ACTES DU
CINQUIEME COLLOQUE
DU GROUPE INTERNATIONAL

PROCEEDINGS
OF THE CONFERENCE
OF THE INTERNATIONAL GROUP

PME

PSYCHOLOGY OF MATHEMATICS EDUCATION

Vol. 2

GRENOBLE - FRANCE

13 - 18 juillet 1981

פנק הטלפון

Actes du
cinquième colloque du
Groupe international

Proceedings of the
fifth conference of
the International group

PSYCHOLOGY OF MATHEMATICS EDUCATION

Grenoble, France
13 - 18 juillet 1981

organisé avec le concours des organismes français suivants :

organised with the support of :

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE
MAISON DES SCIENCES DE L'HOMME
SOCIETE MATHEMATIQUE DE FRANCE
MINISTERE DE L'EDUCATION

Editeur
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38041 GRENOBLE CEDEX
FRANCE

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(dans l'ordre du programme)

QUELQUES ORIENTATIONS THEORIQUES ET METHODOLOGIQUES DES RECHERCHES FRANCAISES EN DIDACTIQUE DES MATHEMATIQUES

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Research has developed in France along specific lines. This specificity is due to our philosophical and scientific traditions. For instance, the concept of "epistemological obstacle" (Bachelard) is hardly known outside France; the work of Piaget is not interpreted in the same way: although Piaget's work is our main reference, we are not satisfied with some of his views, for instance about stages, about mathematical knowledge, about logicism. New theoretical concepts are presented such as conceptual field, theorem in action, relational calculus, didactic variables and effects, system of signifiers, didactic contract. Also some methodological issues are analyzed; the stress is laid upon the necessity of using a variety of methods, and upon the importance of experimenting in the class-room. Class-room experiences require several conditions for them to be fruitful and repeatable. Illustrations of the theoretical concepts and the methodological views are given and will be explained orally in greater detail.

Les recherches se sont développées en France, depuis une quinzaine d'années, sur des bases théoriques et méthodologiques sensiblement différentes des bases sur lesquelles elles se sont développées dans les autres pays. Cela tient principalement à des traditions philosophiques et scientifiques différentes, dont je donnerai deux exemples:

- des philosophes des sciences très influents en France comme Gaston Bachelard et Georges Canguilhem sont presque inconnus à l'étranger, en tout cas des professionnels de la recherche en éducation mathématique.
- les chercheurs français tirent de l'oeuvre de Jean Piaget des leçons différentes de celles qu'on en retire en général dans les pays anglo-saxons.

Pourtant nous partageons avec de nombreux chercheurs dans le monde les préoccupations suivantes:

- comment les connaissances mathématiques se développent-elles, chez l'enfant et l'adolescent en particulier?
- à travers quelles situations les concepts et les procédures mathématiques prennent-elles leur signification?
- quelles conditions didactiques et psychosociales faut-il rassembler pour assurer la transmission et l'appropriation du savoir, c'est-à-dire la reconstruction du savoir par celui qui apprend.

Certaines de ces questions ne sont pas nouvelles et l'on connaît par exemple la diversité des réponses apportées par les psychologues à la première question. Bien que, parmi ces réponses, l'approche de Jean Piaget nous apparaisse la meilleure et la plus féconde, nous n'en sommes pas satisfaits:

1. Jean Piaget s'est désintéressé de l'acquisition des connaissances scolaires. Il a plutôt cherché à caractériser le développement des instruments généraux de pensée, qui lui apparaissent relativement indépendants des connaissances scolaires.

2. Jean Piaget s'est intéressé davantage aux structures pouvant caractériser un stade donné de développement qu'à l'évolution adaptative des connaissances dans une situation ou un ensemble de situations où elles sont fonctionnelles.

3. Jean Piaget a séparé d'une manière exagérée la connaissance mathématique et la connaissance de la réalité physique. Je pense notamment à ce qu'il a écrit à plusieurs reprises sur l'abstraction simple et l'abstraction réfléchissante, la première portant selon lui sur les propriétés des objets et étant à ce titre constitutive de la physique, la seconde portant sur l'action du sujet sur les objets et étant spécifiquement mathématique.

4. Jean Piaget a privilégié les opérations et les structures logiques et contribué ainsi à minimiser les contenus de connaissance, que ces contenus relèvent de la physique ou des mathématiques.

Ces difficultés et l'insatisfaction générale dans laquelle nous laissent les recherches psychologiques sur l'apprentissage, nous ont conduit à définir de nouveaux cadres théoriques:

CHAMP CONCEPTUEL

La première grande question concerne les choix à faire pour découper à bon escient les contenus de connaissance mathématiques et en étudier de manière féconde la didactique et l'acquisition. Il n'est pas raisonnable d'étudier séparément l'acquisition de concepts (et de procédures) qui, dans les situations rencontrées et traitées par les élèves sont difficilement dissociables. Par exemple il serait aberrant de conduire des études séparées pour l'acquisition des concepts de multiplication et de division, de fraction, de rapport et de nombre rationnel, de fonction linéaire et n-linéaire, d'analyse dimensionnelle et d'espace vectoriel, puisque dès les premiers problèmes de type multiplicatif qu'il rencontre (proportions, surfaces, volumes), l'enfant est confronté à des relations qui relèvent de l'ensemble de ces concepts.

D'autre part, il est raisonnable, si l'on veut étudier la psychogenèse des contenus de connaissance, de découper la connaissance en domaines assez larges pour pouvoir en étudier l'évolution chez l'élève sur une assez longue période de temps et à travers un ensemble de situations diversifiées. La psychologie génétique nous apprend en effet que les connaissances se développent lentement; cela est vrai pour les contenus de connaissance comme pour les instruments logiques de la pensée.

Ce sont ces deux préoccupations (interconnexion des concepts et évolution psychogénétique) qui m'ont conduit à définir la notion de champ conceptuel: un champ conceptuel est un espace de problèmes ou de situations-problèmes dont le traitement implique des concepts et des procédures de plusieurs types en étroite connexion. Les deux champs conceptuels auxquels je me suis personnellement le plus intéressé sont les "structures additives" d'une part, les "structures multiplicatives" d'autre part. Evidemment ces deux champs conceptuels sont en relation l'un avec l'autre et en relation avec d'autres champs conceptuels comme l'espace, la dynamique, la logique des classes, etc. Mais ils m'apparaissent avoir une unité suffisante pour justifier des études distinctes. Ils me permettront d'illustrer brièvement la notion de champ conceptuel:

- exemple des structures additives: les premiers principes concernant l'addition et la soustraction, notamment l'augmentation et la diminution d'une collection et l'itération + 1 commencent à être appréhendés dès l'âge de 3 ou 4 ans, ainsi que l'a bien montré Rochel Gelman. Mais le développement des structures additives passe ensuite par le comptage, la conservation des quantités discrètes et continues, la numération, le traitement de relations différenciées (addition de mesures, transformation d'une grandeur, comparaison, composition de transformations, etc.) dont les plus complexes ne sont pas comprises par la majorité des élèves avant l'âge de 15 ans. On peut conduire, dans le champ conceptuel des structures additives, des analyses hiérarchisées sur la difficulté relative des différentes classes de problèmes, sur la complexité conceptuelle et la disponibilité des différentes procédures de traitement d'une même classe de problèmes, sur la signification et l'utilisation des différents systèmes de représentation symbolique. Il est maintenant démontré que l'acquisition des structures additives s'étend sur une période du développement de l'enfant et de l'adolescent supérieure à dix années, et passe par un réseau de chemins qui n'a que des rapports lâches avec la théorie des stades de pensée. On peut repérer dans cette acquisition, des sauts qualitatifs importants (comme celui de l'inversion d'une transformation) ainsi que des "obstacles épistémologiques" durables comme celui de la composition et de la décomposition de fonctions, ou comme celui de l'identification de l'ensemble des nombres naturels avec l'ensemble des nombres relatifs positifs.

- exemple des structures multiplicatives: Gerald Noelting a montré que la même situation de comparaison de rapports pouvait être utilisée entre l'âge de 2 ou 3 ans et l'âge de 17 ans et qu'entre ces deux âges se succédaient une dizaine d'étapes différentes. Le champ conceptuel des structures multiplicatives intègre le concept de rapport et le relie aux concepts de proportion et de fonction linéaire, ainsi qu'aux concepts de fraction et de nombre rationnel. On peut aisément repérer, dans ce champ conceptuel, des relations particulièrement difficiles, comme celles de produit et de quotient de dimensions que l'enfant rencontre dans des problèmes réputés élémentaires. On peut également identifier des procédures plus disponibles que d'autres, comme celles qui utilisent les propriétés de l'isomorphisme f(x)=ax, ainsi que des représentations symboliques plus naturelles que d'autres, comme les tableaux de proportionnalité simple ou double. Des "obstacles épistémologiques" particulièrement difficiles à surmonter existent, notamment dans la construction de l'ensemble des nombres rationnels, et dans l'utilisation des propriétés des fonctions bilinéaires, trilinéaires et n-linéaires (grandeur proportionnelle à plusieurs autres grandeurs indépendantes entre elles).

Pour résumer, la notion de champ conceptuel permet d'étudier d'une manière mieux intégrée le développement simultané et coordonné des différents concepts nécessaires à la compréhension d'un ensemble organisé de classes de problèmes, des procédures permettant de les traiter, et des sytèmes symboliques permettant de les représenter. Les hiérarchies que ces études permettent de mettre en évidence ne forment pas un ordre total mais un ordre partiel, contrairement au modèle de la théorie des stades. La notion de champ conceptuel réhabilite les contenus de connaissance, trop souvent minimisés ou effacés par l'approche structuraliste.

THEOREME EN ACTE ET CALCUL RELATIONNEL

La solution de problème est la source et le critère du savoir. C'est dans la solution de problème, ou plus généralement dans le traitement de situations-problèmes que sont élaborées les notions, et que sont abstraites les propriétés pertinentes. C'est aussi dans la solution de problème que sont éprouvées les connaissances opératoires. Le psychologue et le maître peuvent se former une image des connaissances et représentations des élèves à partir des observables dont ils disposent, c'est-à-dire des actions du sujet en situation et des témoignages symboliques que le sujet fournit de son activité: formulations verbales, dessins, schémas, écritures ...

Les différentes réponses et solutions apportées par les élèves peuvent être considérées comme engendrées par des règles de production, ou procédures. Il est méthodologiquement décisif d'identifier ces règles ou procédures. Mais on ne peut comprendre leur signification que si elles sont rapportées aux relations auxquelles elles s'appliquent. En d'autres termes, il faut les considérer comme des "théorèmes"implicites. Le concept de "théorème en acte" désigne les propriétés des relations saisies et utilisées par le sujet en situation de solution de problème, étant entendu que cela ne signifie pas qu'il est pour autant capable de les expliciter ou de les justifier.

Le concept de "calcul relationnel" désigne de son côté les compositions déductives (et les inférences) qui rendent compte de ses productions. Un exemple permettra d'éclairer ce point: plusieurs auteurs (Freudenthal, Lybeck, Vergnaud) ont relevé le fait que, dans les problèmes de type multiplicatif, les élèves utilisent plus naturellement les relations entre grandeurs de même nature que les relations entre grandeurs de nature différente; pour calculer la distance parcourue par un train rapide, en 36 minutes, sachant qu'il parcourt 40 kilomètres en 16 minutes, de nombreux élèves (entre 11 et 13 ans) font la décomposition suivante:

$$36 = (2 \text{ fois } 16) + 4 = 2 \text{ fois } 16 + 1/4 \text{ de } 16$$

donc la distance parcourue est 2 fois 40 + 1/4 40 = 80 + 10 = 90.
En procédant ainsi, les élèves appliquent le théorème

$$f(\lambda x + \lambda' x) = \lambda f(x) + \lambda' f(x)$$

C'est un "théorème en acte", car il est rarement explicité, et c'est aussi une production inventive des élèves car un tel théorème ne leur a jamais été enseigné. Ce théorème en acte est le produit direct de la conceptualisation des élèves dans le champ des structures multiplicatives. Il repose sur deux théorèmes en acte plus élémentaires

$$f(x + x') = f(x) + f(x')$$
$$f(\lambda x) = \lambda f(x)$$

et sur leur composition.
Le concept de "théorème en acte" renvoie donc à celui de "représentation calculable implicite" (ou encore de théorie implicite).

VARIABLES DIDACTIQUES ET EFFETS DIDACTIQUES

Jean Piaget a le mérite d'avoir présenté, avec la théorie de l'équilibration majorante, une théorie cohérente de l'évolution des connaissances: la connaissance passerait d'un état d'équilibre à un autre par un déséquilibre de transition au cours duquel les relations prises en compte par le sujet dans l'état antérieur seraient mises en contradiction, soit par la prise en considération de relations nouvelles, soit par une tentative nouvelle de les coordonner. Cette phase de conflit serait surmontée au cours d'une phase de réorganisation et de coordination qui aboutirait à un nouvel état d'équilibre. Si l'on applique cette théorie aux connaissances mathématiques, on est amené à considérer que les situations-problèmes présentées aux élèves constituent un levier important pour faire évoluer leurs représentations et leurs procédures. Guy Brousseau a développé à ce sujet une "théorie des situations didactiques" dans laquelle il fait jouer un rôle important à la valeur des variables numériques utilisées et à la quantité d'information à traiter. Les procédures et les notions des enfants se forment en général pour des valeurs petites et entières des variables numériques; le changement de valeur est de nature à faire échouer les procédures "locales" des élèves et à les obliger à élaborer des procédures plus puissantes. Il existe d'autres variables de situation, sur lesquelles il est possible d'agir pour produire des effets didactiques. On peut citer en premier lieu la nature de la tâche cognitive, c'est-à-dire la nature de la classe de problème et des opérations de pensée nécessaires à sa solution: par exemple dans les problèmes de type additif et pour de jeunes élèves (6 à 8 ans), les questions portant sur l'état initial d'une grandeur connaissant la transformation qu'elle a subie et son état final, sont de nature à obliger l'enfant à analyser davantage les propriétés des transformations et à découvrir la propriété d'inversion.

	état initial	dépense de 4 francs	état final
exemple :	$\boxed{?}$	$\xrightarrow{\quad -4 \quad}$	$\boxed{7}$

De même l'accroissement du nombre de données à traiter et la présence de données inutiles sont de nature à rendre fonctionnelle et économique l'utilisation d'un diagramme, d'un tableau, d'une équation. Un bon exemple de l'utilisation des variables de commande didactique est fourni par le travail d'Annie Bessot et Françoise Richard, sur la combinatoire; un exemple sur les proportions sera présenté oralement.

Toutefois, le changement des variables de situation, au lieu de permettre le passage à des procédures de niveau supérieur, peut entraîner la régression à des procédures moins puissantes, mais plus fiables, ainsi que le montrent les recherches de Claude Comiti et Annie Bessot. Plusieurs autres travaux présentés à PME (Douady-Perrin, Artigue-Robinet) s'intéressent ainsi à l'effet didactique de la manipulation des variables de situation. Ces effets ne sont pas automatiques, l'état des connaissances antérieures du sujet est évidemment un élément déterminant de l'apparition ou de la non-apparition de ces effets. Les effets didactiques sont plus aisément constatables au plan des procédures de traitement qu'au plan des concepts. Pourtant on peut repérer certains exemples d'élaboration de concepts nouveaux en situation: l'exemple de la composition de transformations additives montre que les élèves parviennent à traiter certaines situations conceptuellement complexes avec des moyens plus rudimentaires, en opérant des glissements de sens adéquats; on peut alors choisir les variables de telle manière que les glissements de sens soient rendus impossibles et (espérer) provoquer ainsi l'élaboration d'un nouveau concept, celui de la composition de transformations (exemple donné oralement).

SIGNIFIANTS ET REPRESENTATIONS SYMBOLIQUES:

Les mathématiciens font un usage de plus en plus systématique, notamment dans l'enseignement, de représentations symboliques canoniques (graphiques, tableaux, diagrammes, ...). Remarquons au passage que les égalités et les équations forment également une représentation symbolique ainsi que le langage naturel lui-même. Faute de faire suffisamment la distinction entre le concept et sa représentation, c'est-à-dire entre le signifié et le signifiant, il arrive fréquemment qu'on prenne les symboles et les opérations sur ces symboles pour l'essentiel de la connaissance et de l'activité mathématiques, alors que cette connaissance et cette activité se situent principalement au plan conceptuel. Des questions importantes se posent au plan des signifiants. Par exemple, les différents systèmes de signifiants ne mettent pas également en évidence les différentes propriétés des relations mathématiques qu'elles symbolisent: l'algèbre se prête bien aux manipulations de symboles et à l'utilisation des opérations sur les nombres, les graphiques permettent mieux l'estimation des ordres de grandeur, la croissance, la décroissance, la continuité, etc. A l'inverse, ils ne se prêtent guère au calcul. Quelques recherches ont été conduites sur les diagrammes et sur les tableaux, de manière à déterminer quels aspects des relations ces représentations permettaient de mieux symboliser et de traiter (exemples donnés oralement).

L'étude des systèmes de représentation symbolique montre aussi que leur lecture et leur utilisation pose des problèmes spécifiques: par exemple le placement de données numériques sur une droite graduée, suppose des opérations de pensée beaucoup plus complexes qu'on ne l'imagine habituellement et s'avère relativement difficile pour des élèves de 10 à 13 ans (exemple donné oralement). Enfin la formulation en langage naturel de certaines relations mathématiques et de certaines propriétés soulève à elle seule un problème considérable. Colette Laborde et Michel Guillerault traitent cette question dans leur recherche. Pour ma part, je me contenterai de rappeler ce que j'ai dit tout à l'heure sur les "théorèmes en acte".

Utiliser une"propriété correcte" au cours de la solution d'un problème est essentiel. Mais cela ne fait pas une théorie mathématique explicite. Seule la formulation dans le langage naturel, et éventuellement dans un système symbolique adéquat, permet d'objectiver clairement les propriétés utilisées, de distinguer les propriétés correctes des propriétés fausses, et de leur apporter un début de validation. Une partie importante de la didactique consiste justement dans ce souci d'explicitation, qui est d'ailleurs indissociablement lié à la construction des objets mathématiques proprement dits comme ceux de nombre, d'ensemble de nombres, de fonction, de relation, d'espace, de plan, de droite, de groupe, etc.

La didactique repose donc sur deux pieds principaux:

- la construction de situations significatives pour l'élève et productrices d'effets didactiques;
- l'explicitation des relations traitées, l'analyse de leurs propriétés et la construction des objets mathématiques pertinents.

L'utilisation de systèmes symboliques spécifiquement mathématiques, ayant une syntaxe explicite, apparaît alors indispensable. Mais elle ne doit pas masquer le problème plus fondamental qui lui est sous-jacent, celui du concept.

CONTRAT DIDACTIQUE

Je ne peux pas terminer cette analyse rapide de nos cadres théoriques sans mentionner ce que Guy Brousseau a appelé le "contrat didactique", c'est-à-dire l'ensemble des attentes implicites qui règlent le fonctionnement de la classe et les rapports entre le maître et les élèves. De nombreux malentendus surgissent en effet à ce niveau, qui sont responsables d'interprétations erronées des énoncés de problèmes et des demandes du maître. La notion de contrat n'est pas une notion purement sociale, elle implique directement le contenu des connaissances. Je laisse à Guy Brousseau et à d'autres le soin d'illustrer ce concept.

ORIENTATIONS METHODOLOGIQUES

Les questions méthodologiques pourraient à elles-seules justifier un long exposé. Je m'en tiendrai ici à deux idées principales:

1. la nécessité de recourir à une diversité de méthodes;
2. la nécessité de développer l'expérimentation à l'intérieur de la classe.

L'enseignement et l'acquisition des connaissances mathématiques constituent un objet d'étude trop complexe pour qu'on puisse espérer le comprendre avec une seule approche méthodologique. Plusieurs d'entre nous ont donc fait le choix de l'aborder de plusieurs manières: entretiens individuels, expériences papier-crayon obéissant à un plan expérimental strict, expériences dans la classe, analyses de manuel, recherches sur l'histoire de l'enseignement des mathématiques, questionnaires destinés aux maîtres, etc. La finalité de ces différentes méthodes n'est pas la même.

Les entretiens individuels de type clinique et critique, tels que Piaget les a développés, demeurent à mon avis une méthode essentielle pour l'analyse des difficultés conceptuelles et pour l'analyse des procédures par lesquelles les élèves traitent une situation donnée. Ils sont indispensables pour éprouver le solidité des conceptions des élèves (grâce à l'utilisation de la contradiction) et pour analyser l'évolution des procédures et des conceptions en situation. Ils permettent de mieux faire la part entre les conduites vraiment significatives d'une conception, et les artefacts ou autres réponses anecdotiques.

Il s'agit cependant d'une méthode coûteuse en temps, et il serait déraisonnable de se priver des expériences papier-crayon, qui ont le mérite de toucher de larges échantillons, et de permettre ainsi, lorsqu'elles ont été planifiées dans ce but, de faire des comparaisons multiples qui seraient impossibles avec les entretiens individuels.

Par exemple, Graciela Ricco, André Rouchier et moi-même avons conduit 80 entretiens individuels pour étudier la compréhension des propriétés de trilinéarité du volume, à travers une série d'items où les gestes, les dessins et les silences des sujets étaient importants. Jamais nous n'aurions pu obtenir les mêmes informations avec une autre méthode. Inversement, nous avons conduit une étude papier-crayon sur la règle de trois, qui permet un grand nombre de comparaisons en fonction des valeurs numériques choisies: cette étude aurait été dispendieuse avec la méthode des entretiens.

Que représente l'expérimentation en classe par rapport à ces méthodes que je viens de rappeler et que tout le monde connaît?

L'expérimentation en classe, comme toute expérimentation, n'a de sens que si on la conduit avec une problématique claire en tête, c'est-à-dire si l'on a fait un choix réfléchi du thème et des aspects étudiés. Il est vain de se livrer à une observation du travail en classe, si l'on a pas élaboré soigneusement les situations et les questions proposées aux élèves, et si l'on n'a pas cherché à expliciter, aussi clairement que possible, les objectifs didactiques qu'on se donne, les hypothèses qu'on fait sur ce qui va se passer: par exemple sur le volume en classe de cinquième (élèves de 12 à 13 ans environ) nous avons construit une séquence didactique que présentera André Rouchier: pour chaque séance, nous avons réfléchi longuement à la situation présentée, aux objectifs que nous voulions atteindre, aux conduites que nous attendions et aux effets que nous espérions produire. Ni le choix du matériel, ni le choix des valeurs des variables, ni l'enchaînement des questions, ni leur formulation ne peuvent être laissés au hasard. C'est à cette condition seulement que nos observations prennent un sens et sont suffisamment précises. Sans cette condition, l'observation demeure vague, peu fiable et peu répétable.

L'enjeu scientifique est en effet important: est-il possible, au niveau d'une séquence didactique relativement longue et complexe, et au niveau de toute une classe, d'observer des régularités dans les événements qui se produisent, régularités interprétables par l'analyse des conditions qui leur ont donné naissance, pouvant comme telles être considérées comme des faits didactiques. Lorsque les psychologues ont développé la technique des entretiens individuels cliniques, ils ont été confrontés à un problème analogue, et ce n'est qu'avec de multiples précautions qu'ils ont pu mettre en évidence des faits psychologiques incontestables et répétables, au moins à un certain niveau de lecture des observations.

On peut espérer qu'il en ira de même dans la recherche en didactique et qu'il sera possible d'établir des faits didactiques, c'est-à-dire des faits concernant la transmission et l'appropriation du savoir par l'action finalisée de l'école. Le critère de la répétabilité est un critère exigeant. Il est impossible de le satisfaire à la lettre, mais il est peut-être moins malaisé qu'on le croit de s'en approcher, à un certain niveau d'interprétation des événements. En effet, lorsqu'on répète la même séquence didactique avec plusieurs classes d'élèves, on constate des régularités très importantes, notamment au niveau des distributions de conduites , et les variations prennent elles-mêmes un sens par rapport à ces régularités. La répétition permet en tout cas d'améliorer la formulation des objectifs et des hypothèses et de remettre en chantier les propositions didactiques qui ont fait l'objet de l'expérience.

L'expérimentation en classe n'est pas pour autant la voie royale de la recherche, d'une part parce qu'elle ne permet pas, même avec de bons moyens d'enregistrement, d'analyser dans le détail tous les processus en jeu, d'autre part parce qu'elle est d'autant meilleure qu'elle peut s'appuyer sur les résultats obtenus par d'autres méthodes (entretiens individuels, expériences planifiées). En retour elle permet de déceler des phénomènes qu'il serait intéressant de regarder avec la loupe des entretiens individuels. En tout cas, on ne voit pas comment la recherche en didactique pourrait faire l'économie de l'expérimentation en classe.

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Research on Mathematical Learning and Thinking in the United States[1]

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Any survey of the educational research scene in the United States is bound to be not only limited, but also rapidly outdated. The scene is complex, and it is volatile. These observations are only slightly less true if one's view is restricted to research on mathematical learning and thinking. Before one movement has begun to fade, another comes along, and although there are important ways in which the scene remains relatively constant over time, US research on mathematical learning and thinking is affected by the swirling currents in the surrounding seas of educational research. The metaphor of the pendulum is often invoked to characterize change in mathematics education in the United States, but I find that metaphors relating to weather and the ocean capture more of its rhythm and unpredictability.

If we look, for example, at the learning theories that have been used as bases for mathematics teaching, we see that the tides of theory can change quickly. When Howard Fehr (1953) surveyed the then-prominent learning theories of conditioning, connectionism, and gestalt psychology to identify implications for classroom practice, he did not note the wave of cognitive psychology that was about to sweep American educational research. When Lee Shulman (1970) used the issue of discovery learning to contrast the implications for mathematics teaching of Bruner's cognitive developmentalism with Gagné's neobehavioralism and Ausubel's cognitive theory of meaningful verbal learning, he did not anticipate that within a few years not only would

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[1] Paper presented at the meeting of the International Group for the Psychology of Mathematics Education, Grenoble, 13-18 July 1981. I am grateful to Tom Romberg for suggestions about the contents and to Jim Wilson for comments on an earlier draft.

discovery learning be largely abandoned as an issue in American mathematics educational research, but Ausubel and Bruner would no longer be active in educational research and Gagné would have recanted much of his neobehavioralist position. At present, we are enjoying the brisk breezes of information-processing psychology, but who can say what the weather is just over the horizon?

As one measure of the difficulty I have in seeing what is happening in the mathematics education research scene, let me cite an observation I made eight years ago concerning the annual growth in the number of research studies (Kilpatrick, 1973). I noted that there had been a jump of 44 percent in one year, from 1970 to 1971, in the number of dissertations dealing with research related to school mathematics listed in the annual survey of the *Journal for Research in Mathematics Education*. A look at the annual surveys for the years from 1970 to 1980, however, shows how much of an aberration that jump was. The rate of growth in the number of dissertations cited has been much more modest over the decade, and there are some signs that it may be slowing down. Somewhat more steady has been the growth in the number of journal articles published, and there has been a marked increase in the number of journals that publish articles on mathematics education research. I have not looked closely at the data, but my impression is that the greatest growth over the past few years has been in the number of authors and journals from outside the United States.

My record as an analyst, then, is not perfect. Furthermore, one can argue that a native of the culture is the wrong person to ask for an insightful picture of its condition. It is with some reservations, therefore, that I offer the remarks that follow. I shall first attempt to identify some current problems in US research in mathematics education that arise primarily from the context in which it is embedded. Much of our research has been heavily criticized for ignoring this context, so I will make sure it gets appropriate attention by treating it first.. Then I shall attempt to sketch some theoretical and methodological issues in US research on mathematical learning and thinking. Finally, I shall attempt to make the discussion more concrete and focused by examining how some of these problems and issues are reflected in US research on a topic I am especially familiar with--mathematical problem solving. It goes without saying that all of these views are my own; I have no idea how widely they might be shared either within the culture or outside it.

Current Problems in US Research in Mathematics Education

Problems of Climate and Morale

The climate for research in mathematics education in the US today is not an especially healthy one. Much of the torrent of research that appeared in the 1960s and 1970s was powered by federal dollars appropriated to improve the mathematics curriculum and the teaching of mathematics--although it should be quickly added that only a tiny fraction of those dollars actually supported the research. That stream of dollars has all but dried up. Americans increasingly entrusted educational affairs to the federal government in the past three decades principally because of their concern about manpower needs (Spring, 1976). That tide seems to have turned, however. Although the concern remains, people have lost faith in the government's ability to meet educational needs through its policies and programs. The widely held belief that educational research efforts have been mostly ineffective suggests that, even if more money were to become available to improve American education, little of it would likely be earmarked for research. Education is turning to the states and to private enterprise for increased funding, but these agencies have a meager record of supporting research in mathematics education, and I see little likelihood that this will change soon.

One might take heart from Heinrich Bauersfeld's (1979) observation that

> it is in the periods of deep economic depression . . . that the creative framework and fundamental research are prepared which are to build the substance of the following period of ascension and flower. (p. 199)

Perhaps, as the frantic data gathering slows down, more penetrating and scholarly analyses will appear in the literature of US mathematics education. Perhaps. But we face a more serious problem than economic depression; we in the United States appear to have lost much of our faith in the public school's power to transform our people and our society. A devaluation of education as an institution and as a profession pervades American life, demoralizing teachers and researchers alike. The duration, depth, and ultimate impact of this depression of the spirit are impossible to foresee.

Problems of Identity and Status

Bauersfeld (1979) also noted that the mathematics educator must "develop his own self-concept" (P. 210). This is happening in the United States to some extent, but the community of researchers in mathematics education lacks the coherence and identity a true community ought to have. It is not simply that a comprehensive professional organization is lacking; the Special Interest Group for Research in Mathematics Education of the American Educational Research Association, together with the North American branch of PME, fill much of the need for organization. And it is not simply that doctoral dissertations that are the candidate's first and last venture into research continue to flood the terrain--although that is a serious problem. The root problem runs deeper and touches on matters of professional identity and status. It is perhaps not surprising that, with some notable exceptions, US mathematicians view research in mathematics education with a disinterest that borders on disdain: the status differential between the mathematician and the educationist is deeply embedded in our culture. Harder to explain is the (largely covert) disparagement of each other's work that one sees between researchers in mathematics education and researchers in educational psychology in the United States. Despite much borrowing across the fence between the two fields, these neighbors--again with some exceptions on both sides--strongly resent encroachments on their territory. Relations between researchers in mathematics education and mathematics teachers are somewhat more cordial, probably because neither poses much of a threat professionally to the other. All of this territoriality on the part of US researchers in mathematics education might be more productive of a group identity if the territory itself were more clearly marked out. Uncertainty about its boundaries, however, is pervasive.

Problems of Purpose and Effectiveness

Some of the uncertainty as to what constitutes research in mathematics education in the United States arises from accusations that it has been ineffective in changing school practice. The consensus on this point seems to be that too many researchers have been studying the wrong things in the wrong ways. Researchers are (perhaps overly) disillusioned with traditional methods of educational research, but they are uncertain about how to use newer methods appropriately and effectively. Many people are convinced that the wrong research questions have been asked, but few examples of the right questions have been proposed. Lurking below the surface is the issue of how research should affect educational practice. Outside of mathematics education one hears the argument that, on the one hand, the best educational research is both pure and applied (Greeno, 1978) and, on the other hand, the researcher and the practitioner ought to expect little from each other (Phillips, 1980). Inside US mathematics education, one hears mostly silence on this issue.

Theoretical Issues in US Research on Mathematical Learning and Thinking

The preceding gloomy litany of problems provides a backdrop against which to view current theoretical issues, as I see them, in US research on mathematical learning and thinking. To get a better sense of how US researchers are dealing with theory, I examined the 38 articles in the ten issues of the Journal for Research in Mathematics Education from July 1979 to May 1981. Of these articles, 35 had authors with US affiliations only. I looked at each article to see if an attempt had been made to link the question under investigation to some explicit theoretical context. For 20 of the articles, I could find no such attempt. Six of the remaining articles dealt with aptitude-treatment interactions, and although the authors seldom specified the theoretical bases for their activities, I gave them credit for some contact with theory. Of the remaining nine articles, three made more than casual use of Piaget's genetic epistemology, two were linked with expectation theory, and one concerned each of the following: causal attribution theory, a theory of attention, an application of graph theory to the representation of cognitive structure, and information-processing theory. I may have been too harsh in some of my judgments, but I conclude from this and other observations that a lack of attention to theory is characteristic of US research in this field. Further, when theoretical constructs and contexts are used, they are not "home grown"--they are today, as they have been for years, borrowed from outside mathematics education. As Donald Sanders (1981) has noted:

> [We have a] tendency to approach educating through constructs rooted in psychology or the social sciences rather than through theories or constructs fitting to phenomena as they appear in educational settings. . . . Educational research rooted in the theories and paradigms of related disciplines may advance those disciplines, but it does not necessarily advance scientific knowledge of the process of educating. (pp. 9-10)

One of our greatest needs in research on mathematical learning and thinking is for conceptual, theory-building analyses of the constructs and assumptions we are using. Such analyses must, of course, be informed by empirical data--and, paradoxically, "despite rampant empiricism . . . there is very little trustworthy data representing the facts of the educating process" (Sanders, 1981, p. 9). Another need is to shift more of our attention to the study of mathematics learning and thinking as they occur in school. We must take full account in our research of the multiple contexts in which both learning and thinking occur. Each is embedded in interacting systems of the pupil's cognitions, the subject matter, and the social setting. We have tended to concentrate on at most one of these systems, and we have neglected interactions within the system, not to mention interactions between systems. For example, the study of concept learning has too frequently been pursued by US researchers as though the learning of a mathematical concept were uninfluenced by the learner's knowledge, the existence of related mathematical concepts, or the teacher's beliefs about the concept. To design and conduct studies that can handle the complexities of these multiple contexts is perhaps the biggest challenge we face. A final need is to devote some attention to scholarly inquiry into and reflection on our own research activities. Scriven (1980) terms this "self-referent research," and at first it may seem just another gimmick to find something else to do research on. But just as metacognition--cognition about one's own cognitions--is indispensable for intellectual growth, so some metaresearch efforts are required for the growth of our field.

Methodological Issues in US Research on Mathematical Learning and Thinking

The key methodological issue, as I see it, is how to take full account of the complexity of mathematical learning and thinking by adopting and adapting new methods of investigation without at the same time compromising our standards of quality in conducting and reporting research. US researchers are actively exploring approaches such as the teaching experiment (Kantowski, 1978), the case study (Stake, 1978), and the ethnographic study (Rist, 1980). Each approach holds promise, but no approach will suffice by itself. Some researchers are advocating more "artistic" approaches (Eisner, 1981), but it is an open question whether such approaches can be pursued successfully by people who have been trained as scientists and not as artists. The oldest refrain in research in mathematics education is the call for replication studies. The new refrain ought to be a call for convergent lines of investigation in which studies explore a question from a variety of perspectives using a variety of methods.

US researchers are presently rather disenchanted with quantitative methods of data analysis--although one might not necessarily conclude that from reading the articles in the Journal for Research in Mathematics Education. There is some danger, I think, that the reaction may be too extreme, and the quantitative baby may go out with the statistical testing bath water. Rather than abandoning quantitative methods in favor of qualitative ones, our efforts should be directed towards enriching both. In particular, new techniques of exploratory data analysis and presentation should be explored by researchers in mathematics education, and they should consider techniques for the reanalysis of data and the meta-analysis of studies.

Also, the development of instruments to assess mathematical learning and thinking ought to be given priority as a valid, vital research enterprise. We have been too prone to investigate those constructs for which we have instruments at hand rather than taking the time and effort to develop new instruments for assessing our constructs.

Research on Problem Solving in Mathematics

The designation of "problem solving" as the catchword of the 1980s in US mathematics education ought to gladden researchers concerned with mathematical problem solving, and one can expect that their ranks will swell in the next few years, but the problems of morale and identity touch them, too. The profession seems to have convinced itself that problem solving is important in school mathematics; the public, however, may not understand, or accept, this argument in the same way. Claims that research is needed to understand how children solve mathematical problems and how to teach them to solve such problems are likely to fall on uninterested ears. Some US researchers on problem solving in mathematics have formed informal networks of association that have been developed through working groups of the Georgia Center for the Study of Learning and Teaching Mathematics and through the National Science Foundation's meetings of project directors. These networks are fragile, however. I am not sure they can withstand the impending heavy weather of further reductions in federal funding.

Researchers on problem solving are as guilty as anyone in US mathematics education of failing to develop clear theoretical rationales to support their work. The name of Polya is invoked as though that absolved the writer of any responsibility to provide a scholarly analysis of problem solving. The term "process" is used as though its meaning in the context of mathematical problem solving had received a clear exegesis. Mathematics educators should critically examine the possible contributions of information-processing psychology to our theory-building work--to see whether, as Resnick and Ford (1981) assert,

> for the first time, psychology has a language and a body of experimental methods that is simultaneously addressing both the skills involved in performance and the nature of the comprehension underlying that performance. (p. 197)

The stress that information-processing psychology gives to the role of memory is reflected in recent work such as that of Ed Silver (1981), and researchers such as Alan Schoenfeld (1981) are attempting to adapt the information-processing notion of "executive decisions" to the analysis of mathematical problem solving. We still lack, however, those "home grown" adaptations of the information-processing metaphor that would illuminate the specifically mathematical characteristics of mathematical problem solving.

With regard to methodology, I would respectfully suggest that the proliferation of schemes for analyzing "thinking aloud" protocols is not getting us very far. It would be good to see more attempts by researchers in mathematics education to check results obtained from protocol analysis against results obtained by other methods, such as reaction-time studies. More studies ought to employ multiple methodologies--in the spirit, if not on the scale, of Krutetskii's (1976) work. (Some examples of recent studies in which multiple methodologies were used can be found in Harvey & Romberg, 1980.) Serious attention should be given to the tasks used in problem-solving research. Some recent work has been done on this issue (Goldin & McClintock, 1979), but better instruments for assessment are still needed.

I end this cursory survey on a note that I wish could resonate through the ranks of mathematics educators who do research in the United States--and perhaps it is needed in other countries, too. Research on mathematical problem solving is beginning to move out into the classroom, and the work of such people as Mary Grace Kantowski at Florida, Perry Lanier at Michigan State, Dick Lesh at Northwestern, and Nick Branca at San Diego State bears witness to the wisdom of this move. But we still need to reconceptualize our approaches so that mathematics teachers can become full partners in the research enterprise. As observers such as Sanders (1981) have pointed out, in no profession other than education does one find such a sharp differentiation between the community of researchers and the community of practitioners. The time seems ripe, if we are to move ahead confidently and productively, for research in mathematics education to develop its self-concept as an endeavor demanding stronger links with practice as well as better scholarship.

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Towards a Research Consensus in Some Problem Areas in the Learning and Teaching of Mathematics

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A significant turning point in the history of research in any araa occurs when from the chaos of competing ideas about a problem area, a single paradigm emerges which implicitly defines for practitioners the legitimate problems and methods of research. I believe that current work on several problem areas within the learning and teaching of mathematics should be viewed as foreshadowing the emergence of a firm research consensus in these areas.

TOWARDS NORMAL SCIENCE

To build this argument, I follow Thomas Kuhn's description of the "route of normal science." In his now classic treatise on the growth of science, The Structure of Scientific Revolutions (1979), Kuhn argues that the road to a research consenus in any area is arduous. In the absence of a paradigm or set of organizing principles, all facts that could possibly pertain to a problem area are likely to seem equally relevant. As a result, early fact-gathering is a nearly random activity. Furthermore, in the absence of a reason for seeking some particular form of instruction, early fact-gathering is usually restricted to the wealth of data that lie ready at hand. Thus, facts accessible to observation and experiment are pooled together with data retrievable from reports of classroom teaching, curriculum development, or program evaluation.

Although early fact-collecting has been essential to the origin of many significant sciences, one somehow hesitates to call the resulting literature scientific. Research in mathematics education during the decade of the 60s was largely that type of random fact-gathering. For example, in 1969, for the Review of Educational Research issue on science and mathematics education, I argued that research at that time could "be characterized as large in quantity, poor but improving in quality, and diverse" (Romberg, 1969, p. 473). The diverse problems being studied at that time made it difficult to classify studies even into reasonable problem areas. Hence, the copious literature simply lacked an implicit body of intertwined theoretical and methodological beliefs that permitted easy selection, evaluation, and criticism. Ralph Tyler echoed this argument in his summary for the National Conference on Needed Research in Mathematics Education.

> I conclude that we seem to have reached a consensus on a broad definition for research in mathematics education and upon the value of such research for the practical enterprises of teaching and learning in mathematics. There still remain two frequently voiced criticims: (1) current research is not sufficiently discriminating in selecting problems that are significant; and (2) current research seems to be scattered in the sense that it is difficulty to put together the various investigations and their findings in a way that provides the cumulative development of knowledge, thought to be characteristic of the natural sciences (Tyler, 1969, p. 135).

As the decade drew to a close, salesmanship gave way to questioning, and in some cases, to careful inquiry. We began to realize that learning and teaching in schools is complicated. The changes in content alone (called the "modern math" revolution) were not sufficient to produce drastic changes in mathematical learning.

The next decade might be characterized as a period of confrontation, confusion, and reflection. For example, by 1975, the Subcommittee on Research at the National Institute of Education sponsored Euclid Conference on Basic Mathematics Skills and Learning argued that we had now gone beyond fact-gathering.

> We agree with the critics that past research in this area (mathematics education) has been inadequate. Past studies can be characterized as a plethora of piecemeal studies rather than sets of studies reflecting scholarly chains of inquiry. Too many studies have been based on an inadequate conceptualization of the problem being investigated, and have employed poor instrumentation and inappropriate methodology. We feel that these faults of past and current research are the typical characteristics of emerging fields of inquiry. On the other hand, it is clear that we now know more about the teaching and learning of mathematics than we did some fifteen years ago before there was substantial federal support for educational research. In particular, we have eliminated some options which seemed at one time to be viable but proved to be unproductive and we have developed a much more sophisticated research methodology and have identified some potentially promising directions for research. For example, we know not to rely solely on simplistic frameworks such as behavior modification of discovery learning to solve our problems, not to use standardized performance tests as sensitive dependent variables, and not to rely on quasi-experimental designs from agriculture as canons of research methodology. In summary, we feel that a promising start has now been made both in terms of research completed and in terms of improvements in research methods (Romberg et al., 1975, pp. 2-3).

The literature of the past decade was filled with arguments such as: behaviorism vs. constructivism, quantitative vs. qualitative methods, advocates of open schooling vs. traditional, proponents of activit instruction vs. seatwork, and so forth. The claims, counterclaims, and arguments of researchers must have been confusing to teachers, administrato and graduate students. But, it really should be no surprise that, in the early stages of the development of any science, scholars confronting portions of the same range of phenomena arrive at different descriptions an interpretations. However, during such a period of confusion, there is envitably reflection. Some scholars are able to sift through the morass and through reflection piece together concepts and propositions which encompass and clarify the conflicts. What happens then is a group previously interested in the general study of a field, like mathematical learning and teaching is transformed into a scientific group committed to a specific problem area like adolescent learning of rational numbers. By the end of the past decade, I am suggesting that for some problem areas, within mathematics education, consensus is beginning to emerge, as in counting and early learning of numbers, learning of rational numbers, and verbal problem solving.

Kuhn argues that such consensus has two essential characteristics. First, the synthesis by bringing clarity to a confused area is sufficiently unprecedented to attract a group of adherents. Simultaneously, the synthesis is open-ended, leaving all sorts of problems for the redefined group of practitioners to work on.

Elsewhere, I have argued that "the key step in (the) evolutionary sequence from myth and tradition to theory is model building" (Romberg, 1981, p. 186). The purpose of building a model is to unpack the broad assertions about a problem in order to elucidate the key variables and the relationships between them. Going from macro-assertions to micro-analysis is the critical step.

The development of a model starts in some empirical situation that presents a "problem" for which an "answer" can be very misleading. One must recognize that real situations rarely appear well-defined and are often embedded in an environment that makes it hard to obtain a clear statement of the situation. Formulating the problem involves specifying the assumptions, concepts, and principles one believes are operating in the real situation. Such specification must of course be selective by its bias, and most often deals with only a small part of the larger problem. This simplification or idealization is important since the general problem is usually exceedingly complex and involves many processes. Some features of reality will appear significant and many irrelevant. A model's validity rests on its ability to represent the situation initially described. What I am claiming is that for some problem areas in the learning and teaching of mathematics we are now in the model building stage.

In summation, I believe, the "route to normal science" in any area often can be viewed in terms of six stages:

Stage 1: fact-collecting. Here random studies are carried out with little focus and no sense of order.

Stage 2: confrontation, confusion, and reflection. That is a period of sorting out facts, deciding on problems, arguing about well variables, etc.

Stage 3: model building. This is a time for organizing ideas about well identified variables, etc. (The current stage addressed in this paper.)

Although not relevent to this paper Kuhn suggests there are at least three other stages that follow;

Stage 4: paradigm selection.

Stage 5: normal science.

Stage 6: scientific revolutions.

Also, one essential characteristic marking the distinction between the notion of model and the classical notion of theory is that the point of a model often lies more in whether it illuminates the nature of a phenomenon than in whether it is true. There is nothing outrageous in the fact that models that generate very different propositions about the same phenomenon can coexist without invalidating each other.

Finally, whereas a theory is conceived of as having a certain minimum of generality, instances can be quoted of mathematical models that explain only <u>specific facts</u>. This notion of model contravenes the postulate that science deals only in generalities. We can go even further and say that recent progress in the social sciences is partly connected with the fact that it has been considered worthwhile to analyze things that, without necessarily being unique, are specific. Admittedly, this kind of model usually includes some general propositions, but these propositions by themselves are inadequate for explaining an observed reality. In summary, if a model is falsifiable, uniquely fits the phenomena under study, and has some generality (often called a "fitting" model), it is a theory. Otherwise, it is a "guiding" model. It is important to understand that this characterization of modeling is utterly at variance with the epistemological notions offered by classical physics, to which the social and human sciences clung slavishly throughout the whole of the nineteenth century and part of the twentieth. Because having a "fitting" model implies properties such uniqueness, generality, and verifiability, at present it is unsuitable for describing the results of attempted formalizations applied to the analysis of mathematics teaching and learning. Only by posing "guiding models" will we ever be able to build a theory of mathematical education.

AN EXAMPLE: TEACHING MATHEMATICS IN ELEMENTARY CLASSROOMS

Several problem areas in mathematics education could have been chosen to illustrate the trend toward research consensus. For example, Carpenter, Blume, Hiebert, Anick, and Pimm (1981) have just completed an extensive review of early number research and I (Romberg, in press) have argued that consensus is nearing in that area. Similarly, Swafford (1981) has traced the evolution of assessing attitudes toward mathematics and model building has begun (see Nimier, 1980).

For this paper, I have selected classroom instruction, in particular, what teachers do in elementary schools. Space does not permit a detailed consideration of all the issues associated with this development but let me pose the problem and trace the trend toward consensus in this area.

To me the problem is self-evident. Teaching little children is hard work; but the teaching of mathematics in most primary school classrooms after modern mathematics programs were introduced was not just hard it was poorly done. The writing of such schooling critics as Charles Silberman, Crisis in the Classroom (1970); John Holt, How Children Fail (1964); Carl Bereiter, Why Is Teaching Mathematics So Awful?; H. W. Sobel, "The anacronistic practices . . ." (1969); or Morris Kline Why Johnny Can't Add (1973) to name a few are filled with examples of bad mathematics teaching in primary school. Perhaps it is impossible to teach the initial ideas of mathematics well, given the state of our knowledge about mathematics and how children learn. And, probably the teaching of mathematics in primary school has always been poor.

But, it has become a problem of research interest as a result of the "modern mathematics" movement. The intentions of the curriculum reform movement were honorable, but in retrospect, the reformers in attempting to organize mathematics, failed to see the complexity of the pedagogical problems they faced. Because proposed curriculum activities were not often understood by teachers, the resulting instruction too often became chaotic, stultifying, and at best dull and non-motivating.

During the 1960's researchers interested in this problem area found little in the existing research to help clarify the problem. The publication of the Handbook of Research on Teaching (Gage, 1963) helped scholars to see the complexity of the problem. Reviews during the era, such as those by Davis (1967) and Fey (1969), indicated the lack of consensus on how to study the problem of classroom teaching, what was involved, etc.

During the 1970's, two lines of research, best summarized by Bellack (1978 and 1981), became prominent. In the first approach, rooted in behavioral psychology and called "process--product research" (Dunkin & Biddle, 1974), observers count frequencies of classroom behaviors (processes). Then these frequencies are correlated with student performance (product). Gage's recent book, The Scientific Basis of the Art of Teaching (1979), summarizes this line of inquiry. Recent syntheses of this research suggest that for mathematics in the elementary grades, the classroom with the following characteristics leads to increased academic performance (Rosenshine, 1976; Rosenshine & Berliner, 1976; Medley, 1977; Brophy, 1979):

- The teacher keeps the students on academic tasks, and the content coverage is extensive.
- The teacher and workbook questions are highly structured, and elicit a relatively high rate of correct answers from students.
- The teachers and materials provide immediate, academically-oriented feedback, praising correct responses and exploring incorrect ones.
- Instruction is provided to the whole class or to small groups.
- Teachers monitor student performance during recitation sessions, and provide individualized feedback to students.

The second approach, rooted in sociology and called "interpretive research" (Bellack, 1981), involves observers describing the flow of classroom events as perceived and interpreted by teachers and students. Studies of classrooms, such as: Sharp and Green (1975), Bowles and Gintis (1976), Bossert (1979), Lundgren and Pettersson (1979) and recently Popkewitz, Tabachnick, and Wehlage (1981) picture schools with characteristics quite different from the first approach (Romberg, 1980).

- Schooling is a collective experience for children.
- Time is the major control mechanism of schools.
- The students at a particular age are assumed to be more similar to each other than they are different.
- Instruction within a time segment involves children working on a lesson which addresses competition, evaluation, order and control.
- Overt knowledge and skills to be transmitted to the children are expressed in terms of cognitive terms rather than social vocational development.

These five characteristics are traditional elements which make an America school a school. The job for teachers then is to assign lessons to a class of students, start and stop the lessons according to some schedule, explain the rules and procedures of the lesson, judge the action of the students during the lesson and maintain order and control throughout. For students the job is to be active participants in each lesson, attend to the explanation of rules and procedures, work independently on tasks, and try to be successful.

Also, during the 70's a lot of other activities not directly part of either main approach contributed to the problem area such as: program evaluations in mathematics classrooms (e.g., Romberg, 1976); retrospective accounts of curriculum implementation (e.g., Shulman & Tamir, 1978); studies of organizational decision making (e.g., Anglin, 1976); and analytic critiques of research (e.g., Doyle, 1978).

Today, many of the ideas from these approaches and independent activities are being brought together in "time-on-task" models. Borg (1980) has traced the historical background of such models. One such model has been used in the Beginning Teacher Evaluation Study (BTES) (Fisher et al., 1980) and more recently in the IGE Evaluation Project (Romberg, 1976), Coordinated Studies in Mathematics (Romberg, Small, & Carnahan, 1979), and the Sandy Bay Study (Romberg & Collis, 1980). Excellent reviews of this approach have been written by Westbury (1979), and Anderson (1980). In fact, the steps associated with an instructional model based on this had been outlined recently by Good (1981).

The past 20 years of research on classroom teaching in mathematics has followed the stages outlined above. Early research dealt with describing the phenomena of classrooms and how they operate. This was followed by considerable controversy, reflection, alternate approaches, etc. Now I believe there is emerging a consensus associated with models of how time is spent in classrooms.

CHARACTERISTICS OF TRANSITION TO MODEL BUILDING

To summarize this argument about the route to normal science, I would like to emphasize three characteristics of this evolution. The first, is the shift from "macro-analysis" to "micro-analysis." Model building includes a very careful micro-analysis of the processes that are involved in a particular problem area. Studies of "good" and "bad" teaching give way to detailed analysis of specific aspects of teaching (decision making, planning, instruction, etc.).

The second characteristic is a discipline shift. In this case from mathematics to psychology and sociology. The mathematics to be taught and the way it is to be organized as the areas of principal concern has been replaced by behaviors and intentions. I must admit this shift in some cases has been over done as Buchman (1980) has recently argued.

The third characteristic is the shift from individual research to collaborative programs. Lakatos (1978) has outlined the importance of chains of research which link to form a larger conceptual framework about a particular phenomena. Problems such as the study of classroom learning are complex and go beyond the capabilities of an individual researcher to investigate. Collaborative teams with members having different conceptual backgrounds (i.e., mathematicians, sociologists, psychologists, educators, teacher trainers, etc. working together) are needed.

CONCLUSION

From the time when I entered the educational research arena (nearly 25 years ago) what constituted a reasonable research agenda and how one went about doing research has changed. I believe we are near a significant turning point in the history of research in mathematics education. We have gone from a variety of atheoretic, fact collecting studies to a period of model building. We have learned to appreciate the contributions that other scholars (particularly psychologists and sociologists) can make. This is an international trend which I believe is already producing important research for the area of the teaching and learning of mathematics.

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WHAT IS A GOOD ENVIRONMENT FOR THE INTELLIGENT LEARNING OF MATHEMATICS? DO SCHOOLS PROVIDE IT? CAN THEY?

Richard R. Skemp, University of Warwick.

A ICME IV (1980) on a ébauché un nouveau modèle d'intelligence, tout en montrant le rapport avec le travail déjà fait sur la comprehension des mathématiques. On a fait ressortir la fonction adaptive de l'intelligence vis-à-vis de l'environnement physique et social. On a présenté les mathématiques comme un exemple puissant, et utile à plusiers points de vue, de cette activite de l'intelligence humaine qui contribue puissamment à notre maitrise de l'environnement physique, et qui nous rend capables de coopérer avec les autres. Elle est aussi un exemple de notre puissance creatrice. C'est surtout pour ces raisons-là que les mathématiques sont si importantes dans le monde d'aujourd'hui.

Sur plusieurs plans, l'environnement des écoles diffère de celui du monde de travail, où les mathematiques sont exploitées d'une façon pratique. En particulier l'école ne fournit à l'élève aucunne des trois situations majeures dans lesquelles les mathématiques peuvent être aperçues comme une source de pouvoir et de devertissement.

Un caractéristique important de l'intelligence, c'est qu'elle nous donne le pouvoir de construire (c'est a dire de développer et de mettre à l'épreuve) des structures conceptuelles qui, a leur tour, nous permettent d'inventer des stratégies ou des méthodes pour exécuter une grande variété de tâches. On a distingué trois façons de développer, et trois façons de mettre à l'epreuve. Mais de ces six façons, deux seulement sont employées dans les méthodes conventionelles d'enseigner les mathématiques.

Puisqu'on doit toujours s'adapter à un environnement, nous devons nous demander si l'adaptation à l'environnement d'une école aide aux enfants d'apprendre les mathématiques. Je reconnais que les écoles diffèrent beaucoup entre elles, mais j'arrive, à contre-coeur, à la conclusion que beaucoup d'écoles ne constituent pas un bon environnement pour apprendre d'une façon intelligente les mathématiques.

Pourraient-elles le faire? Oui, si elles le désiraient vraiment. Mail il ne suffirait pas de ne changer que le programme scolaire.

The present paper is a direct sequel to the one I presented last year at ICME IV (Skemp, 1980). In that paper, I described how I saw mathematics as one of the most important examples of the functioning of human intelligence, by which (both in general and in this particular case) we are better able to do three things:

(i) to achieve goals in the physical world

(ii) to co-operate with our fellow beings

(iii) to create.

At the end I wrote: "I've been describing the way mathematics is for people like air navigators, scientists, technologists, bankers, industrialists, creative mathematicians. For many children, this is not the way it is. They see it variously as a job to be done, a threat to be averted, a complicated exercise for fulfilling teachers' expectations, or (in Erlwanger's memorable words) as 'a set of rules for making arcane marks on paper'." It is this contrast that I want to explore further today: between mathematics as a source of power for those who use it successfully in the adult world, and mathematics as at best a chore and at worst a source of failure as it is for so many in the world of school. Mathematics is indeed one of the most successful products of human intelligence, cumulatively over many generations. But what has been coming through to me ever more strongly since writing the ICME paper, is that the uses for whose achievement mathematical schemas are so powerful a source of help find little or no counterpart in the child's school situation. It's not just that the uses are different: it is that the <u>kinds of use</u> are different. Some good examples of these differences are given by Davis (1980) in his thoughtful essay "The Possibly Elusive Content that needs to be Learned", the whole of which is highly relevant to the problems on which I shall be focussing today.

In my talk today I shall be taking a wide-angle view, focussing on global features rather than detail, with situational influences and implied goals rather than the overt and explicit contents of teaching, important though the latter are also. My justification for this is that details may change their meaning in different contexts, and the most well-intentioned and well-devised teaching can fail to achieve its goal unless account is taken also of the wider context in which this is taking place, and especially of the reality of the situation as it is construed by the learner. To adapt an aphorism I heard many years ago: "What you are doing shouts so loud, I cannot hear what you say."

A human child is at the most learning age of the most learning animal that this earth has yet produced. Yet daily mathematical instruction for about 36 weeks a year for about 10 years produces the truly dismal kind of result of which we are repeatedly made aware by surveys in my own country (and the story seems to be the same, with minor variations, elsewhere): namely that children have learnt to perform routine processes well, but are bad at adapting these to the requirements of new situations. Adaptability is a key feature of intelligence, so the inference seems clear to me. Lack of adaptability suggests strongly that the mathematical learning which has this as its outcome is not making full use of children's intelligence. Mathematics is not being taught in ways which bring about intelligent learning, or it is not being taught in an environment favourable to intelligent learning, or both.

As conceived in my own model (Skemp, 1979), intelligent learning consists of

(i) the construction of appropriate schemas;

(ii) devising from these particular plans for particular tasks. These may include routine plans (algorithms) for frequently-encountered tasks.

When the tasks are non-routine, (ii) becomes problem-solving. The first of these may be called "knowing-that", and the second and third "knowing-how". We also need "being able" - efficiency at putting these plans (methods) into action accurately and quickly. These are often called "skills", but for me skill implies adaptability, so I would only accept this term if conceptual back-up is also available if needed. It is the parent schemas from which new plans can be derived when necessary which are the sources of adaptability. So we need now to think about the processes by which these schemas come into being.

My own position here is a constructivist one, that conceptual knowledge cannot be transmitted directly. Only the learner himself can construcr these schemas within his own mind.

Teaching is an intervention in this invisible activity. A learner can be greatly helped by a teacher who has a good mental model of the processes by which the learner constructs his schemas. But if this is not the case, I can see no reason to expect that teaching will do good at a level better than chance. And this may not be far from the way things are.

So now let me offer part of my own model of schema construction.

SCHEMA CONSTRUCTION

SCHEMA BUILDING	SCHEMA TESTING
Mode 1 from our own encounters with actuality: experience.	Mode 1 against expectations of events in actuality: experiment.
Mode 2 from the schemas of others: communication.	Mode 2 comparison with the schemas of others: discussion.
Mode 3 from within, by formation of higher order concepts: by extrapolation, imagination, intuition: creativity.	Mode 3 comparison with one's own existing knowledge and beliefs: internal consistency.

Figure 1. (from Skemp, 1979)

As I use the term, construction includes both building and testing. The left hand column describes ways by which a schema is changed; the right hand column, ways in which we check whether a change is for the better. These are the methods which our intelligence has at its disposal for schema construction: and they work best when used in combination. The question is now, to what extent do they describe the methods by which mathematics is learnt in school? As I continue, you may like to fill in your own assessments in the table below, allotting marks from zero to ten.

	USED AT ALL		USED WELL	
	Building	Testing	Building	Testing
Mode 1				
Mode 2				
Mode 3				

Table 1

Building mode 1. This is used freely in the first few years of school, and then usually replaced by work on paper, which is seen as its purpose. For example, structural material such as multibase, abacus, are used for developing the concepts involved in place-value notation, and as a means for obtaining answers to questions such as 375 + 283, 375 - 283. The aim is for children to be able to do without help of this kind. Mode 1 building is often put to good use in these early years, and my complaint is only that it is discarded too soon.

Testing mode 1 is the reverse process - using a mathematical model to predict an outcome with physical materials and events, and then putting this prediction to the test. This may be done as a means of checking up on one's model, as for example by dropping a stone and a feather simultaneously in vacuum. This is the world of the experimental scientist, heavily dependent on mathematical models. Or it may be as a means for achieving a goal in the physical world. For example, if we fly on such and such a bearing we expect to arrive at Newfoundland. Again, the navigator is heavily dependent on a mathematical model.

Certainly, experimental error can be a source of difficulty if we want to introduce mode 1 testing into the classroom. But my complaint is that teachers do not even try. They don't use mode 1 testing even when experimental error would present little or no problem. If structural materials are used to build mathematical models, it is often possible to use these models to make and test predictions of outcomes with these physical materials. In some initial trials of this I have been impressed by the pleasure shown by children when their prediction was confirmed by the outcome.

Overall, I would see the development of mode 1 testing as closely related to the development of closer links between mathematics teaching and science teaching.

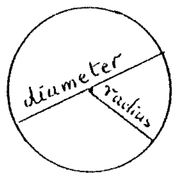
Mode 2 building. This is the main method used in the majority of classrooms. Usually it is done very badly. Here is an example from a text book which is popular in England. It comes in a chapter on area. This began by explaining why, for a rectangle,

area = length x breadth,

followed by exercises in calculating areas of figures which were either rectangles, or could be divided into rectangles. Without any further explanation:

Circles

> The circumference of a circle (that is its perimeter, or the length of its boundary) is found by measurement to be a little more than three times the length of its diameter. In any circle the circumference is approximately 3.1416 times the diameter, which is roughly $3\frac{1}{7}$ times the diameter. Neither of these figures is exact, as the exact number



> cannot be expressed either as a fraction or a decimal. The number is represented by the Greek letter π (pi).
>
> Circumference = πd or $2\pi r$
>
> Area = πr^2

Examples of this kind of "telling, not teaching" abound. Contributions for my collection will be welcome!

What is wrong with the approach which this typifies, and what needs to be done instead for communication to be a successful contributor to schema building, I have discussed at length elsewhere (Skemp, 1971).

Testing mode 2. This refers to the testing of schemas. Indicating, e.g. by a tick or a cross, whether an answer is right or wrong is not testing mode 2. Nor is indicating whether a method is correct necessarily testing by mode 2, since it may be confined to showing whether or not a rule has been correctly followed. If however a method is discussed in relation to a parent schema, this I would call mode 2 testing, in combination with mode 3 - consistency with one's own existing knowledge and beliefs. If we consider this a little further, we shall see that fruitful discussion depends on the presence of shared schemas, or the discussants are not talking about the same things.

A very basic way of helping to ensure that discussants are talking about the same thing is to bring in mode 1 testing also. Thus, both discussants will be looking at the same physical objects and events. Even so, a shared schema is still necessary also, for without this they will not give the same interpretations to what they see.

There are two other features of discussion which make it valuable in schema construction. Successful communication involves the attachment of symbols and the clear formulation of one's ideas as preliminaries for communication, and these in turn favour reflection. The need sometimes to defend one's ideas also promotes reflection, with the object of being in a position to justify one's ideas in relation to an accepted body of knowledge.

Thus I would see discussion as a most valuable mode of schema construction, especially when combined with mode 3. And since mode 3 is the most sophisticated kind of testing, I would wish sometimes to support discussion with mode 1 testing.

To what extent do we find discussion of the kind I have been describing - a genuine interaction of mathematical ideas, of the kind we ourselves enjoy with colleagues and graduate students - in school mathematics lessons? Before you answer, bear in mind that this would involve teachers listening to children's ideas (not just checking their answers), and children listening and responding to each others' ideas. How often do we find this taking place in the average mathematics lesson?

<u>Building mode 3</u>. The two clearest examples of this in mathematics learning are extrapolation, and the formation of higher order concepts. Much of mathematical learning requires one or both of these activities. Extrapolating is done by teachers all the time, but seldom in such a way that the pupils also extrapolate in their own minds, since they are not told what is happening. Consider multiplication. First they learn to multiply the natural numbers. (Often this is taught as repeated addition, which makes the concept particularly hard to extrapolate.) Then they are told how to multiply positive and negative integers,

fractions, matrices. What they are not told is that "multiplication" now means something different. The concept itself has been extrapolated, and not just the method for doing it. Without this essential information it is unlikely that pupils will achieve relational understanding of this new meaning. So when they are asked "Which is larger, 2n or n+1?" they give replies such as "2n, because multiplying always makes bigger." (Hart, 1981).

The formation of higher order concepts involves the process of repeated abstraction: the discovery of regularities among objects and events which are not physical, but mental, objects and events. The constructivist principle shows particularly clearly here, since every learner has to make these discoveries afresh. Yet the need for good teaching is also here particularly great, since even the most gifted pupils, leave alone the average, are unlikely to be able to repeat, unaided, and in a few years, discoveries which have taken some of the best brains of mankind collectively many centuries to discover. So we have a paradox. Children can't be told this knowledge, and they can't construct it unaided. To resolve this paradox, children need teachers who are able indirectly to direct the mental creativity of others. How many teachers of this kind do you know?

If your answer is the same as mine, more's the pity, because mode 3 building is one of the most exciting and powerful activities in mathematics when it happens successfully. By helping it to happen, teachers might gain from their pupils real gratitude, and even devotion.

Testing Mode 3. This is quite a sophisticated activity, involving reflection not only on mathematical ideas in themselves but on relationships of implication between them. (If ABC is a triangle in which $A\hat{B}C = A\hat{C}B$, then ...). Formal proofs are an example of this in its fully developed form. Memorising formal proofs (yes, this does happen) is an example of its gross mis-use. Its proper use would be greatly encouraged by discussion, initially quite informal, in which the participants together examined statements, methods, results, using consistency with accepted mathematical ideas as criterion for acceptance or refutation. For this, relational understanding is a necessary but not sufficient condition. By taking part in this activity over a period of years, pupils might arrive at a stage where the activity itself, independent of its particular content, could be conceptualised and reflected on. Only in this way are they likely to reach an understanding of the nature and purpose of formal proof. I doubt if this is achieved very often. As a good use of mode 3 testing, I would certainly also accept the less sophisticated activity of informal discussion, which I see as a valuable means of leading pupils to an intuitive understanding of the nature of proof. How much of this takes place? I suspect that it varies greatly among classrooms. In those where text books and work cards are the predominant method of teaching, there can be very little.

*   *   *   *   *

Now to the uses of mathematics, as experienced by children in schools. I said earlier that not only were the uses different from those in the adult world, but the kinds of use. The importance I attach to this is based on the assumption that we construct different kinds of schema for different kinds of use. Hence, only the right kinds of use will favour the construction of sound mathematical schemas. Again, you may like to record your own assessments as we go along.

Uses of mathematics as experienced by children in schools.

(i) to achieve goals in the physical world

(ii) to co-operate with others

(iii) to create.

Or do they experience it as

(a) a job to be done

(b) a threat to be averted

(c) an exercise in fulfilling teachers' expectations

(d) a set of rules for making marks on paper?

Table 2

In group (i), mathematics provides an inexhaustible source of models from which particular plans of action can be derived for the achievement of a wide variety of goals in a wide variety of conditions. Even the same model can, by changing the units, be put to a variety of different uses. E.g.

$d = vt$ $D = kd$ $E = IR$

are these different uses of the same mathematical model.

In your judgement, is this how mathematics is made to appear to school children? Does it actually give them greater power to predict and control their environment, in their own here-and-now experience? Please consider your verdict.

Uses in group (ii) relate to the interpersonal, the social uses of mathematics for co-operation in science, technology, industry. Again, I can find no analogous use in the classroom. The kinds of co-operation which take place (e.g.) between designer and engineer, air navigator and ground controller, lens designer and computer programmer, just are not found.

Uses in group (iii). Here I refer to the uses of mathematical schemas as agents of their own growth. By reflecting on our schemas, extrapolating, imagining "What if ...", and sometimes also by intuitive leaps, existing knowledge can be used to create knowledge. It is closely related to mode 3 building. Though particular characteristic of pure mathematics, there can be a payoff also in mode 1. I find it exciting that an extrapolation of the idea of square root to that of a negative number has useful practical applications in electronics. Can we find similar excitements for children?

* * * * *

Having filled in my own assessments in tables 1 and 2, I find the results both depressing and hopeful. Depressing, because it seems to me that we're setting children a learning task which makes considerable demands on their intelligence, but we are not providing them with a situation in which they can use their intelligence to best effect. Rather the opposite. We are asking them to make an important and difficult intellectual journey, not just on level ground but a climb to considerable heights of abstraction, with their legs and their hands tied. I am hopeful also: for if we untie them, there is reason to think that we shall be pleasantly surprised at the progress which children then make. But the changes which will be necessary are more than changes in curriculum: they are major changes in our conception of what is a good learning environment. And if the majority of teachers are themselves the results of learning situations of the kind I have described, how can we persuade them first to untie themselves?

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COMPLEMENT AU VOLUME I
DES ACTES

PSYCHOLOGICAL BLOCKS TO LEARNING OF MATHEMATICS IN REENTRY WOMEN

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ABSTRACT

Les campus universitaires des Etats-Unis voient aujourd'hui à l'intérieur de leur domaine un nouveau groupe d'étudiants: des femmes qui choisissent de recommencer leur instruction universitaire quand elles sont déjà d'un certain âge. Un grand pourcentage de ces femmes souffrent des angoisses à propos des cours de "maths" requis, et elles perçoivent les unités de valeur en mathematiques comme un obstacle assez décourageant quad il s'agit de réaliser leurs projects. Des tests psychologiques administrés à un groupe de ces femmes, qui ont participé dans des travaux pratiques destinés à enlever leurs angoisses à propos des "maths,", ont, en réalité, montré qu'il s'agissait de' hauts niveaux d'inquiêtude quand elles devaient faire face à des phrases contenant des termes de mathématiques comme symbolisme. Lés résultats ont aussi indiqué la préférence de ces femmes pour des renseignements absorbés mieux par la vue que par le son, ainsi qu'une préférence pour la participation plus que pour l'observation. Les processus de raisonnement de ces femmes avaient tendance à réflêter l'application des règles apprises auparavant. Le laboratoire contre les angoisses au sujet des mathématiques était conduit par des psychologues et par des enseignants de mathématiques qui travaillaient ensemble. Le contenu du laboratoire se concentrait sur la capacité des femmes de survivre sous pression et sur leur façon de résoudre leurs problèmes dans une ambiance peu menaçante. Les évaluations faites aprés indiquaient un effort positif de la part de ces femmes pour dominer leurs angoisses.

College campuses in the United States today are seeing within their student body a new group of students: women who choose to resume their education in their middle years ("reentry" women). These women, many of whom have raised a family, are now returning to school with great determination to pursue a new and meaningful career. San Diego State University, a large public university in Southern California recently conducted a new student orientation program especially for reentry students. A large percentage of the women indicated at that time that they would like to pursue a career that involves mathematics, but they perceived this need for mathematics as a formidable roadblock in their plans.

To assist these women in achieving their goals, several women faculty organized a workshop modeled after the experiences reported by Shelia Tobias. Nineteen reentry women participated in the week-long workshop which met for two hours each day. The women ranged in age from 27 to 62 with a median age of 40.5 and an average age of 44. The women entered San Diego State University in the Fall of 1980 after a sustained absence from academic life. Their declared majors are in the areas of Social Work, Speech Pathology, History, Psychology, Anthropology, Zoology, Microbiology, Nursing, Art, Business Administration, and Liberal Studies. Eleven of the group are currently registered in a mathematics course.

In conjunction with the workshop each woman was administered a mathematics anxiety rating scale as well as a cognitive style mapping questionnaire. Results of the math anxiety scale indicated that a high percentage of these women were comfortable with doing arithmetic operations or problems where an arithmetic operation is clearly indicated. However, many of them showed high levels of anxiety on statements where a reference was made to mathematics using symbolism. Nearly all of them registered high levels of anxiety when a reference was made to evaluating their mathematical skills.

The cognitive mapping questionnaire indicated that certain personality and learning characteristics could be identified as being typical of this group of reentry women. Identification of their characteristics served not only to describe the group and support initial perceptions, but also to point out possible reasons for learning difficulties and/or anxieties as well as to indicate activities which could be incorporated into the workshop and from which the women would be likely to profit.

This group of reentry women endorsed statements reflecting a strong commitment to personal principles, a good knowledge of themselves, an awareness of time, reflected in a sense of the importance of meeting time deadlines and an ability and/or a preference for working independently on projects or problems. All of these attributes are consistent with characterizing this group as being dedicated, serious students with clear goals to which they are committed.

These women also tended to be uniform with respect to certain learning characteristics which are related to academic pursuits. They tended to prefer visual input for both verbal and numeric information. Furthermore, input of numeric information by hearing was overwhelmingly the least preferred modality of information processing. Of the five senses, information processing by the sense of touch was predominately preferred to any other of the five senses. They also indicated a strong preference for learning by doing rather than learning by observing. They tended to take a non-assertive role in their relationship with others, avoiding or feeling uncomfortable with a role which called for persuasion or convincing others of their viewpoint. They tended to deemphasize the role of their peer group in influencing their actions or providing assistance for them. Finally the reasoning processes of these women tended to reflect primarily the application of previously learned rules.

The workshop organized to assist these women was conducted by three professionals trained in psychology or counseling and two mathematics educators. Activities were divided approximately evenly between the counseling group and the mathematics group.

The counseling group presented activities which were designed to provide the women with experiences in managing stressful situations and to encourage acknowledgement of their anxieties. These included activities such as a demonstration of relaxation techniques, a discussion of the socialization of women and its possible effects on their attitude toward mathematics, clues on overcoming test anxiety and a discussion of cognitive restructuring of self-imposed belief systems.

The mathematics group engaged the women in activities designed to convince them of their ability to learn unfamiliar mathematical language as well as some problem-solving techniques. Geometric transformations were used in a very intuitive manner to create figures which would tesselate a plane and several geometric theorems were demonstrated by using transformations. Games and mathematic puzzles were used to demonstrate some of the problem- solving techniques found in Polya's work. Activities were chosen which provided the women with an opportunity to seek solutions by manipulating objects or drawing diagrams. Interactive participation by the women was encouraged and did, indeed, take place.

During the final sessions of the workshop each woman conferred with a member of the staff who provided an interpretation of the cognitive style mapping administered earlier in the week. Each also had a conference to assess their mathematics needs. Following the workshop tutors were provided for those women currently enrolled in a mathematics course. Care was taken to insure that the tutors were cognizant of the special problems of reentry women. This group of women have subsequently formed the nucleus of a network designed to provide support for reentry women.

An evaluation instrument administered at the conclusion of the workshop indicated a very high degree of satisfaction on the part of the women. The statement in the evaluation which elicited the greatest positive response was "The workshop gave me a basic feeling of support." A follow-up investigation of these women currently registered in a mathematics course will be done at the conclusion of the Spring term to determine their accomplishments and possible changes in attitude toward mathematics.

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COGNITIVE FUNCTIONING, CLASSROOM LEARNING AND EVALUATION: TWO PROJECTS

by

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ABSTRACT

Dans ce mémoire, les auteurs veulent discuter les progrès de deux études collaboratives.

Le premier programme s'intéresse au rapport entre la faculté de transformations cognitives (CPC) et le niveau de réussites aux problèmès d'addition et de soustraction pour un groupe d'enfants agés de 4 à 8 ans en Tasmanie en Australie. D'abord, on a groupé les enfants selon leur CPC--un indice composé des résultats de deux séries de tests, le premier qui a mesuré "M-space" et le deuxième qui a mesuré le nivean du développement cognitif. Ensuite, on a examine le rapport entre le groupe dans lequel l'éstudiant a été placé et sa réussite à propos d'autres variables. Les résultads préliminaries mon trent que:

(i) il y a une augmentation significative du niveau de réussites sur les problèmes mathématiques qui accompagne chaque augmentation du CPC.

(ii) les enfants de CPC différent utilisent des stratégies differentes.

(iii) le niveau de travail actif ("engagement") s'élevè selon l'augmentation du CPC.

(iv) les étudiants ayant un CPC élevé s'appliquent aux tâches quelle que soit l'activité du professeur.

(v) le pourcentage d'étudiants qui utilisent les algorithmes enseignés pour trouver la solution d'un problème ne s'élevè as selon l'augmentation du CPC.

Le deuxième programme de recherches vise au développement d'une série de "super-items" (Cureton, 1965) pour examiner la capacité des étudiants agés de 8 à 17 ans à résoudre des problèmes mathématiques. La technique utilisée est fondée sur un procédè qui vient d'être développer (Biggs & Collis, 1981); ainsi, on peut analyser les réponses des étudiants selon les caractéristiques de leur structure. Dans ce programme, on a renversé le procédé habituel: on cherche à développer des questions d'une structure particulière afin que la réponse correcte soit un indice d'une aptitude à résoudre les problèmes mathématiques d'un tel niveau.

Nous avons choisi les critères suivants selon les quels nous avons développé les quatre questions posées vis-à-vis "la tige" de chaque problème.

1. une, seulé structure: l'étudiant doit utiliser un seul renseignement qui est presénté d'une facon bien claire dans la tige.

2. plusieurs structures: l'étudiant doit utiliser au moins deux opérations complètes mais differentes à propos des renseignements séparés dans la tige.

3. un rapport: l'étudiant doit utiliser au moins deux opérations complètes mais differentes qui montrent une compréhension intégrée de la plupart des renseignements dans la tige.

4. une abstraction extensive: l'étudiant doit utiliser un principe abstrait qui est évoqué par les renseignements dans la tige entiere.

Au moment actuel, on vient de développer les problèmes et en automne on espère les examinee d'une facon empirique a propos d'un èchantillon assez large d'étudiants américains.

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To this point, my attention has been devoted mainly to children's processing capability whereas my co-author, Professor T. A. Romberg, has had extensive experience in experimental classroom research and analysis. The joint projects that we have undertaken have underlying them the notions that not only might more rapid improvement in classroom instruction result from the collaborative approach but also that new light might be shed on fundamental issues in our own special areas of interest. Currently, we are engaged in two joint projects which I wish to report in this paper. The first, and larger of the two projects, is still being analyzed but enough has been done to give a preliminary flavour of what the final results are likely to be. The second has only just reached the end of stage 1 of a two-stage project but the approach and preliminary results are of sufficient interest to warrant reporting at this conference.

Project No. 1: Cognitive Level and Performance on Addition and Subtraction Problems

The intention in this project was to endeavour to classify a population of children aged 4 to 8 years into groups according to their Cognitive Processing Capabilities (CPC)--this was done by giving two batteries of tests one to measure M-space (Case, 1978) and one to measure Cognitive Developmental Level. The tests in both batteries were intended to bear implicitly on the early learning of mathematical material. The CPC measures and their descriptions were to be obtained by combining the information obtained from the two batteries of tests. Following this classificatory procedure, the relationship between each distinct group with particular CPC characteristics and various experiences incorporating mathematical content in the beginning school years was to be examined. A preliminary examination of some of this data was reported by my co-author at Berkeley last year (Romberg & Collis, 1980a). Let us summarize the results of grouping according to CPC measures.

It was found that, using Factor Analytic techniques and a cluster analysis procedure (for details, see Romberg & Collis, 1980b, c), the population of 4 to 8 year olds could be assigned to six groups with the associated characteristics summarized in Table 1.

The tables following Table 1 indicate the direction in which the analyses are leading us with respect to the relationship between CPC level and the following five variables:

(i) achievement on elementary addition and subtraction problems,

(ii) strategies used by pupils on elementary addition and subtraction problems;

(iii) pupil engagement on school tasks;

(iv) teacher actions: the effect on pupil engagement;

(v) pupil use of addition and subtraction algorithms.

All the results quoted below are from the total population and are independent of both age and grade or class.

Table 1

M-space Groupings with Associated Characteristics

Group	M-space Measure	Characteristic
1	1	elementary qualitative comparisons only, lack quantitative and logical ability
2	2	qualitative correspondence, lack specific quantitative and logical skills
3	2 *S+	high qualitative correspondence, have certain specific quantitative skills (i.e., counting for specific purposes), do not reach criterion on logical skills
4	3 *S-	high qualitative correspondence, high on quantitative skills, do not reach criterion on logical skills
5	3 *S+	ceiling on qualitative correspondence, high on quantitative skills, high on logical skills
6	4 *S-	

*S+ and S- represent the presence/absence respectively of a spatial ability as measured by one of the tests.

(i) Achievement: Table 2 shows the percentage correct by CPC level on addition and subtraction problems using numbers up to 20. The results are the combined scores for two tests, one in which physical material was available and one in which physical material was not available. The results show a significant increase in achievement by CPC level--the biggest gains being made between levels 1 and 2 and again between levels 2 and 3.

(ii) Strategies: Table 3 shows the percentage of the various kinds of strategies used by the same children on the same problems as were involved in the results in Table 2.

Table 2

Achievement on Tests C+ Tasmanian Data
(% Correct, Total Population)

CPC Level	% Correst Responses	No. of Trials Involved
1	22	180
2	65	450
3	81	396
4	83	264
5,6	96	252
Total	72	1542

Table 3

Pupil Strategies on Tests C+ Tasmanian Data
(% of Times Strategy Used; Total Population)

CPC Level	Direct Modeling[1]	Counting	Routine Mental Operation[2]	Non-routine Mental Op.[3]	Inappropriate
1	28	0	1	0	70
2	36	18	13	6	27
3	18	33	26	10	13
4	11	30	35	14	9
5,6	13	40	42	6	0

[1] Using physical material, e.g., counters, fingers, etc.

[2] Using known number facts or relationships

[3] Innovative use of number facts or relationships

There are several interesting features of Table 3 which need careful consideration, three of which will be mentioned here. First, there is a very significant drop in the use of inappropriate strategies employed from levels 1 through to level 3; second, the use of direct modeling goes down as CPC level rises and the use of counting and routine mental operations rises; third, the reduction in use of inappropriate strategies is spread over the other categories, with counting and routine mental operations taking the largest and almost equal shares.

(iii) Pupil Engagement: Table 4 gives us information on the percentage of class time during which the children at the various CPC levels are engaged on the school task in hand.

Table 4

Pupil Engagement on Task
Tasmanian Classroom Observational Data
(% of Time Pupil on Task; Total Population)

CPC Level	% Time On Task	% Time Off Task
1	64	36
2	65	35
3	70	30
4	76	24
5,6	87	13

It appears from these results that those pupils who can best afford to be off task spend more time in this state than their colleagues with higher CPC levels. The data is being broken down for more detailed examination at this time and will be reported more fully in our final report. However, one piece of analysis which is available and is of interest in this context is that <u>when engaged</u> the level 1 group spend 86% of this time on content and 14% on being given directions as to what to do--the percentages in the same categories for level 5, 6 are 91% and 9%. In other words, there does not seem to be a lot of difference between the levels in what the engaged time is spent on but there is a large difference in the amount of time spent attending to classroom tasks from level to level.

(iv) <u>Teacher Action and Pupil Engagement</u>: Table 5 gives us information on percentage of time that the pupil remains engaged on his/her classroom task when the teacher does various things.

Table 5

Teacher Action vs. Pupil Engagement on Task
Tasmanian Classroom Observational Data
(% of Time Pupil on Task; Total Population)

CPC Level	TEACHER ACTION				
	Speaking to Individual Pupil	Speaking to Small Group	Speaking To Large Group	Not Speaking But Interacting	Not Speaking Not Interact
1	67	55	65	62	51
2	61	71	77	63	60
3	69	82	78	63	66
4	73	72	87	76	76
5,6	79	92	95	89	91

It can be seen from this summary that regardless of what the teacher is doing the children with higher CPC levels tend to spend more of their time on the school task in hand. In addition, some of the patterns of behaviour suggested in the table may repay closer examination, e.g., the teacher not speaking nor interacting with anyone seems to have the greatest distraction from task engagement for the lower CPC level pupils.

(v) Pupil Use of Algorithms: Table 6 shows the percentage of children, by CPC level, who, having learned the algorithms for addition and subtraction, actually used them to obtain the answer to a problem.

Table 6

Use of Known Algorithm to Solve Problem Tasks D and E
Tasmanian Data
(% Using Algorithm, Total Population of Subjects Who Had Learned Algorithm)

CPC Level	% Using Algorithm	% Using Inappropriate Strategy	% Using Counting
2	24	37	8
3	19	19	22
4	20	19	18
5,6	25	3	32

Children at all CPC levels use the taught algorithm infrequently, between one-fifth and one-fourth of the number of times when it is appropriate. They appear to prefer to fall back on more "primitive" strategies such as counting which they have used successfully previously.

It can be seen that the data on these tests parallel those in Table 3 in that, with rise in CPC level, the use of inappropriate strategies decreases significantly at the same time as use of counting strategies increases. It is of interest to note that when the children cease to use inappropriate strategies they do not, in the main, turn to the algorithm which has been taught as the appropriate strategy. In fact, for this population, the use of the algorithm does not increase significantly with increasing CPC level. It is interesting to speculate on the reasons for this. Perhaps the emphasis on understanding the relationship between the algorithm and its application is misplaced at least at this early stage; perhaps we should treat problem solving strategies and algorithmic procedures as discrete entities, teach them separately and worry about bringing them together at a later stage in the child's mathematical development.

Project No. 2: Cognitive Level and Assessment of Students' Problem Solving Mathematical Capability

This project has set out to combine an evaluation technique designed by two Australian researchers (Biggs & Collis, 1981) with a mathematical item type proposed sometime ago by Cureton (1965) in order to devise multiple-choice items which could be used to test validly a student's capacity for solving mathematical problems. Based initially on the cognitive development literature, Collis and Biggs devised a response model which enabled a child's response to a particular question to be classified according to the way in which it was structured. The response model or SOLO* Taxonomy proposes that there are five basic

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*SOLO: Structure of Learned Outcome

categories within the concrete operational mode of functioning (i.e., 7 + years to 15 + years). These basic categories* are set out diagrammatically in Figure 1.

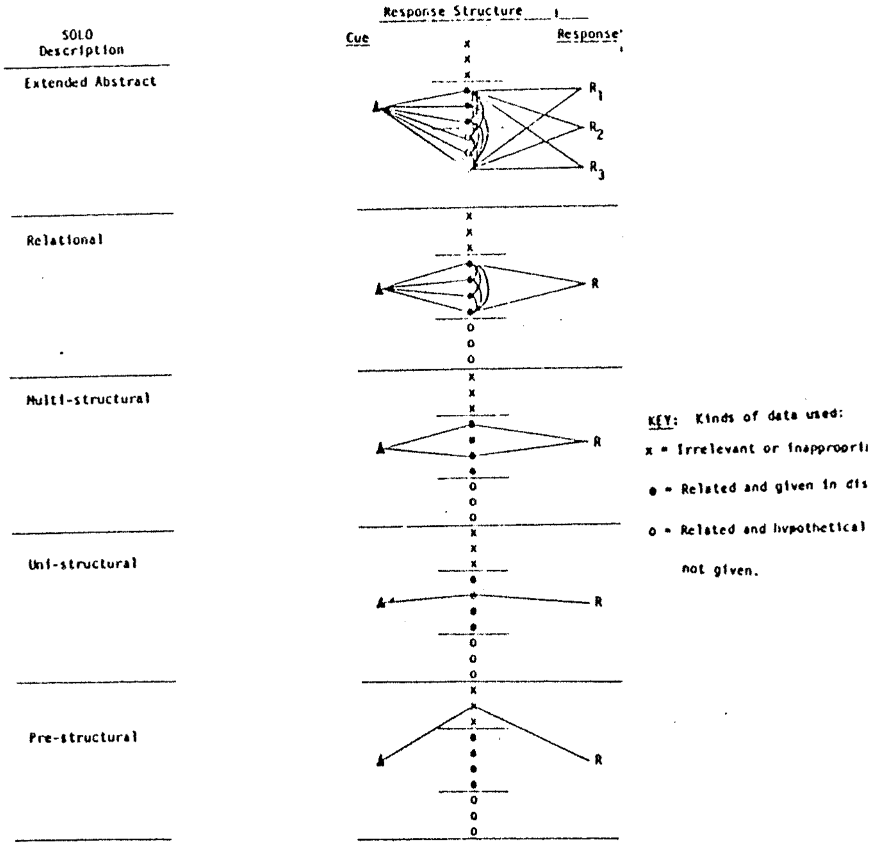

Figure 1. SOLO response description.

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*It should be noted that Biggs and Collis (1981) do not confine their use of this model to mathematics--it can be applied across the whole curriculum. Moreover, the model is equally applicable in other modes of functioning, viz., sensori-motor, pre-operational and formal--the concrete mode is dealt with here because it covers the range of compulsory schooling in most countries. For full details, the reader is referred to Biggs, J. B., & Collis, K. F. Evaluating the Quality of Learning: The SOLO Taxonomy, Academic Press Inc., New York. (In press, expected release Nov. 1981.)

The above diagram is meant to cover the general case. For particular content areas, certain idiosyncracies peculiar to the area need to be taken into account. For mathematics, the following can serve as a highly condensed summary of a response model which is meaningful within the context of school-based mathematical material.

Summary of Response Modes

UNI-STRUCTURAL RESPONSES

Marked by a single direct relationship to concretely (either physically or iconically) available criteria.

MULTI-STRUCTURAL RESPONSES

The ability to handle multiple operations with small numbers by a series of meaningful closures, for instance, may be seen as analogous to using a sequence of given, but unconnected, propositions to support a particular judgment in other content areas.

RELATIONAL RESPONSES

The individual relates elements within the immediately available concrete system and forms generalizations on this basis.

EXTENDED ABSTRACT RESPONSES

Acceptance of lack of closure, use of the reciprocal operation and ability to work with multiple interacting and abstract systems involve a comprehensive use of the given data together with related hypothetical constructs and abstract principles.

Superitems

Cureton (1965) seems to have been responsible for coining this term to describe sets of questions which were asked about a particular problem situation. Typically, the problem situation would be described in the stem which would consist of a paragraph describing the problem and the items would consist of a series of questions which could be answered by reference to the information in the stem. Cureton's basic interest was methodological but others (e.g., Wearne & Romberg, 1976) have used the notion since to develop tests of mathematical problem solving. This latter work showed that the tests were useful because they provided more information about the student as well as a more refined measure of the child's problem-solving ability. However, they did not give information about the level of a child's reasoning in respect of each problem situation. It is this latter aspect that this project is designed to shed light on.

It is hypothesized that, by using the SOLO technique in reverse, as it were, one ought to be able to design items such that a series of questions on the stem would require a more and more sophisticated use of the information in the stem in order to obtain a correct result. This increase in sophistication should parallel the increasing complexity of structure noted in the SOLO categories.

Criteria for Construction of Superitems

Clearly the construction of these items consists of two parts, (a) the writing of the stem and (b) the construction and writing of questions to reflect the SOLO levels. The former is concerned with content validity. This has been achieved in the present study by outlining six items for each of the content areas set up by the National Assessment of Educational Progress (NAEP) publication on mathematics objectives (NAEP, 1977) and carrying out several content vs. category checks using math teachers, educators and mathematicians. The latter requires that suitable criteria be set up to enable four questions to be asked which would not only require a knowledge of the information in the stem but would also be such that a correct response to each question would be indicative of an ability to respond to the information in the stem at least at the level reflected in the SOLO structure of the particular question. To achieve this last the following criteria were set up for designing the questions:

1. Uni-structural: Use of one obvious piece of information coming directly from the stem.

2. Multi-structural: Use of two or more discrete closures directly related to separate pieces of information contained in the stem.

3. Relational: Use of two or more closures directly related to an integrated understanding of the information in the stem.

4. Extended Abstract: Use of an abstract general principle or hypothesis which is derived from or suggested by the information in the stem.

In each superitem, the correct achievement of question 1 would indicate an ability to respond to the problem concerned at at least the uni-structural level. Likewise success on question 2 corresponds to an ability to respond at multi-structural level and so on.

Conclusion

The above is meant to outline the first stage in the development of a critical aspect of a set of superitems to test mathematical problem solving. The next stage will be to subject the hypotheses generated to empirical test and this will be undertaken in the Fall semester of this year.

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MATHEMATICAL VERNACULAR : AN INTRODUCTION

Jan Donkers

1. Introduction

In studying problems of language in mathematics education one frequently comes across discussions about problems that are mainly of a socio-psychological character. As it is indicated in a survey paper by Austin and Howson [1], little research has been done about formal aspects of language in mathematics education. In this paper we shall deal with such formal aspects. These formal aspects apparently have a more than formal significance.
We shall briefly discuss the so-called Mathematical Vernacular, sometimes abbreviated as M.V. By this we mean a system of rules and conventions that enables one to present a mathematical text in an umambiguous manner and such that the structure becomes immediately clear.
The ideas have originally been developed by Prof. N.G. de Bruijn in his lectures on Language and Structure of Mathematics at the University of Technology in Eindhoven (cf. [2], [3]). In Eindhoven we have started a studygroup to elaborate De Bruijn's ideas and to gain some experience in applying his ideas to mathematical texts (cf. [4]).
This paper discusses some experiences with the translation of mathematical texts and describes a few of the main rules, notations and conventions.
We shall do this by means of an example: a translation into M.V. of a mathematica text (see Appendix 1). We have chosen a part of text from Rudin's Principles of Modern Analysis (see Appendix 2), because it demonstrates some features of M.V. quite well and because it appears to be a well known and clearly written book, which, in our opinion, is frequently used in the English speaking countries.
This text is called Rudin's text. It is not exactly his text because we have put together by cutting and pasting from his book. We are mainly interested in the proof of theorem 1.20(a), the so called archimedian property for the reals.
In the proof some definitions and a part of a theorem are used, so there is a translation of these as well. Translated are only those parts of the definitions which are relevant to the proof. So we have examples of two important specimens of mathematical texts, namely a definition-text and a proof-text.
We do not intend to criticise the book nor the part of text that we have taken from it. We only use this as a typical example to illustrate the main features of M.V.

2. Some general remarks

We shall make a few remarks to begin with.

(1) We use the well known logical operators $\neg$, $\wedge$, $\vee$, $\rightarrow$ and the logical symbols $\forall$, $\exists$, and $\exists!$ (the last one for unique existence).

(2) Everything in M.V. is written in some context. What we mean by context is the following:

(i) all the introduced variables, still valid at a given moment (every variable with a description of its meaning by specifying its domain)

(ii) all the assumptions, valid at a given moment.

To be able to read (and understand) a text, it is necessary that the words and symbols which have been defined previously are known. Such words and symbols are not part of the context. For example: the symbol $<$ stands for an order relation on a set, $P(A)$ means the power set of a set A; ϵ has the meaning "is element of", etc.

We mark the beginning as well as the end of the lifetime of every assumption and every variable as follows: a variable is introduced by using a pointed flag and the introduction of an assumption is made by using a rectangular flag. The length of the flagstaff gives the duration of the validity (briefly the lifetime) of the variable or the assumption respectively.

(3) Several times we used the word translation, but by this is not meant a literally translation. So it would be better to speak of rewriting the given text in M.V. At some places the M.V. text differs from Rudin's text, but we shall come back to this point later.

3. The context-structure

On the first two lines of the M.V. text we see two pointed flags. In the first flag a variable-pair of type "ordered set" is introduced and in the second flag a variable of type "subset of S".

On line 8 as well as on line 18 there is a rectangular flag, marking the introduction of an assumption. Note that in contrast with the pointed flags, in which it may happen that symbols appear for the first time, nothing new is inside a rectangular flag. This means, every symbol, word or sentence, used inside a rectangular flag is already known, because it has been defined previously

Looking at the flag-structure, we see that the staff of the first flag spans lines 1 to 11. The other two flagstaffs of the first 11 lines end on line 10. They mark the context. So on every line it is immediately clear what is in the context (of that line). On lines 12 and 13 the context is empty.

What happens if a flagstaff ends? A characteristic of a pointed flag is that it can be followed by a generalization. Indeed, the sentence following the end of a flagstaff of a pointed flagg often starts with "for every ..." or with a universal quantifier. For an example we refer to line 33.

From Rudin's text it is not clear whether we have to write theorem 1.20 (a) with or without a context. The proof starts using x and y as if they were mentioned before. So it suggests a non-void context (that means, implicity using a pointed flag). On the other hand, the theorem is stated by means of an implication and suggests a generalization. It is quite usual in mathematical texts, to state a theorem in a non-void context. In a non-void context, the M.V. text of theorem 1.20(a) could be:

Let $x \in \mathbb{R}^+$, $y \in \mathbb{R}$

Theorem: $\exists_{n \in \mathbb{Z}^+}[nx > y]$

Proof:

Suppose ....

....

....

The end of a flagstaff of a rectangular flag is often followed by an implication. In case of a contradiction it is followed by the negation of the assumption, as is the case on line 32.

On line 26 appears a peculiar flag. It is a pointed-rectangular flag and is called a double (or mixed) flag. A double flag indicates what is called existential elimination. The general scheme for existential elimination looks like this:

$\exists_{x \in A}\ [P(x)]$

Let $x \in A$

Suppose $P(x)$

....

Q

Q

On the line following the existential proposition, there appears a pointed flag with the introduction of the variable x with its type description. On the next line, the assumption $P(x)$ is introduced inside a rectangular flag. Within these two context flags the proposition Q is derived, which does not contain the variable x. Hence we obtained, outside the flags, the following two propositions:

(1) $\exists_{x \in A}[P(x)]$

(2) $\forall_{x \in A}[P(x) \rightarrow Q]$

From these two propositions we may conclude to Q <u>outside</u> the context.

The scheme for the existential elimination is an important one, because it is applied frequently in a mathematical text. Many times a text is obscure, because one does not realize that one is dealing with existential elimination. Here is an example:

$\exists_{x \in \mathbb{R}} [x \geq 1]$, then $x > 0$ ...

In the M.V. text, there is an example of existential elimination on the lines 25 and following. This has to be read as follows:

25 then $\exists_{m \in \mathbb{Z}^+} [\alpha - x < mx]$

Let $m \in \mathbb{Z}^+$

Suppose $\alpha - x < mx$

...

30 contradiction

31 contradiction

Since existential elimination occurs so often and since P(x) can be a long expression, it is easy to have a shorthand notation. We use the double flag with the standard text "choose ... accordingly" for short.

4. Various kinds of definitions

The symbol ":=" on line 3 indicates a definition. In M.V. the things we define can be: statements, substantives and names.
The definition on line 3 is an example of a substantive definition. This is expressed by using the capital S, the so-called substantive binder (or quantifier). What is a substantive? In the sentence: "p is a ...", we use at the open place a substantive. For example: "p is a real number". We consider "real number" as a substantive instead of as being a compound of an adjective and a substantive. In "p is a positive real number less than 100", the combination "positive ... 100" is a substantive. Using the binder S, we write this substantive as

$$S_{x \in \mathbb{R}} \ (0 < x < 100).$$

The usage of the substantive binder helps to make a text clear. Sometimes it makes things easier to handle, as is the case with

$$\text{divisor of } n := S_{k \in \mathbb{Z}} \ (\exists_{m \in \mathbb{Z}} [km = n]) \ .$$

The usage of this binder S has the advantage of giving a clear signal.
As to the definition on line 3, one is probably more familiar with an expression like: "an element β of S is called an upper bound of E, if ... ".

The definition on line 4 is the definition of a name. The arrow is an operator that assigns a set to a substantive. So B is the name of the set of all upper bounds of E. One may write B := {x∈S|x is an upperbound of E} as well.
On line 5 appears the definition of the statement: "E is bounded above".
The name-definition on line 10 appears within more context than the other definitions. This expresses that one has to show that there is exactly one least upper bound of E. In that case we can speak of the least upper bound. So if the set of least upper bounds of E is not empty, then it is a singleton and we give its element a name: sup E.
In M.V. we don't accept things like "sup E exists" as in Rudin's definition 1.10 because we do not want to give names to objects which have not been proved (or explicitly assumed) to exist. As a consequence the M.V. text of line 11 deviates from Rudin's text.
It is obvious that the definitions made in the first part of the M.V. text may be used outside the given context. Sometimes an extra reference is needed, as is the case with the lines 20, 21 and 22.

5. Concluding remarks

Up to now, we have given a very brief explanation of the main notations and conventions in M.V.
In our opinion, using M.V. has certain advantages.

(1) It clarifies the context structure as well as the logical structure of a mathematical text, expecially with respect to definitions and proofs.

(2) It clarifies the way the language is used in a mathematical text (e.g. substantives - statements - names).

(3) It seems a relevant aid for the teacher when he is preparing a lesson. It can clarify some obscure passages in a mathematical text and raise some hidden thoughts to a level of consciousness.

It will be obvious that a textbook entirely written in M.V. would be a very dull one. M.V. is not well suited for communication between mathematicians when they discuss a mathematical problem, for example at the blackboard or by mail. Of course, during moments of confusion or misunderstandi in such discussions, one might want to use M.V. anyway. But we think, if one has some experience with M.V., one gets another view on mathematical texts. In any ca in is because of our experience with M.V., that in reading an ordinary mathematical text, we begin to see the difficulties earlier and more clearly.

Acknowledgments: The author wishes to express his gratitude, to Prof. N.G. de Bruijn for his stimulating discussions, to the members of the studygroup, L.S. van Benthem Jutting, W.H.J.H. van Meeuwen, R.P. Nederpelt, C.P. van Nieuwkasteele and J.T. Udding, and to Prof. D.A. Klarner for reading the first draft of this paper.

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Eindhoven, May 1981

1. Let $(S,<)$: ordered set
2. Let $E\in P(S)$
3. upper bound of $E := S_{\beta\in S}\,[\forall_{x\in E}(x\leq\beta)]$
4. $B :=$ (upper bound of E)$\uparrow$
5. E is bounded above $:= B\neq\emptyset$
6. least upper bound of $E := S_{\alpha\in B}\,[\forall_{\gamma\in S}(\gamma<\alpha\rightarrow\gamma\notin B)]$
7. (clearly there is at most one least upper bound of E)
8. Suppose E has a least upper bound
9. then $\exists!_{\alpha\in S}$ [α is a least upper bound of E]
10. sup E := the least upper bound of E
11. $(S,<)$ has the l.u.b. property $:= \forall_{E\in P(S),E\neq\emptyset}[B(E)\neq\emptyset\rightarrow$ E has a least upper bo
12. 1.19 Theorem: $(\mathbb{R},<)$ has the l.u.b. property.
13. 1.20(a) Theorem: $\forall_{x\in\mathbb{R}^+}\forall_{y\in\mathbb{R}}\exists_{n\in\mathbb{Z}^+}[nx>y]$
14. Proof:
15. Let $x\in\mathbb{R}^+$, $y\in\mathbb{R}$
16. $A := \{z\in\mathbb{R}\mid\exists_{n\in\mathbb{Z}^+}[z=nx]\}$
17. then $x\in A$, so $A\neq\emptyset$.
18. Suppose $\neg\,\exists_{n\in\mathbb{Z}^+}[nx>y]$
19. or $\forall_{n\in\mathbb{Z}^+}[nx\leq y]$
20. then y is an upper bound of A
21. hence A is bounded above
22. define (th. 1.19) $\alpha :=$ sup A

(with respect to $(\mathbb{R},<)$)

23. then $\alpha-x<\alpha$
24. hence $\alpha-x$ is not an upper bound of A
25. then $\exists_{m\in\mathbb{Z}^+}[\alpha-x<mx]$
26. choose m accordingly
27. then $\alpha<(m+1)x$
28. but $(m+1)x\in A$
29. so α is not an upper bound of A
30. contradiction
31. contradiction
32. $\exists_{n\in\mathbb{Z}^+}[nx>y]$
33. conclusion: $\forall_{x\in\mathbb{R}^+}\forall_{y\in\mathbb{R}}\exists_{n\in\mathbb{Z}^+}[nx>y]$.

1.7 Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is *bounded above*, and call β an *upper bound* of E.

1.8 Definition Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.
(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the *least upper bound of* E [that there is at most one such α is clear from (ii)] or the *supremum of* E, and we write

$$\alpha = \sup E.$$

1.10 Definition An ordered set S is said to have the *least-upper-bound property* if the following is true:

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

1.19 Theorem *There exists an ordered field R which has the least-upper-bound property.*

1.20 Theorem

(*a*) *If $x \in R$, $y \in R$, and $x > 0$, then there is a positive integer n such that*

$$nx > y.$$

Proof

(*a*) Let A be the set of all nx, where n runs through the positive integers. If (*a*) were false, then y would be an upper bound of A. But then A has a *least* upper bound in R. Put $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A. Hence $\alpha - x < mx$ for some positive integer m. But then $\alpha < (m + 1)x \in A$, which is impossible, since α is an upper bound of A.

From: W. Rudin, Principles of mathematical analysis (3rd edition).

QUERIES AROUND THE NUMBER CONCEPT

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Résumé :

Questions par rapport à la notion du nombre.

Il y a des problèmes irrésolus autour de la notion du nombre telle qu'elle est connue par les petits enfants.
Leurs idées se distinguent beaucoup des nôtres.
Comment découvrir les idées des enfants ?
Comment les comprendre ?
Comment appliquer ces résultats dans l'enseignement ?

On présentera trois sujets :

I. l'observation mutuelle qui est notre méthode de recherche ;

II. une énumération de sujets possibles de recherche qui nous frappaient ;

III. une discussion détaillée de trois parmi ces sujets :

a) le comptage acoustique ;

b) le comptage de mouvements au lieu de celui d'objets ;

c) l'entendement enfantin du nombre
décrit mathématiquement et psychologiquement.

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