	50	0	20	35
2	13	70	66	66	66	43	59	62
3	11	86	71	75	75	64	79	75
4	8	88	75	94	75	81	81	82
5,6	7	86	86	100	86	100	100	93

are easier than the other three types, but the patterns across cognitive levels remain the same. However, since this relationship could be confounded by a grade effect, the comparable data just for the second and third grade children was also examined. Children in Grade 2 were at cognitive levels 1, 2 and 3 while in Grade 3 they were at levels 2, 3, 4, 5, and 6. The data for these grades is not presented here. However, overall performance does not appear to be effected by grade and the patterns of percent correct across cognitive levels at each grade level are very similar to that for the total population.

Table 2 presents a summary of the strategy data for the six groups of children in the total population with differing cognitive levels. Again, relationship between cognitive level and use of strategy is striking. The inverse relationship between level and use of inappropriate strategies and the direct relationship for use of counting are apparent. Again, since these results could be confounded by grade effect, these were examined for both the second and third grade

Table 2. Cognitive Level by Strategy Category: Percent of Use

Cognitive level	n	Strategy Category			
		Inappropriate	Direct Modeling	Use of Counting	Mental Operation
1	5	53	37	6	5
2	13	24	45	15	16
3	11	14	30	29	28
4	8	8	22	36	33
5,6	7	0	27	49	24

children. The data is not presented here. However, the inverse relationship for use of inappropriate strategies appears to be a cognitive level effect because similar percentages of use appears at both grade levels. But, the choice of direct modeling vs. counting appears to be a grade effect. Second graders use direct modeling most often, and third graders use counting strategies irregardless of cognitive level. It also should be noted that there is no apparent grade effect on use of mental operations. This observation is surprising because of the increased emphasis in Grades 2 and 3 on remembering facts. It may be at the beginning of the school year (the time of the first interview) children who know facts fall back on counting or modeling skills as a check. Increased use of mental operations, particularly in third grade, should be apparent in the later interviews.

We also assumed that the comparison task demands a higher level of M-space than the separating task even though both can be solved by subtraction. Table 4 presents a summary of the strategies third grade children use for those two subtraction problems. The different problems clearly elicit different strategies by all the children. On the difference between children in cognitive levels 2 and 3 are clear on these problems (e.g., no level 2 child counted down on either problem type or matched on the comparison item). Other interesting data, such as level 3 children not using physical separation on the comparison tasks or using counting down more than counting up on the separating tasks appear to be ideosyncratic to the small population.

In summary, the relationships between childrens' cognitive level and their performance and use of strategies is certainly suggested in this data. The primary factor for increased performance is clearly related to a reduction in the use of

Table 4. Cognitive Level by Specific Strategies for the Separating and Comparison Problems for Third Grade Children: Percent Used

Cognitive Level	Inappropriate		Direct Modeling				Counting				Mental Operations	
			Separating		Match/Add on		Down		up			
	Sep	Comp	Sep	Comp	Sep	Comp	Sep	Comp	Sep	Comp	Sep	Comp
2	38	38	13	13	13	0	0	0	13	38	25	13
3	19	25	31	0	0	19	31	0	6	31	13	25
4	6	13	19	19	0	25	31	6	31	13	25	25
5,6	0	0	36	14	0	21	21	0	21	50	21	7

inappropriate strategies. However, the differential use of correct strategies seems in some cases to be a function of item type and in others is related to grade. Analysis of the remaining data, in particular the patterns of growth, should illuminate the relationships discussed here.

REFERENCES

Romberg, T.A. & Collis, K. The Assessment of Children's M-space. (Technical Report #540) Wisconsin Research and Development Center for Individualized Schooling, The Research Committee of the University of Wisconsin Graduate School, The University of Tasmania, Hobart, Tasmania, Australia, May 1980a.

Romberg, T.A. & Collis, K. The Assessment of Children's Cognitive Processing Capabilities. (Technical Report #539) Wisconsin Research and Development Center for Individualized Schooling, The Research Committee of the University of Wisconsin Graduate School, The University of Tasmania, Hobart, Tasmania, Australia, June 1980b.

STUDENT TEACHERS AND THE RELEVANCE OF EDUCATIONAL PSYCHOLOGY

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This paper presents a set of observations and questions arising from the training of mathematics teachers. Concept formation and concept development are discussed in the context of students' own learning and knowledge. Do we have theories or research findings which can help student teachers to overcome some of the persistent difficulties known to be shown by school pupils?

Those involved in training mathematics teachers for their practical classroom role often find that their students have great difficulty in perceiving (or constructing for themselves) the relevance to their teaching of theoretical studies in psychology. In the U.K. most teachers of mathematics for ages 11 and upwards take a mathematics (or mathematics related) degree and then add one year of professional training. There are advantages and disadvantages in this 'consecutive' form of preparation as against a 'concurrent' form. One disadvantage is that their courses in psychology are bound to be brief and introductory and are unlikely to be taught by those with experience of the complexities of the mathematics classroom. In particular, the unsatisfactory state of theories of learning leaves students with difficulties in applying their knowledge directly to their task. Such help as they get must come from their methods tutor, who is responsible for their preparation for the classroom and for their supervision in practice. He should provide a working model of the theories in application.

We have been warned (Bauersfeld, 1976) that it is too early to look for a general theory of the learning processes in mathematics. In any case, a single theory is likely to be too general to be very useful in view of the fact that mathematics includes a range of diverse activities. If theories can be regarded as ways of structuring our universe <u>for some purposes</u>, rival theories can be sustained without creating a clash and students can be encouraged to try contrasting methods with an open mind.

The source of this discussion was in classes in mathematical education, the aims of which are to encourage reflection on the nature of mathematics and on ways of "learning" it; and to reveal, discreetly, gaps in the students' knowledge and understanding. Gaps there certainly will be because students will have backgrounds in pure mathematics, applied mathematics, statistics and operational research, and engineering. The British tradition of linking school mathematics with applications, especially mechanics makes this inevitable.

There is frequent criticism that degree studies concentrate on executive ability and research-directed activities without finding time for reflection on the processes and concepts involved, but there are many students who would not find such reflection interesting. Perhaps countries with a stronger philosophical basis to their studies do not have this problem. A recent work (Murray, 1978) makes a new contribution in concept development in areas of applied mathematics. It will provide working material in these areas, of the kind suggested in this work.

The chain of activities was suggested by study Units on a Classroom Test, Concept Formation, Identifying and Analysing Mathematical Concepts and Schematic Analysis (Mathematics Teacher Education Project, 1980). The practical starting point is the Test. Students are given the scripts of four sixteen-year old pupils, who took a class test on equations and inequations ostensibly of a normal revision and feedback kind. The task is to construct a marking scheme that is 'fair' and yet penalises faulty steps. In fact both the 'test' and 'scripts' are constructed to reveal difficulties and expose unsatisfactory methods of solution. It should be said that the test would only be appropriate for exceptionally well prepared school pupils. The test is to solve the following:

1. $\frac{6}{x} = \frac{3}{5}$

2. $\frac{1}{x-2} = \frac{2}{x-2} + 3$

3. $\frac{x}{x-2} - 7 = \frac{2}{x-2}$

4. $(x+2)^2 + (x-1)^2 = (x+1)^2 + x^2 + 4$

5. $\frac{1}{2x-3} + \frac{x}{4x^2-9} = \frac{1}{8x+12}$

6. $\frac{2}{x} \leqslant \frac{4}{3}$

7. $\frac{3}{4x+1} \geqslant \frac{2}{x-5}$

These include statements which are true for all, some and no values of x; also some which can introduce extraneous roots if techniques are used carelessly; and others with tempting traps for beginners. We hope it is not due to unique deficiencies in the British education system but some students also stumble into the traps. Of course, they can recover easily but it is enough to start the discussion, especially illuminating for those so impeccably efficient that they make no mistakes and see no possibility for them.

Many educational endeavours seem to involve us in conflicting aims. Here the aim is to persuade students that elementary mathematics is not necessarily trivial mathematics. Another is to encourage them to review their elementary work in the light of their superior, professional mathematician's

make balanced judgements between pedantry and a confident but appropriate exposition for their pupils.

There are no close analogies for the process of comprehending and manipulating high order concepts but some writers e.g. (Skemp, 1979) suggest the helpful ones of learning to ride a bicycle or to swim. Those who can do these things find it very difficult to behave as though they could not. In this latter work Skemp draws together recent contributions on qualities of understanding. Students find it easier to consider their own understanding in the light of these views than the understanding of children, until they have built up reasonable close contact with their classes. They are, in effect, tracking their way down the orders of concepts rather than up. It drives them to consider more critically such of the standard texts as they may have encountered in their general courses. For example, translating this extract (Ausubel, 1978) into actual examples in their own mathematical experience provides real difficulties: "As a concept is acquired certain characteristic changes take place. It becomes increasingly less global, less impressionistic, and less diffuse; the learner focusses progressively on more salient critical attributes..... The identification of relevant criterial attributes similarly becomes more precise and refined; non-criterial attributes are sloughed off and new criterial attributes are added. Distinctions from related concepts tend to become sharper. ..... new contextual variants of the concept are acquired with the acquisition of greater sophistication in the same and related disciplines".

These are generalised descriptions of the process and while it may not be possible to recall all the stages in developing a particular concept or related group of concepts it is worth trying to identify the 'criterial attributes' and the 'culturally standardised consensus'. In fact, this blinding clarity is elusive and psychology looks to its origins in philosophy. Discussion of the tests usually starts with claims that questions 3 and 4 do not involve equations, which leads into discussions of what an equation is. Without prompting, the arguments often develop along the lines of the discussion on logical understanding in (Skemp, 1979). Taking 'equation' as a concept we begin to uncover the criteria required in the Ausubel extract above and to identify the 'experiences' or 'examples' which could lead to construction of the concept in the standard way.

There is the persistent difficulty that much of the students' own recent learning has been in a formal mode. They have reached the stage where they handle abstractions very freely and manipulate their symbolic language with ease. It is sometimes disturbing to probe deeper into their understanding

and it needs handling carefully. It can lead to a strong suggestion that in our search for conceptual clarity with children we are asking higher standards than we have maintained in our own learning. A parallel course to the one outlined here considers the historical development of mathematical ideas - not considering history from 5000 b.c. to the present but by selecting a few topics and concepts and tracing them backwards to see how they reached their present shape. Historical scholars tend to disapprove of this, regarding it as 'convergent history' which sees the present situation as inevitable. However, the main point of both these courses is to motivate students to further study by illustrating relevance to their teaching.

In the Units on Concept Analysis and Schematic Analysis (Preparing a Teaching Sequence) there are useful discussion papers (Heritage, 1975; 1976) which point out the practical usefulness of the process and outline some starting ideas. The unit on Concept Formation allows an opportunity to review the students' knowledge of the psychology of learning and gives an opportunity to begin study of more specialised Mathematical Contributors, such as Dienes.

The students concerned are introduced to active researchers working on practical classroom matters so that they can begin to relate their discussion to school and college problems. Bauersfeld's suggestions in the work quoted (1976) are for continuing detailed study over the range of interests, in the absence of general theories. This work is going on quite fast as the evidence of the Proceedings of this shows in the three years between these major conferences. For nearly every topic considered in the way suggested one begins to find immediately relevant contributions; for example on equations there are brief but highly pertinent research reports e.g. (Kieran, 1979).

The authors will consider some topics in open discussion and report on their relevant findings from work with pupils aged 10 - 20.

REFERENCES

Ausubel, D.P., Educational psychology; a cognitive view. Holt, Rinehart and Winston, 1978.

Heritage, R.S., What is a mathematical concept? Mathematical Education for Teaching. 2, 1, 1975.

Heritage, R.S., Schematic analysis and mathematics teaching. Mathematical Education for Teaching. 2, 3, 1976.

(Shortened Versions of the two articles, above are reprinted in Mathematics Teacher Education Project, 1980).

Mathematics Teacher Education Project. London: Blackie, 1980.

Kieran, C., Constructing meaning for first degree equations in one unknown. Proceedings of the Third International Conference for the Psychology of Mathematics Education. Coventry: 1979.

Murray, F.J., Applied mathematics; an intellectual orientation. N.Y; Plenum, 1978.

Skemp, R.R., Goals of learning and qualities of understanding. Proceedings of the Third International Conference for the Psychology of Mathematics Education. Coventry: 1979.

CRITICAL MOMENTS IN TEACHING MATHEMATICS

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This process-training study was designed to identify critical moments in teaching mathematics. Both internal evidence of cognitive processing and affective discomfort of the teacher and external evidence of student difficulty or insight, teacher behavior and characteristics of the instructional flow were used in describing the critical moments. There were four general categories according to the nature of student occlusions (insight or difficulty) and the pervasiveness of the occlusions (isolated incidents or situations).

Teaching style, instructional goals, nature of occlusions, attributions of student behavior, and teachers' knowledge of mathematics were significant determiners of variations across types and teachers.

What makes teaching difficult? What are the critical moments in teaching mathematics? With the resurgence of interest in cognitive processes brought about by the interest in information processing, researchers have endeavored to classify and quantify teacher thought, particularly as it relates to decision making (Peterson and Clark, 1978, Marland, 1977, Morine and Vallance, 1975). There has been some lack of fit between what teachers were found to do and the classical decision-making model in which deliberation over alternatives is a necessary ingredient. Instead there has been support for describing teaching as a reactive and retrieval-oriented process with much of it being handled on a less recallable level, a more automated process. With teacher thought as the prime target of investigation, however, such studies have given little attention to what or how teachers were teaching. Content and laboratory settings were often used only to stabilize the environment; there has been little done to integrate the thought processes with behavior in any systematic manner. Such is the purpose of this study.

Critical moments are those incidents involving student difficulty or insight and teacher elective action for which there is cognitive and affective evidence of teacher difficulty or discomfort. Critical moments provide a strategic research cite for examination and contrast across teachers and teaching situations. The purpose of this study is to identify critical moments and to provide a thick description of the overt behaviors and covert thoughts about student occlusions, elective actions, and the instructional flow in which they occurred.

Process tracing case studies were done with three teachers (Teachers X, Y, and Z) in grades 4-6, teaching units about rational numbers, fractions or decimals, to their own classes. The experience of the teachers ranged between 5 and 12 years. Each was chosen on the basis of reputation for being a good teacher and because of prior involvement with the USMES project, a natural selector of teachers with an interest in improving mathematics instruction and a willingness to be observed and questioned.

To provide opportunity for natural variation to occur during instruction, each teacher was videotaped and asked to participate in stimulated recall for the duration of each lesson for each of the six or seven days. Teachers were not asked to respond to a fixed list of questions at predetermined times, but to try and divulge all that they could about what they remembered thinking as they were teaching. During much of the instruction time there was little for the teachers to recall, and at other times it was necessary to stop the tape to have time to talk. The process of stimulated recall was in no way untainted, as there was evidence of reflective and retrospective interference, fatigue, and facilitator interference. Nevertheless, the teachers were quite helpful in indicating which thoughts and feelings had occurred during actual instruction, and they appeared to be quite open in divulging all that they could remember.

Classroom interaction was analyzed as to the nature and distribution of student occlusions and elective actions and other characteristics of the instructional flow. Student occlusions consisted of difficulties: patterned and nonpatterned errors, expressions of confusion, and teacher-signaled difficulties. Insights included pattern recognition, explanations, and alternatives. To be classified as insight a student suggestion or question needed only to have mathematical validity; there was no assessment made of student knowledge. Elective actions were categorized as exploiting, alleviating, or avoiding moves. Exploiting moves were directed to the entire class with strategies of display and seek or exemplify and try. Except for individual assists which were private, alleviating moves were public but directed to the individual. Techniques included redirecting, reasking, explaining, probing, telling, and exposing. Avoiding moves were usually directed to an individual and were handled by delaying, rejecting, ignoring, or simply acknowledging. Instructional flow was characterized according to external evidence of content, organization, style, instructional goals, and the internal evidence of thought. Teacher comments made during recall were matched with the incident of student occlusions and elective actions to determine what characterized critical moments.

As did instruction, critical moments showed variation across circumstances and across teachers. Critical moments arose over student difficulties and insights and different elective actions, with most actions requiring teachers to retrieve knowledge of mathematics or methods of teaching it. Some critical moments arose in connection with isolated incidents or episodes of student occlusions, while others were of a more general nature associated with a situation rather than specific occlusions. For this reason critical moments are reported in four categories listed in order, beginning with the most frequent: Type I, Isolated Student Difficulties; Type II, Isolated Student Insights; Type III, Situational Student Difficulty; and Type IV, Situations with No Student Occlusions. Ten critical moments were identified for Teacher X during 20 activities, four of Type I and 2 in each of the other categories. Teachers Y and Z each had 5 critical moments distributed over 13 and 8 activities, respectively, in all but the fourth category.

For Teacher X a Type I Critical Moment occurred when she found herself unable to adequately explain something to a student she perceived to be having difficulty in understanding. Each critical moment began with a student occlusion in which the nature of the difficulty or concern was readily apparent: three patterned errors and one question about a specific procedure. Teacher X's concern for the students was reflected in her choice of action, to offer assistance with the new content through an exploiting move or by probing and explaining. What produced the critical moments was that Teacher X provided more than one opportunity for the student to respond, and was confronted with recurring signals of difficulty which she attributed to a lack of understanding. Eventually, unable to resolve the student's misunderstanding, Teacher X would break off her efforts to help with delay tactics, encouraging the student to keep puzzling or promising to return to the idea. Her reasons for the break were a fear of confusing the student and an admission that she could think of no other alternative actions to try. Emotionally she felt frustrated and inadequate at not being able to help the students. It was what Peterson and Clark (1978) described as a Path 2 decision.

Teacher Z experienced essentially the same Type I Critical Moment, but with substantially less teacher-student interaction. Despite an immediate impulse to provide some explanation at the board, Teacher Z could think of nothing else except to tell the student what the answer should have been and hope that it was sufficient. Evidence that the misunderstanding still remained came from

Teacher Z's detection of a nonverbal signal; he did not provide the student with other opportunities to respond.

Teacher Y's Type I Critical Moment came from what the students could not do. Even after seven tries, she was unable to elicit the desired response to a planned question, but the recurring evidence of student difficulty came from many students rather than any one individual. Teacher Y abandoned that question and other similar questions for the duration of the lesson. Unlike the other teachers, Teacher Y did not attribute this to a lack of understanding on the part of the students, but to forgetting, interference of a prior response, and the inhibiting nature of the large group as opposed to a small one. She continued to believe some students were capable of answering the question. Teacher Y did admit to spinning her wheels when trying to come up with yet another way to approach the question, but abandoning the quest for the fraction did not bother her as she believed there would be other opportunities for students to be exposed to the idea.

Type II Critical Moments occurred in response to insightful student occlusions. Having been caught by surprise, the teachers found themselves confused in responding to the students or in thinking about how to respond. For Teacher X this amounted to a "goof" in the exploiting of a student suggestion, which produced a momentary but strong, emotional reaction. Again, what heightened her anger over the mistakes were a fear of confusing students and a lack of adequate alternative knowledge. Teacher Y, as with most critical moments, was less bothered by finding herself in much the same situation. None of the teachers reported being fearful of making mistakes in front of students. Teacher X even stressed that it made for better rapport with students, but this did not prevent her "gut" reactions from surfacing at the time she made mistakes.

Teacher Z found himself being asked to do a type of problem he had not noticed when preparing for the lesson. As he reported doing on other occasions as well, Teacher Z had looked only at the first few problems and had not noticed variations in the exercise. What produced the critical moment was that Teacher Z could not recognize the procedure being suggested by the textbook and could think of only one, unrelated method for performing the task. And, fearful of getting into something he did not already know how to do because of past experiences which led to greater confusion, Teacher Z avoided the problem and told the class that he was not prepared to do it.

The Type III situational critical moments for Teachers Y and Z were quite similar. They each became aware of student difficulty during an activity and were unable to figure out what was causing it. Each perceived the difficulty as being pervasive throughout much of the class. Teacher Y also spoke of prominent individuals viewed as being good students or as barometers for the performance of others in the class as having trouble with the task. After considering inattentiveness and too fast a pace as possible reasons for the continuing student difficulty, Teacher Y finally asked the students if they were confused. When told the Cuisenaire rods were causing the difficulty, she interpreted it to mean students could not remember the rod-fraction relationships and gave them a code which eliminated the need for a translation process. Teacher Y rarely attributed any student difficulty to a lack of understanding.

Teacher Z saw student inattentiveness and lack of volunteering as signals of his Type III Critical Moment. This was an extremely uncomfortable situation for him as he was unable to alter the situation or to even think of ways in which he might or what might be causing it. At the same time he was finding the lesson more complex to present than anticipated. Focusing more on his presentation he gave students fewer opportunities to respond and was extremely verbal, but Teacher Z saw little of the relationship between what was happening to him and how the students were responding.

Teacher X experienced a somewhat different Type III Critical Moment; essentially they were pacing dilemmas, as were her Type IV Critical Moments. Teacher X found herself confronted with a divergent response from the class during practice-oriented activities in which nothing new was being presented. Most of the class seemed to be successfully solving problems while a few were repeatedly experiencing difficulty. The routinized manner in which the activities were being taught and the amount of time it took to perform the task enabled her to move about the room monitoring student progress at the same time she was conducting class. With the Cuisenaire rods being used, difficulties were easy to spot, and she attempted to alleviate the difficulties with individual assists, some of which slowed and even stopped the instructional flow. It was in trying to meet the needs of both groups that Teacher X experienced tension and discomfort. She felt as though she was wasting the time of the successful students, and she was dissatisfied with the way she was trying to hurry others in an effort to have them catch up. Relying on the work periods for helping individuals, the other two teachers never experienced this dilemma. But the fear of having other

students left with nothing to do was something Teacher Z talked about as preventing him from continuing to help some students as a constant pressure to keep moving on.

Differences in the critical moments experienced by teachers reflected differences in their teaching styles, instructional goals and expectations, as well as the nature and distributions of student and teacher behaviors and circumstances within the instructional flow. With regard to instructional goals, for example, Teacher X wanted students to "experience" fractions, which meant to develop and apply concepts in concrete or pictorial situations. Representing the other extreme, Teacher Z's desire to have students understand decimals meant to be able to work problems in the book. Teacher Y wanted to provide assistance in the transition between the two by introducing concepts with concrete experiences and relying on number patterns to solve problems.

Teacher X displayed the greatest variation of elective actions, as did her students, in the type of occlusions. Her instuctional style was to rely heavily on student input and to avoid giving lengthy explanations. Teacher Z, by contrast, was highly verbal and tended toward lecturing and demonstrating solution procedures, asking only for brief responses from students. Teachers Y and Z were more likely to press for desired responses from individuals, while Teacher X was more likely to exploit or probe student errors, concerns, or suggestions.

Type I and II, the isolated critical moments, were more prevalent with student difficulties evoking the stronger discomfort for the teachers. Recurring evidence of student difficulty was a most important precipitator of critical moments, as was perceived teacher confusion. The most common covert difficulties in teaching the mathematics across teachers and types of critical moments were fears of confusing students or of wasting their time and admissions of a lack of alternatives, a limited knowledge of mathematics.

While the latter suggests the need for further teacher education, a caveat needs to be raised. If the goals, beliefs, attributions and propensities for action and instructional style are ignored, the impact of increased mathematical knowledge may be less effective. The same could be said about other intervention as critical moments revealed a complex relationship of many variables.

What remains to be explored is the variation and generality of these critical moments across teachers and situations and the effect of intervention. Already underway is an ecological experiment to determine the impact of teaching an unfamiliar unit on the critical moments for the same three teachers.

References

Marland, P.W. A study of teacher's interactive thoughts. Unpublished dissertation. University of Alberta, 1977.

Morine, G. & Vallance, E. Beginning teacher evaluation study. Special study B: A study of teacher and pupil perceptions of classroom interaction. (Pre-publication copy). San Francisco: Far West Laboratory for Educational Research and Development, 1975.

National Institute of Education. Teaching as clinical information processing. National Conference on Studies in Teaching. Report of Panel 6 (Shulman, L., ed) Washington, D.C., 1975.

Peterson, P.L., and Clark, C.M. Teachers' Reports of Their Cognitive Processes During Teaching, American Educational Research Journal, 1978, 555-565.

Shroyer, J.C. What Makes Teaching Difficult? The Critical Moments in Teaching Mathematics. Unpublished dissertation, Michigan State University, 1980.

CLASSROOM CONDITIONS FOR LEARNING MATHEMATICS

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The nature of research is such that in order to carry it out effectively, the researcher must remain somewhat detached from the object of his study. Even in anthropological educational research where one is adopting a sort of 'live-with-the-subjects' position one must keep the detachment clearly in mind. This detachment is the source of both the strength of educational research and also its potential weakness. At a conference such as this there is little need to discuss the strengths and value of research, but there is certainly reason to consider any weakness, and this paper considers the problems caused by the distance between research into the learning of mathematics, and the classroom.

Research into the learning of mathematics is often stimulated by the desire to improve the quality of mathematics teaching. To that end reports of such research usually include a section on 'implications for the classroom'. For example, Kelly and Philp's (1975) data "suggests that Melpa children are not conscious of the patterns of classification of their own langauge and that it would probably be useful to attempt to make them aware of these before imposing foreign and alien patterns such as are found in the mathematics and science concepts of western technology. Domino and card games could possibly help to achieve this as early as the first school grade" (p.196). Bell (1976) says: "the ability to generalise and prove might be improved by activity with the type of problem described here" (p.36).

Even if the implications are not spelled out, the inference of implications is suggested. For example Good et al. (1978) showed that low SES fourth grade pupils achieved more in their mathematics lessons when the teachers supervised and monitored pupils' seat-work more actively and directly controlled the teacher-pupil interactions. Praise also appeared to be more important in low SES than in high SES classrooms.

Now I am not seeking to be critical of these researchers, indeed I chose these examples because they are researchers whom I do respect. However I am concerned about "implications for the classroom".

The problem is that classrooms appear not to be particularly appropriate environments in which to learn mathematics. Classroom learning can be

characterized by the following constraints:

a) It must take place in a limited time. Usually a lesson in a secondary school lasts for 30-40 minutes, at the end of which the learner must end that activity and move to another subject area.

b) It is often incomplete learning. This is a corollary of the above constraint. Because of a limited time allowance not all learners will be completely successful nor will they always finish their assignments. Even if they are encouraged to work longer on the problems by, for example, 'homework', many are insufficiently motivated to do so.

c) There are multiple objectives. One learns, and is expected to learn, many things at the same time during mathematics classes. Not only are there multiple mathematical objectives, there are also social and general educational objectives.

d) The conditions are 'noisy'. Noise can be aural, visual and also social, and it can distract. If mathematical learning depends heavily on a reflective approach to the content, reasonably noisefree conditions are important. Many learners in today's classrooms seem over-stimulated by their conditions.

e) The atmosphere is one of mutual evaluation. One works "in a crowd" as Jackson puts it, and the constant evaluation by peers and teachers alike creates enormous pressure. It matters particularly in a subject so 'visible' as mathematics where right/wrong evaluations can be relatively easily made. Furthermore the authority of other people can overshadow the authority of the mathematics.

f) Presentation sequences are a compromise. As individuals are unique, no one sequence of presentation of ideas will satisfy all learners. This is clearly the case with heterogeneous groups but it is also, less obviously, the case in so-called 'homogeneous' groups.

g) Teaching is a stressful occupation. Given the above constraints, learning in classrooms can obviously be accompanied by considerable stress. For the teacher also, particularly a sensitive and aware teacher, the classroom can be a stressful place, and stress is not a favourable condition for clear-thinking and wise decision-making.

THE PROBLEM

No doubt classroom learning can be characterized in many other ways but these suffice for our concern here. We have on the one hand research which is attempting to develop knowledge about how children learn mathematics, and about the conditions which could encourage successful learning. On the other we have the classrooms, with all their complexity and even perhaps,

chaos. In fact the contrast seems to be growing. Research is developing rapidly and our knowledge of learning is becoming more and more sophisticated. Meanwhile classrooms are becoming more of a challenge for teachers and many feel that the quality of teaching is declining.

We know, in addition, something of what results from this contrast - half-digested and poorly understood ideas, unpleasant and emotive feelings about mathematics, and much knowledge about power relationships and social groups.

More than that, we know of the dissatisfactions which surround mathematics teaching at present - parents dissatisfied with what teachers are doing, teachers dissatisfied with what researchers are doing, and researchers dissatisfied with the teachers' unwillingness to read, and learn from, research.

SOURCES FOR NON-SOLUTIONS

Given that the problem is as massive as is suggested, no one solution is likely. Nevertheless if we agree to recognise that such a problem does exist we can explore several possible sources for resolving the crisis. First however let us consider a few 'non-solutions'.

a) Ignore or by-pass the teacher. This non-solution is based on several axioms not the least of which is that teaching is too complex and important an activity to be left in the hands of the teacher. One can cite 'teacher-proof' text-books and the flight into technology as prime examples of this non-solution. The teacher as merely one example of "an instructional delivery system" has no appeal to this researcher.

b) Abolish the classroom. As the classroom is such an unhelpful place for learning mathematics (so the argument goes) who needs the classroom? Why not create instead individualised learning laboratories which could maximize the mathematical capabilities of the individual pupils? This argument I reject, as it does not face squarely the issue of 'multiple simultaneous objectives' which characterizes real learning.

c) Stop all non-classroom based research. Again this is an over-simplistic response, and ignores the strength of research which is detached from the classroom action. If all that was studied was what "is" how could we ever explore what "could be"?

d) Make every teacher a researcher. Whilst having some sympathy with this idea I reject it here because firstly it absolves the researcher from all responsibility, and secondly it deflects the teacher from his prime purpose, namely teaching.

SOURCES FOR SOLUTION

a) We undoubtedly need more study of learning under classroom conditions. If we consider the previous list of classroom constraints on learning it will be clear that our knowledge of these is woefully limited. Frequently the conditions thought desirable by a researcher to encourage particular learning are incompatible with those necessary for achieving other goals. For example, the more guided the discovery the more likely it is that the learner will discover the idea being taught, but at the cost of not developing his own problem-solving strategies.

Perhaps, even, the conditions which we would like to see in a classroom are impossible to create there. A good example was the rush to programmed instruction on the basis of reinforcement theory.

If more learning researchers chose the classroom as their strategic site for research we would understand for more about the mechanisms and potential of actual classroom conditions.

b) We need more interventionist research. Whilst not wishing to ban all naturalistic learning research, it is the case that teaching is intervention. Developmental research, as an example of the study of natural learning does not directly help interventionist decision-making. Furthermore, in the classroom, interventions are not merely provided by the teacher - other learners intervene just as frequently, and perhaps more significantly at times. Yet studies of the effects, constraints, and, indeed, values of such interventions are few and far between.

Learning research does appear to move from the real world into the individual mind, in some cases never to reappear. Insofar as research into mathematics learning purports to be concerned with education, it must pay due attention to the social aspects of learning. 'Meaning', in a classroom, has more to do with social negotiation and social validity than with individual schemas.

c) Our constructs need constant overhauling. Research develops, and is inspired by, constructs. The danger is that they will be researchers' constructs only and that they will lack the necessary face-validity for teachers. The problem begins with the fact that researchers work with constructs while teachers, are, generally, concerned with children. Researchers study learning while teachers work with learners. To be provocative, I have listed some examples of constructs which I think have low face-validity for teachers and some which I think have high face-validity

(although I have not formally researched this area):

Low face-validity	High face-validity
algorithms	strategies
concepts	learning styles
stages	rules
schema	explanations
learning hierarchies	skills
mastery learning	errors
meaningful learning	applications
transfer	motivation
abilities	ability
ego-enhancement	learning-rate
concrete/abstract	feedback
convergent/divergent	insight
drive	practice

d) We should listen to teachers more. Teachers are the mediators between researchers and children's learning, so if researchers are concerned about the quality of that learning they must be concerned about the mediator. Constructs have already been pointed to, but many other aspects are important to know about - teachers ideas concerning objectives, and the decisions of compromise between multiple objectives; perceived problems in the classroom; difficulties with materials; criteria of success; rewards and satisfactions; ...

Such aspects could affect a researcher's choice of test material, of criteria to be achieved, of sample, of research site, of mode of intervention, of conditions to be compared, of stimulus material ...

The problem is not how to achieve the best artificial dissemination after the research is completed, but how to ensure the validity of the research before it begins. Cane and Schroeder(1970) found teachers clearly expressing this idea.

e) Researchers should be more involved in in-service teacher education. This is where real negotiation of meaning can take place and is the best context for teachers to both understand and influence research ideas. It is also a time when teachers can engage in research activity and reflect on the relationship between their teaching and that research. However the real value in bringing teachers and researchers together in in-service

education is that the researcher is forced to make contact with teachers' own theories. Many researchers believe that the classroom is the ultimate test of their ideas, but they may forget that they must first pass the mediator-test. In other words do their ideas make sufficient contact with the teachers' theories to influence those teachers' behaviours? Successful in-service education thereby educates the researcher as much as the teacher.

CONCLUSION

Mathematics education is a human, social process. Just as children develop their own problem-solving strategies so teachers develop their own personal techniques for dealing with classroom situations (Bishop, 1976). Lortie (1975) points out that teachers feel that "their work environment has never permitted them to show what they can really do" (p.235). Thus they reject ideas from research and elsewhere which do not address themselves to the teachers' perceived problem-space. Jackson (1966) says that rather than putting theory into practice, what we do is to put theory into practitioners (insofar as we can "put" anything into anybody). This writer has no doubt that the teacher is the key person in mathematics education, and research which ignores this fact stands a good chance itself of being ignored.

REFERENCES

Bell, A.W. A study of pupils' proof-explanations in mathematical situations. Educational Studies in Mathematics, 1976, 7, 23-40.

Bishop, A.J. Decision-making, the intervening variable. Educational Studies in Mathematics, 1976, 7, 41-47.

Cane, B. and Schroeder, C. The teacher and research. Slough: National Foundation for Educational Research, 1970.

Good, T.L., Ebmeier, H. and Beckerman, T. Teaching mathematics in high and low SES classrooms: An empirical comparison. Journal of Teacher Education, 1978, 29(5), 85-90.

Jackson, P.W. The way teaching is. Washington: National Education Association, 1966.

Kelly, M. and Philp, H. Vernacular test instructions in relation to cognitive task behaviour among Highland children of Papua New Guinea. British Journal of Educational Psychology, 1975, 45, 189-197.

Lortie, D. Schoolteacher. Chicago: University of Chicago Press, 1975.

MATH ANXIETY AND ELEMENTARY SCHOOL TEACHERS

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Based on the assumption that the population of elementary school mathematics teachers contains some math-anxious members, a survey was conducted to investigate the relationship between this anxiety and four other characteristics of the teacher: sex, career inhibition, type of training, and recency of training. An inventory was constructed and distributed to elementary school teachers; 230 usable returns were received. Data was analyzed by means of a Chi-Square statistic ($\alpha = .05$). Results showed a significant relationship between math anxiety and career inhibition and between math anxiety and perceived type of mathematics training. No relationship was found between math anxiety and sex or between math anxiety and recency of training. A series of follow-up interviews with twenty of the respondents supplied more information regarding their attitudes toward mathematics.

The newly-labeled phenomena of math-anxiety, defined literally as uneasiness or apprehension regarding mathematics, has recently been the focus of much investigation. Fears and self-defeating attitudes of the math anxious have been identified and ample evidence has been presented that emotions as well as intellect play a role in the learning of mathematics (Lazarus, 1974; Tobias, 1976, 1977, 1978).

Various themes have recurred in the discussions of math anxiety. One theme is that math anxiety is primarily, although not exclusively, a female trait; evidence attributes this not to any innate difference between the sexes in math potentiality, but rather to culturally-induced expectations and experiences (Ernest, 1976; Fennema & Sherman, 1974, 1977a, 1977b; Maccoby & Jacklin, 1974). Another frequent theme is that math anxiety acts as an inhibitor of career choices, the necessity of having to study or use mathematics foreclosing otherwise appealing careers (Ernest, 1976; Sells, 1978). Several studies cite instances of attitudes toward mathematics (often negative ones) being linked with the attitudes, approaches, or behaviors of teachers (Ernest, 1976; Mathison, 1977; Tobias, 1976, 1977, 1978).

The present study represents an initial attempt to investigate the presence of math anxiety and its correlates in elementary school teachers. The study also investigated the linkage of math anxiety with the teachers' mathematics education.

Taking our lead from the current literature and from our own interest in the process of education, we (Dr. Annette Chavez and myself) designed a two-part study. The first part measured the relationship between math anxiety in elementary school mathematics teachers and four other teacher characteristics: sex, career goals, type of mathematics training, and recency of mathematics training. The second, less formal, part of our investigation consisted of follow-up interviews during which we sought further information regarding the teachers' attitudes toward mathematics.

Four hypotheses were proposed for examination during the first stage of the study:

Hypothesis 1. Math anxiety is more prevalent among females than among males.

Hypothesis 2. More teachers classified as math anxious will show evidence of inhibited career goals than those classified as non-anxious.

Hypothesis 3. Fewer teachers who identify their mathematics training as having stressed understanding will show math anxiety than those who identify their mathematics training as having stressed computation.

Hypothesis 4. Fewer teachers with recent training in mathematics will show math anxiety than those whose training is more distant.

INVENTORY DEVELOPMENT AND USE

Since no existing instrument appeared appropriate for our population, we found it necessary to construct an inventory for the purpose of testing the above hypotheses. A few items were adapted from the Fennema-Sherman Mathematics Attitudes Scales (1976). Content validation was based on classification of the items by faculty members in mathematics and education.

The inventory contained 36 Likert-type items with five answer choices provided for each. Seventeen of the items measured math anxiety. (Example: Compared to other subjects, my anxiety level is high when teaching math.) Eleven of the items measured inhibition of career goals. (Example: I would refuse a higher-paying job if it required taking more math courses.) For those items on which agreement indicated anxiety, subjects were assigned scores ranging from 5 for Strongly Agree, to 1 for Strongly Disagree. The reverse scores were assigned on those items where disagreement indicated anxiety. A similar method of scoring was used on the items measuring career inhibition. A split-half reliability coefficient was computed for the math anxiety part

of the scale and also for the career inhibition section. The correlations were found to be .92 and .82 respectively. Teachers were also asked when they took the majority of their college mathematics courses and whether their studying of mathematics (high school and beyond) involved more emphasis on learning computation or on understanding mathematical principles.

On the basis of responses to the inventory, teachers were classified as (a) math anxious, non-anxious, or neutral and as (b) inhibited, non-inhibited, or neutral regarding career goals. On the anxiety scale, possible scores ranged from 17-85. Subjects scoring in approximately the upper 40% of this range, represented by raw scores between 57 and 85, were classified as math anxious. Those receiving scores of 46-56 were classified as neutral, and those scoring in approximately the lower 40% of this range, 17-45, were classified as non-anxious. A similar breakdown was used to classify subjects on the scale measuring inhibition of career goals. Scores of 37-55 were used to classify the subjects as inhibited, 30-36 as neutral, and 11-29 as non-inhibited.

Teachers were also classified according to recency of mathematics training, with training since 1967 being considered recent, and training prior to this as distant. The final classification was according to the teacher's perception of the type of mathematics training he or she had received.

SAMPLE

The survey was conducted among elementary school mathematics teachers in five northern Kentucky counties. Participation of the teachers was voluntary; of 500 inventories distributed, 230 usable returns were received.

RESULTS

Using the classifications described above, 36 teachers, or approximately 16% of the sample, were classified as math anxious.

A Chi-Square statistic (α=.05) was computed to test for a relationship between math anxiety and each of the variables mentioned in the hypotheses. Results were as follows:

Hypothesis 1 was not supported. No significant relationship was found between sex and math anxiety (see Table 1).

Table 1

Number and Percent (given in parentheses) of Males and Females Showing Math Anxiety

	Anxiety Category			
Sex	Anxious	Neutral	Non-Anxious	Row Total
Male	2 (8)	2 (8)	21 (84)	25 (11)
Female	34 (17)	40 (20)	129 (63)	203 (89)
Column Total	36 (16)	42 (20)	150 (66)	228 (100)

This lack of a significant relationship between sex and math anxiety runs contrary to the common notion that math anxiety is more prevalent among females; however the result may well be a function of the population of this particular study.

Hypothesis 2 predicted that more subjects classified as math anxious would show evidence of inhibited career goals than those classified as non-anxious. Chi-Square for this analysis was clearly significant at the .05 level, with the data showing that the differences were in the direction hypothesized (see Table 2).

Table 2

Number and Percent (given in parentheses) of Anxious, Neutral, and Non-Anxious Teachers Showing Career Inhibition

	Career Inhibition Category			
Anxiety Category	Inhibited	Neutral	Non-Inhibited	Row Total
Anxious	27 (73)	10 (27)	0 (0)	37 (16)
Neutral	9 (21)	22 (52)	11 (26)	42 (18)
Non-Anxious	4 (3)	28 (18)	119 (79)	151 (66)
Column Total	40 (17)	60 (26)	130 (57)	230 (100)

Hypothesis 3 stated: fewer teachers who identify their mathematics training as having stressed understanding will show math anxiety than those who identify their training as having stressed computation. The statistical analysis showed signigicance at the .05 level, with the difference in the direction hypothesized (see Table 3).

Table 3

Number and Percent (given in parentheses) of Teachers, According to Perceived Type of Training, Showing Anxiety

	Anxiety Category			
Perceived Training	Anxious	Neutral	Non-Anxious	Row Total
Stressed Computation	29 (20)	31 (21)	87 (59)	147 (67)
Stressed Understanding	8 (11)	9 (13)	55 (76)	72 (33)
Column Total	37 (17)	40 (18)	142 (65)	219 (100)

Finally, we hypothesized that fewer teachers with recent training in mathematics would show math anxiety than those whose training was more remote. Chi-Square was not significant here, nor was there evidence of even a non-significant trend (see Table 4).

Table 4

Number and Percent (given in parentheses) of Teachers, According to Recency of Training, Showing Anxiety

	Anxiety Category			
Recency of Training	Anxious	Neutral	Non-Anxious	Row Total
Distant	14 (15)	16 (17)	62 (67)	92 (40)
Recent	23 (17)	26 (19)	87 (64)	136 (60)
Column Total	37 (16)	42 (18)	149 (65)	228 (100)

Various questions could be raised about this result: Would the same have been

true during the peak period of "modern" math? Does teaching a subject alleviate anxiety about the subject matter? Whatever the explanation, the data in our study shows no relationship between when the subjects received their math training and math anxiety.

TEACHER INTERVIEWS

To obtain more information regarding teachers' attitudes toward mathematics, a follow-up series of 20 interviews was conducted. The interviews focused on four major areas:

1. Feelings about mathematics both as a student and as a teacher and factors contributing to these feelings;
2. Relationship between attitude toward mathematics and career choices;
3. Sex-role stereotyping in relation to math;
4. Suggestions for preventing and alleviating math anxiety in students.

The information gleaned from these interviews was considered anecdotal; no statistical analysis of the data was attempted.

FEELINGS ABOUT MATHEMATICS

It was evident in the interviews that events and feelings from elementary school days were subject to vivid recall. These consistently gave evidence of attitudes toward mathematics being linked to attitudes and behaviors of mathematics teachers. In spite of some negative experiences, the teachers interviewed were generally positive about teaching mathematics and felt secure with the content they were teaching.

CAREER CHOICE

There was no evidence that attitudes toward mathematics had been a primary consideration in choosing teaching as a profession. However, some careers, such as accounting, business, medicine, psychology, veterinary work and computer programming had been considered but abandoned because of the threat of mathematics.

SEX-ROLE STEREOTYPING

Most of the teachers remembered their own schooling as including the "math is for boys" stereotype. Evidence of current sex-role stereotyping was minimal, but not totally non-existent. Some teachers still feel that "girls always have to struggle harder with math than boys do." However, evidence

of a possible reverse-bias at the grade-school level was also present: several teachers commented that girls tend to perform better in all subjects, including math.

PREVENTING MATH ANXIETY

Suggestions for preventing and alleviating math anxiety largely parallel previous research findings. The use of manipulatives and games, individual help and encouragement, avoiding emphasis on speed, checking for acquisition of skills before moving on, and using math when teaching other subjects ("How long ago was this book written?") were all suggested. Many teachers stressed creating a climate where it's acceptable to ask questions and make mistakes. One not entirely facetious suggestion was to require all teachers to take a course "over their heads" in order to acquire greater empathy for students.

CONCLUSIONS

Although the limited sample and the self-report nature of the study prevent broad generalizations, the information gleaned does suggest implications of a psychological nature as well as directions for further research and intervention. The presence of a supportive teacher and a healthy classroom climate are seen as crucial for fostering positive attitudes toward mathematics. Less anxiety also seems to result when mathematics is taught for understanding. Future research could profitably investigate the relationship between teacher and student math anxiety. However, it should be emphasized that one cannot assume that all teachers who disliked mathematics as students, or even those categorized as math-anxious, dislike <u>teaching</u> mathematics. Many, in fact, seem eager to break the cycle of poor attitudes engendering poor attitudes and are determined to provide their pupils with positive experiences in learning mathematics. Intervention programs could include efforts to assist these teachers as well as programs aimed at counteracting math anxiety in students and adults.

REFERENCES

Ernest, J. Mathematics and sex. American Mathematical Monthly, 1976, 83(8), 595-614.

Fennema, E. Sex differences in mathematics-learning: Why??? The Elementary School Journal, 1974, 75(3), 183-189.

Fennema, E., & Sherman, J. Fennema-Sherman Mathematics Attitudes Scales. JSAS Catalog of Selected Documents in Psychology, 1976, 6(2), 31. (Ms. No. 1225)

Fennema, E., & Sherman, J. Sex-related differences in mathematics achievement, spatial visualization and affective factors. American Educational Research Journal, 1977, 14(1), 51-71. (a)

Fennema, E., & Sherman, J. Sexual stereotyping and mathematics learning. The Arithmetic Teacher, 1977, 369-371. (b)

Lazarus, M. Mathophobia: some personal speculations. National Elementary Principal, 1974, 53, 16-22.

Maccoby E., & Jacklin C. Psychology of sex differences. Palo Alto, California: Stanford University Press, 1974.

Mathison, M. Curricular interventions and programming innovations for the reduction of mathematics anxiety. San Francisco, California: Annual Convention of the American Psychological Association, 1977. (ERIC Document Reproduction Service No. ED 154 330).

Sells, L. Mathematics--a critical filter. The Science Teacher, 1978, 45(2), 28-29.

Tobias, S. Math anxiety. Ms Magazine, September 1976, pp. 56-59; 92.

Tobias, S. Math anxiety. Teacher, November 1977, pp. 71-74.

Tobias, S. Overcoming Math Anxiety. New York: W. W. Norton & Co., Inc., 1978.

Werner Lorbeer

Karriereeffekte in der Entwicklung der Mathematikeinstellung

Die Entwicklung der Einstellung zur Mathematik unterliegt in der Schule einer mehrfachen Dynamik. Einmal erweitert sich die Auffassung des Schülers von dem was Mathematik ist durch sein Fortschreiten im Curriculum. Dies geht einher mit einer reifungsbedingten Aufweitung seiner Zeit- und Zweckbezüge und damit seiner veränderten Einschätzung des Faches in Relation zu eigenen zukünftigen Zwecken. Zum anderen erlebt der Schüler die Mathematik als Erziehungsmittel. Dadurch sollte die Einstellung zur Mathematik abhängig werden von der Dynamik der Schülerkarriere. In einer Längsschnittuntersuchung an 80 Gymnasiasten wurde diese Karrierehypothese bestätigt. Als methodisch geeignetes Hilfsmittel erwies sich die Kohortenanalyse, die allgemeiner als Methode zur Evaluation curricularer Innovationen in zyklisch organisierten Institutionen empfohlen wird.

Einstellungen zu Fachdisziplinen sind ein wichtiges Produkt der schulischen Erziehung. Da Erziehung sich der Mathematik als Hilfsmittel bedient, um ihre Ziele zu erreichen, wird die Einstellung des Schülers zur Sachstruktur der Mathematik nicht nur durch deren Erscheinungsbild in Unterricht und Alltag, sondern auch durch die Formen der Instrumentalisierung der Mathematik im Erziehungsprozeß geprägt (SUYDAM 75, WAGNER 75).

Die Mathematikdidaktik muß sich also bei der Erforschung der Einstellungsentwicklung der Schüler nicht nur mit dem Bezug des Subjekts zur Mathematik auseinandersetzen, sondern auch mit den verschiedenen thematischen Färbungen dieses Bezuges, die durch seine Einbettung in den Erziehungsprozeß im Rahmen von Organisationen entstehen.

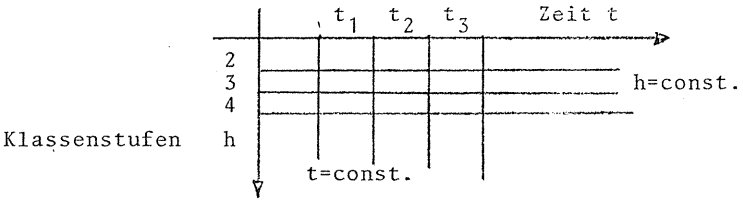
Thematische Färbungen der Einstellung zur Mathematik ergeben sich aus der Instrumentalisierung der Mathematik im Selektionsprozeß des Bildungswesens (WAHL 75, WEINER 75) oder aus der in der Adoleszenz beginnenden zukunftsorientierten Bewertung der Mathematik nach ihrem subjektiven und objektiven Nutzen für den Alltag (KARL in KEIL 77, KARL, LORBEER, BOHM 80).

Der folgenden Darstellung liegt ein eindimensionales Konstrukt der Einstellung zur Mathematik zu Grunde. Gemessen wurde der "Spaß" an der Mathematik. Unabhängig, aus welchem Gegenstand der Mathematik oder aus welchen Motiven er gewonnen wurde.
Hypothese:
Wir wollen durch eine empirische Studie nachweisen, daß die Entwicklung des "Spaßes" an der Mathematik nicht nur ein Problem von Adoleszenzkrise und Stoffdarbietung im Unterricht ist, sondern entscheidend geprägt ist von der "Karriere" des Schülers in der Institution Schule.

1. Methodische Überlegungen zu Einstellungsmessungen in Institutionen mit zyklischer Organisationsform

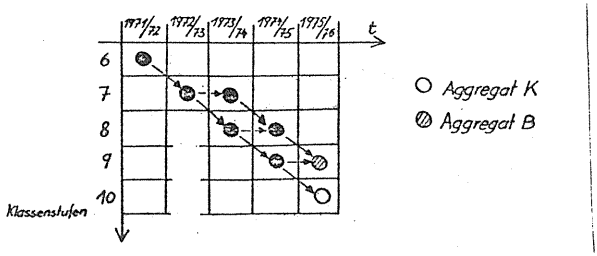
Institutionen stehen häufig vor dem Problem, die Wirkungen curricularer Veränderungen, wie Umstrukturierungen in der Schulorganisation, Einführung neuer Medien etc. in ihrer kurzfristigen wie langfristigen Wirkung auf die Einstellung zur Mathematik einschätzen zu müssen. Wir schlagen dazu folgende Erhebungsmatrix vor.

Erhebungsmatrix



In diese Erhebungsmatrix lassen sich modellhaft die verschiedenen Typen der Querschnittsanalysen einordnen. t=constant ergibt die häufigste Querschnittsanalyse. Sie wird angewandt, um die Verteilung eines Merkmals auf verschiedene Altersschichtungen zu erfassen. Die Querschnittsanalyse h=constant wird verwendet, um die Frage zu beantworten, ob sich die in die Organisation neu eintretenden Mitglieder in irgendwelchen Merkmalen verändern. Schließlich läßt sich die Längsschnittanalyse durch Erhebungen längs der Diagonalen darstellen. Doch bringen diese Darstellungen die Dynamik der Wanderung einer Eintrittskohorte durch die Institutionen noch nicht zum Ausdruck.

Erhebungsmatrix zur Karrierehypothese



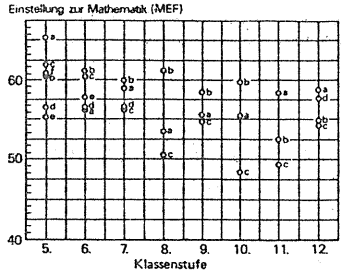
Das dargestellte Meßmodell für die Karrierehypothese ist der Intrakohortenanalyse angepaßt. Die neuen Mitglieder der Institution werden als Personenaggregat aufgefaßt, das durch die historische Situation des Eintritts in die Organisation definiert wurde, kurz als Kohorte (RYDER 65/68). Sie erfährt nun auf ihrem Lebensweg durch die Institution charakteristische Sozialisierungen, die von der Art der Karriere abhängen. Die zeitlich geordnete Abfolge von Interventionen, die zur Zuweisung und Übernahme institutionalisierter Positionen führen, nennen wir Karriere (GOFFMAN 67, WELLENDORF 73, RIDDER 74). Befindet sich eine Organisation in einer stabilen institutionellen Umwelt, so kann man die Wirkung unterschiedlicher Karrieren auf den Schüler nachweisen. Der Untersuchungsplan erlaubt also, die Wirkung der Karriere unabhängig von Effekten der Adoleszenz und historischer Verwerfungen nachzuweisen.
Die Kohortenanalyse hat sich bei meinen Studien zur Evaluation computerunterstützten Unterrichts bewährt. Es konnte gezeigt werden, daß sich die Geschichte eines Forschungsprojekts im Rahmen einer Kohortenanalyse als quasiexperimentelles Untersuchungsdesign hoher Reliabilität und Validität auffassen läßt. (LORBEER 79).

2. Die Wirkung von Alter und Karriere auf die Entwicklung der Einstellung zur Mathematik

Die Ausgangssituation zur Untersuchung der Karrierehypothese war eine große Querschnittserhebung t=constant an einem Gymnasium mit dem Ergebnis, daß mit zunehmender Klassenstufe der Spaß an der Mathematik nur wenig, nicht signifikant im statistischen Sinn, abnahm. Größere Unterschiede zeigten sich in derselben Klassenstufe zwischen den Klas-

sen, was auf starke Interaktionseffekte zwischen Lehrer und Klassengruppe und Schülern untereinander hinwies.
Extrem ungünstig liegende Klassen sind solche, die durch Eingriffe der Schulleitung in ihrer sozialen Struktur verändert wurden.

Einstellung zur Mathematik, Querschnitt t=const.



Klassenmittelwerte der Querschnitterhebung am Ende des Schuljahres 1974/75. Höherer Skalenwert bedeutet günstigere Einstellung, bei 50 liegt die theoretische Skalenmitte; a bis f Parallelklassen (aus *Karl* in *Keil* 77, S. 460).

Dieses Bild der Querschnittserhebung könnte so gedeutet werden, daß die institutionellen Interventionen auf die Entwicklung der Mathematikeinstellung keinen Einfluß haben.
Doch die Betrachtung der Einstellungsentwicklungen bezogen auf eine Kohorte, die sich in ihrem Sozialisationsprozeß auf dem Weg durch die Institution befindet, weist sehr deutlich auf eine solche Wechselwirkung zwischen institutionellen Interventionen und Einstellungsentwicklung hin.

Einstellungsentwicklung im Längsschnitt
bezogen auf die Karrieretypen "erfolgreich"
und "nicht erfolgreich"

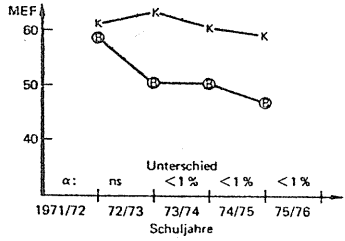


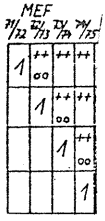
Abbildung 2. Entwicklung der Einstellung zur Mathematik. Vergleich von K-Gruppe und B-Gruppe. $N_K = 58$, $N_B = 22$ (aus *Lorbeer* in *Keil* 77, S. 200).

Aggregat K wird gebildet aus den Schülern der Kohorte, die während des ganzen vierjährigen Untersuchungszeitraums eine im Sinne der Schule "erfolgreiche" Karriere hinter sich brachten. Aggregat B aus solchen, die einen Bruch in ihrer Schulkarriere aufwiesen. Die Schüler der beiden Aggregate K und B unterschieden sich zu Beginn des Untersuchungszeitraums nicht.
Dies änderte sich bereits nach einem Schuljahr. Von nun an unterschieden sich die beiden Aggregate signifikant. Übrigens wies Aggregat K über alle vier Jahre hinweg keine statistisch signifikanten Einstellungsänderungen auf.

3. Instabilität negativer Einstellungen und die Auswirkungen auf das Selbstkonzept des Schülers

Neben der objektiven und statistischen Wechselwirkung von Karriere und Mathematikeinstellung interessiert auch, wie der Schüler eine solch gravierende Einstellungsänderung zu seinem Bild von sich als Schüler in Beziehung bringt.
Eine einfache Hypothese wäre, daß sich eine stabil ablehnende Einstellung ergibt. Dies hieße, daß es möglich sein müßte, die Einstellungen der Schüler über größere Zeiträume hinweg zu schätzen. Doch bereits die Berechnung einfacher Korrelationen der Meßwerte der Längsschnittuntersuchung innerhalb der Aggregate K und B ergibt Resultate, die diese einfache Hypothese die Äquilibration zerstören.

Korrelationen der Einstellungsdaten der
Längsschnittanalyse innerhalb der Aggregate K und B



\+ Aggregat K 00, ++ $\alpha \leq 0.01$
o Aggregat B 0, + $0.01 < \alpha \leq 0.05$

Es zeigten sich stabile Einstellungen in Aggregat K über den gesamten Untersuchungszeitraum hinweg. In Aggregat B war eine solche Prognose auf die Einstellungsentwicklung nicht möglich. Es war deshalb nötig, die Schüler des Aggregats B nach dem subjektiven Erleben ihrer Einstellungsänderungen zu befragen. Die Hypothese war, daß sich die Attributionstheorie von leistungsorientiertem Verhalten zur Erklärung heranziehen läßt (HECKHAUSEN 76, WEINER 76).
Die schwierigen inhaltsanalytischen Auswertungen wiesen jedoch darauf hin, daß dies nicht generell möglich war. Daß sich vielmehr die Veränderung der Einstellung zur Mathematik häufig unter Auswirkungen auf das Selbstkonzept des Schülers vollzieht. Es werden also nicht nur andere Gegenstände der Wertschätzung und des Interesses gesucht, sondern auch das Konzept der eigenen Tüchtigkeit und Begabung umgeschrieben und damit auch die Attribuierungen des Versagens. Wir glauben, aus diesen Analysen noch weitergehende Folgerungen über die Funktion der Mathematikeinstellung in der Schülerkarriere ziehen zu können. Die Mathematikeinstellung scheint Teil des Stabilisierungssystems der Qualifikationskarriere. Sie kann bis zu einem gewissen Umfang im Sinne einer äquivalenzfunktionalen Strategie durch Umwertung in ihrer Bedeutung und Aufwertung anderer Gegenstände ersetzt werden, bis schließlich doch Korrekturen am Selbstbild nötig werden.

4. Zusammenfassung und Diskussion

Durch die vorgetragenen Forschungsergebnisse wird gezeigt, daß die Mathematik auf die Entwicklung des Selbstkonzepts des Schülers einwirkt. Und zwar nicht nur über das Erlebnis der Sachstruktur der Mathematik, sondern (in ihren pädagogisch nicht erwünschten Wirkungen?) auch über die Instrumentalisierung der Mathematik durch die Institution.

Die Interpretation von Einstellungsdaten zur Mathematik sollte daher immer den ökologischen Gesichtspunkt bei der Genese der Einstellung berücksichtigen. Davon ausgehend schlage ich vor, der Forschung über die dynamischen Aspekte der Einstellungsentwicklung zwei thematische Rahmenfragen zu stellen:

i) Wie differenziert der Mathematikunterricht mit Hilfe welcher mathematischer Strukturen die Einstellung (kognitiv, emotional, konnotativ) zum Gegenstand Mathematik?

ii) Wie beeinflußt der Mathematikunterricht mit welchen mathematischen Gegenständen die Entwicklung der Kognition der Interessen und des Selbstkonzepts.

Damit wären der Einstellungsforschung im Rahmen der Mathematikdidaktik ein genuin mathematischer und ein genuin pädagogischer Themenkreis gesteckt.

Literaturliste

Goffman E.: Stigma. dt. Suhrkamp, Frankfurt 1967

Heckhausen, H.: Ein kognitives Motivationsmodell und die Verankerung von Motivkonstrukten. In: Lenk, H. (Hrsg.) Handlungstheorien in interdisziplinärer Perspektive, 1976

Karl U.: Psychologische Beiträge zum CUU. In: Keil K.-A. (Hrsg.) Das Projekt Computerunterstützter Unterricht, Augsburg 1976, Bibliothek der Universität Augsburg

Karl U., Lorbeer W., Bohm G.: Die Einstellung zum Schulfach als Problem von Wissenschaft und Praxis. Befunde und Überlegungen zur Untersuchung von Schülereinstellungen zum Fach Mathematik am Gymnasium. In: Unterrichtswissenschaft, 2/1979, 116 - 127

Lorbeer, W.: Computer in der Schule? Medienwirkung und Schülerkarriere. Oldenbourg Verlag, München, 1979

Ridder, P.: Die Patientenkarriere. Von der Krankheitsgeschichte zur Krankengeschichte. Stuttgart 1974

Ridder, P.: Politische Folgen der Berufskarriere. In: Soziale Welt, 1974, 334 - 346

Ryder N.B.: The Cohort as a Concept in the Study of Social Change, Am. Soc. Review, Vol 30, (1965), 843 - 861)

Ryder N.B.: Cohort Analysis, in: International Encyclopedia of Social Sciences, Vol. 2, 546 - 550)

Suydam M.: Reasearch on some key non-cognitive variables in Mathematics education. In: Schriftenreihe des IDM, Bielefeld 4/1975

Wagner J.: Schülereinstellungen zur Schule - Ein Überblick über ihre Bedingungen. Psychol. in Erz. und Unterricht, 22 (1975), 351 - 367

Wahl D.: Erwartungswidrige Schulleistungen. Beltz, Weinheim 75

Weiner B.: Die Wirkung von Erfolg und Mißerfolg auf die Leistung. Huber-Klett 1975

Weiner B.: Attributionstheoretische Analyse von Erwartungs X Nutzen-Theorien. In: Schmalt H.-D., Meyer W.-U. (Hrsg.) Leistungsmotivation und Verhalten, Klett, Stuttgart 1976, 81 - 100

Wellendorf F.: Schulische Sozialisation und Identität. Beltz, Weinheim 73

IOWO MATERIAL TESTED

Jan v.d. Brink
transl. Prof. Freudenthal

IOWO developers have always attempted to understand the world of children's thoughts and to link up with it. In the last few years the balance has shifted even more towards psychological research into the ideas around mathematics. A number of instructional principles, which had been assumed, needed being tested and typical mistakes of children needed explanation.

Principles.

If I use the term 'principle' I mean instructional choices one can make or reject such as: leading children into conflict situations, asking them to make predictions, or to think about, say, invisible quantities. If such principles are tested, it is neither the subject-matter of teaching, nor the specific learning situations represented by the IOWO material. The stress is on the choice of instructional principles and the question to be answered is whether they indeed link up with the thinking world of the learners.
In series of investigations 'Wiskobas tested' we have tried to answer this question. Then a few picies will be exhibited.

Wiskobas tested.

The investigations were concerned with small groups of about 10 children both from kindergarten and first grade.
Each problem was discussed individually during about 10 minutes. The interview had the character of a lesson during which both the behaviour of the subject and the experimenter was registered. The experimenter did not restrict himself to registering. The child was taught something. The effort he needs in doing this, is in a sence measury the child's level. Moreover it is a source of didactical knowledge about teaching skills. The experiments were undertaken with a large frequency during the school-year. They were concerned with a variety of mathematical subjects. Understanding long term learning processes. The immediate aim, however, has answering the questions:

a. With what kind of ideas of children can we link up our instruction?

b. How are our own systems depicted in the child's imagery?

Systems

Though children are attracted by the systems used by adults, their own ideas about them may differ from ours.*) This is manifest in each class discussion.

Some of our systems in discussion:

the postman, the bus, and having one's birthday today (an IOWO-principle)

Ten K-children, five years old and ten first graders were asked a few questions about our 'systems', among other things:

- How does the postman know that some letter is for you?
- What bus shall you take to get to your grandma?

The third question is more difficult to translate. The most familiar term in Dutch for having one's birthday today is the adjective 'jarig', which includes neither the component 'birthday' nor 'today'. I have my birthday today (next week) is in Dutch simply: I am 'jarig' (Next week I am 'jarig'). Without this explanation the reports in the experiment would be unintelligible. I have also to add that birthdays are celebrated not only at home but also at school and even at other places where the 'jarige' happens to come; seats and room are decorated, for instance with garlands. **)

The foregoing explains the third question:

- Where and at which moments can you be 'jarig'?

The kindergarten group gave six different answers on the first question (regarding the postman):

- I do not know
- The postman just brings the letter
- By the mailbox
- The street is written on it
- The name is written on it
- There is a number on the letter

There was more dispersion between the answers in the kindergarten group than among the first graders. Most of them (8 among 10) chose the same answer: the postman can see it from the given name, the family name and the address.

> Maureen, from kindergarten, thinks long about the first question and finally says: 'He just puts it into the box,' After such an answer one realises that kindergarten-children cannot read and therefore fix their attention on the acts of the postman.

*) Freudenthal , Ginsburg , Donaldson have stressed this fact.

**) Remarks of prof.Freudenthal.

> Norman, first grader, answers with a number of conditions: 'Because there is a stamp on it and a postmark, and perhaps a name.' And if need be, the postman can ask people where Norman lives.

The second question (the bus) also gets a greater variety of answers in kindergarten than in the first grade.
Five among ten from kindergarten answer by a colour: 'The blue (or the yellow) bus.'
Two give a number. The others do not know it.
The first graders show a strong tendency to look for the number of the bus line - six among ten choose this solution. Two choose the colour, and two take the number of the busstops as a criterion. Nobody says he does not know.

> Natali, first grader, says she knows about the yellow bus, but not about the blue one. She could never pay attention to the number - she says: 'Mam always says "There it comes" and than I cannot see the number anymore.'

Being 'jarig'

The ideas on this subject show a wide range . We investigated 4, 5 and 6 years olds. We asked the subject whether somebody in his class had been 'jarig' and whether this one would also be 'jarig' at home and on street. And what would happen after one night sleeping?
The four years olds were quite consistent in their answers: One was 'jarig' at a certain place (at home and at school but not on street), and this happens no longer than one day.

> John, four years old, says that one cannot be 'jarig' on street. 'You cannot bring seats there. Ans if you are, it is cold. You cannot hang up garlands either.'

Five years olds do not make being 'jarig' dependent on a certain place (home, school, street). Moreover they take postponements into account.

> Bernt, five years old, says: 'One can 'yes and no' be 'jarig' on street. When I was 'jarig' we went to the forest. I was 'jarig' the whole day. Even there.'
> And after a night sleeping? Bernt: 'Yes, if I had not been at

> school (on my birthday), I am 'jarig' at school (the next day).'

Six years olds introduce being 'jarig' as somehow gliding in the time.

> Walter, 6 years old: 'After a night sleeping I am still a little bit 'jarig' because somebody can happen to bring me a present and to wish me well.'

Six years olds clearly distinguish between the birthday and its celebration which also defines the place where one is 'jarig'.
The most striking feature is the development of being 'jarig' in the age 4 to 6 from a rigidly defined concept (bound to a place and to one day) to a more flexible interpretation. The fixation to one day disappears. Even the next day one can be somehow 'jarig', because of the celebration. This is in contradiction to the development of many mathematical notions, which become more restricted and confined, whereas being 'jarig' broadens and is more widely applied.
Being 'jarig' is a good subject for a class discussion because it is linked with a large number of ideas. Provoking questions can be posed like 'can you be "jarig" on street?', 'can you be half "jarig"?', 'can you be "jarig" without a celebration?'

Invisible quantities (another IOWO principle)
'Invisible quantities' is thought as a means to draw the children's attention beyond counting to the cardinal aspect of number.
Moreover acting on invisible quantities requires a clear mental picture of the state of affairs.
We discovered that juggling is a particularly appropriate situation to pursue these aims.
Children can very well focus on quantities in juggling tricks and on the actions performed on these invisible quantities. Moreover the situation is strongly motivating.

The jugglar - a test of an IOWO principle.
The following experiment was carried out with 9 kindergarten children and 8 firstgraders. We took two cardboard boxes - let us indicate them by 'A' and 'B'. In A we put five foam plastic morsels. B was empty (Foam plastic was taken to avoid noise).

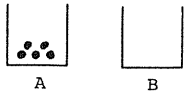


fig. 1.

I told the children I am a jugglar. How many morsels are there in each box, I asked them.
They may look into the boxes.
Then with my right hand I fetched two morsels from A, showed them and told the children I would drop them into B. As a matter of fact I had them sliding along my lefthand and furtively put them back into A, whereas with my righthand I simulated dropping the two morsels into B. So the distribution had not actually been changed: 5 in A, 0 in B.

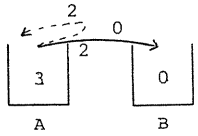


fig. 2.

When I asked how many there were in B, everybody said 'two'. Some children wanted to have a look in B, but they were not allowed to. The question 'how many there were in A' was answered by all of them by 'three', though some had to think about it.
So they imagined the situation:

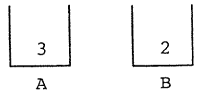


fig. 3.

Now I announced a juggling trick. Pronouncing 'hey-presto, hocus pocus', I pretended transfering one - invisible - morsel from B to A while obviously my hand was empty.

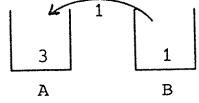


fig. 4.

The children accepted the magic ritual and when asked about the distribution, they said: 'There were one in B en four in A.' I repeated the operation once more so that then the pretended and the actual situation were identical:

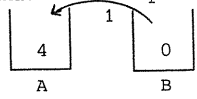


fig. 5.

five in A, none in B. Finally the children were allowed to look into the boxes.

Maureen (K) says when looking into the boxes: 'How is it possible? You have not pulled out anything.'
Kitty (first grade): 'You didn't fetch anything, isn't it? Wait a moment, it looked like that.'

Closer to instruction.

We can draw a few didactical conclusions from this experiment:

- Kindergartenchildren and first graders are interested in juggling. Teachers should learn juggling tricks.
- Children give a meaning to your movements. Movements can represent operations on invisible quantities - a fact that have been stressed by Davydov (His subjects tapped five times on a box with five chips in it (3). One may add that when bringing an invisible quantity into a one - one - relation with a set of fingers, the fingermovements play an important part).
- The juggling trick causes the children to imagine distinctly a situation. This holds in kindergarten as well as in first grade with no exemption at all and in the same measure. So, for the cardinal aspect of number, juggling is a good context.
  The trick can also be played on 4 years olds, provided B is 'emptied' in one blow. 4 years olds are familiar with the idea of nought.

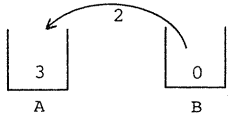


fig. 6.

- There is a conflict between what they *imagine* and what they *see* to happen - imagining that a quantity is transferred and perceiving that it is not. However, it would not be wise to disclosethe secret. Even first graders do not see through it. Let second graders look for an explanation. Then the trick will have served its purpose to support the number concept.
- Finally two practical remarks:
  a. after transferring an invisible morsel we first asked for the number of morsels left in B and then for the new quantity in A. In this order, which corresponds to the movement from B to A, the questions are easier to answer than in the inverse order.
  b. In case of danger that the trick fails suggestive questions are of great help.

Shapes replacing invisible quantities.

Juggling is only one of the means to express invisible quantities. There is a variety of teaching situations where the invisibility is involved.

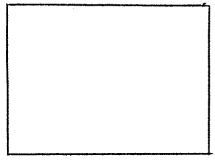


fig. 7.

How many drawings cross each other?

Figures drawn through each other became 'invisible'. The children have to develop strategies in order to separate them.

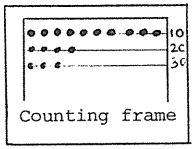


fig. 8.

In order to find out the number of balls on a counting frame, you can count by tens. This kind of counting makes the separate units invisible to the children.

These two examples where the objects are not properly invisible, show once more the part played by true or artificial invisibility in mathematics.

Buses

Another kind of invisible quantities are involved in the so-called busproblems, when the numbers of passengers getting in and out at a stop are orally communicated . The way in which the invisible passengers are realised is characterised by a typical mistake: passengers getting in and out are linked with each other. In the following experiment we try to find out how children explain situations that do not match their ideas. Twelve first graders were given sketches of three situations at one busstop.

1. 'First three got in and then two got out. How many were left in the bus?'
   Eight out of twelve children said there was one passenger left in the bus. As few as three said there were three passengers left in the bus: the ones that got in. One subject answered: five.
   The majority linked the incomers and the outgoers with each other.

In the following question this was impossible. We asked:

2. 'First three got in and then four got out $\xrightarrow{+\ 3}$ $\xleftarrow{-\ 4}$ how many were left in the bus?'
Six among twelve subjects said it was impossible. Three said there were three, that is, the ones that got in. They separated the in- and outgoing people.

Three said the bus was empty, since 3 - 4 is nought.

3. Finally we described a third situation: 'First four passengers got out and then three got in. $\xleftarrow{-\ 4}$ $\xrightarrow{+\ 3}$
How many were left in the bus?'
It is striking fact that now the in- and outgoing people were separated. Ten out of twelve children say: three.

If the problems are tackled by mere arithmetic the situation $\xleftarrow{-\ 4}$ $\xrightarrow{+\ 3}$ provokes a conflict, which can be used to call back the context to the children's mind in general in- and outgetters are not the same persons at one busstop. Learners are prone to automatisms, which can be broken by such problems.

It is clear that testing IOWO-material does not aim at registering the children's mistakes but rather at learning how to teach certain ideas. Psychological research into wrong and right ideas of children may be a service to teaching itself.

recommanded literature

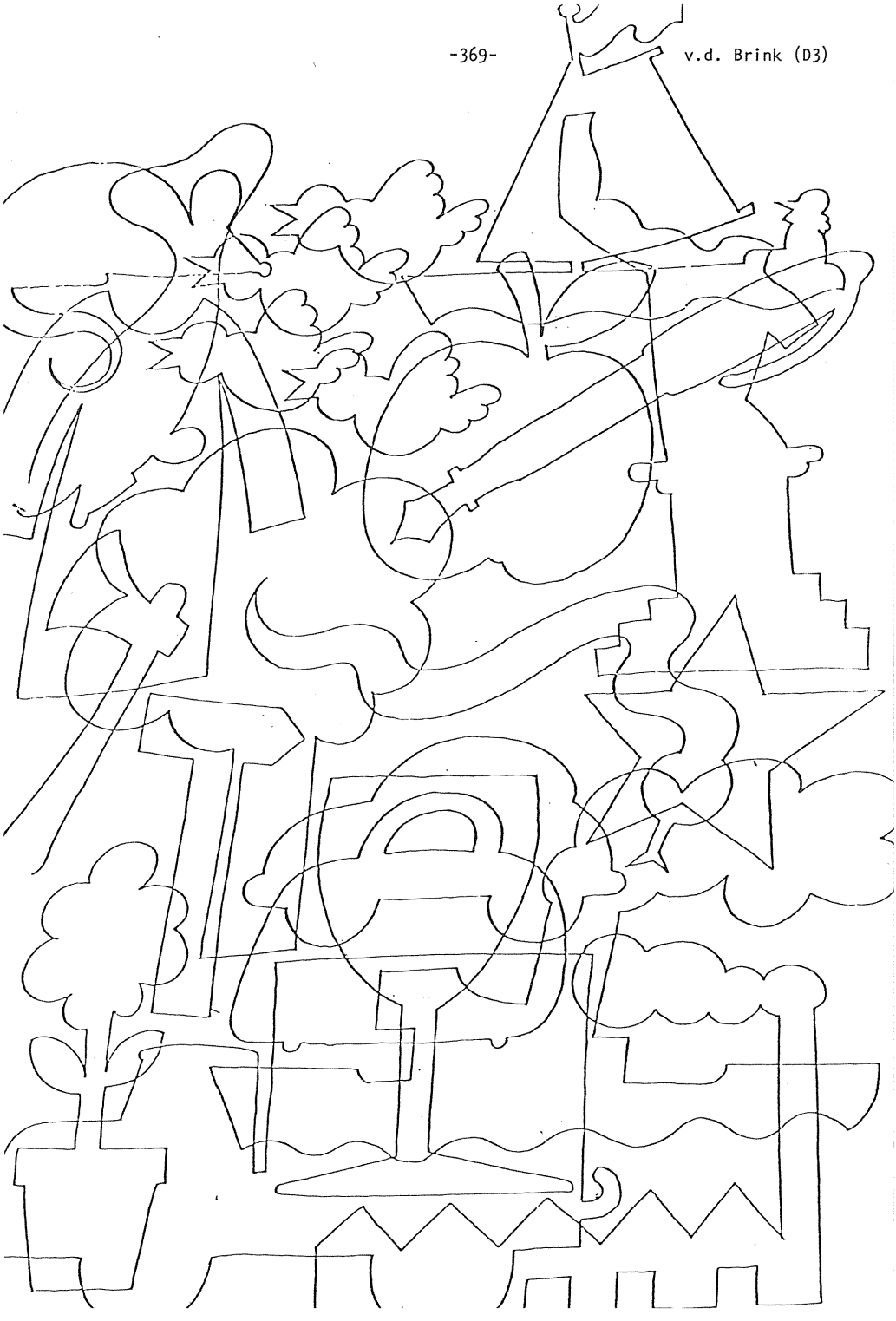
1 Ginsberg, H.: *Children's Arithmetic*, new york 1977.

2 Donaldson, M.: *Children's Minds*, glasgow 1978.

3 Parreren, C.F.van, Carpay, J.A.M. en Davydov, V.V.: *Sovjetpsychologen over onderwijs en cognitieve ontwikkeling*, groningen 1980.

4 Freudenthal, H.: *Weeding and Sowing*, dordrecht 1978.

5 Gelman, R. and Gallistel, C.R.: *The child's understanding of number*, cambridge 1978.



DYNAMICAL MAZES USED TO FAVOUR COMMUNICATION AMONG 7-8 YEAR OLDS.

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Cohors-Fresenborg's dynamical mazes enabled us to let 6 to 8 year olds (1st and 2nd graders) learn to discover the statement of logico-mathematical problems. The solutions of these problems had to be presented in a non-verbal way. The children worked by groups. We used two types of tasks : 1°) reproduction with this equipment of a network presented by a schema, 2°) description and interpretation of the network which has been built. This implies for the children numerous verbal interactions favouring the development of their language and of their socialization.

I. INTRODUCTION.

In previous papers (Lowenthal & Severs, 1979 ; Lowenthal, 1980), one of us (F.L.) formulated the following thesis : "The development of language is one of the main factors of cognitive development of the child. It is possible to favour the development of verbal language by the use of an auxiliary formalism which enables the child to build representations and to manipulate them. It is this manipulation of representations which favours the cognitive development".

So far, only one type of formalism was used (Lowenthal, 1977) ; on the other hand Cohors-Fresenborg (1978a ; 1978b) described a concrete material which can be used by the child to become aware of the notion of finite automaton. One of his aims was to study and to further the children's ability to solve problems. One of the main advantages of the equipment described by Cohors-Fresenborg is the possibility to use it at the three Brunerian levels of representation : enactive, iconic and symbolic (Bruner, 1966). We show here how this equipment can be used as auxiliary formalism and favour numerous interactions between children.

II. MATERIAL AND TECHNIQUE.

The equipment described by Cohors-Fresenborg is made of plastic bricks. These bricks can be connected and form a network. Such a brick is presented as a "piece of railway-line" needed to build a circuit through which a "train" runs. There are several kinds of rails and mechanisms which function when a train goes through them : these mechanisms enable a train to prepare and determine the direction which will be chosen by the next train. All these pieces can be combined to form a finite automaton where each orientation of the mechanisms is an inner-state of the system.

In his first papers, Cohors-Fresenborg used this material with children aged 10 or more, taken individually. He presented the child a logico-mathematical

problem that the child had to solve using the equipment placed at his disposition. The aim was thus to teach mathematics.

We chose voluntarily to reverse the process : we showed solutions to the children and asked them to reconstruct and state the problem whose solution was presented.

1°) Our study concerns only 1st and 2nd graders (6 to 8 year olds).

2°) We chose to let our pupils work by groups of 2, 3 or 4. This makes each attempt of evaluation more difficult but, on the other hand, it makes verbal interactions essential to the group in order to coordinate its endeavours and produce a correct work.

3°) Our reserach comprizes four periods (30 sessions) :

a) During their first school year the children had the opportunity, during fifteen 50-minutes sessions, to get familiar with the material placed at their disposition. We asked them to play freely with the bricks, to dispose them on the appropriate board, while respecting the technical constraints imposed by the material.

b) Since the start of the second school year (2nd grade) we asked the children to build, with the bricks placed at their disposition, networks corresponding to a given schema (see figure 1).

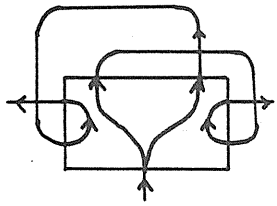


Figure 1.

Each group of children had thus to pass from an iconic representation to an enactive representation of a given finite automaton. The nature of the logical problem associated with this automaton was not specified. The realization of the enactive representation implied manipulations of objects, and was the only way to test experimentally the network and discover the associated logical problem. Moreover, the objects to be handled could not always be placed on the board exactly as they were drawn on the schema. The children had thus to adapt the enactive representation to the constraints imposed by the equipment while respecting the foreseen purpose.

c) During a third period we asked the pupils to test the networks they thus constructed. They had to do it first with one train, then with a sequence of trains numbered 1, 2, 3, ... We then asked them whether they could say what was the networks' use. Thus, one can quickly and experimentally notice that the automaton represented on figure 1 can be used to sort even and odd numbers (the children call this a "counting by 2"). We also asked those children who

to try to create a network for a similar but more complex problem (in the case of figure 1 : "Find a network for a counting by 4").

d) During a last period, we complicated the previous exercise and asked the children to note on a sheet of paper, in some kind of double-entry table : the number of the train (input), the position of the mechanisms when the train enters the circuit, but before it went through them (state) and the exit associated with this train (output). The aim was then to let the children use simultaneously the three kinds of representations : enactive (the network), iconic (the schema) and symbolic (the double-entry table).

III. RESULTS.

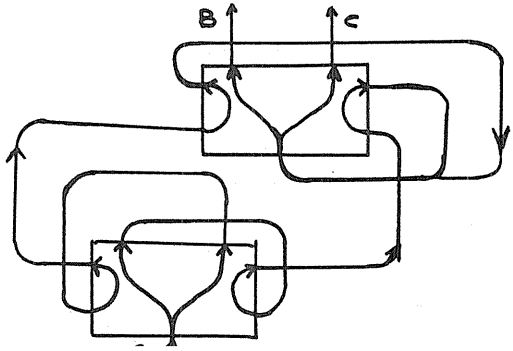
1°) Material constraints.

The children realized very quickly that they could not place the pieces of rail no matter how. They risked indeed either to provoke collisions of trains, or to be unable to let their train run because they did not foresee the space needed to place a piece of rail between the pieces already on the board. After the first 15 introductory sessions, nearly all the children (21 out of 23) knew how to place correctly their bricks.

2°) Verbalization of the action and discovery of the associated problem.

To reach a good result the group of children has to share a limited material. This requires a concertation which favours verbal interactions and thus the development of verbal language. These interactions are of great importance to enable the children to build correctly a network and then discover its use by testing it with trains.

We want to give here two examples. Each team consists of a boy and a girl. The activity takes place at the start of their 2nd grade (October 24, 1979). The children's work is based on the schema shown on figure 2. Both groups were video-taped and knew it.



a) Lény and Catherine build the network. They first discuss about the use of the pieces they must place on the board : they start with the exits B and C. They then go backwards. They frequently look at the schema. The children hesitate about the right way to place one of the mechanisms. Their problem is difficult : how can they let the entry of the second mechanism (which is already on the board) coincide with the bricks foreseen for the exit of the first one (which is not yet on the board) ? After several trials and errors, the children have improved the bricks' position, but several technical problems must still be solved. Catherine suggests to start everything by a mechanism and the entry A, which is easier ; Lény insists on starting with the exits B and C. His technique is more difficult because he must foresee the right spacing between two bricks to let them coincide with the mechanism's exits. Catherine then notices that "one must put correctly the equipment in its box". The children must now solve two problems : firstly, one part of their network is too big to coincide with one of the mechanisms, and secondly the exits have been treated as if they were entries. We help them : we let them notice that, on the schema, there are two mechanisms, but that until now they only placed one on their board. Catherine suggests to move the mechanism which is already on the board : "We must leave space for the second mechanism". One part of the circuit is then correctly connected with this first mechanism. Suddenly Lény says : "I have a fantastic idea, Catherine, one must start with this bazaar (he shows the last mechanism they put on the board, the entrance mechanism), look the arrows goes in that direction". Thereupon Lény demolishes a section of the network which they built but did not orientate correctly : he reorganizes it correctly. The children have now built correctly a large portion of the network, starting from the entrance-mechanism. When they reach the exit-mechanism, which they placed firstly, they correct what they did while saying : "Let's look at the schema". After 19 minutes, Catherine doesn't stop saying : "I found it, I found it". She thinks she understood that a "junction" is needed. Lény, on the other hand, protests : "No, one must start from the lower one, thus, we must start from the entrance-mechanism to build the network". Lény has been able to modify the strategy he used at the start. He has now a good strategy and knows how to inform the others. Suddenly the children realize how to finish their network. They worked alone during 25 minutes.

17 children (out of 23) were able to build a network starting with a given schema. Others had spatial structuration problems.

b) Sarah and Laurent have succesfully built the network associated with figure 2. They study is to discover "what can it be used for". Laurent let correctly a train go through it and explains what he is doing and why. Obviously

the other one. Until now the children use mostly the right part of the network. To do it, one must leave the first mechanism through the left exit. If the children used a second train, they would be able to use the left part of the network : the first train orients the first mechanism to the right and thus direct the second train towards the right exit of the first mechanism and from there towards the left part of the circuit. This perturbs the children who want at all costs to put the mechanism back into its starting position (to the left) and ... they find themselves again, with their train, in the right part of the network. Sarah suggests to turn at a croissing (which is forbidden) or to go backwards (which is also forbidden). Laurent refuses, thinks about Sarah's projects and correctly continues his run. We ask him to let two trains go through, one immediately after the other. He does it and finds out that he came out once through exit C and once through exit B. He explains that this happened because he changed both mechanisms when passing through them with his train : "I also changed here". These children discussed during 15 minutes.

12 children (out of 23) knew how to interprete correctly the network they built and thus how to discover the problem associated with it (e.g. "counting by 2" for the network of figure 1, "sorting multiple of 4 and non-multiple of 4" for the network of figure 3).

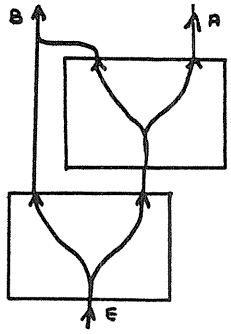


figure 3.

3°) Creation of new networks.

Depending on the nature of the problem, a correct answer was given by all the children or by none of them. This will be studied further in a near future.

4°) Formalization and generalization of the discovery.

If it is certain that 12 pupils succeeded to discover the problem associated with the finite automaton they were building, it is just as certain that the pupils able to express in a formal way their results were not as numerous. As

able to use the double entry table, described previously, to discover its regularities and so the working rules of the finite automaton they study. Two groups of 3 children worked specially well and were able to solve at least partially the problem by drawing up a double entry table corresponding to the automaton they were looking at : all the children of the group watch this table where line after line the following information appears : number of the train, state of the network and associated exit. Using such a table, the children discover often many regularities : without the table, they have nearly no difficulty to discover that the network of figure 3 can be used for a "counting by 4", but they need the table to say, as Youri did : "If I understand properly, when it is right-right, one comes out at A ; the first B, it is left-left ; the second B, it is right-left ; and the third B, it is left-right". He proves thus that he understands the functioning of a finite automaton. The children discover so, on the one hand, that certain numbers are multiple of 4 while others are not, and moreover that among the last ones, one can distinguish numbers of the form 4n+1, 4n+2 and 4n+3.

Children who discover the associated problem but who are unable to draw up the double entry table progress nevertheless regularly and according their own rythm towards the use of symbolic representations.

5°) Team - work and leadership.

Working by small groups is important to favour interactions, but we noticed that some children assume spontaneously the function of leader (all these leaders were self-appointed). When a group does not have a leader strong enough to impose his will to the others, the group does not progress as quickly and reaches less surely a solution : the children do not exhibit the same will to succeed and they become less careful. On the other hand, some groups have leaders and get systematically organized : one of the children will build the network, test it and give the experimental results ; another child will check the schema to be sure that the network is built in conformity with the given schema, later, while the first will test the network, the second will watch the schema to avoid any wrong move ; finally a third one, group-leader, will write down the experimental results as they are given to him.

IV. CONCLUSIONS.

1°) It is obvious that working in groups compels the children to share a restricted equipment to reach a common aim. They must thus act in concert, and this favour verbal interactions and thus the child's verbal development.

2°) The use of a non-verbal auxiliary formalism, such as Cohors-Fresenborg's

could not express otherwise. Indeed the formalism placed here at his disposition is then used as a guide of the child's thought, exactly as the formalism previously used by Lowenthal and Severs (1979).

3°) With the use of this material (the bricks) it is easy to ask the child to solve difficult problems, of which the solution is accessible to him when the problem in stated in a language that the child can understand. This is the case with the non-verbal language of networks. It is also possible, and this is the main part of the present research, to show non-verbal solutions to the child and ask him to infer the problem with which this solution correspond. These problems are presented naturally, in the frame of games and/or created by the children themselves, by their need to generalize results already observed or regularities already noticed.

4°)This material enables finally to confront the child with representations (enactive, iconic and symbolic), different but isomorphic, of a given object. It is thus possible to confront the child very early with problems of invariance of the represented object.

One fact is certain : with few words, a 7 to 8 year old, in a concrete situation, can communicate to someone else the phases of a complex logical operation with the help of an appropriate equipment. This equipment must be the basis of a formalism which suggests a logic in a nearly obvious way. This is the case with finite automata. The child can do this in learning situations only, not in teaching situations.

BIBLIOGRAPHY.

Bruner, J.S. : 1966. On cognitive growth (In : Bruner, J.S., Greenfield, P.M. and Olver, R.R., Studies in Cognitive Growth), New York : John Wiley, 1-67.

Cohors-Fresenborg, E. : 1978a. Learning problem solving by developing automata networks (In : Proceedings of the first MONS Conference on Language and Language Acquisition). Revue de Phonétique Appliquée, 46/47, 93-99.

Cohors-Fresenborg, E. : 1978b. Learning Problem-Solving by developing networks of automata. (In : Proceedings of the 2nd international conference for the psychology of mathematics education. eds. : Cohors-Fresenborg, E. & Wachsmuth, I.) OSNABRUCKER SCHRIFTEN ZUR MATHEMATIK, Reihe D, Mathematisch-didaktische Manuskripte Band 1, 108-115.

Lowenthal, F. : 1977. Games, graphs and the logic of language acquisition - A working hypothesis. Communication and Cognition, 10, (2), 47-52.

Lowenthal, F. : 1980. Language learning and logic. (In : Proceedings of the Cognitive Development Research Seminar, LEEDS, 1979). In print.

Lowenthal, F. & Severs, R. : 1979. Langage, jeu et activité mathématique - Un essai à l'école primaire. Educational Studies in Mathematics, 10, 245-262.

IMPROVING TEACHERS' ABILITY TO VISUALIZE MATHEMATICS

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Many elementary teachers lack the mathematics necessary to creatively teach mathematics or to nuture in their students a positive attitude toward mathematics. The present course was designed to improve elementary teachers' mathematics abilities and attitudes by improving their spatial skills. The course focused on three types of spatial tasks, those involving visual imagery, mental rotation and mental transformation. Employing a problem-solving approach, approximately two-thirds of the class time was spent on developing spatial skills, one-third on improving attitudes. The pre-post test results show that the elementary teachers significantly improved their spatial skills, their problem-solving skills and their attitudes toward mathematics. Informal observations indicated that the teachers changed their mode of teaching mathematics after undergoing such personal change.

Women are entering the universities lacking the mathematics proficiency to insure a full choice of career options (Sells, 1973; Remick and Miller, 1978; NAEP, 1979; Armstrong, 1980). Many of these women go into elementary education, one of the few fields that does not require four years of high school mathematics. They study little mathematics while preparing to become teachers and enter their profession lacking the background to convey a positive attitude toward mathematics to the students they teach. Their students initially like mathematics but grow to dislike it (Ernest, 1976; NAEP, 1979), and as soon as it becomes optional, females choose not to study it. They go on to enter universities without the mathematics necessary to enter many fields of study. Many become teachers, and the cycle continues. This cycle could be broken through intervention at the pre-college level. Young women could be made aware of the necessity of studying mathematics. They could be taught mathematics in such a way that they would believe that they could, indeed, learn mathematics. Their teachers, both elementary and secondary, would have to have the mathematics background requisite to a positive attitude toward mathematics. Such teachers would be able to not only teach mathematics creatively, but would also be able to instill and nuture a positive attitude toward mathematics in their students, who, in turn, would, when given the option, choose to continue their study of mathematics.

The lack of mathematical proficiency in women has two components. First, women avoid mathematics; that is, when the study of mathematics becomes optional,

females choose not to study it (Fennema and Sherman, 1978; Armstrong, 1980; Lantz, 1980). Another facet of this problem involves problem-solving skills. While females perform as well as males in most areas of mathematics, they perform significantly less well than males in problem-solving and application (Armstrong, 1980; NAEP, 1979). Research into the causes of both mathematics avoidance and inadequate problem-solving skills yields similar findings. Two factors, performance on spatial tasks and attitudes toward mathematics, are associated with both problems. Performance on spatial tasks is significantly correlated with mathematics avoidance (Fennema and Sherman, 1977; Armstrong, 1980), as well as with problem solving (Schonberger, 1976; Fennema and Sherman, 1977, 1978). Similarly, attitude toward mathematics is significantly correlated with both mathematics avoidance (Fennema and Sherman, 1977; Lantz, 1980; Armstrong, 1980) and problem-solving skills (Fennema and Sherman, 1977, 1978).

There are at least three types of visual spatial tasks, those involving visual imagery, mental rotation, or mental transformation; however, the majority of research relating performance on spatial tasks to mathematics performance has been focused on tasks involving mental transformation. While there are many studies documenting the relation between mental transformation and mathematics performance, little is known about the relation between visual imagery or mental rotation and problem-solving skills. On the other hand, many attitudes toward mathematics have been investigated in terms of their impact on mathematics performance. Two, confidence in one's ability to do mathematics and an awareness of the usefulness of mathematics, have been shown to be consistently related to mathematics performance, including problem-solving skills.

The present course was designed to improve elementary teachers' spatial skills, problem-solving skills and attitudes toward mathematics. The 30-hour course focused on all three types of spatial ability. While the increase in confidence to do mathematics was a primary goal, the course also dealt with topics ranging from the usefulness of mathematics to sex-role stereotyping in the classroom.

Using a problem-solving approach, the course proceeded along four continua: sequencing types of spatial tasks, sequencing types of materials, sequencing modes of presentation, and sequencing the strategies of instruction. The initial focus was on tasks of visual imagery (no movement of the image), followed by tasks of mental rotation (movement of the complete image) and finally tasks of mental transformation (different movements of different parts of the image).

Each new type of spatial task was first mastered with two-dimensional objects, then with three-dimensional objects. The mode of presentation of each type of spatial task with each set of dimensional objects proceeded from work with concrete objects only, to matching objects to pictures, to finally work involving just pictures of objects. Within each mode of presentation, for each set of dimensional objects and for each type of spatial task, the strategy of instruction moved through the following sequence: a) free exploration of the materials; b) sharing your discoveries with the class; c) working through a structured analysis of the materials, d) some practice with new materials, and e) time to reflect on and synthesize the strategies used. All work was done in teams. The focus on partners was two-fold: 1) to stimulate the teachers' own learning strategies and 2) to help them learn to attend to the differences in individuals' frames of reference, so they, in turn, could better deal with the differences in their students' frames of reference.

One-third of each class session was spent in discussing the processes of learning mathematics, including the skills of 'partnering'. It was through these discussions, concerning both the delights and frustrations of learning mathematics, that the elementary teachers learned to cope with the frustrations and anxieties associated with such learning. They learned to cope with, rather than avoid, mathematics, and by doing so, regained confidence in their ability to do mathematics.

The validation of the program occurred during its second year of operation. Three classes were independently evaluated using a pre- and post-test design employing a paired-t-statistic. Three types of tests were administered: three spatial tests, a college-level algebra problem-solving test, and five scales of attitudes toward mathematics, including a measure of confidence and a measure of usefulness. Significant improvement in all categories was made by each class.

TABLE 1

CHANGE IN ATTITUDES TOWARD MATHEMATICS

1979 Winter Class

Fennema-Sherman * Scales	Mean Scores			
	Pre-test	Post-test	Difference	Significance
Anxiety	1.63	1.47	- .16	$p < .01$
Confidence	3.80	4.17	+ .27	$p < .01$
Effectance	3.85	4.06	+ .21	$p < .05$
Usefulness	4.35	4.44	+ .09	$p < .10$
Male Domain	.37	.26	- .11	$p < .01$

*Fennema-Sherman Mathematics Attitude Scales
University of Wisconsin
Madison, Wisconsin

TABLE 2

CHANGES IN COGNITIVE ABILITIES

1979 Winter Class

ETS Tests *	Mean Scores			
	Pre-Test	Post-Test	Difference	Significance
Mathematics Aptitude (Problem Solving)	6.72	8.52	1.80	$p < .001$
Hidden Figures (Visual Imagery)	7.90	10.79	2.89	$p < .001$
Cube Comparisons (Spatial Rotations)	6.86	10.31	3.45	$p < .001$
Surface Development (Spatial Transformations)	19.24	21.07	1.83	$p < .10$

*Factor Referenced Cognitive Tests
Educational Testing Service
Princeton, New Jersey

TABLE 3

CHANGES IN COGNITIVE ABILITIES

Validation Class - 1

ETS Tests*	Mean Scores			
	Pre-Test	Post-Test	Difference	Significance
Mathematics Aptitude	5.06	6.71	1.65	$p < .005$
Hidden Figures	6.47	7.94	1.47	$p < .10$
Cube Comparisons	7.18	9.41	2.23	$p < .01$
Surface Development	12.41	15.41	3.00	$p < .01$

Validation Class - 2

ETS Tests*	Mean Scores			
	Pre-Test	Post-Test	Difference	Significance
Mathematics Aptitude (Problem Solving)	6.00	7.48	1.48	$p < .005$
Hidden Figures (Visual Imagery)	7.76	10.00	2.24	$p < .005$
Cube Comparisons (Spatial Rotations)	8.48	9.86	1.38	$p < .05$
Surface Development (Spatial Transformations)	16.00	18.95	2.95	$p < .05$

Validation Class - 3

ETS Tests*	Mean Scores			
	Pre-Test	Post-Test	Difference	Significance
Mathematics Aptitude	5.65	7.65	2.00	$p < .001$
Hidden Figures	6.30	8.25	1.95	$p < .05$
Cube Comparisons	7.40	9.30	1.90	$p < .005$
Surface Development	14.70	19.20	4.50	$p < .001$

*Factor Referenced Cognitive Tests
Educational Testing Service
Princeton, New Jersey

The focus on spatial tasks yielded three major benefits. The first was the obvious one, the improvement in one's spatial abilities. The other two were more subtle. Since a focus on spatial tasks was a completely new experience to the elementary teachers, a true problem-solving approach could be employed. Mathematics educators recognize that once a student "knows what the answer should be," a real problem-solving approach is abandoned in favor of a strategy to "find something that yields the right answer." A focus on new material, such as spatial tasks, eliminated the possibility of "already-known answers" and allowed the teachers to really problem solve.

Another subtle benefit arising from a focus on spatial tasks also stemmed from the fact that the spatial material was new. The teachers really did experience the frustrations and anxieties associated with learning, and since the material was not seen as remedial in any way, these frustrations and anxieties were not associated with past failure but were seen to be quite normal. This perspective on learning anxieties facilitated the learning to cope with such anxieties, and thereby improved the teachers' attitudes toward mathematics, particularly those concerned with confidence and failure due to lack of intelligence.

Each teacher was visited in the classroom at least twice. In addition, the teachers were required to keep a math journal. The teachers reported (and the staff observed) such changes as: 1) attention given to students' feelings and frustrations associated with learning mathematics; 2) more confidence placed in children's ability to learn, even if it took some children much longer than others; 3) a greater willingness to let children learn on their own rather than just follow the teacher's pre-set directions; 4) more attention given to the use of visualizing mathematics; 5) more use of concrete materials in general; and 6) more use of math stations and small group instruction.

REFERENCES

Armstrong, J.M. Achievement and participation of women in mathematics: An overview. Education Commission of the States, 1980.

Ernest, J. Mathematics and sex. The American Mathematical Monthly, 1976, 83, 595-614.

Fennema, E. and Sherman, J. Sex-related differences in mathematics achievement, spatial visualization and socio-cultural factors. Journal of Educational Research, 1977, 14, 51-71.

Fennema, E. and Sherman, J. Sex-related differences in mathematics achievement and related factors: A further study. Journal for Research in Mathematics Education, 1978, 9, 189-203.

Lantz, A.E. Determining the importance of factors influencing the election of mathematics courses. Final report to NSF Grant: SED 78-17103, 1980.

National Assessment of Educational Progress. Mathematical applications, Report no. 09-MA-03, 1979.

National Assessment of Educational Progress. Attitudes toward science, Report no. 08-S-02, 1979.

Remick, H. and Miller, K. Participation rates in high school mathematics and science courses. The Physics Teacher, 1978, 280-282.

Schonberger, A.K. The interrelationship of sex, visual spatial abilities and mathematical problem solving. DAI, 1976.

Sells, L. High school mathematics as the critical filter in the job market. In Developing opportunities for minorities in graduate education. Proceedings of the Conference on Minority Graduate Education at the University of California at Berkeley, 1973.

THE TRAINING OF TEACHERS IN THE USE OF CLINICAL METHODS *

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Teachers have to go beyond the students' written work in order to determine the ground upon which they can help them build new concepts and also to monitor how the learners negociate these constructions. In order to achieve this, the present article suggests that teachers be trained in the use of clinical methods such as the "The Clinical Interview" and the "Teaching Experiment". Illustrations of the pedagogical relevance of these methods are given and some exploratory work in this direction is reported.

INTRODUCTION

> We have to abandon the image of the unchangeable subject, mathematics, which is passed on to the student by the teacher. On the contrary, the subject matter varies and changes in the course of the teaching process and as well in the individual learning process.
> Furthermore, the students develop interpretations and strategies on their own, which may remain hidden for a long time, although active, if not detected by means of a mistake or an unusual solution. The extent to which a teacher can adapt to such situations characterizes the quality of teaching.
> (Heinrich Bauersfeld, Proceeding of the Third International Congress on Mathematical Education, Karlsruhe , 1976)

As pointed out by Bauersfeld, the learning of mathematics is not merely a matter of exposition by a teacher in front of a class. The acquisition of mathematical knowledge involves much more the student's construction of concepts and the relevant skills. Such a perspective, far from ignoring the teacher, places him in a more demanding role, that of guiding his pupils in their constructions.

The very idea of helping the learner build some new concept implies the existence of a basis on which it can be erected. Thus, the teacher needs to find

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a simpler or equivalent notion existing in the pupil's cognition to which the new concept can eventually be linked. A constructivist approach takes the student's knowledge as a starting point and seeks to expand it gradually until it can be connected to the new material. However, at each step in such a construction, the instructor needs some feedback in order to assess the learner's interpretation. This interpretation can seldom be evaluated solely on the basis of performance on a written task. Erlwanger (1975) has shown that children can invent a multitude of idiosyncratic rules in order to provide *the right answer*. Access to their hidden interpretations and strategies can only be gained through clinical methods. These clinical methods can help the teacher probe the student's mind in order to determine at first the ground upon which to build and then to see how the learner *negociates* his construction of the new concept. The continual feedback available in individualized interviews allows the instructor to determine immediately at which point the student's interpretations begin to diverge from his own and thus provides both teacher and pupil with an opportunity for adjustement.

The need for teachers to familiarize themselves with clinical methods becomes particularly evident in the light of recent investigations regarding the meaning of mathematical symbols. Behr, Erlwanger, and Nichols (1976) and Ginsburg (1977) have shown that children interpret the symbols used in arithmetic differently from adults. At the secondary level, Wagner (1977) has shown that a change in letter in an equation can affect nany students. Herscovics (1979) has evidence that students can indeed acquire mathematical concepts but find it difficult to express them symbolically. All these results point to the features of mathematics which happen to be major learning obstacles for the student, that is, its abstract and symbolic nature. Teachers may feel perfectly at ease with the notation they introduce. However, their own formal thinking may prevent them from assessing the cognitive strain imposed on a non-formal mind by mathematical symbols and the manipulation of these symbols (Shayer et al, 1976-78). The resulting communication gap can be overcome by the teachers' use of clinical methods which can sensitize them to their students' thinking processes.

Another reason giving some urgency to the teachers' training in clinical methods is the trend in some countries to eliminate streaming in the public school sector. Whereas low achievers used to be directed into lighter courses, their inclusion into regular classes confronts the teacher with a

much more heterogeneous group of students making it increasingly difficult to finds ways of reaching all their pupils. Clinical methods can give them the necessary tools for diagnosis and remediation.

Although it is usually in the context of diagnosis for remediation that clinical methods are discussed, their use in uncovering the thinking of all students makes them relevant to everyday teaching. Current psychological techniques used to penetrate the pupil's mind are the *clinical interview* which is essentially a diagnostic tool (Opper, 1977; Easly, 1977) and the *Soviet teaching experiment* (Menchinskaya, 1969; Kantowski, 1979) which provides the clinical interview with an instructional dimension allowing for the study of the student's thinking while he is being taught.

Following a brief description of these two techniques and some examples showing their relevance, we will discuss some exploratory work on their use by elementary and secondary school teachers. This work seems to indicate the need for a prior training of teachers in the practical applications of models of understanding and learning models which can then provide the clinical methods with a conceptual and a pedagogical frame of reference.

THE CLINICAL INTERVIEW

This technique, developed by Piaget, has been specially conceived to bring to light some evidence regarding the intellectual processes being used (Opper, 1977). It takes the form of a dialogue in which the interviewer questions his subject on his actions, his hypotheses, his predictions, the generality of his concepts, and the reversibility of his operations. Should intersubject comparisons be desired, standardized or at least semi-standardized versions of the interview are needed: same material same manipulations, same questions. However, some flexibility must remain allowing the researcher to explore new and interesting avenues or to clarify answers considered ambiguous.

The clinical interview lends itself particularly well to the analyses of errors and the evaluation of mental blocks in the construction of a concept or the development of a mathematical skill. It is well known that most errors made by students are not due to their lack of attention but to false rules which they have invented and which seem perfectly logical to them (Erlwanger, 1975; Ginsburg, 1977). It is equally well known that at the basis of any difficulty in developing some mathematical skills one finds gaps

in the student's construction of the underlying mathematical structures. In the case of errors, the pupils can tell us the rules which they use, but when the students simply cannot perform some task which has been taught, we can scarcely expect them to tell us about the structures which they have not yet built.

However, in order to help the student complete the construction of the necessary schemas and also to correct his false rules, the teacher needs to assess the level of his cognitive development. We think that it is only through the subtle and flexible questioning of the clinical interview that the teacher can hope to encompass the student's level of cognition and unveil the causes of his difficulties. The following example will confirm its relevance for teaching.

The graph of an equation in two variables is defined as the infinite set of points in the plane whose coordinates are the solutions of the equation. Thus, the graph of a linear equation is a set of points which result in a straight line. This does not cause any problem to those who can conceive of the straight line as a set of points. However, this is not the case for a majority of students (Kerslake, 1977). Indeed, if students are asked how many points there are in a give straight line segment, they may very well answer NONE, ONE, TWO or THIRTY! Clinical interviews of a few students in grade 9 have shown that these answers were far from being silly, but reflected their perception: the subject answering NONE could not see any relationship between points ans a straight line; the one answering ONE explained that he was thinking of the midpoint, and for the pupil answering TWO, they were the two endpoints of the line segments (Herscovics, 1979).

In trying to probe the subjects' understanding of point and continuity, we have repeated one of Piaget and Inhelder's (1947) experiments. The task involved the partitioning of a straight line segment and the repetition of this operation, infinitely many times, in order to imagine the shape of the end-product of the subdivision. Following is an excerpt of an interview with a subject who has just performed some subdivisions:

Q. How long can you keep doing this (subdividing)?

A. *Hum, not too much longer.*

Q. How many times?

A. *About 4 or 5 maybe.*

Q. Why can't you keep doing it more than 4 or 5?

R. Eh ! It'll get too small to work with.

Q. Can you think of that as a piece of elastic that you can stretch in your mind?

A. *Well... If it stayed stretched, we could keep cutting it until it gets to the same size again.*

. . . . . .

Q. Now, what do you think would be left at the end?

A. *Something so small, I guess we wouldn't be able to see it.*

Q. In that little bit that's going to be left, how many points do you think there will be?

A. (surprised) *Points?*

Q. Do you think it will have any shape?

A. *It would, if you could look under a microscope or something.*

Q. What kind of shape do you think it will have?

A. *A square.*

Q. A square?

A. *I think so, because a line has a width....*

. . . . . .

Q. Here are two points. How many points do you think you could put between those?

A. *Same size?... About 30 maybe.*

Q. And if they were very small?

A. *If they're half that size, 60 of them.*

Q. But if they were even smaller?

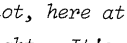
A. *You could, I guess, you could put thousands of them.*

. . . . . .

Q. What would I get at the end?

A. *A line.*

Q. Here you tell me that you'd get a line by putting in the points. But you said that a line is not made up of points.

A. *Well, eh, it's hard to say what a point is really! 'Cause some people mark a point like that* (makes a dot with his pencil) *A point means the centre, like that. So I think like a point-line kind of waves* (draws ). *It is not, here at the top* (shows a line traced with a ruler), *it's all flat, right. It's all the same all the way along. But if you have a point and another point, it goes waving*

Q. Why?

A. *Well, if it's two circles...*

Q. So you think of a point as being a circle?

A. *Yeah.*

It is quite obvious that this student (considered very able by his teachers) had not abstracted the notion of point and confused it with that of a small circle, that is, a dot possessing a dimension. This kind of confusion led him to state that *many straight lines could go between two points* which he explained by drawing:



The information gathered in such an interview reveals the student's thinking and can prove to ve baluable for the teacher in preparing his classes. For instance, in an experiment on the teaching of graphs (Herscovics, 1979), subjects were asked to *cover* the straight line with points. This remedied their initial inability to perceive the straight line as a set of points. Two or three such examples proved sufficient for the necessary abstraction.

This last example has shown how a clinical interview can have a practical application in planning for instruction. Some learning may also occur during the clinical interview when the subject is questioned on a topic new to him. However, this is seldom by design on the part of the interviewer. Should learning and teaching become the focus of the study, the teaching experiment, another clinical method, should be considered.

THE TEACHING EXPERIMENT

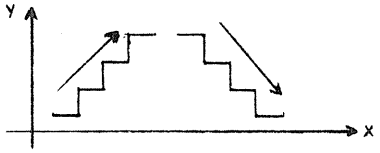
The method known as the *Soviet teaching experiment* adds to the clinical interview an instructional dimension and discloses the child's schemas while they are being constructed. According to Menchinskaya (1969, a.b.), it aims at reproducing systematically the thinking processes and at studying the *very process of learning as it takes place under the influence of pedagogy*. The data are gathered in a clinical framework by registering the written and verbal answers which are then analysed according to qualitative criteria.

The teaching experiment is longitudinal and involves several sessions during which the plan of instruction is constantly modified to reflect the gathered observations (Kantowski, 1979). This method allows the experimenter to discover the cognitive changes which occur during the lessons and at the same time followsthe same child in the evaluation of his mental processes.

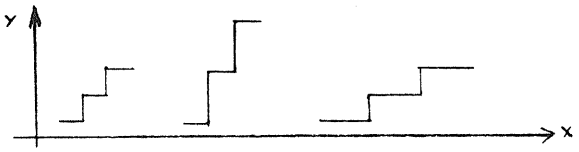
We can illustrate the value of the teaching experiment by describing four pedagogical problems encountered while teaching the concept of slope (Herscovics, 1979). These four problems may appear trivial but each one of

them might constitute a serious obstacle to the student.

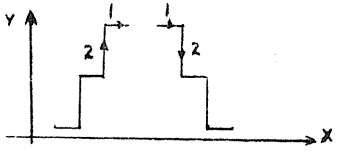
1. Students in grade 9 were asked to draw a rising staircase in the Cartesian plane. Since they were familiar with the positive direction of the axes, no possible confusion was expected. However, the subjects mentioned that they might just as easily go up as well as go down the same stairs, thus indicating that they were focussing on the staircase without viewing it in the context of the plane. The usual convention had to be stated explicitly.



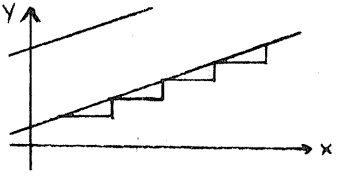
2. They were then asked to draw three staircases by varying the sizes of the threads and risers: staircases where the riser was respectively equal to, double, and half the tread. After ordering them according to their steepness, the subjects were queried about the concept of slope which they related to the relative sizes of the tread and the riser. In asking them to express this relationship, we expected them to represent it as a ratio. However, this was not the case: they used equations instead ($2t = r$ for the second staircase). Of course, the transition from their equation to a ratio was quite easy.



3. The students realized that a convention was needed to distinguish between the slope of a rising staircase and one going down. They readily accepted the usual positive and negative convention but were forever forgetting to use it when measuring the slopes of given staircases. It was only by reversing the problem (drawing staircases corresponding to given slope) that the convention was mastered.



4. The students were presented with a straight line in the plane and were asked *Does it make any sense to talk about the slope of a straight line?* Although they realized that it was related to staircases, they could not connect it to their previous work. Even the leading request *to attach a staircase to the line* had no tangible results, stairs being drawn at random in the plane. When the interviewer drew the required one, the definition of the slope of a straight line became obvious: it's the slope of the attached staircase.



These four minor problems illustrate how difficult it is to predict at which point in the exposition a break down in communication between teacher and students will occur. The teacher who prepares a lesson presumes that it can be followed step by step. However, the formal nature of his thinking may often prevent him from judging when and where he might lose contact with his pupils. By trying out a lesson on a few students, the teacher may discover some unexpected learning difficulties and as a result readjust his pedagogical intervention.

Having described the clinical interview and the teaching experiment and illustrated their pedagogical relevance, we will now discuss their effect on teachers.

SOME EXPLORATORY WORK

In some courses for practicing elementary and secondary school teachers we have found that these techniques were within their reach (Bergeron, Herscovics, 1979). Many of them, some with years of teaching experience, claimed that they had *learned to listen* to their students. Teachers seemed more attentive to their mistakes and more willing to provide help. The clinical interview was used efficiently to uncover the pupil's hidden rules and the teaching experiment served to implement some planned remediation.

The remedial work recommended by the teachers varied widely and was a clear reflection of their teaching. Some teachers at the elementary level were excessively skill-oriented in their teaching of arithmetic, while others were quite knowledgeable of the cognitive difficulties involved in learning sophisticated concepts, such as a place-value notation and the reversibility of operations, underlying these skills. Some secondary school teachers were very formal in their approach to instruction, limiting themselves primarily to their preferred mode of representation, the symbolic one. Others were quite aware of the learning problems involved in constructing meaning for algebraic concepts and were constantly justifying their algebraic operations while reminding their student of the significance attached to the algebraic symbols. By and large, teaching and remediation reflected the teacher's own appreciation for the complexity of the concepts he was introducing and the complexity of the learning processes involved.

Although most teachers felt that they were *teaching for understanding*, only

a few had reflected on what they meant by it. Perhaps it is as a result of their initial training that *understanding mathematics* was too often translated into *using the correct rule and finding the right answer*. Such a limited interpretation does not lend itself to a cognitive analysis of the subject matter taught, thus limiting the scope of the clinical methods. This has led us to believe that their introduction must be preceded by some experience with models of understanding. These models identify different modes of understanding and, in applying them to a given concept, teachers can become more aware of its sophistication. Their analysis can give them the necessary perspective to view the relevant *rules and answers* in a broader conceptual framework.

The relevance of the modes of understanding to the learning processes can best be appreciated in a psycho-pedagogical context. Although many teachers have taken a course in educational psychology, few of them can relate it to the practical problems of learning and teaching. Thus, we feel that the introduction of models of understanding ought to be followed by some exposure to learning models. By applying these to a specific concept, teachers can seek out the various representations of this concept and use them in the constructive process involved in its assimilation.

For the clinical methods to be used to their full potential in both teaching and remediation, it seems that they must be preceded by sufficient experience with models relevant to understanding and to learning. That these models are accessible to teachers and have some practical value is evidenced by the results reported in the following article "The Training of Teachers in the Use of Models of Understanding and Learning Models" which cannot be dissociated from the present one.

REFERENCES

BAUERSFELD, H. Research Related to the Mathematical Learning Process, in Proceeding of the Third International Congress on Mathematical Education: Athen A. and Kunle H. (Eds) Karisruhe, 1976.

BEHR, M., ERLWANGER, S., NICHOLS, E., How children view equality sentences. PMDC Technical Report no 3. Florida State University (ERIC document reproduction Service no Ed 144 802), 1976.

BERGERON, J.C., HERSCOVICS, N. L'expérimentation didactique soviétique, une méthode de recherche à la portée des enseignants: Conférence donnée conjointement au congrès annuel de l'Association Mathématique du Québec, Hull, 1979.

EASLEY, J.A. On clinical Studies in Mathematics Education, in Mathematics Education Information Report: ERIC/SMEAC, Columbus, Ohio, 1977.

ERLWANGER, S.H. Case Studies of Children's Conceptions of Mathematics, in The Journal of Children's Mathematical Behavior, 1975, 1 no 3 157-283.

GINSBURG, H. Children's Arithmetic, New York: Van Nostrand, 1977.

HERSCOVICS, N. The Understanding of Some Algebraic Concepts at the Secondary Level, in The Proceedings of the Warwick Meeting of the IGPME (International Group for the Psychology of Mathematics Education), 1979.

KANTOWSKI, M.G. The Teaching Experiment and Soviet Studies of Problem Solving, in Hatfield L., Bradbard, D. (eds): Mathematical Problem Solving: papers from a research Workshop, Hatfield, L. and Bradbard, D. (eds). ERIC/SMEAC, Columbus, Ohio, 1979.

KERSLAKE, D. The Understanding of graphs, in Mathematics in School, 1977, 6, 2, pp 22-25.

MENCHINSKAYA, N.A. Fifty years of Soviet Instructional psychology. The psychology of mastering concepts: Fundemental problems and methods of research, Soviet Studies in the Psychology of Learning and Teaching Mathematics, 1969 a, b, Vol 1 Kilpatrick, J., and Wirszup, I. (eds). Stanford :SMSG.

OPPER, S., Piaget's Clinical Method, in The Journal of Children's Mathematical Behavior, 1977, 1 no 4, pp90-107.

SHAYER, M., KUCHEMANN, D.E., WYLAM, H. The distribution of Piagetian stages of thinking in British middle and secondary school children 11-14 to 16 year-olds and sex differentials in Br. J. Educ. Psychology, 1976, 46, 62-70.

SHAYER, M., WYLAM, H. The distribution of Piagetian stages of thinking in British Middle and secondary school children 11-14 to 16 year-olds and sex differentials, in Br J. Educ Psychology, 1978, 48, 62-70.

WAGNER, S. Conservation of equation, conservation of function and their relationship to formal operational thinking. 1977. Unpublished doctoral dissertation, New York Univ.

THE TRAINING OF TEACHERS IN THE USE OF MODELS OF UNDERSTANDING AND LEARNING MODELS*

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*It has been argued** that training teachers in the use of clinical methods must be considered if they are to appreciate the child's own thinking processes while analyzing the pedagogical value of their intervention. In addition to clinical methods it is suggested that teachers need a frame of reference for their observations and analyses. We propose as a starting point a model of understanding and a learning model, the uses of which are illustrated with the concept of area.*

MODELS OF UNDERSTANDING

In a workshop involving about 30 elementary school teachers and some mathematics consultants (Herscovics, 1978), a brief exposition was made of the available models of understanding. They were first introduced to Bruner's model (1960) which describes two modes of understanding by contrasting intuitive thinking (global and implicit perception of a problem, unawareness of the processes used in getting the right or wrong answer) with analytic thinking (steps are explicit, full awareness of the relevant information and operations). The second model was Skemp's (1976) which distinguishes between instrumental understanding (rules without reason) and relational understanding (knowing what to do and why). Finally, they were presented with the Byers and Herscovics model (1977) which is not only a synthesis of the previous two models but furthermore distinguishes between mathematical content and mathematical form. This allows for the inclusion of analytic understanding into the broader category of formal understanding which is described by:

- the ability to connect mathematical symbolism and notation with relevant mathematical ideas;
- the ability to combine these ideas into chains of logical reasoning.

In trying to apply these models to a specific concept, the teachers in this workshop selected the notion of area. They were quite aware that the learning

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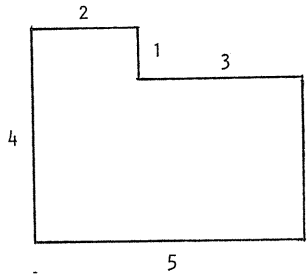
** See preceeding article: "The Training of Teachers in the use of clinical methods"

of this concept is far from being a trivial endeavor and that many children develop numerous misconceptions about it. Agreeing that most pupils will give 20 as the area of a 4 by 5 rectangle, the participants were not surprised by the fact that even in the upper grades of the primary level, many children will give the same answer for the area of the following figure:



In their analysis of this error, teachers thought that these students exhibited an instrumental understanding of the area of a rectangle which they generalized incorrectly to other shapes. The teachers felt that a multiplicative approach to area worked wonders for rectangles but broke down in the case of figures which needed partitioning and recombining.

They concluded that introducing the notion of area through highly specialized forms such as squares and rectangles may in fact encourage misunderstanding and interfere with the child's construction of the general concept. Thus they recommended that the concept of area be introduced in the most general context using an arbitrary non-rectilinear figure.

Teachers were aware of the widespread confusion, even among adults, between surface and area. Defining area as the measure of a surface, they identified four modes of understanding which they described as follows.

An intuitive understanding of this concept is based on both the idea of covering and of quantity but without that of specific measure. It involves the notion of space-filling and many activities can help develop the child's awareness : coloring in of figures, cutting of material such as cloth, covering books, etc. Of course, partial coverings or superimposed coverings are not acceptable. Questions concerning the invariance of area need to be raised, i.e. *Will the covering of a given figure require more paint if it is partitioned?*

A relational understanding would be revealed by a measured quantification of the notion of covering. A non-rectilinear closed form would be covered with unit squares which would then be counted. Since such a covering cannot be exact, the answer found would only be an approximation which could then be improved by the choice of smaller measuring units. However, as with everything involving cognitive processes, nothing is ever certain and this procedure of *covering and counting* can be learned instrumentally. This can be

verified indirectly by providing the child with less than the number of squares needed to cover the given figure. If he then gives the number of squares provided as the area of the figure (which is only partially covered), this can be construed as an instrumental understanding (counting squares). Of course, some children can overcome this problem by first counting the units in the partial covering and then reusing them for the area of the remaining region. This is relational understanding indeed!

Some participants in this workshop, having had a course in calculus, found it easy to describe a formal understanding of area. They pointed out that it could be defined by a Riemann integral. Of course, this was not to suggest that the limit of a sequence of Riemann sums should be introduced at the primary level!

The teachers were quite pleased with the results of two hours of work. Many of them felt that they had been enriched by this experience and had gained a better appreciation for the concept of area. Most of them thought that the general concept of area not only could,but indeed should be taught, before applying it to rectilinear figures. Of course, having described the various modes of understanding seemed to make this all the more feasible. As the results of this workshop seem to indicate, the analysis of a concept in terms of modes of understanding tends to shift the teacher's preoccupation from skills to that of the underlying cognition. The models of understanding can provide a framework and the vocabulary for discussing the comprehension of mathematics. However, they do not deal explicitly with the acquisition of comprehension, this acquisition being more specifically the domain of learning models. These learning models can provide the teachers with the pedagogical framework needed in the preparation and the analysis of a teaching experiment. We illustrate this with the next example.

LEARNING MODELS

Since the purpose of introducing learning models is to train teachers in observing the cognitive processes of their students, the only models that are relevant are those based on cognitive psychology. Various models can be applied to different school levels and to different mathematical activities. However, since we wish to continue with our example of area, we will use a model particularly useful for describing the construction of concept at the primary level. This model is based on Bruner's (1966, 1973) theory of intellectual development.

Bruner's theory can be described in terms of modes of representation which he perceives as enactive (based on concrete operations) iconic (involving images), and symbolic (the symbols not being images of the object). Each one of these means of representation contains different amounts of information and knowledge but their use must take into account the age and maturation of the learner. Bruner thinks that the young child, in the usual course of his intellectual development, moves from enactive through iconic to symbolic representations. He warns against a premature stress on symbolism, for a child lacking sufficient experience with the other two modes would be missing the imagery upon which to fall back should his symbolic transformations fail to succeed in problem solving.

The notion of area, discussed in the last section, will be used to illustrate a learning scheme based on these three modes of representation.

The covering and cutting activities are without any doubt at the level of enactive representation. Even when involved in measuring, the child would initially need the opportunity to cover many irregular shapes with small squares which he would then count. He would thus be able to measure the area of a surface subjected to partition, translation or rotation and, the discovery of its invariance under these transformations would help in his assimilation of this concept. It is only after experiencing a great variety of concrete activities involving covering and measuring that a child should be introduced to an iconic representation.

Such an iconic representation would be the covering of a given figure with a grid (images of squares) and the counting of the squares in the covering. The use of finer grids would lead to improved approximations. Symbolic representations ought to be limited to the more specialized shapes.

Following the acquisition of the general concept of area, the child can apply it to a rectangle by counting the squares needed to cover it. But, a premature introduction to the formula would limit its justification to giving him the right answer without explaining why (i.e. an instrumental understanding of the formula). A better understanding can be achieved by providing a large enough rectangle where the counting of squares is a sufficiently lengthy task creating the motivation for a more efficient method. Since each row contains the same number of squares, their sum could be obtained by repeated addition which translates into multiplication of the number of rows by the

number of columns. This would justify the validity of the formula AREA = BASE X HEIGHT, whence a formal understanding of the area of a rectangle.

The acquisition of a formal understanding of the area of a rectangle can in turn be used to justify the formulas for the area of other geometric figures such as the triangle, the parallelogram, the rhombus, etc. For instance, it is easy to have a child construct a rectangle having the same base and the same height as a given triangle. It is then just a matter of cutting up the pieces of the rectangle and to superimpose these on the triangle to discover that the area of a triangle is exactly half the area of the rectangle.

This example shows how a learning scheme can quite easily be developed on the basis of Bruner's theory. However, the prior identification of the different modes of understanding facilitates this task. That representation and understanding are related is evident. Bruner himself has linked enactive and iconic representations to intuitive understanding, and the symbolic mode to what he termed *Analytic thinking*.

This kind of exposure to both models of understanding and learning models may provide the teachers with a conceptual and pedagogical frame of reference in which the clinical methods can be used to study the cognitive processes involved in the learning of mathematics.

CONCLUSION

We have tried to justify the need for training teachers in the use of clinical methods. In our opinion, such training could improve significantly the quality of their teaching.

Our approach is based on a conception of the learning of mathematics as a constructive process in which the teacher is both the architect and the engineer This is somewhat contrasting with the experience of the last twenty years that can be characterised by an emphasis on curriculum development. That this work on curriculum did not yield the expected results only points out that no program or textbook can cope with all the learning problems encountered in a classroom. Only a teacher can.

In order to cope with these learning problems, the teacher has to be autonomous and innovative. Some training in the use of clinical methods may

provide him with the means. Indeed, we believe that these methods are essential to appreciate the child's own thinking processes and to analyze the pedagogical value of one's intervention.

The models of understanding and the learning models give the teacher a frame of reference for his anslyses and provide him with a more subtle form of evaluation going beyond the mere verification of skills at the instrumental level. Such an evaluation is essential for the teacher who has to work with a heterogeneous class and who must concern himself with the optimal growth of each student. The integration of children with minor learning difficulties into the regular classes further increases the demands made on him.

Our experience in programs involving both future teachers and practicing teachers seems to indicate that in order to attune them to their students' cognitive processes, any introduction to clinical methods must be preceded by some work with models of understanding and learning models. This was expressed by some teachers in a course on the Psychology of Learning Mathematics in which the first four weeks were spent with a psychologist on clinical interviews and the following month on models of understanding with one of the co-authors. They felt that had the order of topics been reversed, they *would have known what to look for* in their interviews with children.

We are presently embarking on a three year research project in which we plan to study systematically the training of teachers in the use of the models and of clinical methods. We wish to determine their impact and hope to develop some criteria for evaluation. We will be happy to communicate with interested colleagues and would be pleased to receive their criticisms and suggestions.

REFERENCES

BRUNER, J. The process of education: Cambridge, Harvard, 1960.

BRUNER, J. Toward a theory of instruction: Cambridge, Harvard, 1966.

BRUNER, J. The relevance of education: New York, Norton, 1973.

BYERS, V., HERSCOVICS, N. Understanding school Mathematics in Mathematics teaching, 1977, 81, 24-27

HERSCOVICS, N. Les modèles de la compréhension: Workshop, Association pour l'amélioration des Mathématiques à l'Elémentaire, Piédmont, Québec, 1978.

PIAGET, J., INHELDER, B.: La représentation de l'espace chez l'enfant: 3ième édition, Paris, Presses Universitaires de France, 1947.

SKEMP, R.R. Relational and Instrumental understanding, in Mathematics Teaching, 1976, no 77

CLINICAL INTERVIEWING: ITS POTENTIAL TO REVEAL INSIGHTS IN MATHEMATICS EDUCATION

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The paper discusses the potential of the clinical interview to provide insights on individual concepts, conceptions of mathematics and idiographic rationality systems. Discussions of each component are followed by exerpts of clinical interviews which illustrate them.

Recently, some of the most compelling and disconcerting findings in mathematics education have been revealing a dramatic gap between mathematics students' possession of algorithmic techniques and their understandings of the processes which underlie those techniques (Davis and McKnight (1979) Erlwanger (1975) Peck and Jencks (1979)). In this paper, I will suggest that the clinical interview is uniquely suited to explore this gap, and that through the interview three basic components are revealed: 1) the extent and diversity of individual concepts, 2) various conceptions of mathematics and 3) idiographic systems of rationality. Following the discussion of each component exerpts of clinical interviews illustrating each component are provided. Afterwards, the components are joined to describe why remediation of basic concepts is often so difficult in mathematics.

INDIVIDUAL DIFFERENCES IN CONCEPTS

One advantage of the clinical interview is that it does not assume a basic unity among students' concepts. Most traditional measurements of student learning in mathematics assume that individual differences in students' concepts either vary insubstantially or are unimportant in their influences on the mathematics studied. Mathematical individual differences are frequently reduced by mathematics teachers to dividing students into categories of high, average, or low ability. Those abilities then are assumed to be a function of the speed, efficiency and accuracy of mastering a set of fixed, nonproblematic concepts. In contrast, if one assumes that there are a variety of ways of understanding a concept mathematically, individual differences in mathematics become the diversity in students understandings of concepts or of mathematics itself. The clinical interview provides a means for searching for and exploring these individual understandings. Not only are delightful and perplexing student understandings revealed, but often these, in turn, reveal surprising insights into mathematics.

To illustrate, consider the following exerpt from a clinical interview of a ninth-grade student on the topic of fractions:

Interviewer: When you see the fraction, one over three, what do you think of?

Student: Two. One third is like that, so you get two left over. (drawing: )

Interviewer: What about 3/4?

Student: Point twenty five. (drawing: )

Interviewer: That's why you are calling it .25 when you were doing it before (Earlier, in a mixed sequence of fractions and decimals, he had consistently transformed 1/4 to .75 and 3/4 to .25) What about 3/5?

Student: 50 or 2. (drawing: )

Interviewer: 2 because there are two left over?

Student: Uh huh

Interviewer: Why 50?

Student: and 50 from 125

Interviewer: Where do you get 125?

Student: Well, I'm going by 25's and 5 x 25 = 125. Hold it, yeah.

Interviewer: So it's 2 left, or 50. Why 125?

Student: I don't know. I just do it all the time whenever I add it.

This student consistently attended to the part of the region which was empty, rather than which was darkened. Only in the translation of decimals to fractions does his misconception become apparent for within a system of numbers he operates appropriately. An error in changing .75 to 1/4 or .25 to 3/4 is likely to be passed over by a teacher as careless since the answer was a multiple of .25 or 1/4. The inability to modify his unit (.25 to .20 for fifths) is more likely to be taken seriously, but without having a teacher who has the time or awareness to listen carefully to the student, often the error is corrected without determining what the student's conception is.

The example just cited illustrates something which we might feel comfortable labeling a misconception; however, there are also examples in which various interpretations of concepts exist which can only be labelled as alternative conceptions. In these, the student provides a consistent and feasible alternative way of understanding the concept. It is these alternative conceptions which often reveal interesting angles on mathematics.

The importance of the example lies in the fact that in six hours of interviewing with each of 21 ninth grade students this basic diversity in understanding concepts is <u>not</u> unusual, nor is it an extreme example of a deviation.

One phenomena which seems to be happening in schools is that formal algorithmic techniques are being taught, practiced, tested and to a large extent, mastered as strings of symbols which have no informal representations to support them. In contrast to a superficial uniformity communicated by mechanical manipulation of strings of symbols, individual diversity in underlying concepts can be both awe-inspiring in its creativity and discouraging in its contortions. Before we can begin to explore the range of it, we must recognize its existence. This can be accomplished through conducting a rather small number of carefully constructed clinical interviews, which will reveal the variety of individual differences and suggest fruitful directions in which to proceed.

CONCEPTIONS OF MATHEMATICS

As one recognizes the diversity in the mathematic concepts of students, one also begins to realize that an individual's concepts taken as a group and combined with the student's purposes and attitudes toward mathematics constitute a whole which can be described as the student's conceptions of mathematics. Erlwanger (1975) described this as follows:

> ...in the course of learning mathematics a child develops his own ideas, views and beliefs about mathematics which can be represented as his conception of mathematics. This conception of mathematics may be regarded as a developing conceptual system of interrelated ideas, beliefs, emotions and views concerning mathematics and learning mathematics that directs and controls his mathematical behavior, what he learns and what he understands. From this point of view, a child's observable mathematical behavior may only be interpreted and explained to the extent that his underlying conception is understood (p. 167).

In addition, in an earlier paper (Confrey, 1978) I stated that how a student perceives a subject: the authority of its claims, the rigidity of its methods, the formality of its structure, the truth of its assertions, and its purposes, and values can be described as a part of his/her conceptions of mathematics and should be measured as an alternative outcome for mathematics educational treatments to achievement and attitude. One method of evaluating such a construct is through direct questioning for which I developed a questionnaire. Also in the IEA study (1979), a section has been provided on conceptions of mathematics which differentiates between students who perceive of mathematics as a process vs. as a fixed set of rules. However, in addition to direct measures, a clinical interview devoted to solving mathematical tasks also provides a unique opportunity to pursue students conceptions of mathematics within a context which may have more contextual validity.

Using the interview in this fashion is accomplished through focusing on moments of change in mathematical understanding; be it transitions, hesitations, confusions or contradictions. Whereas the particular conceptual information displayed at these times is often not reliable, because it vacillates so quickly, the students are very likely at this time to reveal fundamental assumptions and commitments about mathematics generally onto which they hold tenaciously. These parallel the function of what Lakatos (1970) refers to in the philosophy of science as "theoretical hardcore," or the most centrally held and protected axioms of a system.

The following passage from a clinical interview illustrates how these conceptions of mathematics can be revealed through the problem-solving structure of the interview, if the interviewer is sensitive to their existence. Notice this passage also illustrates the gap between acquring algorithmic techniques and understanding. In this case, the student is given two tasks and questions: 1) Change 12/5 into a mixed number. Why does it work? 2) Change 2 1/3 into an improper fraction. why does it work? Her responses were as follows:

Student: Because five into twelve is two times with two left over.
Interviewer: Why does dividing there work?
Student: Because it's supposed to; because that's the way I was taught.
Interviewer: Because someone told you that?
Student: Yeah.
Interviewer: Does it make sense in terms of fractions?
Student: Yeah.
Interviewer: why?
Student: It just does.
Interviewer: You mean it just makes sense to divide 5 into 12 and get 2 with 2 left over. Can you draw me a picture to show that's true?
Student: No.
(She then changes 2 1/3 into 7/3)
Interviewer: Why?
Student: Because (3 x 2) + 1 = 7
Interviewer: Why does it work?
Student: Because (3 x2) + 1 is 7 over 3
Interviewer: Is there a reason why you think 2 1/3 should be the same as 7/3?
Student: I don't know; I don't ask questions.
Interviewer: You don't ask questions? Is that the best way to do math?
Student: Uh huh.

The student accepts that asking questions in mathematics is only going to confuse her and figures that if she can get the right answer then the understanding is unnecessary. "You do mathematics the way you were taught" is the conclusion she has drawn. In other examples, students also reveal a sense of arbitrariness about the rules of mathematics claiming "the man in the book makes them up" or agreeing that the rules for lining up decimal points in addition and "counting in" decimal places in multiplication could have just as easily been reversed. The sum total of a students' thoughts on how mathematics works, how it must be learned, what it is about and why its taught in school form a whole which often stabilizes by high school. It is influenced by the mathematics undertaken and equally as importantly, by how the mathematics is taught. In turn, this conception of mathematics also influences what is learned and how that content is learned. As such, a student's conception of mathematics deserves direct attention from educational researchers and teachers alike.

IDIOGRAPHIC SYSTEMS OF RATIONALITY

A third dimension which can be revealed through clinical interviewing is that students have idiographic systems of rationality which influence their pursuit of knowledge. In conducting a clinical interview, one basic tenet is for the interviewer to probe beyond a response into the tapestry of student understandings until the interviewer reaches an understanding of the "why" which was originally implicit in the student's response. In a vast majority of the cases, a rational set of reasons exist which explain the response, and these reasons exhibit reasonable consistency across concepts and internally to a particular student. The question to be asked here is what does "rational" mean used in such a context. A short answer is that "rational" in this context means that for that particular student, the reason justified the response. It made sense to them, and it is the responsibility of the interviewer to have it make sense to him/her.

For example, when asked to fill in blanks on arithmetic sequences a student gave the following response:

2.0 3.2 4.4 5.6 6.8 8.0 9.2 10.4

Although his comment was that "they skipped seven", his responses were correct. However, when asked to construct his own sequence, his system of rationality

became clearer. He wrote:

2.1 4.3 6.5 8.7 10.9 12.11

This student saw the decimal as separating the sequence into two sequences of whole numbers, and he searched for the pattern and solved each one seaprately. Logically, within his system of rationality what he did made perfect sense, and therefore so did his comment on the missing seven.

Students' reasons for an answer vary considerably from an assertion which may indicate that the only way to get an answer is to proceed as rotely as possible or imitate the teacher, the book or other authority, to circuitous routes of calculation whose sheer complexity would deter most of us. Also, of course, there are responses which are disturbing in their insistence that this just is the answer and beyond that is the realm of nothingness.

In the midst of all this diversity in idiographic rational systems come essential questions which straddle the territory of methodology and content. These include: are the responses rationalizations as a response to the interviewer's insistence to produce an answer, rather than reasons which guided the response? Is the pattern of rationality consistent for an individual across responses? How much of mathematics is done rotely, so routinized that asking such question borders on being nonsensical to the student? The answers to these questions required far more clinical interviewing, and the examination and development of interviewing as a methodology.

The value of the clinical interview in describing systems of rationality is that the interviewer attempts to assess the process by which an answer is formed, and the reason why the student believed it to be so. The approach is associated with the need to understand the "whys" and "hows" of a student's beliefs before asserting confidently that it is warranted knowledge for that student.

REMEDIATION

In the previous sections I have described the potential of the clinical interviews to allow probes into the areas of individual concepts, conceptions of mathematics and systems of rationality. I would like to suggest a way to see how these unite to form a system which is resistant to remediation.

One way to view education is to see it as a process of progressively changing sutdents' concepts. concepts are valuable to education in that they exist at a private (individual) level as well as at a public (consensus) level. In learning mathematics, students' private concepts change to correspond more and more closely to the various public versions; this entails a view of successive approximations which is neither singular in its route or goal nor does it require a common starting point. Various routes to different conceptions exist, both starting and ending at different places.

To accomplish the changes, two processes unite: comprehension (individually making sense of the concept) and justification (subjecting those reasons to public scrutiny). In the process of comprehension, the student's prior knowledge and systems of rationality are paramount. In justification, public standards for evidence are made primary. Lastly, and perhaps, more importantly, the processes are intertwined with the content of the concepts, so that the content and processes respond to each other.

If one accepts this thumbnail sketch of learning, then one can begin to see how individual differences in concepts (both as starting points and as successive approximations) and systems of rationality (as they alternate from being comprehension-oriented to justification-oriented) combine as students' concepts are changed through teaching. All of this is then placed within the context of the conceptions of mathematics, which exerts its own force on those two dimensions, and is, in turn, altered by them.

The result is then an integrated whole. Remediation may on occasion simply require readjustment of a single part of that whole, but one must remember besides possibly delodging or altering connected concepts, that change must also meet the private criterion for rationality and be within the student's general conceptions of mathematics. What is being suggested is that we need to know a great deal more about students' mathematical understandings including the variance in concepts, conceptions of mathematics and systems of rationality and how these interact. It is further proposed that the clinical interview can be a means to finding out some of these answers and gaining generalizability in the findings.

While examinations into these issues continue which they are at Michigan State by us, and by the Hawkins (University of Colorado) and by Jack Easley (University of Illinois) and by Dahlgren and Marton (1978) in Sweden, and another strand needs to be developed simultaneously. The clinical interview is conducted in the relatively unusual situation of one-on-one and students respond differently in these situations than in classrooms. Hence, the information provided in the confines of individual interviews needs to be considered in light of the social influences on teaching and learning in the classroom. Ties to teachers' comments, teachers' methods of instructing and their own conncepts, conceptions of mathematics and systems of rationality need to be undertaken. Curriculum need to be examined for their implicit assumptions. Social and cultural influences of peer groups and public pressures of performance on students undoubtedly exert profound influences on the process of teaching and learning mathematics and these need to be studied and then considered together with interview results in order to develop a fuller picture of teaching and learning mathematics.

REFERENCES

All examples are drawn from the clinical interview portion of the General Mathematics study being conducted at the Institute for Research on Teaching, Michigan State University funded by NSF and directed by Perry Lanier.

BIBLIOGRAPHY

Confrey, Jere. Conceptions of mathematics: an alternative to affective and achievement outcome measures (unpublished) 1978.

Confrey, Jere. Number concepts, conceptual change and the introduction to calculus; Unpublished dissertaion, Cornell University, May 1980.

Dalgren, Lars Owe, Marton, Ference. Students conceptions of subject matter: an aspect of learning and teaching in higher education. Studies in Higher Education. Vol, 3, No. 1, 1978.

Davis, Robert. McKnight, Curtis C. The conceptualization of mathematics learning as a foundation of improved measurement. Development Report Number 4. The Curriculum Laboratory, University of Illinois; October, 1979.

Easley, J.A. Jr. On clinical studies in mathematical education. Information Reference Center for Science, Mathematics and Environmental Education; Ohio State University, 1977.

Erlwanger, Stanley. Case studies of children's conceptions of mathematics - part I. The Journal of Children's Mathematical Behavior, 1975, 1 (3), 157-283.

International Association for the Evaluation of Educational Achievement, Second Study of Mathematics, December 1979.

Lakatos, I. Falsification and the methodology of scientific research programme. In I. Lakatos and A. Musgrove (Eds.), Criticism and the Growth of Knowledge. London: Cambridge University Press, 1970, 91-196.

Opper, Sylvia. Piaget's clinical method. Journal of Children's Mathematical Behavior, 1977, 1, 90-107.

Peck, Donald M., Jencks, Stanley M. "Are common fractions out-of-date? The Elementary School Journal, University of Chicago, Vol. 80, No. 1, 1979.

NATIONAL ASSESSMENT: A PERSPECTIVE OF MATHEMATICS ACHIEVEMENT IN THE UNITED STATES

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This paper examines the curricular implications of the results of the recently completed mathematics assessment of the National Assessment of Educational Progress. Specific areas to be examined include: 1) the need for a broader definition of basic mathematical skills, 2) the importance of students' understanding of mathematical concepts and processes, 3) the need to develop problem solving skills, 4) documentation of the development of mathematical skills over an extended period of time, 5) the need to develop alternative algorithms, 6) the need to increase students' enrollment in mathematics classes, and 7) the need to increase students' active involvement in mathematics classroom activities.

In recent years, concern in the United States about student achievement in mathematics and other basic content areas has been fueled by reported declines in standardized test scores. Such reports have been cited as support by those advocating that instruction in mathematics emphasize the teaching of basic computational skills. Recent results from the mathematics assessment of the National Assessment of Educational Progress (NAEP) suggest, however, that a greater emphasis on the mechanical learning of rote computational skills is just the opposite of what most students need. Performance on basic computational skills is generally good, but there are serious difficulties in most basic non-computational skills. Deficiencies in both computational and non-computational skills appear to be related to the inadequate development of fundamental concepts underlying the skills. This results not only in poorly developed basic skills, but also in the inability of most students to apply the skills they have learned to solve even simple problem situations. The purpose of this paper is to document these and certain related conclusions with selected results from the second NAEP mathematics assessment. Caution must be observed in interpreting these results. The assessment is not designed to identify causes of student performance; so in order to provide some general conclusions, we have frequently

extrapolated beyond the data. Other authors would possibly reach different conclusions. However, the conclusions that we have presented are generally supported by a wide range of exercises in addition to the illustrative exercises reported.

BACKGROUND

National Assessment conducted its second mathematics assessment during the 1977-78 school year. Exercises covering a wide range of objectives were administered to a carefully selected nationally representative sample of over 70,000 students at ages 9, 13, and 17. Item sampling procedures were used so that between 250 and 450 exercises were administered to each age group, with approximately 2400 students responding to each exercise. Consequently, the results provide an accurate sampling of the knowledge of American elementary and secondary students over a broad range of objectives. The assessment also has the advantage of providing analysis of performance on specific objectives. Each exercise is analyzed on an exercise-by-exercise basis to provide a description of students' performance on specific tasks.

The assessment focused on five major content areas: 1) numbers and numeration, 2) variables and relationships, 3) geometry (size, shape, and position), 4) measurement, and 5) other topics, which included probability and statistics and graphs and tables. Each content area was assessed at four levels: 1) knowledge, 2) skill, 3) understanding, and 4) application. Knowledge level exercises involved recall of facts and definitions. This included such tasks as ordering numbers; recalling basic addition, subtraction, multiplication, and division facts; identifying geometric figures; and identifying basic measurement units. Skill exercises involved various mathematical manipulations including computation with whole numbers, fractions, decimals, and percents. Also included were making measurements, converting measurement units, reading graphs and tables, and manipulating algebraic expressions. Understanding exercises tested students' knowledge of basic underlying principles such as the concept of a unit covering in measurement. These exercises were constructed so that students could not simply apply a routine algorithm. Application exercises required students to apply their knowledge or skills to solve some sort of problems. Both routine textbook problems and non-routine problems were included in this category.

COMPUTATIONAL VS. NON-COMPUTATIONAL SKILLS

If students' performance on the second mathematics assessment is a measure of

instructional emphasis, then one must conclude that the focus of most mathematics programs is on the development of routine computational skills. Students demonstrated a high level of mastery of computational skills, especially those involving whole numbers. The majority of students at all age levels demonstrated severe deficiencies in the other basic skill areas.

Almost all students demonstrated mastery of basic number facts. About two-thirds of the 9-year-olds could perform simple addition and subtraction computation using algorithms for regrouping; and by age 13, almost all students could perform simple computations involving addition, subtraction, and multiplication. Most of the older students were successful with the more difficult calculations such as those summarized in Table 1. Students encountered greater difficulty with whole number division and operations with fractions and decimals. Performance was significantly lower, however, on exercises assessing basic non-computational skills.

Table 1

Whole Number Computation

Exercise	Percent Correct	
	Age 13	Age 17
4285 3273 + 5125	85	90
Subtract 237 from 504	73	84
671 x 402	66	77

In general, the only non-computational skills for which students demonstrated a high level of mastery were those involving simple intuitive concepts or those concepts or skills they were likely to have encountered and practiced outside of school. This is reflected in students' knowledge of geometric terms. Students were familiar with common everyday terms like square or parallel, but not with mathematically important terms like tangent and hypotenuse, that are used less commonly in everyday vernacular. Over 95 percent of the 13-year-olds could identify squares and parallel lines, but even by age 17, fewer than 60 percent of the students were familiar with terms like tangent and hypotenuse.

The failure of students to learn basic geometric concepts is illustrated by the fact that only about a fifth of the 13-year-olds and a third of the 17-year-olds could solve a problem involving a simple application of the Pythagorean Theorem.

Performance on measurement exercises generally followed the same pattern as the geometry results. Most students were familiar with measurement concepts and skills that would likely be encountered and practiced outside of school, such as recognizing common units of measure, making simple linear measurements, and telling time. They had a great deal of difficulty, however, with many other basic measurement concepts and skills, especially those involving perimeter, area, and volume (See Table 2).

Table 2

Selected Measurement Exercises

Exercise	Percent Correct Age 9	Age 13	Age 17
Measure length to the nearest inch	81	91	94
Read clock to the nearest minute	59	85	
How many inches are equivalent to 1 foot?		87	92
Area of a Rectangle	4	51	74
Area of a Square		12	42
Area of a Right Triangle		4	18

Another basic skill area in which performance was generally low was probability and statistics. Fewer than half the students at any age level demonstrated even a tentative understanding of most basic probability concepts.

THE IMPORTANCE OF UNDERSTANDING

As results in the previous section showed, students are failing to master a broad range of basic skills. Further, many of the skills that have been learned have been learned at a rote, superficial level. Students' performance showed a lack of understanding of basic concepts and processes in many content areas, such as measurement and computation with fractions.

Almost all students could make simple linear measurements. Over 80 percent of the 9-year-olds and 90 percent of the 13-year-olds could measure the length of a segment to the nearest inch. However, when students were presented with a problem similar to the one illustrated in Figure 1, only about 20 percent of the 9-year-olds and about 60 percent of the 13-year-olds gave the correct measure.

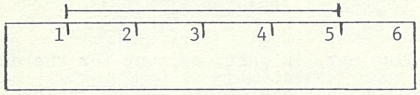


Figure 1. How long is this line segment?

Seventy-seven percent of the 9-year-olds and 40 percent of the 13-year-olds gave an answer of 5. Thus, although most students would line up the end of the segment when they were measuring it, this change in problem context demonstrated that they do not understand the consequences of not doing so.

Students' superficial understanding is also apparent in many computation exercises. For example, students are relatively successful in multiplying two common fractions, perhaps because multiplying numerators and denominators seems to be a natural way to approach the problem. However, the results from the simple verbal problem shown in Table 3 indicate that students have no clear conception of the meaning of fraction multiplication and, therefore, cannot apply their skills to solve even a simple problem.

Table 3

Fraction Multiplication

Exercise	Percent Correct Age 13	Age 17
$\frac{2}{3} \times \frac{2}{5}$*	70	74
Jane lives $\frac{2}{3}$ mile from school.* When she has walked $\frac{2}{5}$ of the way, how far has she walked?	20	21

*Similar to an unreleased exercise

Students' failure to learn basic fraction concepts is also illustrated by performance on several estimation exercises. For example, although 39 percent of

the 13-year-olds and 54 percent of the 17-year-olds could calculate the answer to $\frac{7}{15} + \frac{4}{9}$, only 24 and 37 percent, respectively, could make even a reasonable estimate of $\frac{12}{13} + \frac{7}{8}$. Over half of the 13-year-olds and over a third of the 17-year-olds simply tried to add either the numerators or denominators. This response, which is a totally unreasonable estimate, suggests than many students look for some rote computation rule to apply without even considering the reasonableness of the result.

The importance of understanding may, in part, account for the difference in the level of performance for whole number operations and operations involving fractions and decimals. Most assessment exercises indicate that students have learned the basic concepts underlying whole number computation, and have some notion of the place value concepts involved in the computation algorithms. As a consequence, performance on whole number computation exercises was generally good. As the above examples suggest, however, most students do not have a clear understanding of fraction operations and appear to operate at a mechanical level. This lack of understanding results in relatively poor performance on some fraction computation, and is further highlighted by the serious difficulties encountered in solving simple problems involving fraction operations.

The results reported in this section strongly suggest that students have at best, a superficial understanding of many mathematical concepts and processes. Yet, around 90 percent of the 13- and 17-year-olds felt that developing understanding was an integral part of mathematics learning as evidenced by their agreement with the statement "Knowing why an answer is correct is as important as getting the correct answer." There responses may reflect their actual beliefs, or may in fact, reflect their belief that the statement is one they have heard from their mathematics teachers, and agreement with the statement was the "right" response. The second alternative gains credence when considered in contrast to the fact that around 90 percent of both older age groups agreed that "There is always a rule to follow in solving mathematics problems." The students may be concentrating on mastering rules to the extent of ignoring concomitant understanding, because their experiences dictate that right answers, obtained usually via rules, are rewarded.

PROBLEM SOLVING

One of the consequences of learning many mathematical skills at a rote mechanical level is that students cannot apply the skills they have learned to solve problems. In general, NAEP results showed that the majority of students at all age levels had difficulty with any non-routine problem that required some analysis or thinking. It appears that students have not learned basic problem-solving skills, and attempted instead to mechanically apply some mathematical calculation to whatever numbers are given in a problem.

A word of caution is in order in describing the performance of students on problem solving. Problem solving is often equated with solving textbook verbal problems, but these were not the type of problems that caused difficulty. In fact, students generally were successful in solving routine one-step verbal problems such as those found in typical textbooks. The results summarized in Table 4 are representative of student performance on one-step verbal problems in which the main steps are deciding whether to add, subtract, multiply, or divide and then performing the calculation.

Table 4

Multiple-Choice Subtraction Exercises

Problem/Response	Percent Responding		
	Age 9		Age 13
	With Calculator	Without Calculator	
George has 352 arithmetic problems to do for homework. If he has done 178 problems, how many problems does he have left to do?**			
174*	38	70	82
530	6	8	1
226	6		1
Other subtraction error	10		4
352** - 178			
174*	50		85
530	1		0
226	15		2

*Indicates correct response
**Similar to an unreleased exercise

The verbal problem in Table 4 was presented to one set of 9-year-old respondents without a calculator and to another group of 9-year-olds who had a calculator available. A third set of respondents was presented the same subtraction calculation as a straight computation exercise. As with all exercises on the assessment that required reading, the verbal problems were presented on an audiotape as well as written in the exercise booklet. The results indicate that by age 13, there is very little difference in students' ability to solve the verbal problem and their ability to perform the required calculation. Furthermore, few students at either age level chose the wrong operation.

Although students could successfully identify which operation should be used to solve most simple one-step problems, they had a great deal of difficulty analyzing non-routine or multi-step problems. In fact, given a problem that required several steps or contained extraneous information, students frequently attempted to apply a single operation to the numbers given in the problem. Students' difficulty with non-routine problems and problems with extraneous data is illustrated by the results summarized in Table 5. In spite of the fact that

Table 5

Multi-Step Problems

Problem/Response	Percent Responding Age 9	Age 13	Age 17
Lemonade costs 95¢ for one 56 ounce bottle. At the school fair, Bob sold cups holding 8 ounces for 20¢ each. How much money did the school make on each bottle?*			
Correct response		11	29
Students added, subtracted, or multiplied two of the numbers given in the problem		40	25
One rabbit eats 2 pounds of food each week. There are 52 weeks in a year. How much food will 5 rabbits eat in one week?			
2 pounds	2	1	
10 pounds**	47	56	
52 pounds	16	5	
104 pounds	16	11	
520 pounds	12	23	
I don't know	6	3	

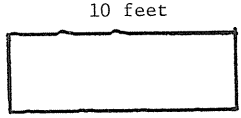
*The actual exercise is unreleased, but is similar in content to the given problem.
**Indicates correct response

the problems involved calculations that were well within the range of computational skill of students, many were unable to solve either problem.

One of the difficulties that students have in solving problems is that they have not developed good problem-solving strategies. A basic strategy that helps in analyzing certain types of problems is drawing a picture. In the pair of exercises in Table 6, the same basic problem was given to one set of respondents with an accompanying figure and to another set of respondents in a problem situation without a figure. The variations in the problems produced a difference of over 30 percentage points in the percent of correct responses for both 9- and 13-year-olds. Since most students could identify a rectangle as shown by results on a simple recognition exercise, many of them apparently did not apply their knowledge by drawing a figure to help them solve the verbal problem. Instead, many of them simply added or multiplied the two numbers given in the problem.

Table 6

Related Perimeter Exercises

Problem	Percent Responding Age 9	Age 13
What is the DISTANCE ALL THE WAY AROUND this rectangle? 		
32 feet*	40	69
16 feet	39	12
60 feet	4	13
Mr. Jones put a rectangular fence all the way around his rectangular garden. The garden is ten feet long and six feet wide. How many feet of fencing did he use?		
32 feet*	9	31
16 feet	59	38
60 feet	14	21

*Indicates correct response

CONTINUED DEVELOPMENT OF MATHEMATICAL SKILLS

Although problem solving and other content areas clearly require an increased emphasis in the curriculum, we do not deny the importance of computational skills. A reasonable level of computational skill is required for problem solving. We are suggesting, however, that problem solving not be deferred until computational skills are mastered. Problem solving and the learning of more advanced skills reinforce the learning of computational skills and provide meaning for their application.

It is important to recognize that most computational skills are learned over an extended period of time. The results summarized in Table 7 suggest that most skills are mastered after their period of primary emphasis in the curriculum. For example, even though a goal of most mathematics problems is that students learn subtraction facts by age 9, there is significant improvement in performance on subtraction exercises from age 9 to 13, and there is even some improvement between ages 13 and 17. Furthermore, many basic misconceptions

Table 7

Improvement in Performance by Age

Problem	Percent Correct		
	Age 9	Age 13	Age 17
Basic Subtraction Facts	79	93	95
Three-digit Subtraction	50	85	92
$\frac{4}{12} + \frac{3}{12}$		74	90
$\frac{1}{2} + \frac{1}{3}$		33	66

also disappear as students progress in school. Although over 30 percent of the 9-year-olds subtracted the smaller digit from the larger in a subtraction exercise that required regrouping, only 5 percent of the 13-year-olds and one percent of the 17-year-olds committed this error.

These results seem to have profound implications for minimum competency programs. They suggest that rigid minimum competency programs which hold children back until they have demonstrated mastery of a given set of skills may, in fact, be depriving them of the very experiences that would lead to mastery of the particular skills.

Although some skills will continue to develop through use in other contexts, this is not always the case. The current high school curriculum does not take into account that many basic skills are not well-developed by the time students begin instruction in algebra and geometry. For example, very few 13- and 17-year-olds have mastered percent concepts or skills, but outside of general mathematics, there is very little opportunity for high school students to extend or maintain their knowledge of percent.

DEVELOPMENT OF ALTERNATIVE ALGORITHMS

The widespread availability of hand calculators has profound implications for the mathematics to be taught in the schools. Over 85 percent of the 17-year-olds in the assessment indicated that they had access to a calculator. This almost universal availability of calculators would seem to have implications for the appropriate level of emphasis that computation should receive and the types of algorithms we should teach.

In spite of the extensive instruction provided on whole number division, only half of the students are reasonably proficient in division by the time they graduate from high school. With a calculator, however, over 50 percent of the 9-year-olds and over 90 percent of the 17-year-olds could divide accurately. This at least raises some serious questions as to whether the time spent drilling on division is a productive use of time and effort that might otherwise be devoted to other topics. Certainly, it is clear that our current approach to teaching division is not effective for most students.

The division algorithm as well as most of the other algorithms that we teach in school are designed to produce rapid, accurate calculation procedures. Given the widespread availability of hand calculators, it would seem that the continued emphasis on developing facility with computation algorithms should not be as high a priority as it was formerly. Certainly, computation is important; but what is needed are algorithms that students will remember and will be able to generalize to new situations, which brings us back to the issue of understanding. Students are more likely to remember and be able to generalize and apply algorithms if they understand how they work. Thus, it would seem appropriate to begin to shift to computational algorithms that can be more easily understood than the current ones, even if they are less efficient.

The results summarized in Table 8 illustrate the potential impact of calculators on our thinking about computation. Students had more difficulty solving this

Table 8

Non-Routine Division Exercise

A man has 1310 baseballs to pack in boxes which hold 24 baseballs each. How many baseballs will be left over after the man has filled as many boxes as he can?

Response	Percent of 13-year-olds Responding	
	Without Calculator	With Calculator
14*	29	6
54-7/12, 54.583, etc.	22	26
1286 (1310 - 24)	8	15
Other	27	25
No Response	14	28

*Indicates Correct Response

problem with a calculator than without using a calculator. Students also had more difficulty comparing and ordering a set of fractions with a calculator than they did without. They apparently did not understand that fractions can also be thought of as quotients, which allows one to represent them as decimals that are relatively easy to order. Students appear to have rather rigid ways of thinking about numbers and operations. Calculators sometimes require alternative interpretations and require that students have a deeper understanding of numbers and how the operations work.

Calculators also place an increased importance on estimation skills and alertness to reasonableness of results. The results of one exercise given to 13- and 17-year-olds illustrate the importance of this skill and the gross errors that can occur when students who are oblivious to the reasonableness of a result use a calculator. Students were asked to find $7 \div 13$ using a calculator. About 20 percent of the 13- and 17-year-olds chose the response 5384615 rather than 0.5384615.

PARTICIPATION IN MATHEMATICS COURSES

A previous section proposed that mathematics learning is a continuous process that encompasses the entire 12 years of elementary and secondary school. Many, if not most, basic skills are not mastered by age 13 and must be reinforced and

developed as part of the high school curriculum. Consequently, if we are going to significantly improve the mathematics performance of high school graduates, we must insure than they continue to take mathematics throughout their high school program. The assessment background data summarized in Table 9 indicate that this is currently not the case.

Table 9

Mathematics Courses Taken by 17-year-olds

	Percent having completed at least 1/2 year
General or Business Mathematics	46
Pre-Algebra	46
Algebra I	72
Geometry	51
Algebra II	37
Trigonometry	13
Pre-Calculus/Calculus	4
Computer Programming	5

STUDENTS' PERCEPTION OF MATHEMATICS CLASSES

The specific types of activities that students participate in when they are enrolled in mathematics classes is also an important factor that influences the nature of their mathematics learning. Results on certain exercises administered during the second NAEP mathematics assessment confirm the findings of other studies that students have a passive role in learning mathematics. This confirmation is based on information provided by students rather than their teachers, and corroborates the results cited earlier.

One set of exercises presented students with a list of classroom activities (e.g., taking mathematics tests) and asked them to rate the activities in terms of how often they thought the activities occurred in their mathematics class. The activities may be described as student-centered, teacher-centered, classmate-centered, and "other," which included activities requiring active student involvement such as using manipulative objects.

Table 10 lists the frequency ratings on the student- and teacher-centered activities. Table headings represent response options for 13- and 17-year-olds to

Table 10

Ratings of Frequency of Selected Classroom Activities

Activity	Age	Percent Responding Often	Sometimes	Never
Student-centered				
Taken mathematics test	9	44	46	9
	13	61	37	1
	17	63	33	3
Done mathematics homework	9	43	45	12
	13	67	29	3
	17	57	36	6
Worked mathematics problems alone	9	71	22	7
	13	81	17	1
	17	80	19	1
Worked mathematics problems at the board	9	39	54	7
	13	33	58	9
	17	27	60	13
Used your mathematics textbook	9	75	18	6
	13	81	14	5
	17	87	11	3
Done worksheets	9	71	27	2
Teacher-centered				
Listened to the teacher explain a mathematics lesson	9	71	27	2
	13	81	16	2
	17	78	19	2
Watched the teacher work mathematics problems on the board	9	78	18	4
	13	76	21	2
	17	79	18	3
Gotten individual help from the teacher on your mathematics	9	21	67	11
	13	17	71	10
	17	18	70	11

the question "How often have you done these activities in your mathematics classes?"; options for the 9-year-olds were "a lot," and "never." As the table shows, most students felt that they spent a lot of time listening to and watching the teacher work and explain mathematics problems. They also felt that they spent a lot of time working alone on mathematics problems from the textbook and, for the

9-year-olds, also from worksheets. The percentages of frequency assigned to these activities were the highest for any of the group of activities.

As a group, the classmate-centered and "other" activities received the highest percentages of "never" ratings for all age groups. Among the classmate-centered activities, around half of all age groups said that discussing mathematics in class occurred often; around 60 percent of the 9-year-olds and 75 percent of the 13-year-olds and 17-year-olds said they sometimes gave help to or received help from their classmates in mathematics. Thirty-five, forty-four, and twenty-eight percent of the 9-, 13-, and 17-year-olds, respectively, said they had never worked mathematics problems with small groups of students.

Most of the older students said they had never made reports or done projects in mathematics classes, and over two-thirds of the 9-year-olds and three-fourths of the older respondents said they had never done mathematics laboratory activities. Further, over half of the 9-year-olds said they had never used objects like counters, rods, or scales in mathematics classes.

These National Assessment results show that students perceive their role in the mathematics classroom to be primarily passive--they are to sit and listen to and watch the teacher do mathematics, and then the rest of the time is to be spent working on an individual basis on problems from the text or from worksheets. They feel they have little opportunity to interact with their classmates about the mathematics being studied, nor are exploratory activities or work with manipulatives frequent activities.

CLOSING THOUGHTS

Although we have identified a number of serious deficiencies in student performance, the assessment results provide some basis for cautious optimism. It is probably fair to say that the focus of mathematics instruction has been on computation. Thus, there is evidence that students are learning what they are being taught. There is also evidence that curricular reforms can have some impact. On exercises that measured change in performance from the first assessment, there were significant gains of 10 to 20 percentage points on exercises that dealt with metric measurement. These results appear to reflect the increased emphasis on metric measurement in the curriculum over that period of time. There is hope that if we reorganize the mathematics curriculum to address the concerns identified in this paper, students' performance will respond accordingly.

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