PROCEEDINGS of the EIGHTH INTERNATIONAL CONFERENCE for the Psychology of Mathematics Education

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A. PLENARY ADDRESSES

RESEARCH IN MATHEMATICS EDUCATION IN AUSTRALIA

A BRIEF ENCOUNTER

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1.0 INTRODUCTION

In recent years valuable reviews of research in mathematics education have been prepared by Shumway (1980) and the Shell Centre for Mathematical Education Bell et al (1980). Both reviews delineate research in mathematics education into a number of important areas: research in learning and teaching; curriculum development and curriculum research; individual differences and attitudes; and research on teacher education including the social context of mathematics teaching. At the time of writing this paper a similar review surveying the Australian scene is in preparation: Blane, Conroy, Jones and Leder (1984) and is planned for publication prior to ICME 5.

This paper highlights aspects of reviews by Jones (1979, 1983) and Conroy (1983) which were prepared as lead-up papers to the overall review. The source for Jones' reviews was confined to research associated with the Mathematics Education Research Group of Australia (MERGA) and covered the period 1977-1983. Conroy surveyed other research in mathematics education from Australian journals and higher degree theses and his review covered the period 1977-1982.

Jones' review of MERGA research identified six broad categories; the number of studies in each category is indicated in Table 1.

Table 1 - Categories and Number of Studies in Each Category (1977-1982) - Jones.

CATEGORIES	NUMBER		
	1977-1979	1980-1983	TOTAL
1. Curriculum	29	26	55
2. Attitudes to Mathematics and the Learning and Teaching of Mathematics	10	6	16
3. Problem-Solving	8	1	9
4. Studies on Learning and Cognitive Development	22	42	64
5. Calculators and Computers	5	1	6
6. Other	5	1	6
TOTAL	79	77	156

Conroy noted that his categories did not match exactly those devised by Jones but there was considerable overlap and the two major areas of interest were the same. A summary of the number of studies associated with Conroy's categories by source (journal, higher degree thesis) is presented in Table 2.

Table 2 - Categories and Number of Studies in Each Category (1977-1982) - Conroy

CATEGORIES	SOURCE		
	Journal Articles	Higher Degree Thesis	Total
1. Curriculum	11	3	14
2. Characteristics of Learners	1	4	5
3. Cognition	6	15	21
4. Learning and Instruction	4	7	11
5. Tertiary and Teacher Education	4	3	7
TOTAL	26	32	58

Two observations can be made from an examination of these tables : (i) major interest in research in mathematics education in Australia has centred on learning and cognition and (ii) MERGA is providing a substantial forum for presentation and dissemination of Australian research in mathematics education.

Since the dominant thrust of Australian research has focused on studies in "learning, instruction and cognitive development" and also because it is particularly germane to this conference, only investigations in this area will be highlighted.

2.0 STUDIES ON LEARNING, INSTRUCTION AND CONGITIVE DEVELOPMENT

The studies have been categorised into four major areas:

Stages of development and information processing;
Acquisition of specific mathematical concepts and processes;
Language and mathematics;
Error analysis and learning difficulties.

In all of the categories considered the scope of studies has to some extent ranged across different cultures located in the Pacific region.

2.1 STAGES OF DEVELOPMENT AND INFORMATION PROCESSING

Any attempt to present Australian research under this heading without setting it in the context of the work of Collis and Halford would be incomplete. Hence, a more detailed overview of their work will be provided.

Collis (1980)

In the area of *cognitive development* a series of studies by Collis from 1969 through 1975 helped to clarify some of the basic concepts of the cognitive development model originally put forward by Inhelder and Piaget (1958) and also had the effect of defining the proposed Piagetian stages operationally in terms of mathematical items. It was found that the differences children exhibited in their reasoning at different stage and sub-stage levels could be directly described in terms of certain interacting constructs : complexity of mental operation e.g. cognitive processing demand; abstractness of elements e.g. size of numbers and complexity of variables; abstract systems e.g. ability to work within an arbitrary set of rules and definitions; operating on operations e.g. use of the inverse in simple equations; tolerance for lack of closure e.g. ability to work with a mathematical statement without feeling the necessity to close the operations immediately; and interacting systems e.g. ability to order different substances by weight and also by volume.

Collis notes that while these constructs are extremely helpful in describing typical developmental stage phenomena, they are not particularly useful in explaining the stage phenomena itself. This leads to his explanation in terms of *information processing*.

Collis had also noticed a recurring and strange phenomena in the individual protocols. For example, when seven year olds were given a problem such as *"what number does 2 + 3 + 4 equal?"*, which was expected to be within their normal competence, many children would not only solve it incorrectly but would also display physical and emotional signs of frustration. In essence, the children seemed unable to take in all the necessary information, to keep an eye on the overall problem to be solved and to operate on the data appropriately, all at the same time.

In his explanation, Collis states that the task lead to an 'overload' situation for children of this age and resulted in them not being able to accommodate the final number 4. Collis argues and is supported by Case (1978), that some time later, at the next sub-stage, the child overcomes the problem because small numbers like "2" and "3" and the operation become entities in their own right. Nevertheless, similar characteristics are demonstrated by students right up to the formal operational level and are related to increases in the amount of given information, greater abstraction in the elements and increasing complexity in the operations. In summary, an increase in processing capacity (with each sub-stage) has a two-fold beneficial effect : access to more effective strategies and the need for less processing capacity.

In an effort to synthesise the work on developmental phenomena, Collis and Biggs (1976) examined task responses across several subject areas. Two domains were posited : the internalized stages (Piaget et al) called the "hypothetical cognitive structure"(HCS) which was a property of the

the individual and the "structure of the learned outcomes" (SOLO) which was associated with the actual response made by an individual and is task specific.

With respect to the SOLO level of responses, five general stages can be distinguished in the acquisition of complex responses, skills or knowledge and these stages form a learning cycle: "prestructural" (non-existent, irrelevant or inadequate attempt to learn a component); "unistructural" (one relevant component acquired); "multistructural" (several components acquired independently of each other); "relational" (components integrated) and "extended abstract" (integration is generalized into a superordinate level with is unistructural with respect to the next mode or level of generality).

In some respects, increasing level of "mode" parallels the Piagetian stages of cognitive development and hence the HCS and SOLO are closely related but not identical. Table 3 outlines this relationship between learning cycles, modes of functioning and cognitive development.

Table 3 - Learning Cycles, Modes of Functioning and Cognitive Development[1].

Mode of Functioning	Learning Cycle (Structure of response within mode of functioning)	
Sensori-motor	*Unistructural*	
	Multistructural	
	Relational	=Prestructural
Intuitive/Pre-operational	Extended abstract	=*Unistructural*
(typical through early		
childhood)		*Multistructural*
	Prestructural	=*Relational*
Concrete operational	*Unistructural*	Extended abstract
(typical through childhood to		
adolescence)	*Multistructural*	
	Relational	=Prestructural
Formal-First order	Extended abstract	=*Unistructural*
(typical of best adult thinking)		*Multistructural*
	Prestructural	=*Relational*
Formal-Second order	*Unistructural*	=Extended abstract
(necessary for basic research)	*Multistructural*	
	Relational	=Prestructural
Formal-Third order	Extended abstract	=*Unistructural*
etc.		*Multistructural*
		Relational

1. Collis, K.F. and Biggs, J.B., Matriculation, Degree Structures and Levels of Student Thinking, The Australian Journal of Education, August, 1983.

The Unistructural-multistructural-relational cycle occurs within a mode : the prestructural end of the cycle extends back into the previous mode; the extended abstract into the following mode.

Collis and Biggs have identified four areas in mathematics education where the SOLO construct can be of immediate value : (i) de-emphasising the notion of a stable intellectual capacity for the child; (ii) evaluating the quality of the children's responses; (iii) assessment of children's work and (iv) the gathering of normative data.

Halford (1978)

Halford has attempted a more sweeping categorisation of the difference between concrete and formal thinking in terms of mathematical items. He proposes that the pre-operational stage is that of unary operations or binary operations. The concrete stage is that of compositions of binary operations. By "composition", it is meant that two binary operations have to be co-ordinated and cannot be closed in sequence, for example, the item (2 □ 2) □ 2 = 6 is classed as concrete since it contains one unknown operation, used twice, whereas the item (4 □ 2) □ 3 = 2 where different operations must be used to fill the two boxes is formal. On Halford's sample the first item was answered correctly by 65% of the pupils aged nine, the second by 65% only at age twelve. Halford states that in solving the first of these two items, the pupil has only to try the four operations in succession in each of the two boxes whereas the solution of the second requires in principle each operation to be tried in the first box together with all possible operations in the second. It is thus yet another kind of "acceptance of lack of closure"; since a temporary choice for the first operation must be made while trials are performed on the second. Halford's classifications tend to predict quite well the order of difficulty of items as shown by testing (i.e. describe the typical stage phenomena), however it is difficult to see how they explain the developmental stages through this single dimension.

In the arena of processing capacity, Halford argues that a span of 2 is required for pre-operational tasks, a span of 4 for concrete operational tasks and a span of 6 for formal operational tasks. By way of comparison, Pascual-Leone emphasises the child's processing capacity rather than storage capacity. In contrast to Halford, the child would be tested on immediate recall of information that first has to be operated on in some way, rather than use a straight forward span test.

Additional Australian Studies

Simon's (1978) study on seriation confirmed Bryant's (1974) general thesis that young children centre upon relative size difference between adjacent elements and do not take into account the absolute size difference between these elements. The latter involves a decentration since A has to be compared with B, B with C, C with D and so on in order to deduce that the size difference is always the same. The differential performance due to variations in the step-size of the standard series also confirmed Bryant's theory with regard to the effects that the limitations of relative codes have upon young children's judgments. Nevertheless, it does appear from her study that when young children are given a small number of elements and a means of responding which does not presuppose operational reversibility on a seriation task, they will show signs of decentering and display a more advanced form of reversibility when the display conditions are favourable. In this instance, it appeared that extreme variations in the step-size enhanced decentration.

A number of studies have made contributions on transitive reasoning. Rosenthal (1977) investigated the Piagetian claim that children cannot solve three-term series problems (of the type: if aRb, bRc, then aRc) until they have reached the formal operational stage and can deal with the task hypothetically, using serial ordering together with implication. In an initial study with 12-year olds, it was found that ability to solve such problems was not associated with ability to solve a formal operations task (oscillation of a pendulum). Children who were able to describe their strategies used either a pictorial-spatial technique or a linguistic strategy of elimination. A follow-up study with seven-year olds showed that ability to solve three-term series problems was related to ability to seriate objects that are physically present, and that success on both tasks tends to depend on the ability to co-ordinate the "more than/less than" relations.

A group of three articles discussed this issue in a somewhat different form. Halford and Galloway (1977) tested 163 children in the age range 4½ to 9 years for transitivity of length (if $a > b$, $b > c$, then $a > c$) and ability to remember the lengths of the three rods taken in successive pairs. Of 107 children who failed to make the transitive inference, only 20 were unable to remember the comparisons. The writer interprets the results as being contrary to the hypothesis that lack of transitivity is attributable to failure to remember the appropriate comparisons. Pre-school children who appear to be able to make transitive inferences may be depending on knowledge of the stimuli used rather than using transitive reasoning. Grieve and Nesdale (1979) dispute the previous argument, questioning whether transitivity had, in fact, been tested. They argue that three issues are involved: (a) *"... to what extent are children susceptible to different forms of illusion at different ages?"* (b) *"... how do children interpret ambiguous questions in the context of different illusions at different ages?"* (c) *"... the nature of tests of transitive inference."*

Halford (1979) gives a carefully argued reply to the Grieve and Nesdale (1979) rebuttal. He assesses the general problem of measuring transitivity and suggests that *"polemics of this kind are inherently unresolvable unless there is a consensus in advance as to the operational definition of the relevant concepts"*. In conclusion, he proposes a test procedure which might achieve such a consensus.

Munro (1980) simultaneously examined changes in the young child's comprehension of verbal and non-verbal reference to temporal sequence. The study showed that children comprehend simultaneity prior to sequence and comprehend "first", "last" and "together" earlier than "before", "after" and "at the same time as". The findings support the notion of a cognitive non-linguistic basis to temporal language acquisition. In a related study in Papua New Guinea, P. Jones (1982) developed a written test based on the concepts of "more", "less" and time, to study the development of ability to read and understand mathematical language. The test was administered to grade 2, 6 and 10 children for whom English is a second language, and also to children for whom English is a native language. He noted that the Papua New Guinea children performed at a significantly lower level than their contemporaries who spoke English as a native language. The retardation was primarily due to a period of two to three years of only marginal improvement in the ability of Papua New Guinea children to read and understand mathematical language.

The neo-Piagetian information-processing approach to the study of learning, cognition and human performance generally was developed by

Pascual-Leone, 1970. Important suggestions for its potential applicability to mathematics learning have been made by Case, 1978, di Ribaupierre and Pascual-Leone, 1979 and Romberg and Collis, 1980.

In a series of studies, J.T. Briggs (1980) evaluated the relevance of Witkin's field-dependence-independence dimension of cognitive style in mathematics education and examined its relationship to the neo-Piagetian theory of Pascual-Leone, together with the concept of embeddedness in these contexts. Briggs (1981) also presented validation and reliability data for an adaptation of Case's Digit Placement Test which could be used in a group situation.

Lewis (1980) tested 38 students aged four to five years in three trial samples of a study to trace the development of understanding of prerequisites of length measuring. It was hypothesized from an analysis of the information processing demand of the tasks that the majority of the children of this age would succeed with certain tasks and that there would be a predictable order of development of such tasks. Tasks as predicted for which a majority of subjects gave correct responses were subitising ($5 \geq n \geq 3$), identification and construction of straight lines, recognition of equality and inequality of length, counting ($n \geq 7$), seriation, seriation of length and measuring by direct comparison.

Grabham (1982) reviewed research on information-processing and considered the extent to which cultural and educational experiences affect cognitive development. She posits that educational and cultural experiences play a significant role in the development of information processing strategies and that effective teaching requires a match between the experiences gained in the learning environment and the cognitive style of the child. She also suggests that care should be taken to ensure that educational practice encourages the use of a wide range of abilities in cognitive processing so these indeed become "aptitudes" in the sense described by Cronbach and Snow (1977). Grabham (1981) examined some relationships between individual aptitudes and early school performances in mathematics. The modes of information-processing of 91 subjects between the ages of 5.7 and 6.3 were examined using Luria's model of brain function as the theoretical basis of the study. The results of a factor analysis of six psychometric tests administered to all subjects indicated the presence of two factors, hypothesised to be the separate contributions of "simultaneous" and "successive" synthesis.[2] A further exploratory study indicated that although neither aptitude for successive synthesis nor aptitude for simultaneous synthesis has productive value performance in verbal subtraction problems, simultaneous synthesis was possibly the predominant mode of information-processing.

Hunting and Whitely (1983) reviewed research related to mathematics achievement of Aboriginal students, studies of cognition, including cross-cultural comparison studies and within culture studies, and aspects of culture and language pertinent to mathematics learning. The paper concludes that an understanding of the nature of an individual's prior knowledge is critical for successful learning. For the Australian Aborigine, there is evidence that sophisticated intellectual functions have been developed for dealing with quantitative and spatial problems arising in traditional environments which teachers should take account of when planning instruction.

2.2 ACQUISITION OF SPECIFIC MATHEMATICAL CONCEPTS AND PROCESSES

This has been an area of substantial investigation and has been further sub-classified into : Number, Rational Number and Proportion, Measurement, Spatial and Visual Concepts, Algebra, Probability, and Problem Solving.

Number and Number Operations

Firth (1981) investigated children's counting processes and more specifically set out to elucidate the conceptual pre-requisites for the acquisition of the concept of numerosity as an attribute of the whole numbers. In the interviewing methodology used, the interviewer acts as teacher: praising, encouraging, repeating and/or rephrasing questions in order to determine the "maximum" understanding of the child. Firth's case study account of "Brenda" (age 5.2) illustrates that she cannot count a number of objects when some of the objects are covered by a cardboard mask, and she is told how many objects are hidden and asked to continue counting. Brenda is not an isolated case, and Firth con-

2. *Although Luria's two modes of information-processing are associated with visual (simultaneous synthesis) and auditory (successive synthesis) perception respectively, in higher mental activities they are not modal specific.*

cludes that this has obvious implications for children's acquisition of numerosity and the operation of addition.

Nason (1982) reviewed research on children's development of the number concept which has been generated either from Piaget's cardinal-ordinal model or Brainerd's ordinal model. As an alternative to these structural models Nason proposes that a more productive line of research would be generated from models which examine processes that underlie the development of cardination, ordination and conservation of number. In particular, Nason suggests that research based on Newell and Simon's[3.] production systems may help to resolve the conflicts within the literature about the development of number concepts.

In a series of teaching studies on numeration, Barr (1977) found that both counting and grouping experiences were important in establishing meaning in two digit numeration. Irons (1980) also noted that clinical research has suggested that a knowledge and understanding of numeration in its own right will not necessarily imply that children use those concepts in other contexts. His study investigated the level of understanding children have of numeration and the extent that they use this understanding in solving addition problems. Conclusions reached were: children did understand and have a functional competency with numeration skills apart from the addition situation; good performance on numeration tasks does not necessarily imply that a child will use or see that numeration skills are to be used in the context of addition problems; for children exhibiting difficulties in addition the most obvious error comes from not being able to apply numeration concepts; and numeration errors continue to cause the greatest percentage of difficulties when compared to total errors.

There have been a number of studies focusing on number operations, both in terms of acquisition and performance. The ACER Maths Profile Series Number Operation Test (MAPS) developed by Collis in 1977, consists of 60 multiple choice items involving the solution of number sentences. The items' complexities depended both on their structure and the nature of the elements. Collis claims that the MAPS test results can be used to identify the Piagetian stage of development of a child in number operations.

3. *This technique helps to capture the regularities in a subject's behaviour by writing a set of independent rules known as a "production system" that expresses what the subject does under what conditions.*

Campbell (1979) investigated the reliability of the MAPS test by comparing the stage of development of primary pupils and primary teacher trainees in number operations with their stage of development in sovling "false conservation" problems (Lunzer 1965). The results indicated that certain MAPS sub-tests were less reliable for upper primary classes than was expected and that both primary pupils and teacher trainees were at a lower stage of development when solving geometrical transformation problems than would be predicted from the MAPS test.

In a related study on number operations, Munro (1978) examined children's thinking in relation to verbal problems on addition and subtraction. In a problem such as *"If Jean began with 4 apples and ended up with 7, how many more did she buy?"*, Munro refers to the term *"markedness"* in the sense that the unknown element may be marked as either start, change, or final element, by using temporal clauses such as *"starts with"*, *"finds...more"*, and *"ends up with..."* respectively. In general, he noted that for verbal problems based on the relationship a + b = c, variations, in the markedness of the unknown element may lead to variations in difficulty. Those forms of the mathematical relationship involving maximum deviation from the given order were most difficult for all subjects. This finding was accounted for by suggesting that subjects solved these problems by re-arranging the data to the given order - a procedure which required increased mental effort and attention.

In a series of studies, English and Irons investigated how children perform processes associated with multiplication. English (1982) investigated year 2 children's interpretations of two multiplication models namely the "set" model and the "array" model. She found no significant differences between children's interpretative understanding of the two models and also proposed, from a number of post-hoc analyses, a model for the child's numerical reasoning of multiplication. Irons (1981) administered a 20 item multiplication test of the basic facts to 108 children in years 3, 4 and 5. Children who scored more than 10 correct were interviewed and tested on 5 division items in order to examine method of solution (counting, multiplication, etc.), accuracy, and speed associated with each strategy. Irons concluded that multiplication when used as a basis for division seems to provide both greater accuracy and speed of recall for division basic facts.

As a result of investigations with primary children, Irons (1977) advocated a "sharing" approach for teaching division both in introducing the concept and in developing the algorithmic processes for whole numbers and decimals. In a related study, Barr (1980) investigated the relationship between children's performance in solving division problems and, on the one hand, the strategies used with concrete materials for finding answers to examples of the division algorism and, on the other hand, the understanding of division as shown by the type of division word problems generated. Barr concludes that it was not possible to identify any particular group who performed better on word problems and as a result suggests that children do not see the algorism, classroom word problems and their own concept of division as being related in any significant way. He notes that currently recommended procedures for teaching division may be encouraging this separateness to some extent.

Rational Number and Proportion

Southwell has been involved in a series of investigations on the operator construct with children in Australia, Canada and Papua New Guinea. Kieren and Southwell (1979) examined differences between children's ability to perform operator tasks when the task was embedded in a function machine compared with a "simpler" approach consisting of patterns of symbolic input-output number pairs. An analysis of variance of correct responses indicated no significant differences due to representation mode. Three levels of rational-number operator development were observed in data from both types of tasks. The authors suggested that understanding of equivalence class and partitioning were the important mechanisms underlying this development. Partitioning refers to the division of a set into subsets. Applying equivalence class thinking to a "one-third" task, a subject who correctly pairs 2 with 6 explains, "divided by 3". A more sophisticated use of the mechanism is required for success on the non-unit fraction task of "two-thirds"; to pair 90 with 60, a student thinks, "divide by 3 and take 2 of them". The general fractional operator appears to require the co-ordination of the partitioning of two subsets of numbers with a multiplicative operation, in this case doubling. This covaried partitioning strategy was used most often by subjects in the machine representation condition. In the pattern representation condition, a pattern explanation frequently accompanied a correct response. Thus, in pairing 24 with

16 in the two-thirds task, the subject would say, *"Well, I know 12 went to 8 so I just doubled to get 16"*. A higher level of performance was observed in the machine group at a younger age compared with the pattern group.

In a related study, Southwell (1980) investigated the operator construct using children from Papua New Guinea and again used a machine representation and a representation involving proportionality. Analysis of the data indicated a considerable delay in the development of the operator construct for Papua New Guinea subjects in comparison with Canadian subjects, however three comparable stages in the development were isolated. Following further investigations in Australia, Southwell (1981) concluded that there were significant gains in performance with the operator construct with delays in performance of one and two years occurring in Australia and Papua New Guinea respectively in comparison with Canada.

Karplus and Lunzer made contributions on proportional reasoning development of children while visiting Australia. R. Karplus (1980) investigated the effect of school, grade, sex, task structure (integral vs non-integral ratios), task demand (comparison vs adjustment of ratios) and task format on proportional reasoning development. The results showed that eighth graders used proportional reasoning more reliably than did sixth graders and boys performed better than girls. The presence of integral ratios made items easier, while format had little effect. Additive procedures were widely used instead of proportional reasoning on items that did not have integral ratios. P.L. Jones (1980) also presented data which revealed that the ability of students in Papua New Guinea to use proportional reasoning is below what is normally assumed of their counterparts in developed countries.

In 1982, Lunzer provided a review of the major contributions that have been made concerning children's attainment of proportionality concepts. In his analysis Lunzer adumbrates that the teaching of proportionality should rely on three principles: (i) begin with a problem which can evoke a strong perceptual schema, allows maximum perceptual feedback and involves easy ratios; (ii) demonstrate the error of the additive solution and the correctness of the ratio solution; and (iii) then demonstrate the correctness of a purely multiplicative solution. In summarising he notes that among the more successfully controlled studies, the work of Wolman and Lawson (1975) and Gold (1978) attest to the value of these three principles.

Spatial and Visual Concepts

Clements (1981) in a review of visual imagery and school mathematics examined : definitions of imagery; problems associated with the externalization of imagery; measurement of imagery ability; attempts to improve imagery skills; the verbalizer-visualizer hypothesis; relationships between visual imagery and spatial ability and research into the role of visual imagery in mathematics learning. He concluded that although previous research has not produced clear guidelines for classroom practice, there is considerable potential for further research aimed at achieving this end.

Willis (1980) examined the nature of spatial ability and discussed genetic and neurological influences on its development. She concluded that while analyses are somewhat speculative they do suggest the need to evaluate the intra-hemispheric as well as the inter-hemispheric processing of cognitive tasks. Clarkson (1981) examined whether the preferred mode of processing mathematical information of some year 10 Papua New Guinea students could be influenced by using questions with diagrams, and whether they were consistent in their methods of solution. It was concluded that diagrams had little influence in their thinking and did not lessen the difficulty of questions.

Two studies associated with teachers are reported in this area. Dettrick (1977/78) examined the development of projective spatial relationships within the concrete operational period and also surveyed students' and teachers' ability to deal with area, perimeter and volume relationships prior to and during a series of workshops for teachers. Dettrick notes that children's ability to coordinate spatial relationships by the age of twelve years is far from complete and expresses concern at the tendency for teaching strategies in mathematics to move away too quickly from concrete physical experience. Interestingly, from the teacher survey, he notes that about 50% of the group sampled either do not have much reason to reflect upon area, perimeter and volume relationships when teaching or that they have been reflecting incorrectly upon the relationships considered in some tasks.

Measurement

Low (1977) examined the ages at which children typically solve measurement tasks involving length, area and volume and considers children's reasoning on these tasks in terms of Piagetian constructs. Particular emphasis is devoted to tasks based on measurement by unit iteration and the reasoning used by children in the formal operation stage. The results like those of Collis suggest that the ability to work within a system does not develop fully for most children until late secondary school. He concluded that we may be teaching too much too soon in the school mathematics curriculum. In a similar vein Harris (1978) concluded that the measurement program prepared for white English speaking school children living in an urban situation is unlikely to be appropriate for introducing the study of measurement to Aboriginal children living in remote tribal communities. Mode of expressing the concepts, their relevance and lack of reinforcement in the Aboriginal community and lack of prerequisite experiences were the major reasons identified.

Algebra

Firth (1977) investigated secondary students' learning of algebra by rules in situations where the rule they were using was found to give incorrect results. His findings indicated that a student's ability to abandon or modify a rule when it was found to be inadequate was closely related to the student's ability to accept and work within the limits of a newly defined system. He notes that the results were consistent with Low's study and the work of Collis and Lunzer, viz. that working within the bounds of a new system depends on the acquisition of a higher level of reasoning not normally reached until about age 16.

Probability

Jones (1977) investigated the effect of grade, I.Q. and embodiments on young children's performances in probability. He concluded that the development of probabilistic thinking had begun for most first grade children and that their thinking could be stimulated better by some embodiments than others. I.Q. appeared to be a potent factor in predicting lower levels of performance in probabilistic thinking.

Problem Solving

While a visiting professor in Australia at the time of the first MERGA conference, Lester (1977) set the stage for a number of studies in problem solving. In his review of the use of "heuristics" in teaching mathematical problem solving he identified four different stragegies: (i) instruction in unitary skills; (ii) instruction in individual heuristics; (iii) instruction in heuristics in combinations; and (iv) extensive exposure to problems without direct teacher intervention or guidance. Lester further noted that there was little evidence to support any one of these approaches over the others.

Mortlock, Kissane and Malone (1979) and Putt (1979) examined the effect of introducing heuristics as part of a strategy for teaching problem solving. Putt noted that heuristic teaching strategies promoted desirable behaviour related to problem solving, but concluded that a strategy involving heuristics, was not significantly better than a strategy which did not include heuristics, but gave students experience in solving process problems. Both strategies were significantly better than a strategy which involved no special instruction in problem solving. The study by Mortlock, Kissane and Malone, however, found that there were no significant differences between three strategies similar to those used by Putt, on measures of problem solving and attitudes. The fact that Putt worked with elementary children while Mortlock worked with seondary students may have implications for the timing of heuristic instruction in problem solving.

In a series of studies, Briggs has investigated relationships between field-dependence-independence, M Space and problem solving performance. Briggs (1982) reported preliminary results of a study which investigated some of the difficulties associated with problem solving performance in the upper primary school. He concluded that strong disembedding ability (associated with field independence) appears to assist children in identifying target words in narrative text and target numerals in number sentences and when information processing demand in word problems is increased field independence appears to be far more important for success than high M space. Moreover, Briggs noted that the generally inferior performance of field dependent subjects in normal or difficult problem solving tasks appears to be able to be improved either through specific instruction in relevant strategies or through increased familiarity and experience with the task situation.

In a more detailed explication, Briggs (1983) investigated relationships between two person variables (FDI and M space) and the performance of children in year 5 and year 7 on school word problems in which the information demands were varied while other relevant characteristics were held constant. Briggs' major conclusions were: (i) the specific details of the context of the verbal statement in a word problem are more potent in relation to difficulty level than the amount of reading and difficulty level increases with increase in information processing demand rather than merely with the increase in quantity of reading required; (ii) problem difficulty appears to increase with increase in potentially relevant information within the problem statement; and (iii) with increasing task difficulty, field independent performance is more important for success than high M ability. In general, substantial increase in information-processing demands may likely have the effect of causing substantial confusion of the kind associated with "embedded" situations.

Galbraith (1979) outlined a project in which the clinical method was employed with processes of mathematical proof for secondary students. In commenting on the clinical approach, Galbraith notes that it provides an opportunity for probing and insistent questioning by the interviewer which is essentially dynamic and consequently more appropriate for investigating process abilities which are themselves dynamic. He further comments that it became very evident that answers with the same surface response are far from equal when the deeper structure of the underlying process is investigated.

2.3 LANGUAGE AND MATHEMATICS

In seminal studies in language and mathematics Omara and Conroy reviewed the literature and proposed directions for future investigations. Omara (1979), as part of her review, suggested a useful five part classification of the language contexts of mathematics: verbal language, computational language, pictorial language, textural language, and test language. Conroy (1979) notes that there are two major directions in the body of research, viz the application to the study of mathematics afforded by theoretical studies relating to language, cognition and psycholinguistics and the aspect of mathematics that is viewed as a language per se and as a means of communication. Particular reference was made to children's use of language in mathematical contexts, encoding skills, reading comprehension, visual imagery and factors uniquely related to second language learners.

Kelly, Philip and Lewis (1982) focused on the relationship between cognitive development and language and examined specifically the relationship between conservation of length and knowledge of the relevant comparative dimensional adjectives. Working with Greek migrant and Australian children aged from 8.3 to 12.6 they administered conservation of length tests and examined these in relation to horizontal decalage, the concept of measurement and hierarchies of dimensional length terms. They noted that (i) the order of difficulty of conservation tasks varies according to the nature of the task, with age being the most important variable irrespective of sex, ethnicity or materials, (ii) males perform better than females and ethnicity is unimportant irrespective of other factors; (iii) no child was able to conserve who had not passed the measurement test; (iv) the order of dimensional adjectives was found to be longer/shorter coming before thicker/thinner, before bigger/smaller before wider/narrower; and (v) knowledge of words in advance appeared to be necessary for success on conservation of length.

Wishart (1977) examined the relationship between the language young infant grade children use and the basic mathematical concepts they hold of the world in which they live. She identified a hierarchical acquisition of language and noted the importance of teachers structuring the learning of mathematical concepts which are associated with various semantic complexities. In a related study, Conroy (1980) contrasted oral language forms of the four operations with notational forms for children in the infants' school. Moroever, he enquired whether there was a hierarchy in the various language forms for the four operations. He concluded that there are indeed hierarchies which reflect the language of the classroom but there are also some language structures peculiar to mathematics which the child has to learn as these do not occur frequently in the child's natural language.

In a series of studies concerned with the effects of language on mathematical achievement, Turner and Newman drew important implications for the teacher and the teaching process.

Turner (1980) investigated aspects of the language of mathematics which affect achievement in mathematics. In particular, he examined language in terms of the learning of mathematics in a second language and compared two groups of secondary students, one with an English speaking background and the other with a non-English speaking background. The mean values for the various tests indicate a reasonable degree of balance between the two groups as far as symbol language was concerned. The non-English group performed slightly better on the formal examination but in the vocabulary and Cloze tests the English group performed better. He concluded that teachers at all levels should take time out to teach the specialist vocabulary of mathematics since it was evident from his study that students were not picking this knowledge up for themselves. Newman (1981) reviewed comprehension of language in mathematics in the teaching context. She concluded that pupils experiencing language problems in mathematics need to be taught to use the necessary comprehension skills in the set reading period or as part of the mathematics lesson.

In a related study involving ESL children, Metcalfe (1983) examined the relationship between knowledge of mathematical language and other variables such as mathematics performance, reading ability and problem solving ability. In particular, Metcalfe investigated some of the relationships between performance and knowledge of mathematical language among a group of year 6 children, many of whom spoke English as a second language. A number of statistically significant differences were found between the ESL children in the sample and a control group of non-ESL children. The findings suggest that a proportion of the ESL children are learning to compute successfully without having a sound basis of underlying meanings and relationships in mathematics. The strong relationship found between reading ability and mathematical performance also suggests that ESL children could be helped to build concept structures if they were encouraged to discuss the meanings of the mathematical words they encounter and to read about mathematics.

2.4 ERROR ANALYSIS AND LEARNING DISABILITIES

This has been a major area of research interest in Australia. A seminal study by M.A. Newman (1977), which resulted in the development of a diagnostic interview model, led to a series of follow up studies by Casey (1978), Clements (1982), Clarkson (1980) and Watson (1980). There have also been other related studies in the area which have focused on remediation of learning difficulties.

Newman's model for error analysis was predicated on the assumption that there is a hierarchy of "performance strategies" which need to be applied to solve written mathematical tasks successfully. Failure to apply the appropriate "performance strategy" at the appropriate time would most likely prevent a pupil from progressing to the next strategy, and therefore prevent the pupil from gaining the correct answer. These performance strategies: Reading Ability (word recognition and symbol recognition); Comprehension (literal understanding and specific terminology understanding); Transformation; Process Skills (arithmetical, spatial and logical operations); Encoding; Careless Error; Motivation (willingness to try to solve the problem) and Question Form were arranged sequentially to form what Newman termed the "Criterion for Error Causes".

Based on this study comparable data (on low achieving primary students) was obtained in the studies by Watson, Clements and Clarkson in addition to Newman. Table 4 presents the percentage of errors classified into the Newman error categories.

Table 4 : Percentages of Errors for Each Newman Error Category

Investigators	Error Categories							
	A	B	C	D	E	F	G	H
Newman	13	22	12	26	2	22	3	0
Watson	15	26	7	47	3	2	0	0
Clements	5	8	25	32	2	28	0	0
Clarkson	12	21	23	31	1	12	0	0

A - Reading Ability
B - Comprehension
C - Transformation
D - Process Skills
E - Encoding
F - Careless Error
G - Motivation
H - Question Form

Newman reported that 35% of the total errors made by low achievers occurred in reading ability and comprehension and hence they failed to arrive at a point in their attempted solution where they could even begin to apply the relevant mathematical knowledge, skills, or principles. Her result was supported by Watson's and Clarkson's study in this aspect, but Clements concluded that students who were weak mathematically or had a poor grasp of mathematical language tended to make more systematic errors.

On careless errors, Watson and Clarkson's results showed a lower percentage when compared to Newman and Clements' studies. Clements hypothesised that mathematically competent and confident children who know their work tend to make a greater proportion of careless errors than other children - a proposition supported by Owston (1981). High achievers tend to make non-systematic errors and low achievers make systematic errors.

Newman's results also showed that most errors for a single category arose at the process skills stage viz 26% of all errors. Similar conclusions can be drawn for the other studies. In exploring and developing the model further with older children, both Casey and Clarkson reported that errors tend to concentrate in both the process skills and carelessness/motivational categories, the latter of which Casey posited to be psychologically based. Clarkson also suggests that the Newman categories may provide a useful means of identifying students whose learning has regressed.

In a further conceptualising study, Ransley (1980) developed a stage model for the problem solving process based largely on Luria's (1973) theory of "thinking as a concrete activity", but also combining features of Newman's model and Le Blanc's (1977) model for problem solving. This led to the development of a diagnostic interview technique utilising the stage categories in his model. Table 5 presents the stage categories together with the percentage of initial errors (i.e. errors which caused initial breakdown) and the percentage of all errors (errors which occurred at other stages as well). Ransley concluded that his data indicates how invalid it is to assume that an incorrect answer to a mathematical problem implies a lack of mathematical knowledge or skill.

Table 5 : Percentages of Errors ofr Each Ransley Error Category

Stage Category	Initial Errors (%)	All Errors (%)
Decoding	24	10.1
Investigations of Conditions	28.6	21.4
Strategy	9.2	12.5
Tactics	6.5	9.3
Encoding of Tactics	10.2	9.7
Processing	13	15.6
Encoding Solution	1.9	4.3
Comparison with Original Conditions	6.5	16.8

In a study focused on the used of materials designed to assist children with learning difficulties, Thornton, G. Jones and Toohey (1982) assessed the effectiveness of materials especially prepared to help learning disabled children organise their thinking about basic addition facts and thus to memorise them more quickly. Findings resulting from the use of the material in 71 schools for year 2 through 6 indicated substantial learninggains for the LD students which reflected similar patterns to those occurring for children in regular classes when highly structured, organized strategies were used in teaching the basic facts.

3.0 CONCLUSION

The review presented in this paper examines research investigations focusing on "learning, instruction and cognitive development" which have been presented or published through MERGA or have been identified by Conroy (1983) in a review of other mathematics education research in Australia. Accordingly, the review does not give a complete picture, but hopefully it does reflect the increasing momentum of mathematics education research in this country in recent years.

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ON SOME OF TENDENCIES IN MATHEMATICS EDUCATION RESEARCH IN JAPAN WITH SPECIAL REFERENCE TO PSYCHOLOGICAL ASPECTS

Tadasu KAWAGUCHI

Japan Society of Mathematical Education (JSME)

GENERAL PRINCIPLE OF THE WAY OF PRESENTATION

One of the most important problems of research in mathematics education is to clarify the developmental aspects of formation processes of mathematical concepts and the functions of various types of mathematical reasoning. In order to promote the research from such a viewpoint, it would be inevitably necessary to explore the problem by using psychological approaches. In Japan also, many contributions were made in this area by mathematicians and psychologists. But it is very difficult for me to introduce here all of their results in detail. Therefore I would like to select the representative models of research among them from the viewpoint of study methods in use and thereby, to show some of tendencies in mathematics education research in Japan with special reference to psychological aspects. The following are the categories of method selected according to this point of view:

1. Control Condition Method
2. Comparison Methods
   (1) Follow-up Research Method
   (2) Control Groups Method
      (a) Same Content and Unequivalent Qualitied Groups Method
      (b) Different Contents and Equivalent Qualitied Groups Method
   (3) Discriminating Method for Outward Correct Answers
3. Original Material Method
4. Open End Approach Method

GENERAL SURVEY OF THE PRESENTATION

The mathematical contents have generally so many variables to be hard for students to learn that it is very ambiguous which variable causes them to stumble in learning. The method of the category 1 is such a way to clarify the effect from some aimed variable by controlling others to be fixed.

Comparison Methods of the category 2 are, in general, of broader meanings but I would like here to limit them to three types (1), (2) and (3) as shown above. The method (1) is the way of checking whether the same results may be gotten or not by follow-up the same program for research made by the senior psychological scientists.

The (a) in the method (2) is the way of investigation on the difference between aspects or achievements of two different qualitied children groups which are respectively trained by using common contents of subject. The method (b) is, on the other hand, the way of investigation on the difference between achievements of two equivalent qualitied groups which are respectively trained by using different learning contents with some common factors.

The method (3) in the category 2 is the way to discriminate the outward correct answers of small size in an achievement test in comparison with the state of other many answers related to the former.

The method of category 3 is, I am sure, one of the most important strategies in mathematics education research to catch the aspects of function of mathematical reasoning achieved by students. Because, for the clarification of developmental aspects of formation processes of mathematical concepts or of functions of various type of mathematical reasoning carried by students, it should be necessary to seize internal thinking processes in students' brains. But it is, of course, impossible to seize them directly. And so the original materials issued form students' activities such as informal drafts or scrawls written by them in a paper are very important for us to conjecture them indirectly. I would like to show an example of the method belonging to such category.

The method of category 4 is a strategy developed for the purpose of motivation, encouragement or evaluation of students' mathematical activities of higher objectives such as problem formation and solving through the processes of abstraction, idealization or simplification applied to the concrete situations in real world. In the such course of activities, some conditions or hypotheses would be set up or modified according to the aimed conclusion. The situation of the problems derived from real world is so open as to afford the manifold development of problem formation that it may be available for students to carry out such activities as generalization or systematization of mathematical contents. Therefore the key point of this method is surely to prepare the open end problems of small scale for students to have sufficient possibilities to get manifold solutions from them not beyond the reach of their ability.

At the Session in the Congress, I should be very happy if I could use the aid of OHP to talk about the concrete examples of research along my program as shown above.

PROBLEM-SOLVING AND SYMBOLISM

IN THE DEVELOPMENT OF MATHEMATICAL CONCEPTS

Gérard VERGNAUD, C.N.R.S., Paris.

Mathematical concepts have their roots, like other concepts, in situations on the one hand, in words and other symbolic elements on the other hand.

Symbols are so important in mathematics, that many teachers and many schoolbooks tend to concentrate upon symbolic activities with written numbers, diagrams, written fractions ($\frac{p}{q}$), algebraic expressions, graphic representations of functions, and so on.

But there is now a tendency among researchers to move their attention to the situations and problems for which mathematical concepts and procedures are useful. This change in the theoretical framework of didactics and psychology of mathematics has several reasons.

- certain symbolic activities are meaningless to many students;
- it is a difficult job to transform a situation or a word-problem into a symbolic representation;
- epistemology of mathematics puts the stress on the nature of practical and theoretical problems that a concept deals with.

These considerations have led some of us to classify situations into categories and subcategories requiring different operations of thought, although the mathematical representation of them may be the same. One finds good examples of that kind of research in three recent books: Carpenter, Moser and Romberg (1981), Ginsburg (1983), Lesh and Landau (1983). For instance addition and subtraction problems split into many classes of poblems whose progressive mastering appears to be hierarchically organized over a long period of time, up to late secondary school for some problems involving just one addition (Marthe, 1982). The same is true for multiplication and division problems.

Many 13-14 year-old students still use primitive conceptions of addition, subtraction or fraction when they have to deal with algebraic equations or rational numbers (Hart, 1980).

By considering problem-solving as both the source and the criterion of operational knowledge, and viewing mathematics as a functional knowledge at the students' level, as is mathematics for mathematicians, physicists and other users of mathematics, one opens an interesting perspective. But it does not mean that symbols and symbolic activities are not important and eventually essential in the formation of concepts. Then come the next questions: how can symbolic representations be functional? How are these functions fulfilled?

The answer to these questions may be easy in some cases, and very difficult in other cases: the conveniency of place value notation of numbers may be experienced by students by letting them operate with other notation systems and try to calculate with large numbers: but the function of equations cannot be so easily experienced: one rarely sees students use equations to solve arithmetic problems although equations are taught at the primary school level: most often students solve the problem first, by choosing the right operations in the right order, and write the equation afterwards. This is not true for equations only. One can observe the same phenomenon for other symbolic representations, such as Euler-Venn diagrams at the primary level. So the function of symbols is not clear. How can we make it clearer?

I can see two main possibilities:

(1) a symbolic representation is made functional by the fact that it really helps students in finding the solution, that they would otherwise fail to find.

(2) a symbolic representation is made functional by helping students in discriminating situations, relationships and operations, that they would otherwise confuse.

These possibilities are not independent of each other.

As one does not see clearly enough the kind of help that one gains from using a symbolic representation when there is only one operation to make, it is a reasonable idea to use problems with many data to see how helpful it is to put them into "representation" and relate them with the questions asked. This is the kind of experiment we have tried, at different levels of age and with different conceptual domains. In simple proportion and multiple proportion problems, we used situations in which students had to deal with up to ten or twelve numerical data, and could be interested in many different questions.

We used double-entry tables for those experiments but the analysis of the different properties of space used to symbolize the different properties of the quantities and magnitudes involved shows that the syntax of the double-entry table may vary a lot.

Let me use three examples

<u>Example 1</u> :

Existence of a relationship - "is the companion of"

	A	B	C	D	E	F
A		X				
B	X					
C					X	
D						X
E			X			
F				X		

Left-margin : the set of persons
Top-margin : the same set of persons
Inside the table : presence/absence of the relationship.

Example 2:

Simple direct proportion of n variables.

length of foundation	volume of concrete	weight of cement	weight of sand	weight of gravel	cost of cement	cost of sand	cost of gravel	cost of concrete
		50			25			
				1000			43	
	1	250	580	1500				c
		g	1000			40		
56	a			b				
385		d	e	f				

This correspondance table reads as follows:

50 Kg cement cost 25 francs
for 1 m^3 concrete, one needs 250 Kg cement, 580 Kg sand and 1500 Kg gravel.

c is the unknown corresponding to the cost of $1m^3$ concrete.
d is the unknown corresponding to the weight of cement necessary for a foundation of 385 m.
and so on.

The top margin consists of the different magnitudes involved.

There is no real privilege of the left margin. Any column can be used as a reference.

c : cost of 1 m^3 concrete.
g : weight of cement necessary for 1000 Kg sand.

The variables inside the table are numerical and they are expressed in different units: it depends on the column.

Example 3

		Number of workers		
		1	2	12
	1			
number of days	5		4150	
	15			a

cost of manpower

This table exhibits the double proportion of cost to the number of workers employed and to the number of days. Knowing the cost of 2 workers during 5 working days, you can calculate the cost of 12 workers during 15 days.

The left margin and the top margin express two different magnitudes but they play a symmetrical role; this is not the case in example 2.

Inside the table: numerical values taken by the variable: cost of manpower.

This syntactic analysis of the meaning of spots can be accompanied by a syntactic analysis of operators (relationships between spots).

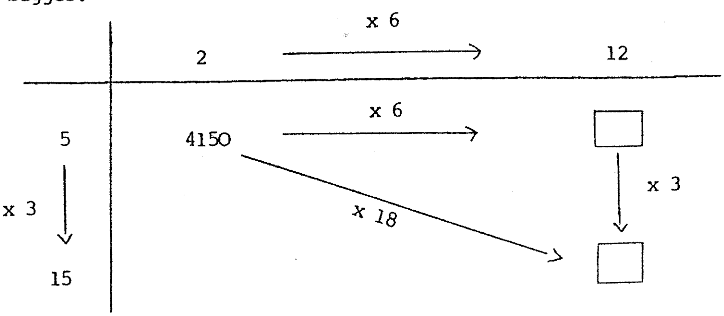
In example 2, one case calculate the cost of 1 m^3 concrete by adding the costs of the cement, the sand and the gravel needed. And the cost of the cement is either "five times more" than the cost of 50 Kg, or the image of 250 by the function f (x) = ½ x.

weight of			cost of	
cement	sand	gravel	cement	
50 x 5 ↓ 250			25 ↓ x 5 ☐	
x			$f(x) = \frac{1}{2}x$	

Multiplying (vertically) by 5 uses a scalar operator. Multiplying (horizontally) 250 by ½ uses a function operator (a quotient: francs per kilogramme).

In example 3, one can calculate the cost of 12 workers during 15 days by multiplying the cost of 2 workers during 5 days by the combination of two operators.

6 times more workers and 3 times more days make the cost 18 times bigger.

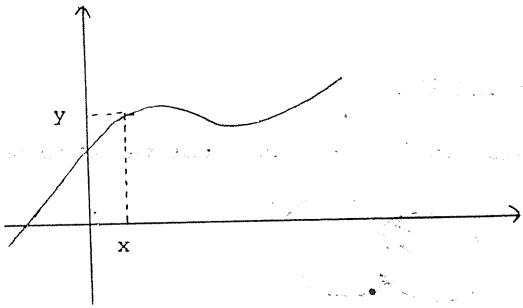


In this last case, the vertical and the horizontal operators are both scalars; they play the same role and they can be commuted. x 18 is also scalar.

Our experiments show interesting effects of these different tables, but they also show some confusions that derive from the different syntax of each symbolic representation. One must deal with these confusions, because such tables are really helpful, especially in showing (example 3) that the cost is proportional to the number of workers when the number of days is held constant, and to the number of days when the number of workers is held constant. This is not a minor point and the idea of "same line" or "same column" helps conceiving the idea of holding a variable constant.

In example 1, the situation is all different as there is no real operator from one spot to another one. The property of symmetry in this case is just particular.

When this kind of dot-representation is applied to numerical coordinates, one gets symbolic representations of functions.



The jump from a discrete to a continuous representation is a very important jump; also the jump from qualitative coordinates to numbers. But it does not change the fact that, altogether, there is no visible arithmetical operation that relates one spot to another one. Graphs are also good at exhibiting continuity and discontinuity properties, growth and rates of growth, maximum and minimum values; they are not helpful for calculations. Example 2 and example 3 tables are better tools for calculations, they are not of any help for topological or differential properties.

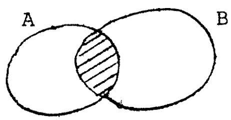
Example 2 and example 3 tables are also very good at showing the type of dependence between variables (simple dependence in example 2, double dependence in example 3), the inversion of scalar operators and function operators, the combination of functions and the combination of scalar operators. In that sense, they have a great discriminative power, and they can help students in disentangling confused notions that they are often unable to analyze properly, even when they are able to make the right arithmetic operation. In other words, not only do symbolic representations help students in dealing with complicated data, and in discriminating analyzed notions, but they also help them in identifying the right relevant relationships as new mathematical objects.

In that respect, the function of symbolic representations is similar to the function of natural language, as the use of different words for different classes of objects, for different actions and different circumstances, and the syntactic organization of sentences is a help for the analysis and the conceptualization of the world.

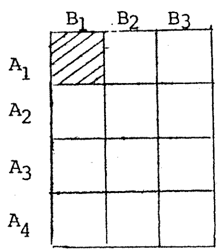
The above examples illustrate different ways of using the properties of space to represent the properties of mathematical relationships. Let me give a few other examples.

<u>Intersection and inclusion</u>

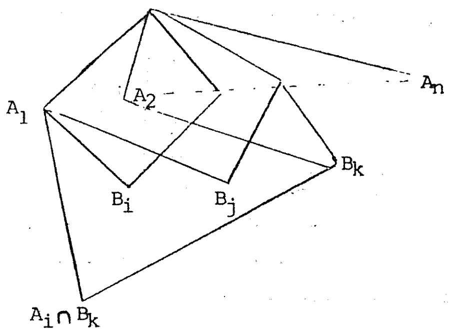
can be represented either by topological properties of space



or by both topological and projective properties of space



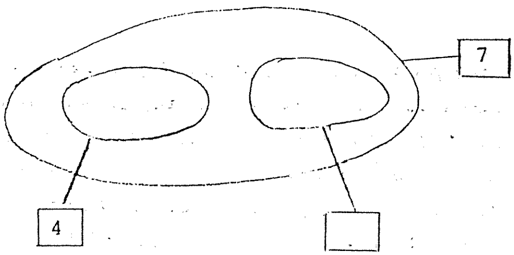
or by partial-order properties of space.



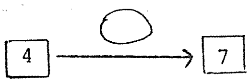
It appears to be the case that not all properties of the signifier (space) are equally adequate to represent properties of the signified (subsets), especially at the primary school level.

<u>Addition and subtraction</u>

Schemas like Euler-Venn diagrams



or arrow diagrams

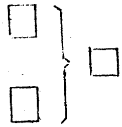


or equations

$$4 + \ldots = 7$$

do not fulfill the same function as some of them are better appropriate for certain relationships and not for others.

Euler-Venn diagrams are adequate to represent the combination of measures. Another adequate representation is the following:



but they are not adequate to represent either transformations (win, loss, expenditure, income, etc.) or relationships (more than, less, debit, credit, etc.) because there is no way to represent negative transformations or relationships in a standard way.

Arrow diagrams do represent transformations and relationships adequately, but they cannot represent the binary character of the combination of measures.

In other words, students need both kinds of symbols, and also both conceptual frameworks of the binary combination and the unary operation. It is well-known that equations are not viewed as binary but rather as unary. As a consequence of this, the equal sign does not represent a symmetrical and transitive relationship but rather an output; students' use of equations violates both symmetry and transitivity.

In summary, the properties of space and the properties of the symbols used in mathematical representations, may convey either adequate and powerful meanings that students are able to use, or deep misunderstandings: these misunderstandings may come from the limited scope of validity of the symbolic shape used, from the gap between the student's way of reading it, and the teacher's one, from the ambiguous value of the same property of space or from the loose connections between the signified and the signifier (as many symbolic representations are taught and learned as meaningful unanalyzed metaphors).

This conclusion might sound pessimistic; in my view, it is not. But we must pay more attention to the careful analysis of the operations needed to read and use symbolic representations.

This analysis cannot be performed without a careful analysis of the concepts and the situations that can be represented. A more functional approach of mathematical symbolism relies upon a more functional approach of mathematical concepts. Not only must symbols reflect the properties of the mathematical objects; the work on symbols must also reflect the ways students work at the problems to be solved.

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B. THEORIES OF TEACHING AND LEARNING

TEACHING AND LEARNING OF APPLICATIONS OF MATHEMATICS: IMPORT OF PSYCHOLOGICAL STUDIES

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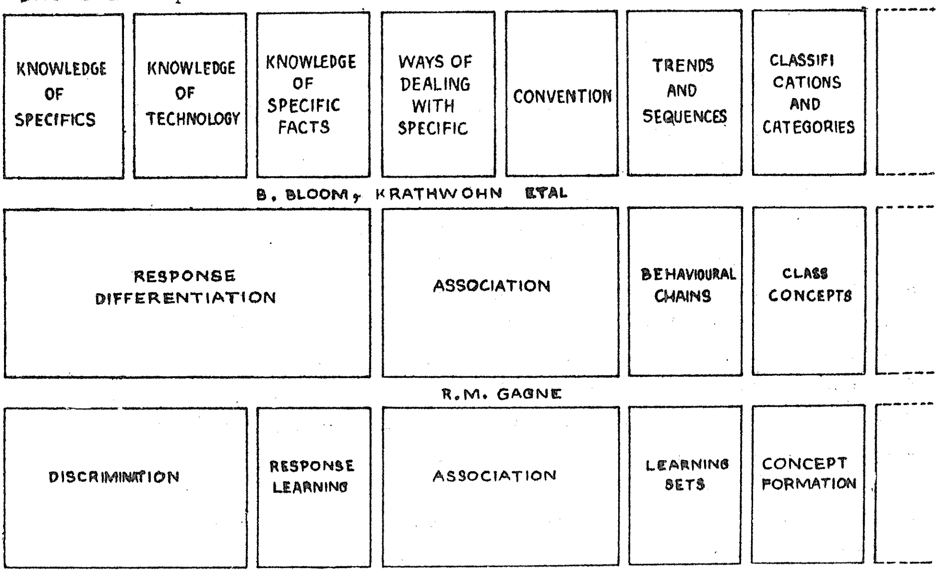
I. INTRODUCTION

An important area of mathematical education is, doubtless, the concern about teaching and learning of mathematical content that go by the labels applications of, applied or applicable mathematics. This becomes all the more deepened in view of fast expanding ambit of this kind of study. Various connected issues, for example, concept formation assume newer dimensions and more so in the context of emerging psychological studies. In this context, the interest, evinced at the international levels and national levels, should be mentioned, vide Bauersfeld (1). Of all areas of mathematical education, the input from studies in psychology is quite significant and more so in respect of learning process. While some areas of mathematics as such have been studied in this direction, the same for applications of mathematics have not been attended to in depth of all theories in learning of mathematics dealing with psychological aspects, those by Piaget, Skemp, Dienes, Bruner, Gagné, have acquired a classical dimension. But these have not been discussed in the context of applications of mathematics, which are fast changing, hence the need for a deeper study in this direction. The ambit of these classical studies being so wide that, for the present, we confine ourselves to investigations in the light of Gagne's and Bruner's theories. The plan of this paper is as follows: First two sections we deal with salients of Gagne's and Bruner's theories and then these are assessed keeping in view the requirements for learning of applications of mathematics.

2. GAGNE'S HIERARCHIAL MODEL OF LEARNING.

The educational psychologists, like Bloom(3). Gagné(10), have a general agreement regarding the psychology of learning. According to them learning takes place in a series of stages or hierarchies. The following diagram shows a comparison of some of these (19).



Gagne's hierarchial model for learning of intellectual skill or capability, which may be a form of problem solving may be given as follows (10).

Higher Order Rules which
require as prerequisites

↓

Rules
which require as prerequisites

↓

Discrimination
which requires as prerequisites

↓

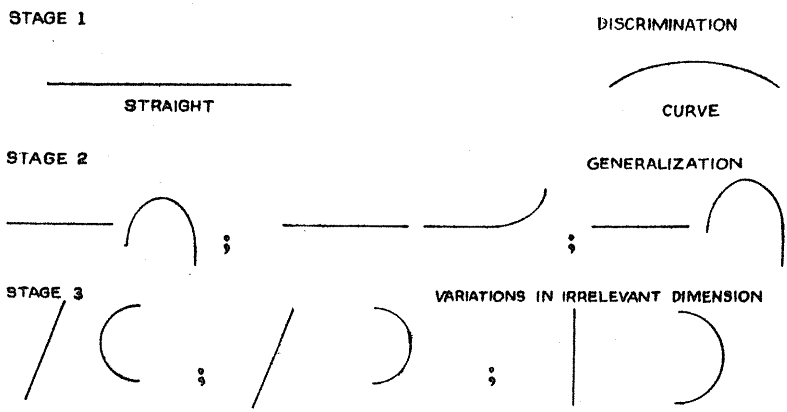
Basic form of learning
Association and chain.

According to Gagné, if the final 'capability' desired is a problem solving capability, then the learner must know a series of hierarchial set of prerequisites such as concepts, discrimination learning, until he ends up with the fundamental building blocks of learning.

Association is the basic process in learning. In different learning situations a learner generally associates or connects a stimulus with a response. This 'associations' generally depends on such experiences where two ideas are associated as stimulus-response, bond-the presentation of one idea tends to arouse a thought of the other. Generally 'association' learning may be of three kinds (a) Signal learning (b) Stimulus - Response learning (c) Verbal association learning. These may be connected again in a sequence.

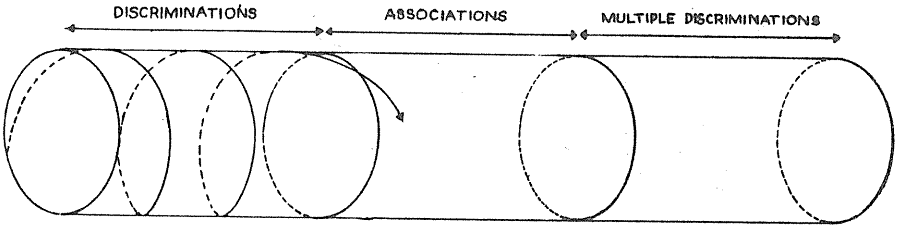
The next important feature of learning is to develop the ability of 'discrimination'. This is the ability to tell the difference between variations in some objects property. At the early stage, the young ones learn to discriminate between the colours, shapes, sizes, numericals, alphabets etc. According to E.J. Gibson (12) discrimination learning leads to perceptual differentiation within five media, object, space, events, representations and symbols. Discrimination learning is often concerned with the distinctive feature of stimulus objects and ensures different responses, inspite of similarities between the stimuli. Discrimination learning is essential for developing the ability to recognize new kinds of stimuli and attainment of concepts.

An individual has formed a concept when he exhibits categorizing behaviour. Concept formation involves both discrimination and generalization. Discrimination ensures different responses inspite of similarities between the stimuli where as generalization ensures similar responses inspite of difference between the stimuli. The following is the example due to Gagne (10), which shows the stages of learning beginning with a discrimination and leading to the learning of a concept of straight line.

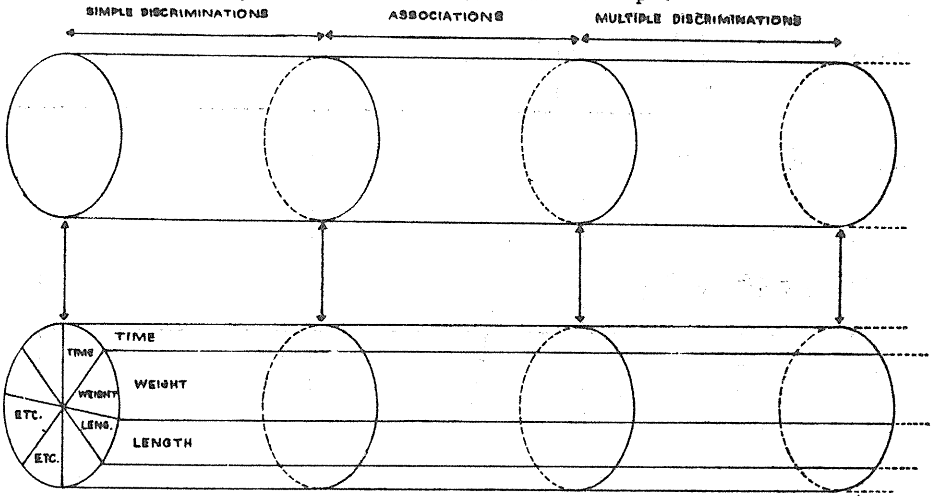


If the learner can identify a straight line, in a brand new example, then we can say that the concept has been acquired. The effect of concept learning is to free the individual from control by specific stimuli.

Hence for early stages of learnings, we are using simple discrimination, concept formation, which has been given by Bill Vaughn (19) in his model showing path of pupil through hierarchial learning.



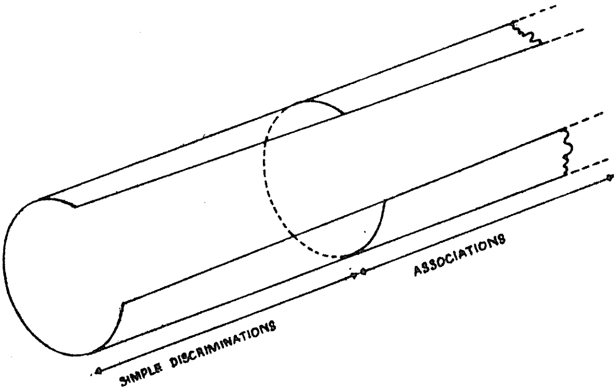
For learning mathematics, one more dimension to the above model is added, which is mathematical concept.



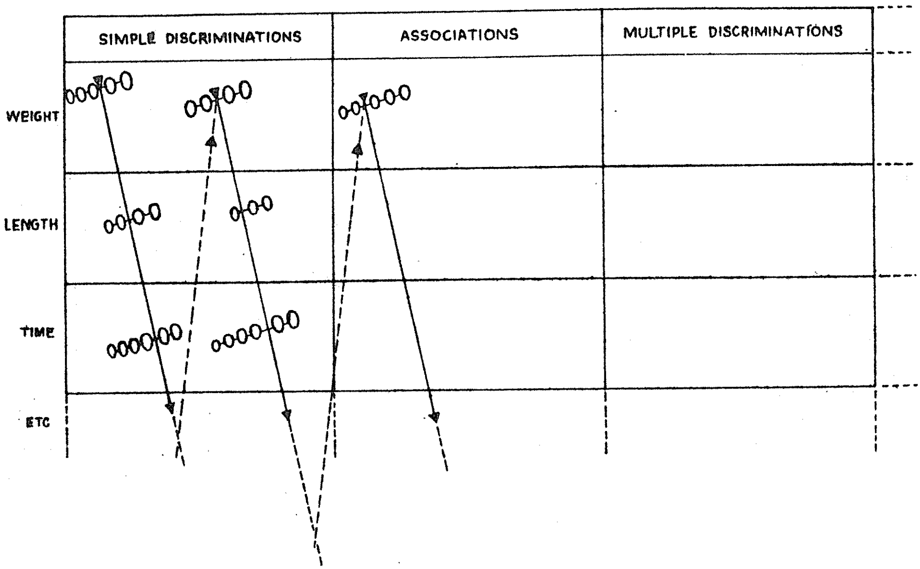
MATHEMATICAL CONCEPT

WITH THIS ADDED DIMENSION, THE MODEL LOOKS LIKE A CYLINDER

With this added dimension, the model looks like a cylinder



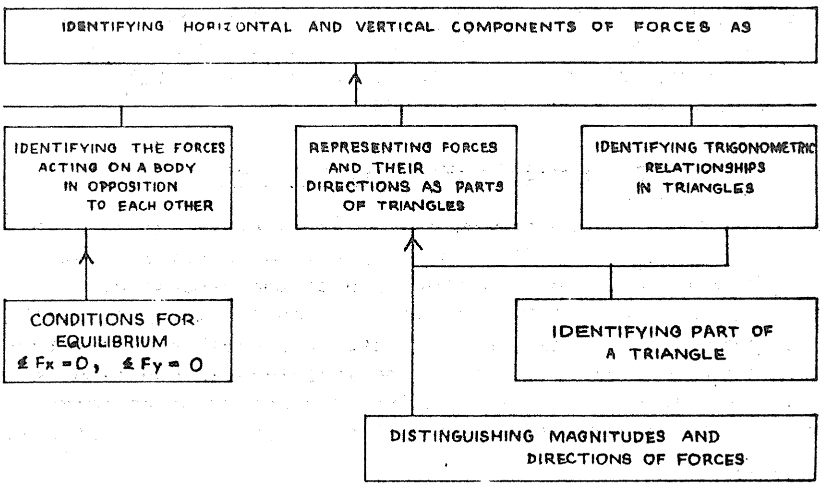
If this cylinder is cut along a horizontal line, model of learning experience in two dimension will appear as below:



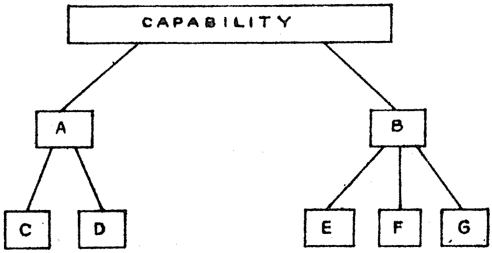
The path of a pupil as he has a series of experiences in each concept is represented by the oblique lines with attached oval shapes, where the oval shapes represent a gradation of experience at each stage.

But the problem is that every concept can not be learned by direct interaction with the learner's environment. Particularly in mathematics, there are concepts which are to be learned by the use of language and symbols only. For example, set of prime numbers which is a defined concept, abstract in nature, may be a thing concept or relational concepts. According to Gagné, a defined concept is a particular type of rule, which governs one's behaviour. A rule is then, is an inferred capability that enables the individual to respond to a class of stimulus situations with a class of performances, such performances being predictably related to the stimuli by a specific class of relation. Hence the psychological organisation of intellectual skills or capability or solving problem may be represented as a learning heiararchy as stated earlier, which are largely composed of rules. Below is given a learning hierarchy showing the analysis of a rule to be learned into prerequisite components (10).

VECTOR RESOLUTION OF FORCE



Hence we can conceive these hierarchies of rules and concepts as a complex pyramid at the top of which is placed, the desired 'capability' of performing certain functions under specified conditions (16).



In Gagne's model, we have found that concepts depend on the discrimination and association. Once concept is formed, it helps to the positive transfer of the learning of rules and these rules direct learning of more complex rules and the capability of solving problem.

3. BRUNER'S THEORY OF LEARNING.

Jeromes Bruner, also propounded a theory of learning which is based on understanding and well suited to mathematics. Bruner's approach to cognition is with the process of knowing and the cognitive growth of a child is related with his perception, skill and learning. He describes 'perception' as a decision making process. The individual's decision is based on certain cues he gain from the object he perceives. The cognitive activity of the child is based on this process of decision making.

Bruner's theory regarding the development of cognitive growth depend on 'representation' and 'integration'. 'Representation' is the ability to represent the recurrent regularities of the environment and the ability to link all the informations from past to present and føuture is the 'integration'.

Bruner has described three levels of representation; enactive, ikonic and symbolic. The first level is enactive, when the child manipulated materials directly. He then progresses to the ikonic level when he deals with mental images of objects but does not manipulate directly. Finally, he moves to the symbolic level, where he is strictly manipulating symbols. Though Bruner's theory of developmental stages can be regarded as Bruner's interpretation of the developmental theory of Piaget, Bruner differs with Piaget regarding the 'conception' of readiness of the children. As he said "we begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (6). He asserts that fundamental principles or structure of any discipline are essentially simple and consequently these simple structures can be taught and learnt in an intellectually honest form through any mode of representation. This becomes a major impetus for the trend towards earlier integration of subject matter to elementary school children.

In his book the "process of education" (5) he stresses on the structure in all teaching and associate it with changing theories of transfer of tranning". 'To learn structure in short is to learn how things are related'. Beside this 'skill' is also an important factor in the cognitive development of the child, which is the result of intentions, feedback and structure.

The basis of Bruner's theory of learning, which is known as discovery learning, is the structure and skill uniquely combined with his developmental model. In some work with Dienes (8), Bruner and Kenney (4) describe how children are led to discover the pattern of results of a quadratic expression, by manipulating first with the Dienes algebraic experience material and then arriving at the symbolic conclusions. He has combined the skill with problem solving activity. To develop skill, it is necessary to arouse intention and interest which determines the end state of the situation and leads the activities to success. His discovery approach is primarily concerned with 'how to know'. He defines discover as a matter of rearranging or transforming evidence in such a way that one is enabled to go beyond the evidence, to assemble new insights. The learning process involves first the corresponds between the intuitive regularities of the child with the manipulation of material regularities. According to Shulman (16). "The discovery involves an internal reorganization of previously known ideas in order to establish a better fit between those ideas and the regularities of an encounter to which the learner has had to accomodate".

4. IMPLICATIONS OF GAGNE'S HIERARCHIAL MODEL OF LEARNING AND BRUNER'S MODEL OF LEARNING IN APPLICATIONS OF MATHEMATICS.

The key terms of Gagne's hierarchial model are discrimination, association, concepts and rules. In the case of learning and instruction in applications of mathematics, these words acquire some definite meanings. For a construction of a model, identification of the problem in terms of a variable is a process which comes close to the idea of 'discrimination', as set out in Gagne's modelling. Also one of the main purposes of study of applications of mathematics is to have encounter with a number of choices of variables and it is there that 'discrimination' has a definite role in relating the variable to the particular choices.

The implications of 'association' due to Gagné in the context of applications of mathematics is a bit farfetched but if it is possible to relate some mathematical structure relevant to a particular model it can only be made effective, through what Gagné calls 'association'. Indeed the axiomatic structure from which the mathematical structure emerges is again an implicit and indirect acceptance of the concept of 'association'. The implication of 'concepts' relating to applications of mathematics is so obvious that they need not be elaborated, but what should be stressed is that applications do not necessarily mean acquisition of skills but largely and essentially development of concept. 'Rules' as put forward by Gagné come up as final attainment of the ability of problem solving in the study of applications of mathematics; of course 'rules' in the area of applications of mathematics may become necessary in conjunction with the other three key processes set out by Gagné. Indeed, modelling assumes the character of a higher order 'rule' if this has a predictive capability going well beyond the stage of a better understanding of the phenomenon of the situation concerned.

Bruner's theory of learning mathematics, also fits in well with the ideas on mathematical modelling. In this process of learning of applications of mathematics, the stages are to rearrange and reorder the existing information, then integrate these with some existing structure (cognitive) and then to reorganise or transfer the above integral combination. One always finds questions like that should be done to start with. And prior to this, is a process of decision making, which in Bruner's term 'cognitive activity', of the child'. The answer is the decision process taken into account, collection and organization of data, that are so important in the business of modelling. The idea of representation occupies an important place in Bruner's scheme. The representation just does not remain there, by itself. It is only when related to another important idea of Bruner namely 'integration', that it becomes effective. The important outcome is obviously a better insight into the nature of phenomenon.

Even though, there are some distinct points, in Bruner's and Gagne's theories, there are many significant overlaps in respect of 'connotations' and 'implications'. This point of view may not be obvious from the standard of instruction to areas other than applications of mathematics. In other words, basically both the theories reflect by and large the essential stages of mathematical modelling, but in different words. The difference between a standard model and non standard model as viewed by either of these two theories, is largely due to concern for applicable areas of mathematics, rather than situation that can be mathematized with the help of conventional concepts and techniques.

5. CONCLUSIONS.

In the foregoing lines an attempt has been made to discuss learning of applications of mathematics, within the confinement of two important theories. This is the first step in this direction. These two theories need to be again assessed for applicable mathematics and the emerging area of mathematical modelling. The other theories would also be taken up in a subsequent presentation. It is believed that the above critique, will provide a deeper and a more correct understanding of the psychological nature of learning of applications of mathematics. It transpires from the above that this area of investigation is basically interdisciplinary in character and another important dimension which should go along with mathematical modelling is the very problem in the modelling in mathematical terms of the psychological implications,learning of applications of mathematics, as is often done in mathematical psychology and psychomathematics. This will be reported in subsequent investigations.

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SHORT AND LONG TERM LEARNING -EXPERIMENTS IN DIAGNOSTIC TEACHING DESIGN

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At last year's PME Conference our research group reported an experiment on teaching the meaning of decimals and the reading of scales, in which a 'conflict' teaching method was compared with one which similarly focussed attention on the known misconceptions, but emphasised correct approaches rather than provoking conflict. Both groups showed significant learning, but the 'conflict' method was superior (Swan 1983). Studies of pupils' approaches and misconceptions in the recognition of operations in verbal problems were also presented, by us, and also by Fischbein. (Greer 1983; Fischbein 1983; Bell, Fischbein and Greer 1984).

Subsequent work has extended this in two ways, first by developing teaching material on the choice of operation and on directional quantities, and secondly by developing a 'diagnostic methodology' that is, a set of principles for designing and using material which will enable teachers to use such material effectively and to design their own, or at least to adapt material in these ways. Thus our most recent experiments have consisted of observing lessons in which our material has been used by teachers who have not been involved in its development. This work is still in progress, but results so far suggest that the amount of cognitive conflict generated in discussion is significantly related to pupil gains on tests; it may be of interest to give an account of the methodology and of the observations made so far.

Early research on learning tended to look at short-term effects or, at most, at retention and transfer of single principles or of a limited set of facts or skills. More recently, several factors, including Piagetian work and survey results, have forced our attention to questions of longer term learning. Similarly, growing awareness of most pupils' very limited success in transferring knowledge learnt in the mathematics classroom to relevant situations in other subjects, or to everyday problems, has made us consider more seriously the deep-lying, well-digested kind of mathematical knowledge which is available for transfer in this way, and how we may help pupils to build up such knowledge.

We know that, except for the most able, pupils' perceptions of problems focus most strongly on the contexts in which they are set, rather than on the structure which they embody (for example, Silver, 1978); and that problems in familiar contexts are often handled differently from those in unfamiliar ones (for example, Wason and Shapiro, 1971). It is also becoming clear (though I am not sure that this has been documented) that situations which are structurally identical when fully mathematised are by no means similar when first perceived; for example, money and temperature problems are differently perceived, although they both involve states and changes in directional quantities. It follows that we need to consider much more seriously than we have previously done, the development of conceptual structures in one context, then in another, then perhaps exploring the isomorphism. The traditional practice in text books of offering sets of problems drawn from a variety of contexts indiscriminately, while it may provide good practice for a pupil who has fairly well mastered the structures, is probably unhelpful at the earlier stages.

The demands of long-term, transferable learning lead us to consider how pupils approach new problems in the fields in which we are interested. Some workers are concerned with the strategies with which pupils approach problems in general; we are concerned with problems which are as realistic as possible, but embody concepts of a particular type, in the case of the present work, additive structures with directional quantities. Recognising the dominance of context, we need to help pupils to respond to recognition of say, a money situation, or a temperature change situation, or a relative ages situation by calling up a schema of concepts such as overdrafts, negative changes, and so on. We need to see what general concepts pupils initially have which govern their approaches to these fields, and to develop or remedy them as necessary.

To become specific, we have tested pupils with a fairly wide ranging set of questions involving directional quantities, and from the common errors (and from discussions with pupils) have identified the following misconceptions:

1 <u>'Count start and finish'</u>

"Norwich has gone up 6 places from 9th position. Where are they now?"
Answer: 4th

This is not particularly related to directionality, but has occured quite extensively in some groups. The pupils do not realise the importance of counting just one of the start or finish numbers when finding a difference.

2 <u>'Up is increase'</u>

"Norwich has gone up 6 places from 9th position. Where are they now?"
Answer: 15th

The normal direction of increasing numbers, from 9 up 6 to 15 dominates.

3 <u>'More means add'</u>

"Liverpool scored 6 more goals this month than they did last month. They scored 13 goals this month. How many did they score last month?"
(Answer: 19 goals)

The difficulty of relating the time-order with the order of number of goals leads to a breakdown, and the word 'more' dominates.

4 <u>'Difference means subtract'</u>

"A traveller went from Dakar, where the temperature was 31°, to Reykyavik, where it was -3°. How much did the temperature fall?"
Answer 28°

31 down to -3 is 31 subtract 3 ie 28; a minus sign has been 'used' and the size of the answer does not cause conflict with expectation.

5 'Position and move confused'

"The afternoon temperature was 8°, but then fell 6° by nightfall. What was the temperature at nightfall?"

Answer: 6°

Linguistically, the confusion is between falling by 6° and falling to 6°. The 'position' interpretation tends to be dominant.

6 'Sign denotes region'

"The temperature changes from -6° to -2°. How much was the change, and was it a rise or a fall?"

Answer: Rise of -4°

The degrees below zero are thought of as negative degrees even when they are moves. Similarly, a journey from a point 6 miles north of a given town to 2 miles north of the same town may be described as a journey '4 miles north', since it is 'in the north'.

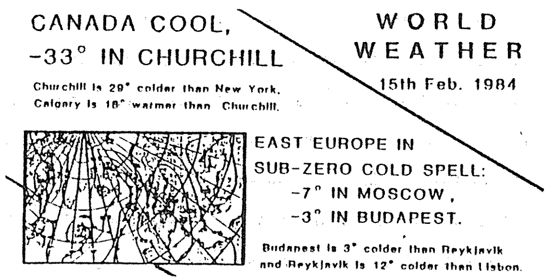
Similar misconceptions occur in temperature, money and journey contexts, but differ somewhat in their degree of incidence and their character. Most of them occur also in dealing with ranking structures, such as Pop Charts or League Tables, where the ordering is like that of the negative number scale.

'Up is increase', 'more means add' and 'difference means subtract' occur because the correct interpretation of the problem requires some reversal of thought from its 'normal' direction, which is a cognitive strain. The misconceptions have been expressed here in the form of the implicit beliefs held by the pupils. It seems necessary to do this in order to enable us to feel the reasonableness of the misconception from the pupil's point of view; we need to do this to have sufficient imaginative identification with the pupil's viewpoint to be able to set up a situation which will convincingly show him what is the correct view. This is the importance, and the difficulty, of the diagnostic step from recognising an error, - a wrong answer - to explaining it by identification of the misconception that is the pupil belief which governs it.

CORRECTING MISCONCEPTIONS

Establishing the correct approach means that the pupil must appreciate that, for example, "sometimes finding a difference involves adding the (unsigned) numbers involved; in particular this occurs when the difference goes across zero". Our work leads us to believe that getting the pupils to express both the false and the true generalisations in some such way, in general terms, facilitates their clarification; this is our aim in discussion.

But this is to jump ahead somewhat. We have experimented with a number of types of lesson and learning task, including (a) written problems with intrinsic feedback but without discussion, (b) conflict-discussion lessons, (c) making up questions, (d) 'Marking Homework' and (e) games. In all of these we have attempted to work in familiar contexts. To begin with the conflict discussion lessons, these have typically had three parts. First there are some questions to answer which explore the topic and which are likely to expose the misconceptions. For example, a lesson on World Weather began from a page of 'newspaper headlines' (see illustration) and asked first for the temperatures in



Reykyavik and Lisbon, and the change on travelling from Budapest to Livingstone. When there has been sufficient time for pupils to write their answers to these, a discussion is begun. The teacher says, for example, 'For Budapest to Livingstone you may have rise or fall of 28°, or 29°, or 33°, or 34°; has anyone any other answer?' He then asks how many pupils have each of these answers, then asks pupils to explain how each answer is obtained, and also asks others if they agree. He aims to arouse and expose as much conflict as possible at this point, and may, if necessary, himself put forward faulty arguments to be challenged. He tries to get pupils to express the false and the true generalisations in general terms. After this discusion the pupils continue with further questions in writing. Feedback of correctness is provided in this case by giving additional clues, such as "in questions 4 and 5, the rise should be 26° more than the fall." (In a similar exercise on League Tables, feedback was provided by getting pupils to fill in their answers on a partially completed table.)

Such lessons have been observed, systematically, and the amount of conflict generated in the discussion estimated by the observer. When compared with other lessons, and in particular with some in which the same written exercises with feedback were used but without discussion, there appeared to be a significant connection between the amount of conflict and the pupil gains. (These were shown by the pre and post testing with a few of the verbal questions designed to show the misconceptions, as illustrated above. In the 'Marking Homework' exercises, a set of questions with mixed correct and incorrect responses have to be 'marked' by the pupils, and the nature of the misconceptions stated. In 'Making Up Questions' pupils are asked (singly or in pairs) to make up 'some similar, quite hard questions' for their neighbours to answer. Both of these exercises are quite difficult but appear to be possible ways of obtaining the degree of reflection on the problems which appears important in increasing insight.

Some specially designed card and board games also provide a setting in which pupils have to make many decisions, involving the concepts and misconceptions, and in which feedback is possible by opponents' challenging mistakes in play.

DISSEMINATION

As we become satisfied that these teaching approaches are valuable, we intend to offer them to other teachers to incorporate into their own styles. Our experiments in this direction are very limited so far, but it is clear that for many teachers these approaches represent substantial changes.

In particular it is important, and difficult, to convey to pupils that in these lessons we are aiming to look carefully at the mistakes they make, to take them seriously as reasonable (though wrong) approaches, and to help them (the pupils) to see how they go wrong and how to put themselves right. This means moving the focus of attention from the answer to the method; pupils are testing out their ideas and their methods and checking whether these work correctly. Pre and post testing, with the expectation of some changes in response, also need repeated explanation to the pupils.

These comments so far are based on a limited amount of experiment - about 50 lessons, comprising varied numbers of realisations of about 8 lesson-plans. Further work is aimed at investigating ways of maximising the effectiveness of the different lesson-components and applying the methodology to some other topics and other conceptual fields. We shall continue to use repetitions of similar lesson plans, to observe lessons systematically, using a proforma, to record the degree of conflict in discussions, and to use fairly frequent pre and post testing to judge effectiveness, the tests comprising probes of recognised misconceptions. Overall pre and post tests will look for transfer to structurally and contextually different problems.

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CHILDREN'S CONCEPT IN ELEMENTARY GEOMETRY - A REFLECTION OF TEACHER'S CONCEPTS?

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INTRODUCTION

During elementary school, children in Israel are required, starting in the first grade, to study geometry one hour every week. The goal of this program is to impart basic geometrical concepts, their properties, and also elementary calculation skills, which are embedded in exercises utilizing the concepts. Elementary school teachers who teach these topics in class therefore have a tremendous opportunity to enhance their students' geometrical concepts.

We have investigated basic-geometrical concepts in school children (grades 5-8) in a series of studies which were started in 1980 (Vinner & Hershkowitz 1980, 1983; Hershkowitz & Vinner, 1982, 1983).

The purpose of the present study was to obtain a better view of the processes of concept formation in children and of the factors which affect their acquisition by studying these same concepts in elementary school teachers. We used the same questionnaire as reported in the aforementioned studies.

The teacher population in our study included two main groups:

1) Student-teachers in their pre-service education, in the first, second and the third year of their studies in the teachers' college (PRE, N=142). These prospective teachers got 2 hours of mathematics every week, little part of which was devoted to geometry.

2) Senior-teachers (ST, N=25); most of them coordinators of math teaching in their schools, but their formal mathematical education was the same as those of the pre-service teachers mentioned above.

We will now analyze the teachers' results and compare and contrast their responses with those of the students. In so doing, the types of questions and concepts assessed will be clear and will illustrate:

a) The concept images of some basic geometrical notions.

b) Factors which influence the concept images.

c) The power of a verbal definition to form new geometrical concepts.

These three aspects were examined in the student population. We now take a deeper look at them comparing the student responses with those of the teachers. Please note that the teachers in this study were not the teachers of the students discussed here.

RESULTS AND DISCUSSION

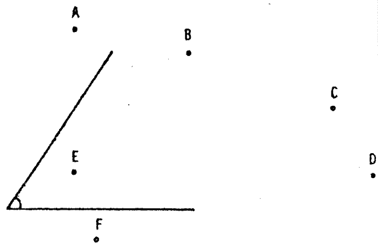
a) *Concept images of basic geometrical notions.*

We will discuss here in detail the results of one example and briefly describe a few of the other examples.

The concept of an Angle

The notion of an angle as a "Euclidean concept" is an abstract one. No matter what definition is used to introduce it, the concept of infinite continuation is needed. If you understand the notion of angle correctly you realize that when drawing an angle on a sheet of paper, you are representing only part of it. This understanding was assessed by the following item:

In the following drawing, circle all the points which are inside the angle.



The results are given in Figure 1.

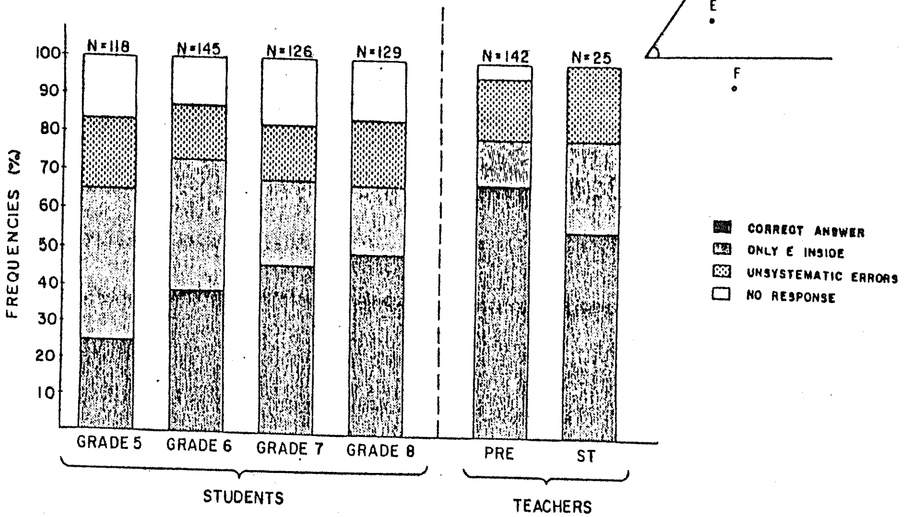
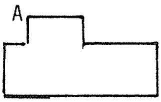


Figure 1 : Distribution of students' and teachers' responses to inside outside angle task.

These results show that less than half of the students conceive the angle as an "infinite entity", (from 25% in the 5th grade to 50% in the 8th grade), and not much more than half of the teachers, (68% of the prospective teachers and 55% of the teachers). The concept image of many students includes only the "finite picture" of the angle, (those who circled only E), but it improves with age, (from 40% having the finite picture in the 5th grade to only 17% in the 8th grade).

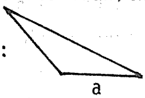
In the group of the senior senior teachers 25% have the same "finite concept" of the angle and probably it is reflected in their teaching.
Half of the sample (students and teachers) were given the following definition in addition to the above item:
An angle is the part of the plane which is between 2 rays emerging from one point.
It is worthwhile to note that neither the students' nor the teachers' answers were improved by that definition.
Two other examples like: drawing an altitude in different types of triangles or drawing all the diagonals from the vertex A in non-connex polygons like the following: 

, show the same pattern: - The teachers' concept images are only a little better than those of the students', but still many teachers have incomplete concept images or concept image which includes wrong elements.
For instance:

i) The teacher drew only "inside diagonals" in the above polygon. When confronted with a re-entrant polygon 

, - the "inside" notion of their concept image of a diagonal was crystalized, for they either drew nothing at all or they drew something like 

They rejected the notion of an "outside" diagonal. Here too the verbal definition of the diagonal did not help improve their performance.

ii) When teachers were asked to draw the altitude to side a in the obtuse-angle triangle: 

, part of them drew the median to a or the perpendicular bisector of a; again they did not accept the idea of an "outside" altitude. The incompleteness of this concept image and its wrong useage were expressed in exactly the same way by the students.

b) *Factors which influence concept image.*

i) <u>The orientation factor</u>

In previous studies we reported that the success of students in identifying a figure, decreased when the figure was rotated from its "canonical position". Figure 2 shows the comparison between teachers' identification to students identification of a rotated right-angle triangle. (In the task they had to identify all the right-angle triangles in a given collection of triangles).

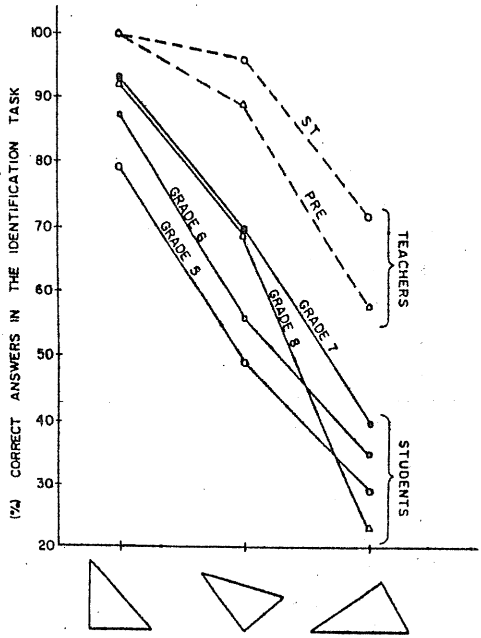


Figure 2 : Success of students and teachers in identification of right-angled triangles.

It can be seen that the teachers performed a little better than the students, but both graphs in Figer 2 have a similar structure, This indicates that the teachers are affected by the orientation factor in the same way as the students: a larger rotation of the triangle makes it more difficult for them to identify it. The orientation effect is a perceptual problem, and it has been examined in many studies. The failure to identify "tilted figures" affects the ability to solve certain geometrical problems.

ii) The "distracting attribute" factor.

When an example of a concept has some dominant non-critical attribute (see Hershkowitz & Vinner, 1983) it inhibits its identification. For example, when an isosceles triangle has also a right-angle, there are difficulties in identifying it as an instance of an isosceles triangle. In this example once again the teachers' behaviour has the same patterns as the students' behaviour. In the student population, the percent successfully identifying "canonical isosceles" triangles was 80% and for isosceles triangles with a "distracting attribute , 60%. In the teacher population it was 96% and 81%,correspondingly.

c) *Verbal definition as a tool for concept formation of new concepts.*

We start with an example previously reported in Hershkowitz and Vinner 1982:

Definition: A bitrian is a geometric shape consisting of two triangles having the same vertex. (One point serves as a vertex to both triangles).

One half of each of our groups (students and teachers) was asked to identify bitrians among other shapes, the other half was asked to construct two bitrians.

The results of the students and the teachers in the identification task are shown shown in Figure 3.

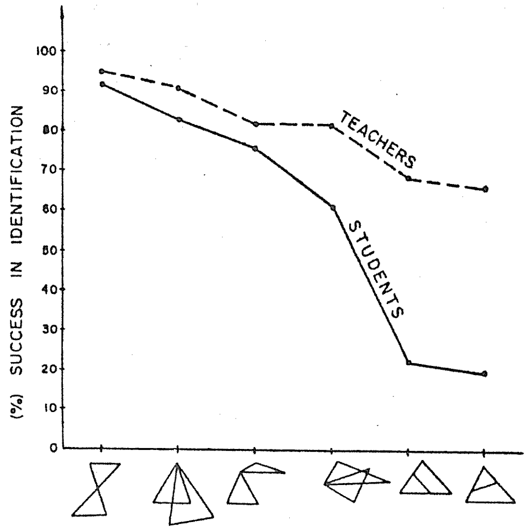


Figure 3: Success of students and teachers in the bitrian identification task (in percentages).

All shapes(which were drawn) were collected and classified. Figure 4 shows the frequencies (in percent) of each shape in both populations.

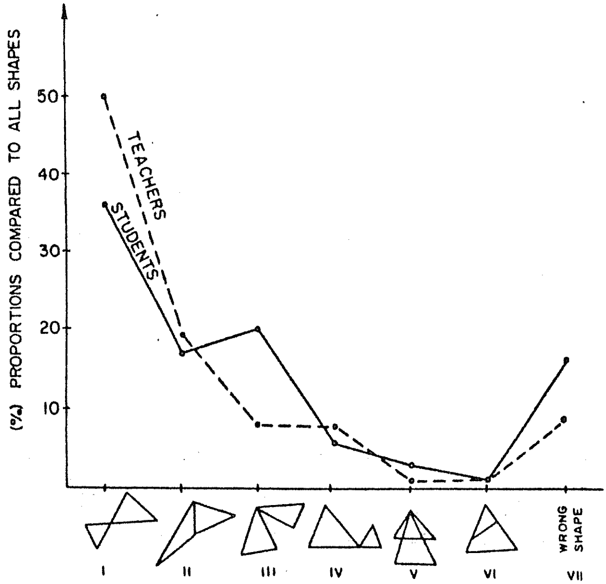


Figure 4 : Construction of bitrians by students and teachers.

It should be noted that; i) 20% of the children did not respond to this item, whereas only 3% of the teachers did not do so. ii) The high frequency of the shape (ii) in Figer 4 may be a result of confusion. The two triangles in the shape have a common side and two common vertices. Since we did not require in our definition "exactly one common vertex" this shape is a bitrian. However, it is also possible that it was drawn because "common vertex" and "common side" were confused. In the two bitrian tasks the teachers and the students started from the same point:- the "concept" was new to all of them, yet each group was familiar with its component parts: triangles, vertices, sides etc...)
The similarity of the results from this point of view are less surprising because teachers are not suppose to teach it. On the other hand it is interesting that the teachers' concept image has the same structure as that of the children: - there were "canonical examples" embedded in the concept image of most individuals, (teachers and students), while other examples were noticeably absent.

CONCLUDING REMARKS

Do children's concepts reflect the teachers' concepts? The first impression is that this is really the case. However, a deeper look at the results shows that the situation might be much more complicated. It is true that the teacher has a major role in developing the concept for the students. A teacher's incomplete or incorrect concept image however will probably be repeated in the students' geometrical thinking. But beyond this, there might be difficulties which lie in the concepts themselves. These difficulties are inherent to the concept and act similarly on teachers and students as evidenced by the results reported here. Among these difficulties are the orientation factor, the existence of "canonical shapes" etc...

From the above analysis two additional points must be made:

1) Teachers lack basic geometrical knowledge and skills.
2) Either they are not aware of their lack of knowledge or they lack analytical thinking ability: teachers who got definitions in their questionnaire did not use them, either because they thought they knew the concept or because they were not capable of using them.

We have seen in this paper that there is almost no difference between teachers and students concerning basic geometrical concepts. Students grow up to be teachers, teach geometry poorly to another generation of students and the cycle is repeated. Breaking this vicious cycle is the major concern of our preservice and inservice teacher training programs.

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COGNITIVE MECHANISMS UNDERLYING THE EQUATION-SOLVING ERRORS OF ALGEBRA NOVICES

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Matz (1979), in developing a process model for high school algebra errors, suggested that algebra novices try "to fit and stretch" their existing arithmetic knowledge to cover unfamiliar algebraic situations. Many of the errors of algebra novices can be explained, according to the Matz model, in terms of unsuccessful attempts to extend their existing knowledge. This model served as a basis for analyzing the equation-solving errors of 12- and 13-year-old algebra novices in a study to be described in this paper. The errors these algebra novices made when trying to solve algebraic equations, which were mostly unfamiliar to them, can be interpreted as unsuccessful attempts at fitting and stretching the arithmetic knowledge they had acquired up to that time.

Six seventh graders of average mathematical ability participated in the study— a three-month teaching experiment on equation solving. This experiment was preceded by a pretest interview with each subject. The interview tasks included a set of 14 equations to be solved. The solving procedures used by the subjects and the relationship between these procedures and subject's views of algebraic letters have been described in a previous paper (Kieran, 1983); however, that paper did not include any discussion of errors and the mechanisms underlying those errors. The latter topic is the focus of this paper -- the cognitive mechanisms underlying the unsuccessful solution attempts of the novice subjects of this study on the pretest equations. Three mechanisms which were found to account for several of the equation-solving errors were: 1) ignoring the unfamiliar; 2) overgeneralizing the right-to-left inversing procedure; 3) applying a limited notion of mathematical inverses.

METHOD

A three-phase study was carried out during the school year 1980-81. The first phase involved preliminary interviews with 10 seventh graders (12 to 13 years of age) to ascertain some of their pre-algebraic notions. This phase was necessary to verify subjects' lack of previous experience with, in particular, algebraic equations and equation-solving (Kieran, 1981). A subset of this group (n = 6) was retained for the second phase of the study, a teaching experiment on equation-solving. The pretest of the second phase included, along with other questions, the following 14 equations, which subjects were asked to solve: $6\underline{b} = 24$, $2\underline{x} - 6 = 4$, $\underline{y} + 596 = 1282$, $16\underline{x} - 215 = 265$, $\underline{n} + 6 = 18$, $13\underline{x} + 196 = 391$, $4\underline{c} + 3 = 11$, $32\underline{a} = 928$, $37 - \underline{b} = 18$, $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$, $30 = \underline{x} + 7$, $3\underline{a} + 5 + 4\underline{a} = 19$, $2 \times \underline{c} + 5 = 1 \times \underline{c} + 8$, $4\underline{x} + 9 = 7\underline{x}$. Subjects' verbal and written responses in this task were recorded, transcribed, and analyzed.

RESULTS AND DISCUSSION

Table 1 indicates how many of the six subjects were unsuccessful in attempting to solve each of the 14 equations of the pretest. As can be seen, certain

Table 1
Number of Incorrect Solutions
on each Equation of the Pretest

Equations	No. of incorrect solutions (n = 6)
$6\underline{b} = 24$	0
$2\underline{x} - 6 = 4$	0
$\underline{y} + 596 = 1282$	0
$16\underline{x} - 215 = 265$	3
$\underline{n} + 6 = 18$	0
$13\underline{x} + 196 = 391$	1
$4\underline{c} + 3 = 11$	0
$32\underline{a} = 928$	0
$37 - \underline{b} = 18$	2
$4 + \underline{x} - 2 + 5 = 11 + 3 - 5$	3
$30 = \underline{x} + 7$	0
$3\underline{a} + 5 + 4\underline{a} = 19$	5
$2 \times \underline{c} + 5 = 1 \times \underline{c} + 8$	5
$4\underline{x} + 9 = 7\underline{x}$	6

equations created no difficulties for subjects, for example, those with a single operation (excluding subtraction) or a double operation with small numbers. In solving these types subjects had been able to successfully stretch their existing knowledge. However, they had not been able to do so for several of the others. Three mechanisms were found to account for many of the incorrect solutions. An approach which will not be discussed in this paper, but which accounted for many incompleted solution attempts, was that of substitution. Though incompleted substitution attempts yielded incorrect solutions, the method itself is not an erroneous one and is therefore not included in this paper.

The Mechanism of Ignoring the Unfamiliar

One mechanism used in the pretest was to ignore, perhaps only temporarily, those equation features which were unfamiliar. Four of the six subjects tried to solve $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$ as if it were the equation $4 + \underline{x} - 2 + 5 = 11$. They substituted different values for $\underline{x}$ until they got the left hand side to equal 11 (by eventually replacing $\underline{x}$ with 4). Since most of their prior elementary school experience had involved equations with only one numerical term on the right hand side, they "solved" the given equation as if it had only one term after the equal sign. After solving for $\underline{x}$ in this way, they then performed the computation of $11 + 3 - 5$; but these "extra" numbers on the right hand side had no bearing on the value of the unknown. One of the four subjects who had been on this wrong track subsequently changed his mind and decided to total the right hand side before trying to substitute different values for $\underline{x}$; he ended up with the correct solution. It is important for later discussion to note that in this equation subjects had noticed the location of the equal sign, but had ignored all terms after the first one on the right hand side of the equation.

Another occasion when the ignoring mechanism was used was with the equation $2\underline{x} - 6 = 4$. Although the subject eventually solved the equation correctly, he had begun by ignoring the 4. He had tried to solve the equation as if it were $2\underline{x} = 6$. His past experience had included only equations with two numerical terms, one before the equal sign and one after. Thus, he "saw" only the first two numbers in this equation.

This mechanism was also at play with the equation 16$\underline{x}$ - 215 = 265, but this time the term not attended to was the 16. Three subjects subtracted 215 from 265 (the error of solving subtractions as if they were additions is discussed in a later section) and declared the result of this single operation to be the solution. As above, this equation was solved as if it had only two numerical terms and one operation ($\underline{x}$ - 215 = 265). The difference between this example of ignoring and the previous one is that in this case the two numbers which were clustered around the equal sign were noticed; whereas in the previous case, the first two numbers on the left side were the ones that were "seen".

The Mechanism of Overgeneralizing the Right-to-Left Inversing Procedure

The results shown in Table 1 suggest that equations with more than one occurrence of the unknown presented a cognitive obstacle to most of the novices of this study. An examination of their elementary school mathematics text books indicates that their prior experience with equations had consisted essentially in equations with one numerical term, an operation (usually addition, but sometimes multiplication), and a placeholder (a letter or a blank) on the left side, and the result on the right hand side of the equal sign. A strategy which had routinely been suggested in the texts to solve such open sentences (those of the form a x $\underline{x}$ = b or a + $\underline{x}$ = b) had been the right-to-left inversing strategy, that is, take the number on the right, inverse the operation between the letter and the number, then take the number on the left. This yielded either b - a or b ÷ a as the computation to be carried out.

That this strategy had remained with several of the subjects was clear from the way that they solved one-operation equations involving addition or multiplication (Kieran, 1983). However, in the pretest, one of them overgeneralized this right-to-left inversing procedure to cover equations of the types a$\underline{x}$ + b + c$\underline{x}$ = d and a x $\underline{x}$ + b = c x $\underline{x}$ + d. The subject simply modified the method to the extent that he increased the number of terms to be inversed. The starting-point for the inversing remained the extreme right-hand member of the equation. Thus, this subject attempted to inverse the multi-operation equation containing two occurrences of the unknown on the left side,

$3\underline{a} + 5 + 4\underline{a} = 19$, without first totaling together the multiplicative terms, "19 divided by 4, subtract 5, . . . subtract . . . divide 3". He was not sure whether to inverse the addition or the multiplication. A dilemma had presented itself: there were two operations left, but only one numerical term. Overgeneralization of the right-to-left solving strategy was also invoked in multi-operation equations containing an occurrence of the unknown on each side of the equal sign, as in, $2 \times \underline{c} + 5 = 1 \times \underline{c} + 8$; again there was the dilemma of not knowing which operation to inverse, since there were more operations than numbers. This overgeneralization mechanism could not be applied, however, to the last equation of the pretest, $4\underline{x} + 9 = 7\underline{x}$; for in this equation the extreme right hand member was not numerical. There was no way to start up the inversing sequence; the subject was clearly blocked and said so.

The Mechanism of Applying a Limited Notion of Mathematical Inverses

A common approach used in equations, such as, $16\underline{x} - 215 = 265$, was to subtract 215 from 265. This was unexpected. Since subjects had solved addition equations by subtracting, we had expected that they would solve subtraction equations by adding. However, for three of them, this ability to apply correctly their knowledge of inversing seemed to depend on the operation to be inversed. Additions were solved by subtractions, but so were subtractions. This kind of response put into question the reversibility of subjects' knowledge of inverses. In other words, they could apply their knowledge that subtraction is the inverse of addition to the solving of equations, but were not able to do it when it was a subtraction that had to be inversed. The same seemed true for multiplication and division. For example, in the equation $\underline{x}/4 + 22 = 182$ which was asked in the posttest, half the subjects tried to solve it by subtracting 22 from 182 and then dividing that result by 4. Thus, a mechanism that seemed to be used by the novices when they were in doubt as to what mathematical operation to apply in the inversing procedure was to solve the equation with the same operation as the given one. This unexpected behavior prompted us to look at a few elementary math text books to see how much prior experience our subjects may have had with subtraction open sentences.

All elementary school children learn a little about the relationship between addition and subtraction, and between multiplication and division. For example, given the triple 5, 7, 12 they would all be able to construct $5 + 7 = 12$, $7 + 5 = 12$, $12 - 5 = 7$, $12 - 7 = 5$. Practically all children entering grade seven have also been taught, at sometime during their elementary arithmetic program, that addition and subtraction are inverse operations to the extent that missing addend open sentences, such as, $\square + 5 = 12$ or $5 + \square = 12$ can be written (i.e., solved) in terms of subtraction, $12 - 5 = \square$. They have also been taught the converse, that subtractions with a missing answer can be rewritten as addition sentences with a missing addend. However, subtractions with one of the first two terms missing, such as, $\square - 5 = 8$ or $10 - \square = 3$, seem rarely to be presented in elementary school math texts. The reason for this may be due to the difficulties children would have in applying the inverse operation principle in the same way across both types of equations. The first subtraction above $\square - 5 = 8$ can clearly be rewritten as an addition with the placeholder in the last position, $8 + 5 = \square$; whereas the second subtraction ($10 - \square = 3$) cannot, $3 + 10 \neq \square$. The only way to write the second subtraction in terms of addition is with a missing addend, $3 + \square = 10$. Seemingly, because of these inherent obstacles, subtraction open sentences are not often experienced by elementary school children.

CONCLUSIONS

These three mechanisms are of interest not merely because they show some of the difficulties which novices have in making the transition from arithmetic to algebra, but also because some of these mechanisms can be related to a particular view of equations and equation-solving. A previous paper (Kieran, 1983) classified the equation-solving approaches of the novices of this study into symmetric and asymmetric. An asymmetric approach usually involved working backwards, going from right to left, e.g., solving $4\underline{x} + 5 = 9$ by subtracting 5 from 9, and then dividing by 4; a symmetric approach, on the other hand, entailed using the given numbers and operations in the given order, as when one solved by substitution or by performing the same operation on both sides of the equation. Associated with the asymmetric approach was a view of the letter in an equation as the result of performing a set of inverse solving operations; associated with the symmetric approach was a view of the letter in an equation as representing a number within the given equality relationship.

The mechanism of ignoring the unfamiliar involved for some subjects ignoring the location of the equal sign and for others the location of certain numerical terms. In ignoring the terms after the 11 in $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$, subjects were in fact noticing the equal sign. The position of the equal sign seemed to have more significance for solvers who preferred symmetric procedures. On the other hand, the subject who "solved" $2\underline{x} - 6 = 4$ as if it were $2\underline{x} = 6$, by ignoring the original location of the equal sign, was a solver who clearly preferred asymmetric procedures. Similarly, it was an asymmetric solver who overgeneralized the right-to-left procedure and tried to solve $2 \times \underline{c} + 5 = 1 \times \underline{c} + 8$ by inversing from the 8. He too was disregarding the position of the equal sign; it was as if he was "seeing" $2 \times \underline{c} + 5 + 1 \times \underline{c} = 8$.

The mechanism of overgeneralizing the right-to-left inversing procedure was clearly related to an asymmetric view of equation-solving. The subject who used this mechanism was one who preferred the asymmetric solving procedure. Though this mechanism was used by only one subject in the pretest, other subjects with asymmetric solving preferences also used it occasionally throughout the study as a means of solving equations of the two types, $a\underline{x} + b + c\underline{x} = d$, $a \times \underline{x} + b = c \times \underline{x} + d$. The existence of this mechanism raises some serious questions with respect to the amount of emphasis we should be placing in elementary school on the use of inversing as a procedure for solving open sentences. It would seem that we should be encouraging greater use of symmetric procedures such as substitution. The presence of this overgeneralized right-to-left mechanism also suggests that the use of the "Think of a Number" game ("I am thinking of a number; if you multiply it by 4 and add 7, the result is 19. What is the number?") as a way of motivating the solving of multi-operation equations may be misguided. It is simply non-generalizable to equations of the two above types; yet it seems to encourage overgeneralization.

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THE INFORMATION PROCESSING DEMANDS OF ENUMERATION; ABLE AND DISABLED ARITHMETIC LEARNERS

JOHN MUNRO

The skills to encode individual quantities, and to enumerate a quantity are essential foundation skills in the construction of mathematics knowledge. Recent analyses have examined developmental trends in the enumeration performance of preschoolers (Gelman & Gallistel, 1978; Schaeffer, Eggleston, and Scott, 1974; Siegler & Robinson, 1982. The present study examines the information processing strategies used by primary-school children to enumerate quantities. The strategies used by able arithmetic learners are used as a basis for analysis of the strategic behaviour of arithmetically disabled learners.

In the learning situation, the pupil is required under two conditions: the latency condition (Klahr, 1973), when the quantity is visable for as long as necessary to permit its enumeration, using physical or environmental enumeration strategies (Mandler & Shebo, 1982) and the threshold condition, when the quantity is visible for a brief duration and a representation needs to be encoded in short-term working memory, in order that it be enumerated using mental enumeration strategies (Mandler et al, 1982). The two conditions in terms of the demands made on information processing strategies. The present study examines performance under each condition.

Enumeration under the latency condition

For conditions in which the numeration time was unlimited, the relationship between the numerosity of a quantity and the time taken for its enumeration has been described by an approximately linear function, with a discontinuity in the gradient at a numerosity value x (Chi & Klahr, 1975), Mandler et al, 1982; Klahr, 1973; Jensen & Reese, 1950). The gradient of the function for arrays of less than $(x+1)$ is generally assumed to be less than the gradient for arrays larger than x . The discontinuity in gradient for adult performance has been reported to occur at numerosity values of three (Chi et al, Mandler et al) four (Klahr, 1973) and five (Jensen et al). Re-analysis of the data provided by Klahr and Jensen et al suggests a discontinuity at a numerosity of three. On the basis of these differences in enumeration rate, two alternative enumeration processes were proposed; a rapid enumeration strategy for numerosities of less than four items, and a slower counting strategy for numerosities exceeding three (Chi et al, 1975; Klahr, 1973).

Rapid enumeration strategies

The rapid enumeration strategy has been known for over a century; Jevons (1871) reported that he could enumerate arrays of up to four items rapidly and accurately. The term "subitizing" has been used by some investigators to describe this phenomenon, although the term was first coined by Kaufman, Reese, Reese & Volkmann, (1949) to describe rapid enumeration under the brief exposure condition, rather than the unlimited duration exposure condition. The present study prefers not to use this term for the rapid enumeration phenomenon under the unlimited condition. Mandler et al's (1982) study supports this preference; these investigators showed that the accurate and rapid enumeration process for brief durations consisted of two components, one that corresponded to the rapid enumeration phenomenon for unlimited exposures, and one that corresponded to the slower counting strategy. The present study uses the term "rapid enumeration strategies" to label the rapid enumeration phenomenon under the latency condition.

What is the nature of the rapid enumeration strategy? Earlier studies have shown that the time taken for rapid enumeration increases with numerosity, albeit at a lower rate than for arrays of numerosity exceeding three. The reported gradient is of the order .05 to .1 seconds per item. This finding suggests a mechanism that involves attending to the number of items available. The mechanism has not been further elucidated.

Counting strategies. When rapid enumeration strategies are not appropriate, the enumerater initiates either the use of counting strategies, or estimation strategies. The counting strategies used while the quantity is visible are termed "environmental" or "physical" counting (Mandler et al, 1982). The counting strategy involves the co-ordination of processes that tag each item, and that generate the sequence of number names (Gelman et al, 1978). Perceptual properties of the array have been shown to affect the enumeration time (Beckwith & Restle, 1966), leading to the proposition that in counting the enumerater may segment the array into small groups, enumerate each group and compute a total either by maintaining a running total or summing across groups at the end (Klahr, 1973; Chi et al, 1975).

Children's performance. The numerosity - enumeration time relationship for children has been shown to be essentially similar to that observed for adult subjects; a linear function with a change in gradient at a numerosity value x in the range between three and four (Chi et al, 1975; Svenson & Sjoberg, 1978). The gradient values for the two portions of the relationship were much larger than for corresponding adult performance. The findings were interpreted as suggesting that children, like adults, use two alternative processes, with an almost errorless rapid enumeration phenomenon for numerosities below four operating about six times more rapidly than the counting process for numerosities exceeding three. Chi et al suggest caution in assuming that the two processes used by children are necessarily identical to those used by adults.

Developmental trends in the acquisition of rapid enumeration strategies for arrays of less than four items has been demonstrated by children aged between three and five years (Gelman et al, 1978; Shaeffer et al, 1973; Cuneo, 1982). Gelman et al noted that counting low numerosities developmentally preceded their rapid enumeration, and used this observation to argue against the notion that the rapid enumeration strategy is based on a low-level perceptual mechanism as suggested by Shaeffer et al. Developmental trends in the acquisition of counting strategies have been reported by several investigators (Gelman et al, 1978; Shaeffer et al, 1973). These investigators generally accounted for trends towards correct enumeration performance, and explanations for the nature of counting errors made. They did not examine trends in the use of counting strategies that led to correct enumeration responses. Differences between adults and children in the nature of the counting strategies used, leading to the observation that nine-year olds count more slowly than adults, have not been empirically examined.

Predictions examined in the present experiment

Rapid enumeration strategies. The present experiment assumes that rapid enumeration strategies are mediated by a cognitive process that permits the enumerater to handle up to 3 numerosity units at a time without tagging each unit with a number. This assumption leads to two predictions:

(1) rapid enumeration is not a function of the perceptual properties of the array, and

(2) arrays consisting of less than four numerosity units where each unit consists of two items, elicit rapid enumeration strategies.

The first prediction derives from the assumption that the cognitive process underlying rapid numeration, unlike that mediating counting, does not involve the subjective organization of the array into smaller numerosity groups. Up to three numerosity units can be encoded by a process that involves noting the presence of each item, but not actually tagging each with a number name. The time taken to enumerate numerosities of less than four identical items is predicted not to depend on the extent to which the arrangement of the items facilitates segmentation.

The second prediction proposes that, because the underlying process is cognitive rather than low-level perceptual, a "numerosity unit" can consist of more than one item. An arrangement of three pairs of items, for example, is predicted to elicit rapid enumeration strategies. An arrangement of two pairs of items, and an additional item (that is, a numerosity of five) is predicted to be enumerated in a shorter time than that taken to enumerate a numerosity of five items not so organized. Further, an array of three pairs of items is predicted to be enumerated in a shorter time than an array of two pairs and one item. The rapid enumeration process is assumed to involve a subvocal enumeration process. The two pairs and one item involves two different numerosity units, and in order to enumerate it, one must switch from an enumeration code of two, to a code of one. This switching of codes is not necessary for the three pairs. It requires additional time.

Counting strategies. The present study examines the assumption that, for numerosities not eliciting the use of rapid enumeration strategies, the enumerater has access to a repetoire of alternative counting strategies, and that the selection of any strategy depends on the extent to which the enumerater can subjectively segment the array. An array of items can be enumerated by tagging each item with a number name in a serial manner with number names allotted to individual items at a regular rate. Alternatively, the array may be enumerated by segmenting it into groups of numerosity two or three, and rapidly enumerating each group, progressively maintaining a running total. This type of enumeration performance involves processing more than one item at a time.

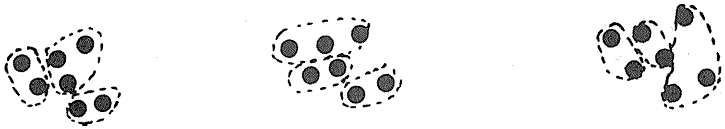
The extent to which an array can be segmented into groups of smaller numerosity will depend on the extent to which items can be seen as distinctive or different from other items. For arrays consisting of identical items, items will be discriminated only in their spatial properties; the spatial arrangement of the items relative to each other will affect the extent to which the array can be segmented. Three spatial parameters are identified as relevant; differences in inter-item distance, and the extent of alignment differences, both in the horizontal and vertical dimensions. Three arrangements or formats for a numerosity of seven are shown in Figure 1.

Figure 1: Three arrangements for a numerosity of seven

These arrangements are assumed to differ in the extent to which each facilitates segmentation. Items in format (a) differ only in the distance from a vertical axis; differences in inter-item distance, and in horizontal alignment, are removed. Items in format (b) differ in inter-item distance and in alignment from both axes. Each item in format (b) is, therefore, more easily distinguished from other items on the basis of spatial properties. This format is, it is proposed, more easily segmented into groups of two or three, and therefore enumerated more efficiently and rapidly than format (a). Further, it is predicted that the extent of spatial difference between items in an array is related to the ease with which the array can be partitioned into the categories of "already tagged" and "to be tagged" items as described by Gelman et al, (1978). Format (a) is predicted to be more likely to elicit serial counting strategies than (b), with each item tagged as a discrete, individual unit. Format (b) is more likely to elicit segmentation strategies.

Format (c) in which the items have been organized in a regular, recurrent, numerosity pattern, involves a property of redundancy not present in the other formats. The prior organization or grouping of the items removes the need for the learner to impose a subjective grouping. The learner who is able to recognize the recurrent pattern, and who can access the appropriate enumeration code, can then increment in steps equal to the numerosity of each group.

Developmental trends in counting strategies. The present experiment proposes a developmental trend in the acquisition of counting strategies. Prior to the acquisition of segmentation skills, the pupil uses serial counting strategies, separately tagging each item and assigning tags at a constant rate. The acquisition of segmentation skills permit the pupil to enumerate in increments of more than one; he organizes the items into groups of numerosity two or three, enumerates each group, and maintains a running total. Initially, it is proposed that the learner is unable to use rapid enumeration strategies to enumerate each group, but rather, tags each item in each group. This stage is seen as a prerequisite to the use of rapid enumeration of each group. Just as Gelman and Tucker (1974) observed that serial counting strategies preceded the use of rapid enumeration strategies for numeraties of two or three, the present study proposes that the child, having decided to segment the array into groups of small numerosity, uses serial tagging strategies to enumerate each group, prior to using rapid enumeration strategies. As suggested by Beckwithe et al, (1966), tags are assigned at a faster rate than that observed when the pupil does not use segmentation skills. A possible reason for this faster assignation of tags may be because the child needs to assign no more than three tags at a time, before encoding the intermediate total and then commencing the next group. The next stage is characterized by the use of segmentation and rapid enumeration processes; the pupil organizes the items into groups of numerosity two or three, rapidly enumerates each, and maintains a running total. Obviously, some arrays can be segmented in several different ways, as shown for a numerosity of seven in Figure 2.



(a) (b) (c)

Figure 2: Various ways of segmenting an array of seven items.

Some organizations of the data may be expected to be enumerated more rapidly and efficiently than others. An aspect of development would involve learning to organize the data as efficiently as possible. It is also likely, in the course of development, that a child may decide to segment an array, and then use both serial tagging and rapid enumeration processes, in enumerating the segments subjectively partitioned from the array. A related aspect of development would involve learning how to apply segmentation processing skills to arrangements in which differences in spatial properties are minimized. Initially, the child may decide to use segmentation strategies only for arrangements in which items differ spatially in vertical alignment. Later, he may learn how to segment arrays that differ only in one spatial property. The present study examines empirically this developmental model of enumeration skill acquisition, and uses the model to analyze the enumeration performance of arithmetic-disabled learners.

The performance of disabled arithmetic learners

Rapid enumeration performance of disabled learners. Mathematics disabled learners have frequently been characterized as having difficulty counting meaningfully; while they can recall number names in note fashion, they have difficulty co-ordinating the recall of number names and their assignation (Johnson, 1979; Bartel, 1975). Inspection of the anecdotal data provided by Bartel suggests that disabled arithmetic learners are able to accurately enumerate arrays of up to five items. This observation is consistent with the accuracy property of rapid enumeration strategies. No reference is, unfortunately, made to the comparative rate of enumeration.

Given that these learners can accurately enumerate small numerosities, it is possible that they do so using inferior or less efficient processing strategies; they may, for example, be more likely to use serial counting strategies. As noted, serial counting strategies are not based on a segmentation process, and the time taken to enumerate using serial counting is proposed to be relatively insensitive to the spatial arrangement or configuration of items, as long as the quantity is within foveal view. It is predicted, therefore that the enumeration performance of numeration of less than four will be independent of the spatial arrangement of items, and that disabled arithmetic learners will take longer to enumerate these numerosities than corresponding able learners.

The use of counting strategies by disabled learners. As noted, many mathematics-disabled learners exhibit a difficulty co-ordinating the recall of number names, and their allocation to items, in terms of Gelman et al's, (1978) framework, the child has difficulty co-ordinating the tagging and partitioning processes; the child had not co-ordinated the assignment of number names with the transfer of items from the "to be counted" category to the "already counted" category. Individual items in the array are defined by relative spatial location. Arithmetic-disabled children frequently have difficulty both with processing this type of visuo-spatial data, and encoding visuo-spatial data in short-term working memory. One may expect that the arithmetic-disabled child is less able to accurately recall the items that he had already tagged; he will exhibit the types of errors observed by Gelman et al (1978) to be made by younger, able learners in the course of enumeration skill acquisition.

In addition, arithmetic-disabled learners may be expected to use less efficient, though correct, enumeration strategies. They may be less likely to subjectively organize the data, and to segment it into groups of numerosity two or three. The observation that many mathematics-disabled learners are unable to visualize clusters of groups within a larger group (Johnson & Myklebust, 1969) or to perform visual closure skills (Ceci & Peters, 1980) support this expectation.

Given these information-processing difficulties, it is likely that the arithmetic-disabled learner enumerates an array by processing each item independently and separately, rather than processing groups of items at once.

While the arithmetic-disabled learner may not spontaneously use subjective organizational strategies, he may be able to enumerate in increments of more than one in contexts in which the items have been organized into groups of a regular numerosity prior to presentation. This expectation is not different from the finding that while mentally-retarded subjects have difficulty subjectively organizing material in a short-term memory task, they can use organizational features provided by the experiments to facilitate recall. Thus, given that the arithmetic-disabled learner can recognize the recurrent numerosity pattern, for example, four pairs of items, it is predicted that the learner can use this repetitive numerosity pattern to initiate enumeration in increments of two. It is also predicted that the arithmetic-disabled learner is more likely to exhibit enumeration performance characteristic of preservation (Homan, 1970) and to enumerate an array consisting of two numerosity groups, for example three pairs and a single item, as if it consisted entirely of pairs; the arithmetic-disabled learner is more likely to exhibit difficulty changing from one enumeration code to another.

METHOD

The present study examined the time taken by able and disabled arithmetic learners to enumerate correctly arrays of up to 15 identical counters arranged in three formats or contexts.

The subjects

The subjects of the experiment were 60 able and 60 disabled arithmetic learners, with 20 of each type of learner at each of the fourth, fifth and sixth grade levels; at primary schools in the Eastern suburbs of Melbourne. The rationale for the selection of the able and disabled learners is described in Munro (1984). In summary the populations from which the disabled and able arithmetic learners were randomly chosen, were selected as follows:

(i) performance on the AM4 IV standardized arithmetic test (performance in stanines 1 or 2, and stanine 5 for disabled and able learners respectively)

(ii) past history of arithmetic performance (rated either as disabled or able arithmetic performers by their teacher)

(iii) average general ability, as indicated by the Tests of Learning Ability 4 (A.C.E.R., 1977), and

(iv) no evidence of physical, mental, sensory or emotional handicap, or environmental or cultural factors likely to affect learning, including prolonged non-attendance at school.

The 20 disabled learners at each grade level were randomly selected from the population of disabled arithmetic learners with approximately equal numbers of males and females. For each disabled learner, an able learner who had been in the same learning environmental as the disabled learner for the past year, had a chronological age of within $\pm$ 3 months of the disabled child, was of the same gender, and had a general ability score of within $\pm$ 2 percentile points of the disabled child's general ability. The median general ability score, gender ratio, and median chronological age for the twenty children in each group is shown in Table 1.

Table 1

Median general ability score, gender ratio, and median chronological age for each category of subject involved in the present experiments.

Characteristic	Able Arithmetic Learner			Disabled Arithmetic Learner		
	Grade 4	Grade 5	Grade 6	Grade 4	Grade 5	Grade 6
Median general ability score	62	64	65	59	60	64
Median chronological age (year, months)	9-4	10-5	11-7	9-6	10-8	11-6
Ratio of females to males	9:11	1:1	11:9	9:11	1:1	11:9

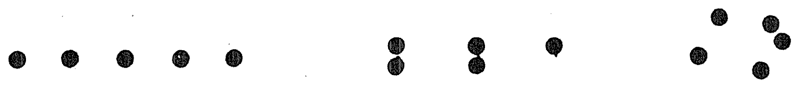
Design

Four independent variables and two dependant variables were examined. The independent variables were grade level (3 levels; fourth, fifth and sixth), type of arithmetic learner (2 levels; able arithmetic learner, disabled arithmetic learner), the numerosity of the quantity to be enumerated, (15 levels), and the context or arrangement of the items to be enumerated (3 levels). In all, 45 experimental stimuli were generated. All stimuli were administered twice to each subject, in an order of presentation determined for that subject.

The Material

The quantities to be enumerated were presented using a slide projector (Kodak Carosel SAV 200). The time taken to enumerate the array was measured using an electronic timer capable of .001 second discrimination (Astol A 1030) a data printer (DPP7E1) and a touch key. In addition a tape recorder was used to record the child's response and to ensure that the touch key was actuated during the child's response.

The quantities to be enumerated consisted of up to 15 identical red counters, arranged either in a linear, a scrambled, or a grouped context. For all stimuli, the visual area of display was 90 x 86 cm^2, and each counter had a radius of 1.98 cm. The linear context was generated by centering the items on a straight line that bisected horizontally the display field. The central point of the array co-incided with the centre of the visual field. All item centres were 5.00 cm. apart. The scrambled context was generated by randomly locating the items within a circle of radius 30 cm. centred on the visual field. The grouped context consisted of pairs of counters, the distance between the centres of the items comprising each pair being 4.00 cm, and the centres of each pair lying on a vertical line. For the grouped arrays of odd numerosity, the un-paid item appeared in the extreme right position, its centre horizontally aligned with the centres of the upper member of each pair. An example of each context, for a numerosity of five is shown in Figure 3.



(a) linear (b) grouped (c) scrambled

Figure 3: An example of each context, for a numerosity of five

In this experiment the child was located so that her or his eyes were 2.20 m from the screen, and were in a horizontal plane with the horizontal bisector of the visual area of display, centred on the visual display. The angle subtended by each counter with the subject's eye was 0.52^{o}, and the largest angle subtended by an array, the array of numerosity twelve in the linear contexts, was 15.98^{o}.

Procedure

Each child was briefed on the task, and familiarized with the equipment and task. A practice session followed the initial exposure. A set of 18 quantity slides, of which 6 were randomly selected from each context was used. For each stimulus the child sat facing the screen 1.98 m away, his head resting on the chin-rest. The child was instructed to look straight ahead at the screen, to keep his head still, and to state the number of items in the array, correctly and as rapidly as possible. The work "count" was carefully avoided. The experimenter signalled the onset of the stimulus by saying "Ready". In addition, in every situation in which the presentation of a set of stimuli was beginning or re-commencing, a stimulus consisting of a centrally-placed cross (X) was shown to assist the subject to attend to the centre of the visual field. At the instant of the child's verbal response, the experimeter pressed the touch key, deactivating the timer. In addition, the quantity was withdrawn from view. The child's response was recorded. Following every second stimulus, the child was asked to describe how he enumerated the array. The practice session terminated after the child had demonstrated enumeration performance over a run of at least ten stimuli that both he and the experimenter judged as satisfactory, in terms of familiarity and understanding of the task conditions and demands.

The experimental stimuli were administered in a random order for each subject, over several consecutive school days. Altogether 90 stimuli were administered, with a block of fifteen presented to each subject on each day. The order of stimulus presentation was individually determined for each child prior to task administration, as follows. Each stimulus was assigned a number from one to ninety. These numbers were randomly sequenced for each subject, and the stimuli administered in that order with the restriction that identical arrays were not consecutive. Each block of stimuli was presented in the physical context and under the instructional conditions used for the practice session. The child was instructed to say the number of counters shown, correctly and as quickly as possible without moving his head. The centrally-placed cross was used to direct the child's vision to the centre of the visual display at the beginning of each test session, and following interruptions. The experimenter said "Ready" prior to the onset of each stimulus. The audio cassette recorder was activated at the beginning of each test session and was used to ensure that the timer was deactivated at the instant of the child's enumeration response. The child's enumeration response for each stimulus was recorded by the experimenters. A rest pause of at least 20 seconds followed each response.

In addition, immediately after the enumeration response for every fourth stimulus, the child was asked to describe how she or he "went about working out" the number of counters in the array. The reports were organized into a series of categories;

(1) the use of serial counting strategies; the child overtly tagged each item at a constant rate,
(2) the use of rapid enumeration strategies; the child ascertained the numerosity accurately and rapidly without tagging,
(3) the use of multiple-incremental counting strategies; the child regularly counted in multiples of two or three, for example two, four, six ...",
(4) the use of segmentation and tagging strategies; the child enumerated one part of the array by tagging, and then enumerated a second part, etc.,
(5) the use of segmentation and rapid enumeration strategies; the child rapidly enumerated one part, then incremented by the numerosity of a second part, etc. for example "three, five, eight, nine",
(6) the child alternated between (1) and (4), and
(7) other strategies.

The performance of able learners are discussed in the following section.

RESULTS AND DISCUSSION

The enumeration performance of able arithmetic learners

The trend in the time taken to enumerate correctly (seconds) with numerosity, for each condition of context and grade level is shown in Figure 4.

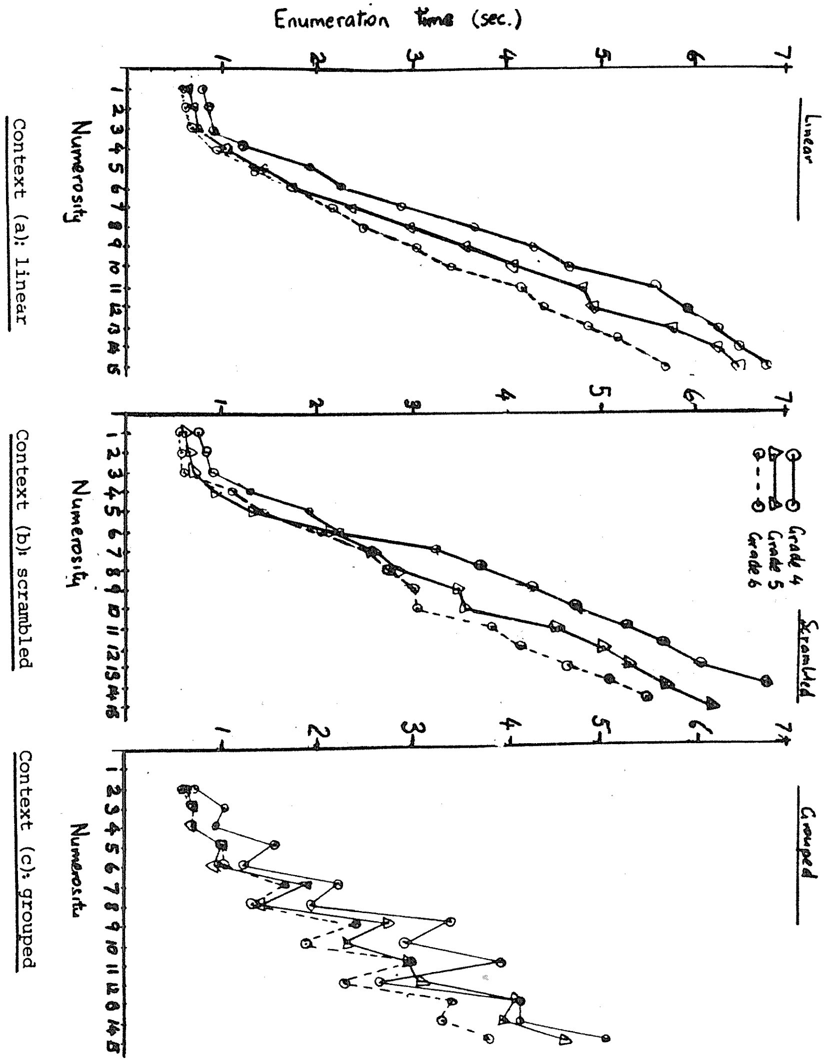


Figure 4: trend in enumeration time with numerosity for each context and grade level

As shown, a consistent trend emerged for enumeration in each context, at the fourth, fifth and sixth grade levels, with enumeration time gradually increasing with numerosity. However, while the linear and scrambled contexts show the characteristic approximately linear curves previously described by Klahr (1973), Chi et al (1975), and Svenson et al (1978), the relationship for the grouped contexts showed a different trend; the mean enumeration time for arrays of numerosity n where n was even, was less than the time taken for arrays of odd numerosity (n - 1), for the numerosity, range $3 < n \leq 16$. This finding conflicts with the claim by Chi et al (1975) that "it always takes longer to immediately apprehend" n + 1 items then n items, for all values of n " (page, 434). Performance in the grouped contexts is discussed in a subsequent section.

The numerosity-time relationship for the linear and scrabled contexts. The numerosity-time functions for the linear and scrambled contexts show the characteristic discontinuity in gradient in the numerosity range $3 < n < 4$ previously by Klahr, (1973), Chi et al, (1975), Svenson et al, (1978). The least-squares fit simple regression procedure (Nie et al, 1975) was used to fit a linear function to the relationship for the ranges of numerosity $n < 4$ and $3 < n < 16$. The overall test of goodness of fit of the regression equation, (F_{reg}) for each numerosity range, was significant, ($p < .01$) for each condition of context and grade level (Munro, 1984). The gradient and intercept for each numerosity range in each context is shown in Table 2.

Table 2

Gradient and intercept for the numerosity ranges $n < 4$ and $n > 3$ for able mathematics learners

Context	Intercept			Gradient		
	Grade 4	Grade 5	Grade 6	Grade 4	Grade 5	Grade 6
Numerosity range 4						
linear	0.72	0.59	0.55	0.07	.06	.04
grouped	0.10	0.51	0.39	0.18	.08	.11
scrambled	0.69	0.47	0.63	0.06	.08	.03
Numerosity range 3 16						
linear	-1.26	-1.33	-0.96	0.61	0.53	0.45
grouped	-0.72	-0.64	-0.41	0.40	0.33	0.29
scrambled	-0.64	-0.90	-0.58	0.50	0.48	0.42

These values of gradient and intercept are compatible with those obtained by earlier investigators; the present gradients are between those recorded for five and nine - yearolds (Chi et al, 1975; Svenson et al, 1978), and those recorded for adults (Chi et al, 1975; Klahr, 1973), for both ranges of numerosity.

Inspection of these finding suggests that the gradient of the numerosity-time relationship for the numerosity range $n < 4$ is less than the corresponding gradient for the range $3 < n < 16$. This difference was confirmed for the linear and scrambled contexts using the procedure recommended by Howell (1982) ($p < .05$). The two-tailed t-value for each condition of context and grade level for able arithmetic-learners is specified in Munro (1984).

The numerosity-time relationship for the grouped context. The nature of this relationship for the grouped context differs from that for the other contexts in that the mean time taken to enumerate arrays of numerosity n where n is even, is less than the time taken to enumerate arrays of numerosity ($n-1$). The question of whether the learner processes each numerosity group as a single enumeration unit, and showed rapid enumeration performance for numerosities of one, two or three pairs was investigated by examining the numerosity-time relationship for even and odd numerosities separately, in each of the grouped context. This relationship for each grade level is shown in Figure 5.

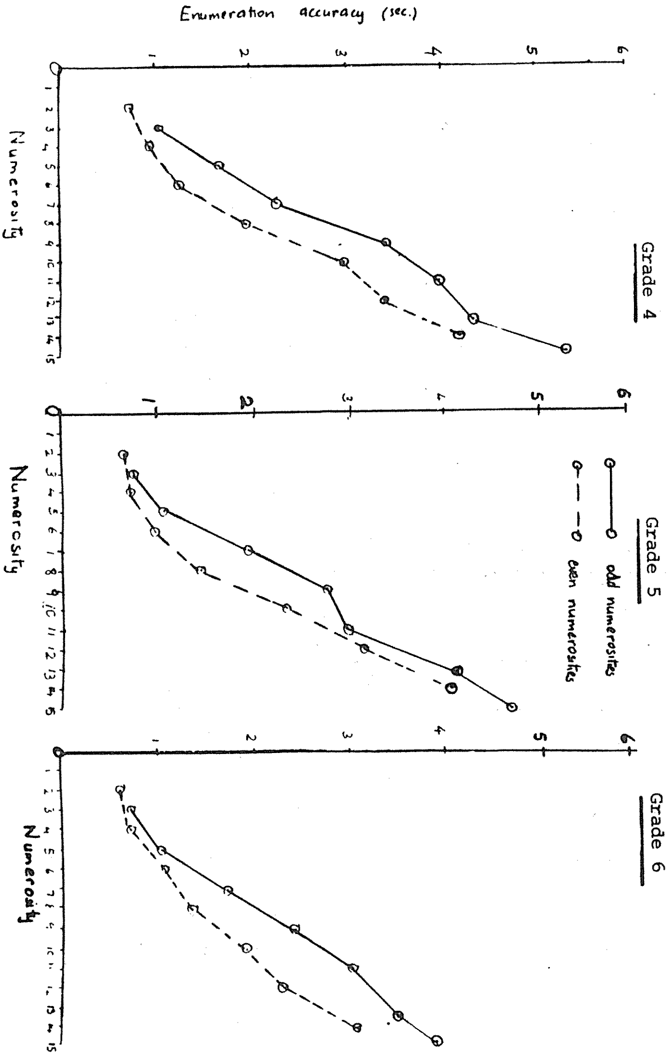


Figure 5: the numerosity-time relationship for the grouped context

Inspection of the numerosity-time relationships for even and odd numerosities indicates a consistent finding. The relationship for even numerosities indicates a discontinuity in the numerosity range $4 < n < 6$, with the relationship for numerosities of $n \geq 6$ parallel to the function for odd numerosities. The relationship for odd numerosities did not reveal a discontinuity for fourth grade performance. At the fifth and sixth grade levels, a slight discontinuity was suggested in the numerosity range $3 < n < 7$. This discontinuity is not as well pronounced as that for the corresponding even numerosities, and is not, therefore treated as a major discontinuity.

The simple regression procedure was used to fit a linear function to the relationship for each range of even numerosity and for the entire range of odd numerosities. The F_{reg} for each range emerged as signifant ($p < .01$). The gradient and intercept for the odd, and even numerosities, for able learners is shown in Table 3.

Table 3

The gradient and intercept for the odd, and even numerosity values for average learners enumerating in the grouped context.

Numerosity values	Gradient			Intercept		
	Grade 4	Grade 5	Grade 6	Grade 4	Grade 5	Grade 6
n = 3,5,7,9,11	0.37	.31	.30	-0.19	-0.26	-0.31
n = 2,4	0.12	0.04	.05	0.51	0.63	0.50
n = 6,8,10,12	0.32	0.35	0.20	-0.56	- .63	-0.22

These findings reveal a similarity between the numerosity-time relationship for even numerosities in the grouped context, odd numerosities in the grouped context and the relationship for the linear and scrambled context. First, the gradient for even numerosities of less than six in the grouped context is similar to that for numerosities of less than four in the linear and scrambled contexts. Second, the gradient of the numerosity-time relationship for even numerosities larger than four in the grouped context did not generally differ from the corresponding gradient for odd numerosities (two-tailed t-test, $p < .05$). Third, the gradient of the numerosity-time relationship for even numerosities larger than four and for odd numerosities in the grouped context is approximately one-half of the corresponding gradients in the linear and scrambled contexts. Fourth, the difference in gradient between the components of the numerosity-time relationship for even numerosities was significant for fifth and sixth grade learners. (Munro, 1984).

The reciprocal of the gradient of the numerosity-time relationship in each condition provides an estimate of the rate of enumeration. Comparison of the enumeration rate in the grouped context with that in the linear and scrambled contexts suggest that the children processed each pair of items in the grouped context as an enumeration unit. Arrays of items in the grouped context for even numerosities larger than six, and for odd numersities, were enumerated at approximately twice the rate of enumeration for the linear and scrambled contexts, while odd and even grouped numerosity arrays were enumerated at the same rate. The additional time required to enumerate the odd numerosities, indicated as a vertical displacement between the graphs for odd and even numerosities, provides an estimate of the time taken to switch enumeration codes, from enumerating in increments of two to increments of one. The difference in gradient between the two portions of the numerosity-time relationship for the even-grouped numerosities suggests the use of alternative enumeration strategies; a rapid enumeration process for even numerosities of less than three pairs of items, and a slower enumeration process for odd numerosities, and for even numerosities of more than two pairs. The rate of rapid enumeration for the even grouped contexts was similar to that for the linear and scrambled contexts, suggesting that repetitive numerosity units of more than one item elicit rapid enumeration strategies in the same way as do individual items.

Two questions emerge from these observations; (1) Why are the rapid enumeration strategies not elicited by even numerosities of three pairs? and (2) Why are the rapid enumeration strategies not elicited by odd numerosities of less than three pairs? Possible explanations are based on developmental effects. First, the mean enumeration times for three pairs of items, recorded in the present study is due, it would seem, to the use of the two alternative enumeration strategies. The subjective reporting of the strategies used by subjects and discussed in a subsequent section indicated that while some subjects maintained that they could "tell immediately" the numerosity of three pairs, others used counting strategies. Thus, just as serial counting was proposed to precede the use of the rapid enumeration process for up to three items (Gelman et al, 1978), so may serial counting in increments of two precede the rapid enumeration of numerosity-pairs. The observed nonsignificant difference in enumeration rate between arrays of less than three pairs and arrays of more than two pairs, in the grouped context at the fourth grade level suggests that arrays in both ranges of numerosity were enumerated using the same type of enumeration process by most fourth-graders. Similarly, the lack of support for the widespread use of rapid enumeration strategies for arrays of three pairs by the oldest subjects involved may be a second aspect of this developmental trend.

A possible explanation for the second question is also based on developmental factors. The graphs of the numerosity-time function for odd numerosities shown in Figure 5 provide evidence of an emerging discontinuity in gradient between the numerosity ranges $n < 7$ and $n > 5$ for arrays in the grouped context. At the fourth grade level there is little evidence of discontinuity. With increase in grade level, the gradient of the numerosity-time function in the range of $n < 7$ becomes progressively smaller, suggesting that an increasing number of children at each grade level are using rapid enumeration strategies to enumerate odd numerosities in grouped contexts.

The findings indicate that able arithmetic learners at the fourth, fifth and sixth grade levels enumerate arrays consisting of a recurrent numerosity group of more than one item by processing each numerosity group as a unit, and incrementing in multiples of the numerosity group. Even numerosities of more than two pairs of items, and odd numerosities were enumerated at a similar rate. The observation that even numerosities of n items were enumerated more rapidly than odd numerosities of $(n - 1)$ items was explained in terms of the time needed to switch enumeration codes. The findings also indicated developmental trends in the use of rapid enumeration strategies for groups of less than three pairs of items.

<u>Developmental trends in the application of counting strategies</u>. These trends were characterized by an increase in the rate of enumeration that is attributed in turn to changes in the ways in which the items are organized during the enumeration process. Variation in the rate of enumeration with grade level for each context, for numerosities in the range $3 < n < 16$ was investigated by analysis of the decrease in gradient with grade level, as described by Howell (1982). The two-tailed t-value for each difference indicated that fifth graders enumerated grouped and scrambled contexts more rapidly than fourth graders ($p < .05$), while sixth graders enumerated linear and scrambled contexts more rapidly than fifth graders ($p < .01$). This finding was interpreted as indicating that the fifth graders enumerated more rapidly than the fourth grade children in those contexts in which it is either easier to subjectively segment or in which the items are maximally organized prior to exposure. The sixth graders, having acquired additional subjective organizational skills, counted more rapidly than fifth graders in those contexts in which the organization of quantitative data was not as easy.

The effect of context on enumeration time. The influence of context and numerosity on the mean enumeration time, for the ranges of numerosity $n < 4$ and $3 < n < 16$ was examined using a repeated measure ANOVA design, with repeated measures on the context and numerosity. The F-ratio for each main and interaction effect is shown in Table 4.

Table 4

The analysis of variance of enumeration time for correct enumeration, for able arithmetic learners, for each range of numerosity at each grade level.

Numerosity range $n < 4$			
Main effects			
Context (C)	F(2,186) = 0.81	F(2,244) = 0.83	F(2,210) = 1.06
Numerosity (N)	F(2,186) = 13.38**	F(2,244) = 14.31**	F(2,210) = 8.61**
Interaction effect			
C x N	F(4,186) = 0.34	F(4,244) = 1.04	F(4,210) = 0.97
		Mean square term	
Residual	MS(186) = 0.31	MS(244) = .07	MS(210) = 0.01
Numerosity range $3 < n < 16$			
Main effects			
Context (C)	F(2,758) = 27.36**	F(2,1099) = 57.34**	F(2,868) = 63.41**
Numerosity (N)	F(11,758)=137.01**	F(11,1099)=234.31**	F(11,868)=234.56**
Interaction effect			
C x N	F(16,758 = 2.61**	F(16,1099)= 3.49**	F(16,868)= 3.47**
		Mean square team	
Residual	MS(758) = 1.03	MS(1099) = 0.64	MS(868) = 0.56

** $p < .01$

These findings indicate that both numerosity and context exert an effect on the time taken for correct enumeration; numerosity for both ranges of numerosity, and context for numerosities in the range $3 < n < 16$. The emergence of numerosity as a significant variable for the numerosity range $n < 4$ suggests at some point in the rapid enumeration process, the use of a mechanism that is sensitive to the number of items in the array. This process was not affected by changes in the spatial properties of the items in the array. The counting process, on the other hand, predicted to be used for the enumeration of arrays in the range $3 < n < 16$ involves mechanisms that were sensitive to both the number of items, and their spatial arrangement. These findings therefore, provide support for the predicted model of enumeration.

The trend in mean enumeration time with context for each range of numerosity is shown in Figure 6.

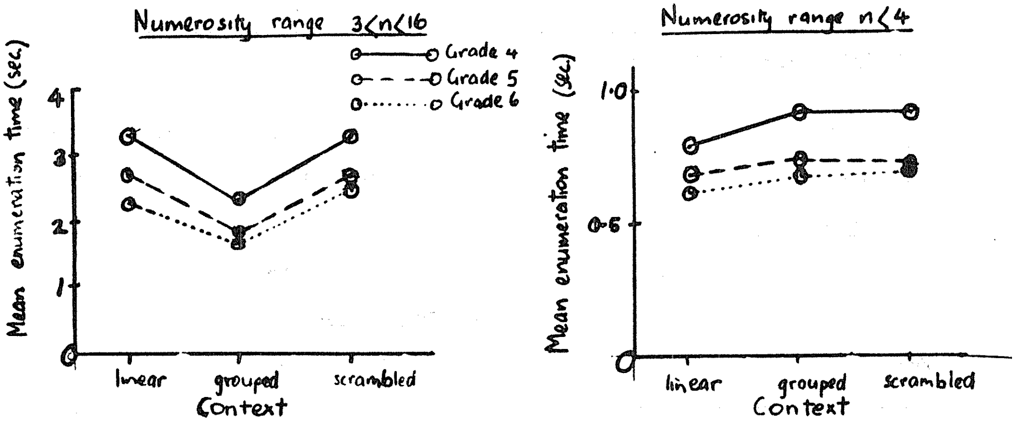


Figure 6: trends in enumeration time with context

These trends were examined using planned comparisons for repeated measures, indicating that for the numerosity range $n < 4$, the three contexts did not differ in enumeration time. For the range $3 < n < 16$ at the fourth grade level, the linear and scrambled contexts did not differ in enumeration time, ($p < .05$) and each task longer to enumerate than the grouped context ($p < .01$). At the fifth grade level the linear context took longest time, and the grouped context took shorter time ($p < .01$), and at the sixth grade level, a pattern similar to that recorded for fourth grade performance was observed.

The strategies reported to be used by children at each grade level assist in the interpretation of these results. The percentage of subjects reporting the use of each type of strategy in each context at each grade level is shown in Table 9.

These findings indicate that, for numerosities of less than four, children report strategies consistent with rapid enumeration. For numerosities larger than three the data suggest a developmental trend. At the fourth grade level, subjects used serial counting strategies more frequently than segmentation strategies, for the two non-grouped contexts. With increase in grade level the frequency of use of segmentation strategies increased first for scrambled arrangements. Further, the likelihood of the co-ordinated use of segmentation and rapid enumeration strategies was greatest at the sixth grade level.

Taken together, these findings support the proposed model of enumeration.
Rapid enumeration strategies. The time taken to enumerate numerosities of less than four items is not a function of the arrangement of items. Most subjects reported being able to "tell immediately" the numerosity of these arrays; they were aware that they did not need to count consciously, in order to correctly ascertain the numerosity. Their performance indicated an increase in enumeration time with numerosity.

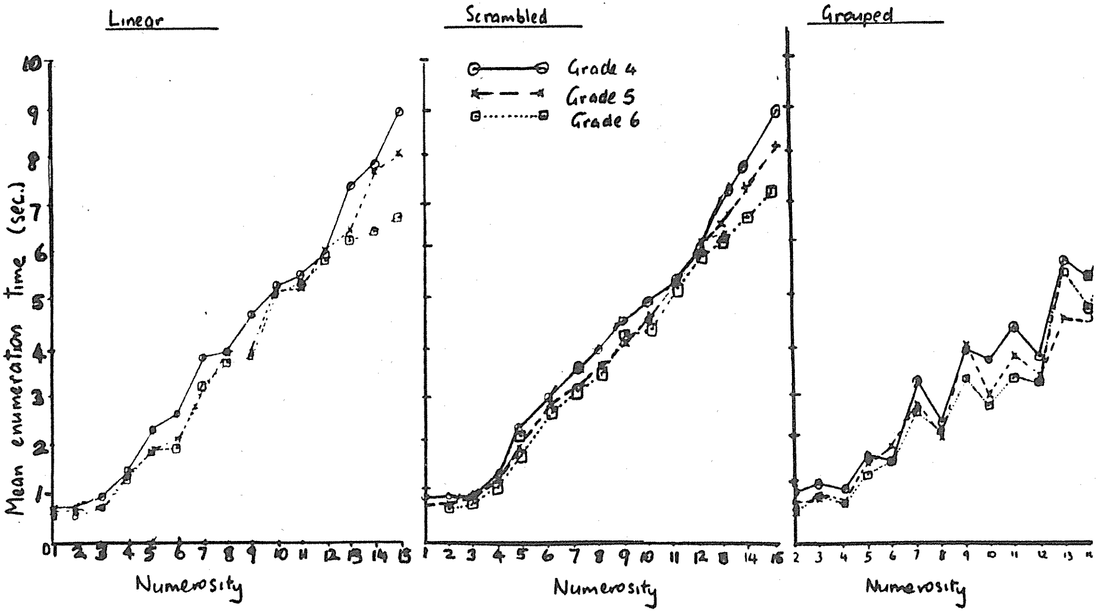
The enumeration of grouped quanties suggested the use of rapid enumeration strategies for numerosities of up to six, that is, three pairs. The "saw-tooth" curve for this context, with odd numerosities taking longer than the even numerosities larger by one supports the claim that the mechanism underlying the rapid enumeration strategy is sensitive to the numerosity of the array. The need to switch enumeration codes was proposed to account for the comparatively longer time taken by odd numerosities. The findings support the proposition by Gelman et al (1978) that the rapid enumeration phenomenon has a cognitive base. They do not support an implication from Mandler et al's (1982) canonical pattern recognition process that triangular representations of three are enumerated more rapidly than linear representations. They suggest that it is not a canonical pattern recognition process that mediates rapid enumeration, but rather a process that recognizes a recurrent numerosity pattern. To have a numerosity of five rapidly enumerated, it was not necessary that the items be arranged in the canonical pattern proposed by Mandler et al, as shown in Figure 7 (a). An arrangement of two pairs and one item as shown in Figure 7 (b) elicited rapid enumeration.

(a) Mandler's canonical pattern (b) The grouped pattern

Figure 7: Arrangements of numerosity five eliciting rapid enumeration

Counting strategies. The findings indicate a developmental trend towards the co-ordinated segmentation and rapid enumeration process proposed by Klahr (1973) and Chi et al (1975). Arrangements that were more likely to facilitate segmentation (the scrambled arrangement) elicited these strategies, and showed a corresponding decrease in enumeration time before those arrangements in which spatial differences were minimized. The latter arrangements continued to elicit serial counting strategies, that attend to each item at a time, rather than attending to groups of items at once. Further, children used a mechanism involving the co-ordination segmentation and serial tagging strategies, prior to the co-ordination of segmentation and rapid enumeration strategies.

The enumeration performance of disabled arithmetic learners.
The trend in the time taken to enumerate correctly (seconds) with numerosity, for each condition of context and grade level is shown in Figure 8.



Context (a): linear   Context (b): scrambled   Context (c): grouped

Figure 8: trend in enumeration time with numerosity for each context and grade level for disabled arithmetic learners.

These data indicate that the trend in enumeration time with numerosity for disabled learners is qualitatively similar to that observed for able learners; the approximately linear function for the linear and scrambled contexts, with a discontinuity in the gradient in the region $3 \leqslant n \leqslant 4$, and the saw-tooth function for the grouped contexts.

The linear characteristics of each relationship was analysed using the simple regression procedure (Nie et al, 1975). The F_{reg} for each condition was significant ($p < \cdot 01$). The gradient and intercept for each numerosity range is shown in Table 5.

Table 5

Gradient and intercept for the numerosity ranges $n < 4$, and for $n > 4$ for arithmetic-disabled learners.

Context	Intercept			Gradient		
	Grade 4	Grade 5	Grade 6	Grade 4	Grade 5	Grade 6
Numerosity range $n < 4$						
linear	0.86	0.75	0.61	0.10	0.08	.07
grouped	0.58	0.14	0.54	0.25	0.16	0.13
scrambled	0.83	0.73	0.62	0.11	0.07	0.07
Numerosity range $3 < n < 16$						
linear	-0.74	-1.30	-1.18	0.58	0.65	0.61
grouped	-0.69	-0.38	-0.34	0.46	0.42	0.39
scrambled	-0.56	- .46	-0.15	0.51	- .54	0.44

The gradient for the range $3 < n < 16$ was greater than that for the range $n < 4$ ($p < \cdot 01$) for the linear, and scrambled contexts, at all grade levels. These findings suggest that the disabled arithmetic learner, like the able learner, uses both rapid enumeration strategies and counting strategies, and is able to use the recurrent numerosity pattern in the grouped context to assist enumeration.

The difference in gradient between able and disabled learners for each condition was examined to indicate differences in enumeration rate. This analysis indicated the two types of learner did not differ in the rate of rapid enumeration ($p < \cdot 01$) for any condition and grade level. Able learners counted more rapidly than disabled learners for the following conditions; fifth graders enumerating scrambled arrangements, and sixth graders enumerating all contexts.

The enumeration performance of the grouped context by disabled learners was examined by analysing separately the trend in enumeration time with odd, and with even numerosities. The trend for each grade level is shown in Figure 9.

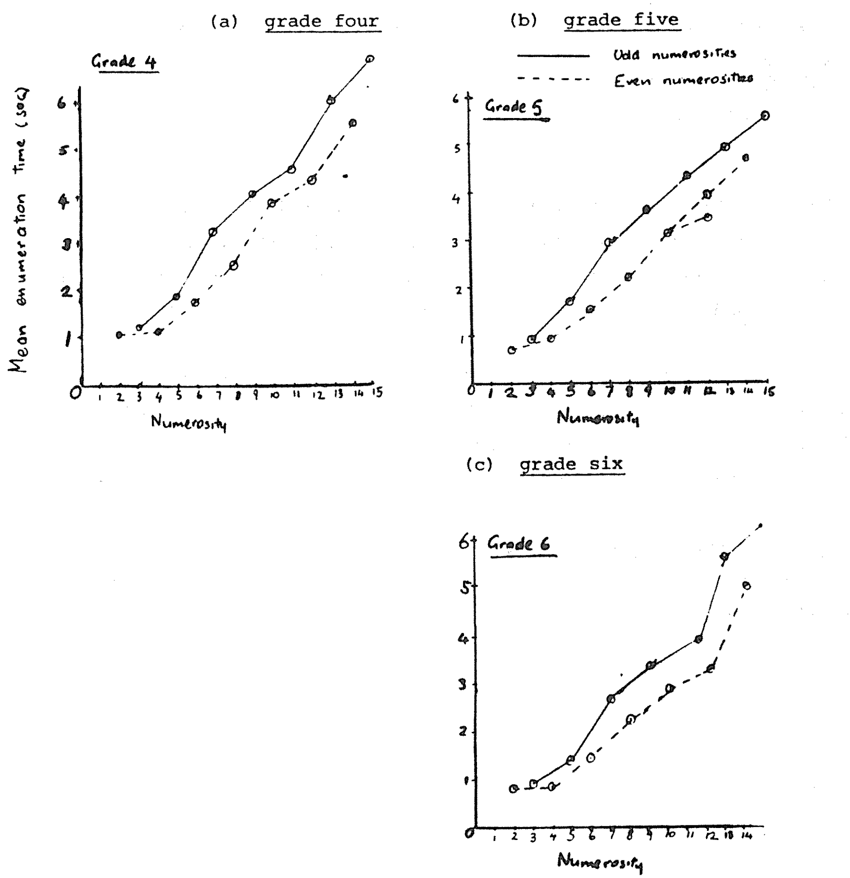


Figure 9. Variation in enumeration time with odd, and even numerosities

Inspection of these data, suggests a discontinuity for even numerosities in the range $4 \leq n \leq 6$. The simple linear regression coefficients for each set of numerosities is shown in Table 6.

Table 6

The gradient and intercept for the numerosity-time relationship, for disabled learners, for odd numerosities, for even numerosities $n < 6$, and for $6 < n < 12$.

Numerosity set	Gradient			Intercept		
	Grade 4	Grade 5	Grade 6	Grade 4	Grade 5	Grade 6
Grouped horizontal context						
n is odd and $n > 3$	0.46	0.38	0.44	0.20	0.74	-0.64
n is even and $n \leq 4$	0.14	0.09	0.08	0.70	0.49	0.46
n is even and $6 \leq n < 16$	0.44	0.42	0.38	- .88	- .93	- .72

The difference between the gradients for the odd numerosities and the even numerosities larger than four was not significant ($p > \cdot 05$), suggesting that, as with average mathematics learners, quantities consisting of either m pairs or (m-1) pairs and a single item, were enumerated at a similar rate.

The two components ($n < 6$ and $n > 6$) of the numerosity-time function for even numerosities differed in gradient for the fifth and sixth graders ($p < \cdot 05$). This finding suggests that disabled learners, like their able peers, use rapid enumeration strategies for grouped numerosities, consisting of one or two pairs.

The effect of context on enumeration time.

The trend in mean enumeration time with context for each range of numerosity is shown in Figure 9.

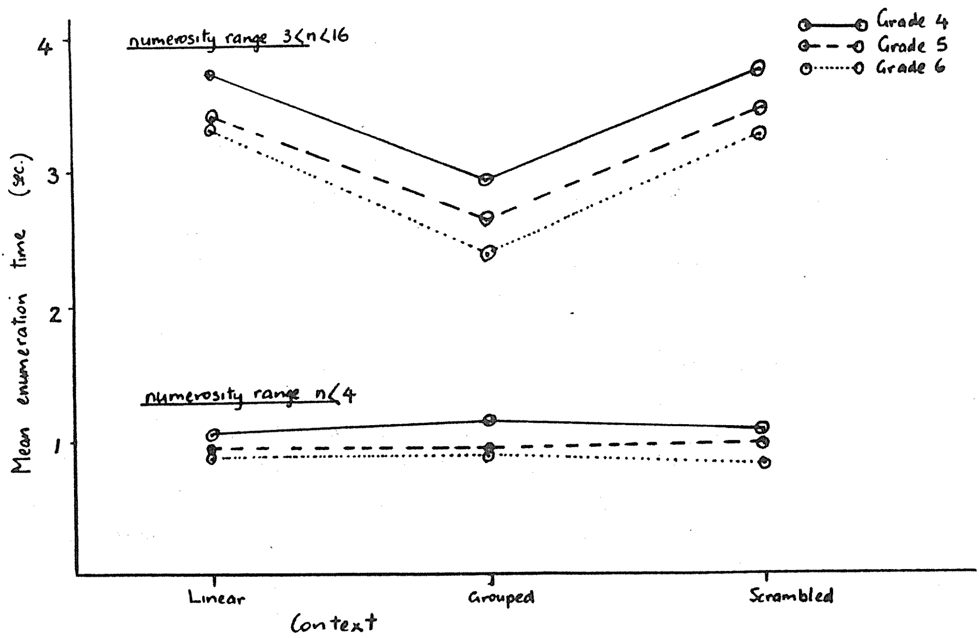


Figure 9: mean enumeration time with context

The ANOVA indicated that the effect exerted by context on enumeration time was not significant ($p > \cdot 05$) for the numerosity range $n < 4$, and was significant for the range $3 < n < 16$ ($p < \cdot 01$) at all grade levels. The planned comparisons procedure indicated that the linear and scrambled contexts did not differ in enumeration time, at all grade levels. The grouped arrangement was enumerated more rapidly than the linear and scrambled arrangements, at all grade levels ($p < 01$). As with the performance of able learners, these findings are interpreted in conjunction with the strategies reported to be used.
The percentage of subjects reporting the use of each type of strategy in each context at each grade level is shown in Table 7.

These data suggest that disabled learners, like their able peers, use rapid enumeration strategies for numerosities of less than four. They differ from their peers for numerosities requiring counting strategies. Whereas able learners report the increasing use of segmentation strategies, disabled learners at all grade levels used serial counting strategies for both the linear and scrambled contexts.

Summary

The present study provides an insight into the relationship between the acquisition of enumeration skills and various information processing skills, both for able and disabled arithmetic learners. The rapid enumeration phenomenon, exhibited both by able and disabled learners for numerosities of less than four items, was showed to have a cognitive base involving specialized enumeration processes that attended to the number of numerosity units presented. The use of counting strategies, for numerosities exceeding three, were shown to be mediated by different information processing strategies. Able learners increasingly shown a tendency to use a procedure involving the processing of more than one item at once; the child was increasingly likely to enumerate by segmenting an array, enumerating each segment, and maintaining a running total. Disabled arithmetic learners, on the other hand, used procedures that involved processing each item separately; they were less likely to spontaneously organize the array into segments, and to enumerate each segment. They were, however, able to use recurrent numerosity patterns to facilitate enumeration.

The study has direct implication for the arithmetic education of disabled learners. The findings indicate that these learners need to learn how enumerate arrays more efficiently; to segment an array, and to enumerate each segment, initially by using tagging strategies. These students may initially learn how to process these arrays using motoric processing strategies, physically segmenting an array, and imposing boundaries around groups of items. Gradually the child may be led to internalize these strategies and describe how they will segment an array visually, before enumerating the array.

Table 8

The frequency of each type of enumeration strategy

(percentage of disabled subjects exhibiting the strategy)

for each context and grade

Strategy	Numerosity range $n<4$									Numerosity range $3<n<16$								
	Linear			Scrambled			Grouped			Linear			Scrambled			Grouped		
	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6
1. Serial counted item by item	6	2	2	7	5	2	0	2	2	86	69	57	72	68	47	17	2	0
2. Did not need to count; could tell "immediately"	85	90	92	79	83	95	93	88	94	0	4	3	0	4	4	0	7	5
3. Counted in increments of two or three	0	0	0	0	0	0	4	7	2	0	6	3	0	6	0	56	63	69
4. Segmented and tagged each item in each segment	3	0	0	9	8	0	0	0	0	0	7	8	10	7	8	9	11	10
5. Segmented and rapidly enumerated each segment	0	0	2	0	0	0	0	0	0	0	4	7	0	4	8	0	0	0
6. Alternated between 1 and 3	6	8	4	5	4	3	3	5	2	9	6	12	11	8	13	11	8	13
7. Other (including estimation responses)	0	0	0	0	0	0	0	0	0	5	4	10	7	3	10	6	9	2

Table 9

The frequency of each type of enumeration strategy
(percentage of able subjects exhibiting the strategy)
for each context and grade

Strategy	Numerosity range $n < 4$									Numerosity range $3 < n < 16$								
	Linear			Scrambled			Grouped			Linear			Scrambled			Grouped		
	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6
	N=60	60	60	60	60	60	40	40	40	97	115	121	117	109	124	94	103	117
1. Serial counted item by item	0	0	0	0	0	0	0	0	0	73	41	23	66	24	19	0	0	0
2. Did not need to count; could tell "immediately"	95	100	100	92	100	100	95	100	100	2	5	6	2	6	8	5	19	25
3. Counted in increments of two or three	0	0	0	0	0	0	5	0	0	0	7	6	3	7	12	75	49	69
4. Segmented and tagged each item in each segment	0	0	0	0	0	0	0	0	0	6	13	26	12	20	17	8	3	0
5. Segmented and rapidly enumerated each segment	0	0	0	0	0	0	0	0	0	3	12	25	5	18	28	3	4	0
6. Alternated between 1 and 3	5	0	0	8	0	0	0	0	0	9	20	14	10	20	14	4	2	0
7. Other (including estimation responses)	0	0	0	0	0	0	0	0	0	7	2	0	2	5	2	5	3	0

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Tests of Learning Ability.

DEVELOPING BASIC CONCEPTS IN ELEMENTARY ALGEBRA

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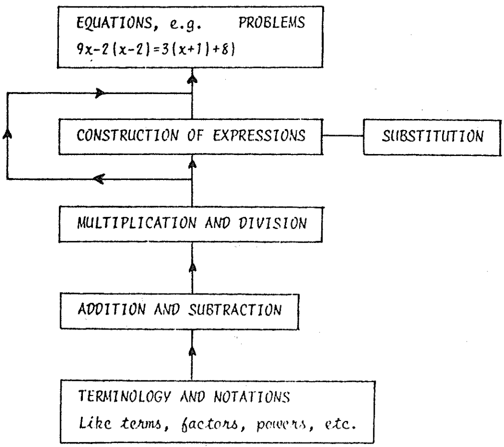
INTRODUCTION

A large scale study of some 40 000 13 - 15 year olds' understanding of the basic concepts and procedures of elementary algebra confirmed misconceptions and difficulties as identified by CSMS and SESM (Hart, 1981; Booth, 1982). However, these misconceptions and difficulties cannot simply be attributed to the cognitive development of students and/or the cognitive demands of the content, because learning is to a large extent dependent on the instruction received. At most one may conclude that present teaching strategies do not lead to an adequate understanding of elementary algebra.

A survey of some 800 teachers' perspective on algebra, objectives with the teaching of algebra and teaching strategies employed, revealed a teaching practice that is nearly totally geared towards rote manipulation of symbols. The typical teaching strategy is based on a linear analysis of the knowledge and skills that are prerequisites for later work:



Such a teaching strategy inevitably fosters the meaning of letter symbols as objects ("fruit salad algebra" is rife), mechanical manipulation of symbols as an isolated skill and a pre-occupation with the application of rules in terms of abstract terminology.

The further research hypothesis of RUMEUS is that given appropriate learning experiences, 13 - 15 year old students can adequately master the basic concepts of elementary algebra. The quest is thus to design an appropriate teaching module. A first attempt at such a module is now briefly outlined.

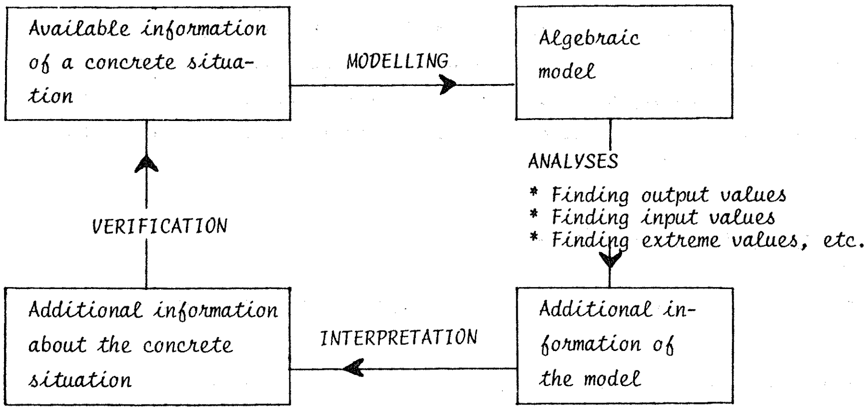
AN ALTERNATIVE TEACHING MODULE

The following served as basis for the design of the alternative teaching module:

- The content should be potentially meaningful to the learner and should create a positive learning set (Ausubel, 1968).
- The meaning (i.e. its relationships with related mathematical concepts) and significance (i.e. its use, importance or function within mathematics, as well as in situations outside mathematics) of the notations, concepts and procedures of elementary algebra should be stressed (Brownell, 1937).
- The basic concepts of elementary algebra (i.e. variable, expression and function) should be the focus of instruction from the beginning as anchoring ideas or scaffolding from which further ideas can be differentiated.

Thinking from a problemsolving or modelling context, the concepts of and relationships between equations, identities and expressions (formulae, functions) were analysed. Both from a mathematical and psychological perspective, the function is seen as a unifying concept with the greatest inclusiveness, generalizability and explanatory power, making possible a teaching sequence where equations, identities and manipulation can find dynamic meanings from and be integrated into the function scheme (Ausubel). As such the model is more extensive and inclusive than that designed by Booth, e.g., in introducing letter symbols at a higher level of interpretation (variable vs. generalised number).

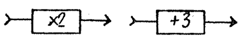
The notations, concepts and procedures of elementary algebra are given significance when used to describe and analyse relevant functional situations in all the phases of the following modelling diagram, and not as isolated skills:

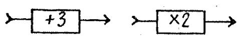


CONSTRUCTING ALGEBRAIC EXPRESSIONS

Constructing algebraic expressions in a modelling context gives sematic meaning to letters and expressions. Further syntactical meaning is given by interpreting an expression as a computing procedure (function rule) which specifies the operations to be performed on the values of the variable in order to obtain values of the dependent variable. Other ways of describing such computing procedures are in words, a flow diagram and a calculator keystroke sequence. Translations between notation systems develop the meaning of expressions and meaningfully introduce brackets, e.g.

Write an algebraic expression for

(a) 

(b) 

Interesting and meaningful work which can develop the meaning and significance of the basic concepts and procedures can be done with algebraic expressions at a simple level, without the interference of notation/terminology and manipulation as hampering factors. In this way algebraic contents are developed gradually, for example in using several preliminary algorithms for the solution of equations.

Constructed or given algebraic expressions can be meaningfully used to obtain additional information.

(a) Finding output values

By substitution of values into the expression, values of the output variable can be determined.

(b) Studying the behavior of expressions

By choosing such context where it is sensible to substitute many values into the expression (e.g. in constructing tables for taxi fares, income tax, post tariffs, etc) and studying the output set (table) as a whole, one can recognise how (by how much) a change in one variable effects related variables. Thus the concept of variable is adhanced and a "feeling" for the function is developed. The point-by-point plotting of the graph of the function now easily follows.

(c) Finding input values

Equations like $3x + 4 = 9 - x$ may be solved by a trial and error method untill equal output numbers are found for $3x + 4$ and $9 - x$, e.g. in tabular form, so developing the concept of equation and the meaning of the root of an equation, while at the same time keeping alive the concept of variable.

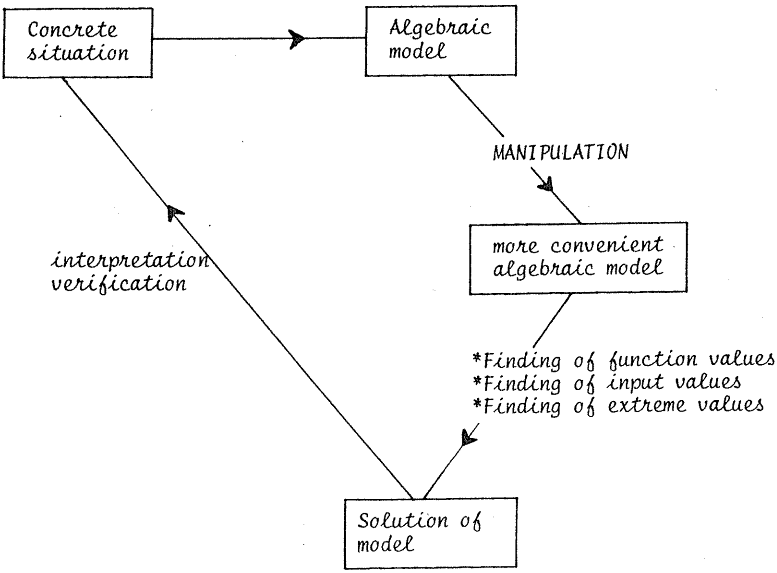
MANIPULATION

The concept of equivalent expressions are developed from tables of values and defined as expressions having equal values for all permissible values of the variable(s), e.g.

x	1	2	3	4	5
4x + 7x	11	22	33	44	55
11x	11	22	33	44	55

The fact that two equivalent expressions always assume the same values, means that the one can be replaced by the other when more convenient. In algebra the purpose of the construction of equivalent expressions (that is manipulation) is to simplify the procedures by which additional information is obtained in modelling situations. In the case of more complex algebraic models the analysis of the model can be simplified if we first replace the model (expression) with an equivalent expression before the solution process in question is applied.

The function and place of manipulation in the modelling diagram is clear from the following scheme:



FURTHER RESEARCH

Early informal evidence show that this kind of teaching module effectively forestalls the kind of misconceptions and difficulties that pupils develop in the traditional teaching of elementary algebra. In our future research this hypothesis will be investigated and the teaching module refined.

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CLASSROOM TASKS, INSTRUCTIONAL EPISODES, AND PERFORMANCE IN MATHEMATICS

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Abstract

Recent time-on-task studies of mathematics classrooms have documented striking differences between how the same mathematics is taught in different classrooms. This paper presents the basis for reaggregating and analyzing time-based data by presenting the premises upon which a model could be built in which specific features of children's behavior, teachers' behavior, and the content taught in class fit together to describe various classrooms, and how these features affect learning of the basic skills needed to solve verbal addition and subtraction problems. Building the model is the first step. The model will then be used to analyze data collected as part of a three-year longitudinal study.

INTRODUCTION

Anyone who has observed how the same mathematical skills are taught in different elementary classrooms has seen that each classroom seems to operate differently from the others. Sometimes the differences are striking and other times they are subtle. Sources for these differences are many: different curricular materials, children with different backgrounds, teacher personality differences, different emphasis on aspects of a "hidden curriculum" and so forth. The separate or cumulative effects on pupil performance of these operational differences is not always clear even though one is sure some classes are "better" than others.

During the past few years, several researchers from different research traditions have been spending more and more time in classrooms documenting various characteristics of what teachers and students do, with the expectation that some characteristics will prove to be importantly related to increased pupil performance. Much of this work is based on John Carroll's (1963) "model of school learning" and has been referred to as "time-on-task" research. Lorin Anderson (1980) and Nancy Karweit (1984) have reviewed that research. Anderson documented the different major "camps" of researchers, summarized their findings, and made a set of recommendations. Karweit reexamined the assimilated research, summarized the problems of interpretation, and highlighted the important findings.

Both, however, concluded that while the research to date has been fruitful, if the potential is to be realized, then research should go beyond descriptions of time use in classes to an interpretation of how and why time is spent differently in classes.

The analysis now being constructed reacts to two related major flaws in most time-on-task studies. First, the descriptions of how time is spent in classrooms lack any explanatory theoretical base to explain variations both within and across classrooms; and second, that the time based variables that are examined are treated as if they operate independently from other variables in classrooms.

The purpose of this paper is to describe an extensive time-on-task data base and the conceptual premises for a model which can be used to reaggretage that data in order to explore relationships between what occurs in mathematics, what was intended, and pupil performance.

THE DATA BASE

Observational time-on-task data were gathered in several classrooms as part of a collaborative three-year longitudinal study on the development of initial addition and subtraction problem-solving skills. The study was conducted by the Mathematics Work Group of the Research and Development Center at the University of Wisconsin (Romberg, Carpenter & Moser, 1978). The potential value of this data base can be demonstrated by comparing what was done in the longitudinal study with Anderson's four suggestions for carrying out viable time-on-task research. Anderson's (1980) recommendations were:

1. "Research should be carried out in classrooms in which the learning tasks are clearly defined" (p. 28). In most past research, the learning tasks have been general (e.g. reading, mathematics, etc.) rather than specific. In the longitudinal study, we studied instruction on addition and subtraction. All instruction was based on the same curricular materials, which were specially prepared for the study. Furthermore, in addition to the usual teacher action and pupil action data collected in time-on-task studies, detailed information was gathered on the actual content covered for each observational minute.

2. "Research examining the role of time-on-task in the learning of different types of teaching tasks (e.g., factual, conceptual, and skill-oriented) under different instructional conditions should be conducted" (p. 29). It is argued that different types of instruction are appropriate for different learning tasks. In the longitudinal study, each learning task was coded in terms of 29 variables. This provides us with a unique opportunity to relate what was actually taught to both pupil actions and teacher actions and in turn to pupil performance on that specific content.

3. "Research examining the patterns or consistency of time-on-task should be conducted" (p. 29). The total number of minutes of "allocated" or "engaged" time on mathematics only illustrates overall variation between classes. Patterns of time on different tasks are likely to be more important in understanding classroom effectiveness. Because of the content detail, we can identify different learning tasks and their patterns.

4. "Research should be conducted which relates actions on well-defined learning tasks to performance on such tasks and not to "the content of ill-defined, large-scale achievement batteries" (p. 29).

In the longitudinal study we gathered both paper-and-pencil objective-referenced test data, and interview data on each child over a three-year period.

To Anderson's recommendations, I would add a fifth:

5. Research which relates classroom tasks to instructional episodes via time-on-task data and in turn relates that data to student performance must be theory based. The potential of documenting what goes on in classrooms will not be realized until the measurements of time are related via some conceptual framework to specific classroom tasks and how they are organized.

To illustrate the potential value of the data base, the following examples of how instruction varies in several classes are presented.

Each of the ten instructional topics designed to teach addition and subtraction for this study is made up of several activities and each activity is separated into one or more parts. The basic observational unit is an instructional part. In total, there are 147 parts, 67 activities, and 10 topics. For each part, information has been coded for 51 variables. Twenty-nine are on the content of the part, six are on observed pupil actions, ten are on observed teacher actions, and six on observed classroom characteristics.

From this data base of 7497 bits of information about curriculum parts, we have constructed a picture of how instruction takes place in each classroom. In this descriptive step, we already know there is considerable variation between classes. For example, the number of minutes allocated to one of the topics, S5 (Solving situations and sentences, 0-20) varied from 525 minutes to 945 in sex classes (see Table 1). Such variation may be important, but we have much more precise content information as well. For example, the number of minutes spent on particular content areas is available. In Table 2, the variation in minutes spent on verbal comparison problems (subtraction) during instruction on Topic S5 varied from 134 minutes to 247 minutes.

We can also determine the relative emphasis teachers put on various aspects of instruction. For example, again in Topic S5, in Table 3 the number of minutes spent on parts designed to encourage student invstigation is shown. The variation in minutes is from 13 minutes to 95 minutes (or from 2% to 13% of allocated time). In Table 4 the number of minutes spent on parts categorized as encouraging discussion is shown. The variation in minutes is from 60 to 184 minutes (or from 9% to 26% of allocated time). Note also that in both Tables 3 and 4, the classes at teh extremes on this topic are the same. This suggests that Class 2 and Class 3 consistently differ in their pattern of instruction with exploration and discussion emphasized in Class 3 but not in Class 2. These examples indicate the richness and variability of this data set.

Table 1

Number of Minutes Allocated to Topic S5 by Six Grade 2 Classes

	Class					
	1	2	3	4	5	6
Number of Minutes	630	630	720	945	680	525

Table 2

Number of Minutes Spent Working on the Comparison (Subtraction) Verbal Problem Objective During Topic S5

	Class					
	1	2	3	4	5	6
Number of Minutes	150	134	247	200	171	145

Table 3

Number of Minutes Spent Working on Parts of Topic S5 Categorized as Encouraging Student Exploration and Investigation

	Class					
	1	2	3	4	5	6
Number of Minutes	63	13	95	83	87	56

Table 4

Number of Minutes Spent Working on Parts of Topic S5 Categorized as Encouraging Pupil/Teacher and Pupil/Pupil Discussion

	Class					
	1	2	3	4	5	6
Number of Minutes	121	60	184	150	164	87

Such descriptive data characterize each class and the process of instruction in terms of a large number of conceptually independent concepts. However, in the life of classrooms, these concepts are not independent, but related. Our analytic task at this step is to put the pieces of seemingly independent information. A typical procedure with time-based data has been simply to add pieces together. However, as Doyle (1977) and Romberg (1980) have pointed out, this procedure fails to capture the intent of a lesson and its episodic nature.

To illustrate this point, the actual time spent in classrooms on various parts can be compared with the recommended time for each part. In this way, a pattern of what content was emphasized in each class can be seen. For example, the teacher in Class 4 in Grade 2 (see Figure 1) believed the students were the "low" group spent more than twice as much time as recommended on six sentence writing objectives. The modification of the instructional matrials toward a skills orientation by the teacher is evident. From available interview data, this teacher believes that children with different levels of "ability" should do different things; more "drill and practice" for "low" students, and more "explorations" for "good" students.

Also, with the detailed data, we are convinced that indices for classroom tasks and distinct instructional episodes can be specified. For example, Table 5 shows the number of minutes spent in each class on the sequential parts of the topic (S5) where manipulatives were expected to be used. The teachers in classes 2 and 6 select and spend time on activities with manipulatives in a different sequence and manner than teachers in the other classes. By putting such information with other descriptive data about each activity, we should be able to describe distinct instructional episodes.

Table 5

Number of Minutes Spent Per Part of Topic S5 Categorized as Providing Opportunities for the Use of Manipulatives

	Class					
Part	1	2	3	4	5	6
B1	25	0	6	15	41	0
C1	29	0	31	54	25	51
D1*	0	0	28	0	0	0
D2	8	0	7	0	0	0
E1	49	55	86	133	68	51
H1	22	0	29	8	39	0
Total	133	55	187	210	173	102

*Note: D1 and D2 are alternate parts to C1; H1 is an optional part.

In summary, the variability of how time is spent in classrooms teaching the same mathematical content is remarkable. Our task is to to identify some key features based on a conceptual basis which will illuminate how mathematics instruction differs in classrooms.

THE CONCEPTUAL FRAMEWORK: ITS BASIS

An adaptation of the "Model of Pedagogy" presented by Romberg, Small, and Carnahan (1979) will be used for this study. This adaptation, shown in Figure 2, posits that the beliefs teachers have about schooling, learning, and mathematics are mediated by the specific content to be taught and reflected in their plans for instruction. In turn, these plans are reflected in both their actions and student actions in classrooms and then in student performance. Furthermore, the character of their plans can be inferred from how they allocate time and give emphasis to various content.

Two problems are now being addressed by the project staff. First, the basis for characterizing teacher beliefs is being addressed. It is anticipated that Anglin's (1976) analysis of instructional decision making (based on Perrow, 1969 and Kliebard, 1972); and Popkewitz, Tabachnick, and Wehlage's (1982) analysis of elementary schools will be used.

Second, we expect to aggregate the available teacher action, pupil action, and content data in order to characterize three features of classrooms:

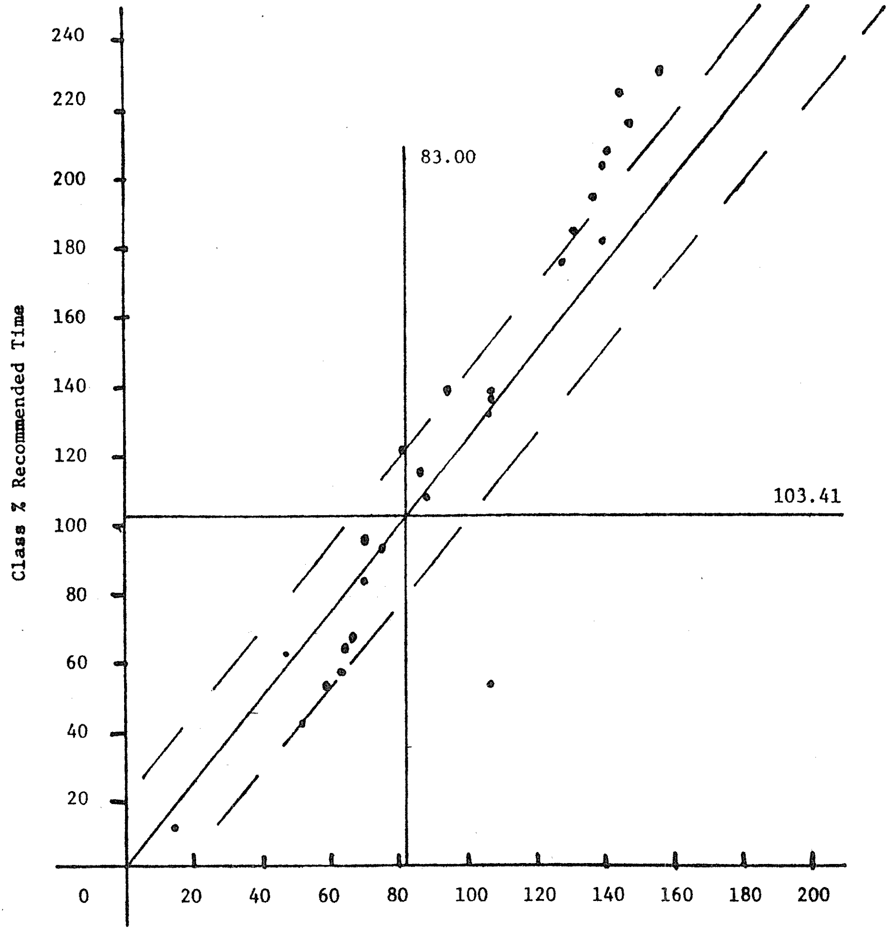
1. Classroom tasks for children. This feature involves creating variables which reflect both the operational definition of pupil work in classrooms and the social relations implicit in carrying out that work.

2. Classroom tasks for teachers. Similarly, this involves creating variables which reflect both the operational definition of pupil work in classrooms and the social relations implicit in carrying out that work.

3. Instructional episodes. This involves identifying different episodic patterns of classroom instruction on the same content in order to describe how knowledge is operationally developed in each context.

SUMMARY

First, a model of mathematical pedagogy will be developed in order to specify key indices of classroom planning and behavior. Second, by treating each class as a "case study", we will attempt to characterize each in terms of key indices about how classrooms operate based on the model. Third, using the basic observational data, we will create new variables or proxies related to the key indices. Last, we will contrast the classes on these variables. The pdoruct of this effort will be a set of reports which should provide us with a better understanding of how classrooms operate.



<u>Above</u>

%	Code
195	Other sentences 0-20
204	Add-simple join 0-20
214	Subtract-simple separate 0-20
218	Subtract-join addend 0-20
225	Subtract-part/part/whole 0-20
231	Add-part/part/whole 0-20

<u>Below</u>

%	Code
57	Explorability

Figure 1

Plot of Class and Grade Percent Recommended Times
for All Codes on the S Topics: Grade 2, Class 4

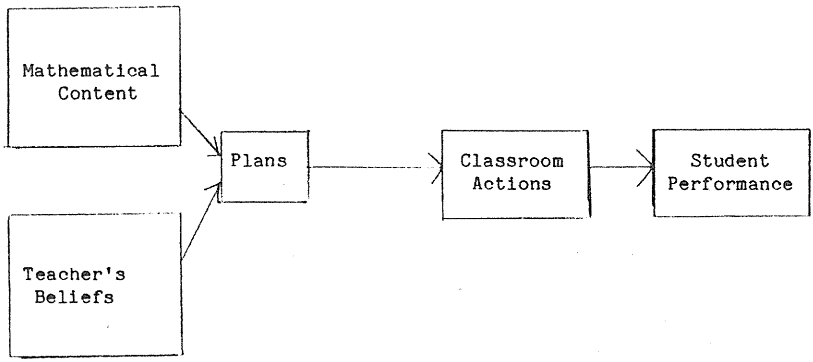


Figure 2

The Elements of a Model of Mathematics Pedagogy

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TEACHING FOR CONCEPTUAL DEVELOPMENT IN MATHEMATICS AND ACHIEVEMENT IN MATHEMATICS

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INTRODUCTION

The need for conceptual development in mathematics as an effective means of teaching the subject, especially at the formative years of schooling, cannot be over-emphasized. Mathematics is a highly structured subject with a hierarchical building-up of concepts (Skemp, 1964).

The conceptual nature of mathematics and the Piagetian stage-dependental nature of the process of acquisition of mathematical concepts and principles (Collis, 1972) makes it imperative for teachers of mathematics to lay greater emphasis on conceptual development in children rather than the development of routine mathematical skills and algorithms. However, there are instances when teachers tend to regard it as more expedient to teach their children routine mathematical skills such as computational ability, algorithms and "short-cuts" for solving routine problems, mathematical formulae and "rules of thumb", etc. Perhaps they are persuaded in doing so by such constraints as limited time available for covering the syllabus, limited resources, or just lack of enthusiasm for conceptual development.

It is therefore, of importance to make an investigation of the extent to which teachers emphasize the development of mathematical concepts and principles in their children, as against the development of routine mathematical skills, algorithms, and rote procedures. This teacher concern for learning with insight and understanding, as opposed to rote memory - called "Theoretical Orientation" by some writers - is considered to be one of the more dominant affective teacher variables related to the teacher effectiveness in mathematics (Begle, 1979).

The chief aim of this study is to examine (by means of empirical evidence) the relationship between this affective teacher variable and the teacher effectiveness in mathematics as depicted by pupil achievement in mathematics.

THEORETICAL BACKGROUND

The conceptual nature of mathematics and the consequent need to teach mathematics as a process of systematic building up of concepts, have been emphasized by many writers (Piaget, 1952; Bruner et al., 1956; Dienes, 1960; Skemp, 1971). That children's successful understanding of mathematics depended on conceptual development, as opposed to rote learning, has been demonstrated by Skemp (ibid, pp.40-42). In a series of experiments, Collis (1972) demonstrated that children's understanding of mathematics is stage dependent, in unison with the well known Piagetian stages of intellectual development.

Hence the importance of teaching mathematics with due emphasis on conceptual development of children, in accordance with the Piagetian stage dependental nature of learning the subject.

The willingness, or the attitude, to teach mathematics in the aforesaid manner is an important affective teacher attribute. It has long been believed that this attribute exercised great influence on the "teacher effectiveness" of a mathematics teacher. ("Teacher effectiveness" has been defined and measured in terms of pupil achievement.)

Begle (1979, pp.38-47) who analysed a vast proliferation of research involving teacher attributes, under the two broad categories: "Background Variables" and "Affective Variables", found that the affective teacher variables correlated many times more highly with teacher effectiveness than the background variables. Although this was true as a comparison, Begle also found that neither of these two types of variables had a strong influence on teacher effectiveness. This finding of Begle, especially in respect of the affective teacher variables, appears to be contradictory to the popular belief mentioned above as regards the attitude for teaching mathematics for conceptual development.

Now, the teacher attitude to teaching mathematics for conceptual development as opposed to rote memory, is one of the more important affective teacher variables listed by Begle. The importance of conceptual development for the effective teaching of mathematics has already been mentioned. Hence the need for further research into the relation between the said teacher attitude and teacher effectiveness in mathematics.

THE OBJECTIVES

The specific objectives of this study are:

(1) to construct an instrument to measure teacher's attitude to teaching for conceptual development as against the development of routine skills and rote procedures;

(2) to measure the said attitude of a sample of teachers teaching mathematics at the Junior Secondary level;

(3) to assess the mathematics achievement of a group of children taught by each member of the said sample of teachers, over a one-year period;

(4) to examine the relation between teacher's attitude to conceptual development in mathematics, and their pupil's achievement in mathematics.

MEASURING ATTITUDE TO CONCEPTUAL DEVELOPMENT

An instrument to measure teacher's attitude to teaching mathematics for conceptual development, had to be prepared. The preparation of this instrument became a formidable task owing to certain constraints. These are described briefly in the following paragraphs.

First, there was the perennial problem of the objectivity of responses to a questionnaire. This was particularly to be expected in the present circumstance, for the attitude dimension under consideration was a subtle issue. It is extremely unlikely that a mathematics teacher would readily admit that he did not teach a topic for conceptual development, if asked directly. Teaching mathematics for conceptual development is not easy. A teacher may be teaching the subject by using the more convenient (although unfortunately, less effective) rote procedures quite intentionally, or may believe that he is teaching for conceptual development while, in fact, he is teaching rote techniques all the time. In either case, it is highly unlikely that the actual position could be uncovered by direct questioning. It was therefore, necessary in the construction of the questionnaire, to write items suitable for probing the attitude dimension under investigation, while at the same time concealing the real issue as best as possible!

The second problem arose in connection with covering the universe of content. In designing an instrument to measure a given attitude, care must be taken to see that the universe of content is adequately covered, and that the items fully span the attitude dimension (Moser and Kalton, 1971, pp.357, 358). There were so many different and yet essential aspects of the curriculum to be considered that it appeared to be necessary to write a very large number of items to cover the universe of content adequately. However, the length of the questionnaire had to be considered, and it was necessary to limit the number of items in the instrument.

The third constraint was the possible "negative discrimination of positive respondents". Sometimes even a teacher with a highly favourable attitude to conceptual learning would adopt on occasion, what may appear to be rote procedures. Now, it is possible for such teachers also to opt for responses of the rote learning type contained in an item. This was a problem besetting the construction of the entire questionnaire.

In the writing of items, the writer was guided by his own previous work (Ruberu, 1980) and the work of Mackay (1971), Koay (1975), Adlem (1977), Northfield (1980) and Gnanarajah (1981).

The attitude dimension under investigation appears at first sight, to be wholly connected with teaching method only. In practical terms however, this is not so; for methods of teaching - especially those devised for conceptual development - are invariably related to the other aspects of the curriculum, viz. objectives, content and evaluation (Beeby, 1970, pp.42, 43). In planning items for the questionnaire, therefore, careful attention had to be paid to

(a) various general aspects of the methodology itself (of teaching mathematics at the Junior Secondary Level), and
(b) aspects of method in relation to objectives of learning/ teaching, content, and evaluation of pupil progress.

Only the more important of these aspects could be considered for the purpose, owing to the constraint (2) mentioned earlier. They formed the domain of the questionnaire.

The initial draft of the questionnaire comprised five different scales: A, B, C, D and E. A was a direct Likert type scale, with 10 items. B was of the multiple-choice type with 10 items. C was also of the multiple choice type (with 10 items), but with a difference. A respondent here had to consider each of these choices separately, and indicate his intensity of preference for it by checking one of four alternatives given against it.

Scale D comprised 10 items which were of the "scenario" type. Each of these was presented in the form of a part of a dialogue between two people, one arguing for and the other arguing against a type of teaching approach. A respondent was required to indicate his/her agreement or disagreement with both of them, or a Likert type response category system.

All items in the above scales were concerned with certain general aspects of methodology and certain aspects of method in relation to objectives and content. The 12 items in scale E were of the multiple-choice type, and were concerned with methods of evaluation of mathematics learning at the Junior Secondary level.

A pilot trial of this initial draft was next planned. However, a pre-pilot scrutiny revealed that certain items of it had to be modified and certain others had to be dropped altogether. The resulting instrument had 10 items in scale A, 8 items in B, 10 in C, 9 in D, and 8 in E. It was then pilot trialled on a parallel sample of teachers teaching mathematics at the Junior Secondary level in 11 schools.

The data gathered from this exercise were used to determine the reliability coefficient of each of the scales and to perform item analyses on them. The results of these analyses were used in conjunction with teacher's observations on items/scales, to further refine the instrument.

Scales A and B had to be rejected outright, for they showed up very poorly on reliability; (Cronbach's α for A was 0.12 and for B was 0.29). Scales C (α=0.57), D (α=0.35), and E (α=0.53) were retained. These three scales (renamed A, B, C), with certain items modified or deleted according to the pilot findings, formed the final instrument. It comprised 26 items, with 9 items in scale A, 7 items in B, and 10 in C.

THE SAMPLE

The aforementioned instrument was used to measure the said attitude dimension of 60 teachers teaching mathematics to children in Year Seven in 12 state schools in Victoria. Six of them were High Schools and the other 6, Technical Schools. These schools were a mixture of upper and lower middle class socio-economic types, in roughly equal proportions. Eleven of them were co-educational schools.

The sample of children chosen for this study were children who studied mathematics in Year Seven during 1983, under the aforesaid sample of teachers. Year Seven was decided upon for this study for two reasons.

(1) A considerable number of new mathematical concepts and principles were introduced for the first time at this year level, and

(2) For the children, this was their first year in the new High or Technical School. There was therefore, no prior "conditioning" of any type by their teachers, and/or the surroundings.

There were 1163 children in the sample, studying mathematics in 70 parallel classes of Year Seven in 12 schools, under the aforesaid 60 teachers.

MEASURING PUPILS' ACHIEVEMENT IN MATHEMATICS

A pre-test and a post-test in mathematics were planned for the above-mentioned children, in order to assess their mathematics achievement during the year. The format of these two tests were conceived to be similar in every respect, as far as was practically possible.

In order to ascertain first hand, the content that should be tested and the depth to which that testing should be done, the writer had many discussions with several senior teachers teaching the said children in a number of High and Technical schools of the sample. Mathematics syllabuses at the Primary level were also studied for the purpose.

Each test was planned to consist of 40 items, requiring an average of about 40 minutes for completion. They were not speed tests, and accordingly, allowance for a certain amount of extra time was mooted for the slow performers. Test items for the two tests were borrowed from materials prepared by the Australian Council of Educational Research (ACER, 1976, 77). Standardized tests of the ACER - the "CATIM" (Class Achievement Test in Mathematics), and the "MAPS" (Mathematical Profile Series) were the sources used. (Permission to use these materials is gratefully acknowledged.)

The pre-test was conducted in the twelve schools of the sample at the beginning of the first school term, before the children were much exposed to mathematics teaching in Year Seven. The post-test was conducted towards the end of the third term. Teachers' assistance was sought in the conducting of these tests, which was willingly given at all times. The writer wishes to acknowledge his debt of gratitude to these teachers for their cooperation in this, as well as in answering the teacher questionnaire.

TEACHER ATTITUDE AND PUPIL ACHIEVEMENT

Copies of the teacher questionnaire issued to the sample of 60 teachers were each fully completed and returned (100% return). They were scored (according to a scoring scheme prepared in advance) and the scores were first analysed for reliability of the instrument. Conbach's α for scale A was 0.53, that for scale B was 0.69, and for scale C was 0.68. α for the entire questionnaire was 0.75, which was quite encouraging.

The sample of teachers was divided into 3 groups of nearly equal size - Low, Medium and High - according to increasing magnitude of their total attitude scores (TAS). A two way analysis of covariance was carried out with TAS and "school type" (High Schools, Tech. Schools) as the independent variables, the mean post test score of the group of children taught by each teacher as the dependent variable, and the corresponding mean pre-test score as the covariate.

The F ratio for the main effect TAS was not significant, (at α=0.05 level). Hence, the hypothesis that high teacher-attitude to teaching for conceptual development was associated with high academic achievement in the pupils (and vice-versa), had to be rejected.

The F ratio for the main effect of "school type" (STP) was significant, (at α=0.01 level). Results of multiple classification analysis showed that higher academic achievement (as measured by the post test) was associated with High Schools. The F ratio for the two-way interaction between TAS and STP was also not significant (at α=0.05 level).

Further analyses involving TAS and a few other teacher variables, and achievement in certain topic areas, have also been carried out.

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THE PROBLEM OF ABRUPTNESS IN MATHEMATICS TEACHING

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PSYCHOLOGICAL PERSPECTIVE

Learning in general and mathematical learning in particular has to be viewed as continual modification of knowledge structures. The whole process can be characterised as a scenario of transitions, thus becoming an avenue for management of transitions to maximise smooth learning, that is, learning with the least resistance. This entails avoidance of abruptness through deliberate and well planned provision of connecting links to facilitate information processing.

The development of mathematical thinking, though sequential stage-wise, is not linear but concentric, radiating streams of refinements. So if what is to emerge later can be indicated earlier, wherever possible and in whatever crude form it would have to be presented, the jolt of abruptness can be made to disappear to facilitate surfacing of connections and acceptance of apparently 'shocking' rules for the sake of preserving stuctures. Moreover, this is pedagogically sound, as it would be in keeping with the historical evolution of mathematical concepts.

The problem of abruptness in mathematical learning has not received enough attention, I believe, in the hands of curriculum planners, instruction designers and text book writers as a result of which the disconnected presentations have militated against integrated thinking and deprived learners of developing readiness for restructuring of acquired knowledge to develop new knowledge in a smooth transition.

A few areas in school mathematics are selected and dealt with in sufficient detail to defend this thesis.

WHOLE NUMBERS TO FRACTIONAL NUMBERS

Many of the properties of fractional numbers and their rules of operation could be introduced while teaching whole numbers and their operations. Division in the set of whole numbers is indicated by $\div$ and if, besides this, the dividing bar notation of the fractional form is also used as an alternative to the symbol $\div$ and if learners are encouraged to explore the patterns of addition and subtraction, multiplication and division of whole numbers with addends and factors expressed in division form using fractional notation, the learners can be seen developing readiness to map dividend, division and quotient onto numerator, denominator and fractional number and appreciate the consistency in the presentation of structural properties. Only in the case of remainder, the change in the system of fractional numbers would be dramatic, as, being zero always, it would cease to vary. Fractional number would then emerge as an entity by anticipation and not by abrupt introduction.

To start with, let the division indicated by $6 \div 2$ have also an alternative form $\frac{6}{2}$ and let us use this alternative form in transforming addition, subtraction, multiplication and division statements in the system of whole numbers. Examples are cited below by way of illustration.

Consider $3 + 5 = 8$. Rewriting 3 and 5 as divisions with the same divisor, say 2, we can write

$$3 + 5 = \frac{6}{2} + \frac{10}{2}$$

How to obtain 8, using 6, 10, 2 and 2 becomes a question for exploration. Students discover that by adding the dividends 6 and 10 only and dividing the sum by 2, 8 can be got.

$$3 + 5 = \frac{6}{2} + \frac{10}{2} = \frac{6+10}{2} = \frac{16}{2} = 8$$

The question that arises in this context is whether it would be possible to get 8 when 3 and 5 are expressed as divisions with different divisors, as for instance $\frac{12}{4}$ and $\frac{15}{3}$. By trial and error exploration, it is discovered that unless the same divisor is used, the addition cannot be performed to get the required sum. Examining the subtraction situation, the experience is seen to be similar.

Now by writing,

$$3 = \frac{3}{1} = \frac{6}{2} = \frac{9}{3} = \frac{12}{4} = \frac{15}{5} = \frac{18}{6} = \frac{21}{7} = \frac{24}{8} = \frac{27}{9} = \text{etc}$$

(multiplication table of 3 in a different form), the principle that when the dividend and the divisor are multiplied by the same natural number or divided by their common factor, the same quotient is obtained surfaces and is spelled out. Later, this gets recognised as the property of equivalent fractions, once the concept of fractional number is presented.

In the case of multiplication, the constraint experienced in doing addition and subtraction is surprisingly found to be absent.

Consider next $5 \times 8 = 40$. Writing 5 and 8 as divisions with the same divisor, say 3, we can write

$$5 \times 8 = \frac{15}{3} \times \frac{24}{3}$$

There is a feeling of anticlimax as $\frac{15 \times 24}{3}$ does not yield the required product 40, but only $\frac{15 \times 24}{3 \times 3}$. The feeling gets heightened when the situation remains unchanged, even if the factors are rewritten as divisions with different divisors.

$$5 \times 8 = \frac{30}{6} \times \frac{16}{2} = \frac{30 \times 16}{6 \times 2} = \frac{480}{12} = 40$$

What remains now to be examined is just division. This operation surprisingly bristles with manipulative complexity after a deceptive start when the dividend and divisor are expressed as divisions with the same divisor. Consider $12 \div 4 = 3$. Rewriting 12 and 8 as divisions, first with the same divisor, say 4 and then with different divisors, say 2 and 5, the oft-repeated thumb rule 'invert and multiply' emerges after considerable trial and error.

$$12 \div 3 = \frac{48}{4} \div \frac{12}{4} = \frac{48 \div 12}{4 \div 4} = \frac{4}{1} = 4$$

(The rule does not emerge at this stage).

$$12 \div 3 = \frac{24}{2} \div \frac{15}{5} = \frac{24 \times 5}{2 \times 15} = \frac{120}{30} = 4$$

$\frac{24 \times 5}{2 \times 15}$ is the same as $\frac{24}{2} \times \frac{5}{15}$ and $\frac{5}{15}$ is the inverted form of $\frac{15}{5}$. Examining if this holds good in the 'same divisor' case, we find $12 \div 3 = \frac{48}{4} \div \frac{12}{4} = \frac{48}{4} \times \frac{4}{12} = \frac{192}{48} = 4$, showing that this pattern of manipulation covers both the cases and hence general.

Finally, in addition to the notation of writing $13 \div 5 = 2R3$, if $13 \div 5 = 2\frac{3}{5}$ (read as 2 and 3 for 5) is also introduced, the transition from the system of whole numbers to the extended system of fractional numbers will be rendered smooth and meaningful. Once the notion of whole and part is introduced, 2 3/5 will take on a new meaning as 2 and 3 fifths, 2 3/5 itself becoming the quotient with null remainder. Allowing familiar things to assume new meanings for greater generality and power with some losses of course has been the underlying spirit of adventure in mathematical thinking.

WHOLE NUMBERS TO INTEGERS

Integers also lend themselves to this kind of anticipatory exercise. Any whole number can be expressed as a sum or difference of two whole numbers and that gives the initial start for managing the transition. A few examples are set out below:

Addition : 7 + 4 = 11				
	7 = 5 + 2	7 = 9 - 2	7 = 10 - 3	7 = 4 + 3
	4 = 3 + 1	4 = 8 - 4	4 = 3 + 1	4 = 5 - 1
	11 = 8 ---	11 = 17 ---	11 = 13 ---	11 = 9 ---
	11 = 8 + 3	11 = 17 - 6	11 = 13 - 2	11 = 9 + 2

7 and 4 are expressed as binary expressions involving addition or subtraction with two numbers. While adding, a sum is known and the first term of its binary expression is also easily obtained. The problem is to fix the second term and it is fixed with respect to the first term in order to obtain the required sum. A study is

made to find out how the second terms in the two addends are computed to get the second term of the sum and this reveals a need to recognise the second terms with their + and - signs as entities and formulate their behaviour in addition. A familiar pattern seen in the addition of two integers emerges.

Subtraction : 9 - 4 = 5					
	9 = 6 + 3	9 = 6 + 3	9 = 13 - 4	9 = 13 - 4	9 = 15 - 6
	4 = 2 + 2	4 = 5 - 1	4 = 6 - 2	4 = 3 + 1	4 = 11 - 7
	5 = 4 ---	5 = 1 ---	5 = 7 ---	5 = 10 ---	5 = 4 ---
	5 = 4 + 1	5 = 1 + 4	5 = 7 - 2	5 = 10 - 5	5 = 4 + 1

The first term in the binary expression for difference is easily obtained as it is ordinary subtraction. Then the second term in the expression for difference is fixed so as to get the required difference. After handling a number of cases, students are excited to face the challenge of discovering the pattern in arriving at the second term of binary expression for the known difference. By handling the second terms of the minuend and the subtrahend, formulation of yet another oft-repeated rule 'change the sign and add' is made. It is also observed incidentally that each of the subtractions with second terms can be associated with its equivalent addition as displayed below:

⊖		⊖		⊖		⊕
6	+	3	is equivalent to	6	+	3
5	-	1		5	+	1
⊖		⊖		⊖		⊕
6	+	3	is equivalent to	6	+	3
2	+	2		2	-	2 and so on.

Multiplication is more challenging than addition and subtraction and yet it is not beyond the ability of upper primary school level students to tackle it and discover the familiar 'rule of signs' in multiplication.

9 x 7 = 63		
	9 = 6 + 3	9 = 11 - 2
	7 = 5 + 2	7 = 4 + 3
	63 = 5(6+3) + 2(6+3)	63 = 4 (11 -2) + 3 (11-2)
	= 30+ 15+ 12+ 6	= 44-8+ 33-6
	9 = 7 + 2	9 = 13 - 4
	7 = 10 - 3	7 = 9 - 2
	63 = 10(7+ 2) -3(7+ 2)	63 = 9(13-4) -2(13-4)
	= 70+ 20-21-6	= 117-36-26---
		= 117-36-26+ 8

Division does not lend itself to transitional management and hence consideration of it is postponed.

STRUCTURAL RELATIONS TO EQUATIONS

Every addition fact yields not more than 2 associated subtraction facts and every subtraction fact in turn yields one addition fact and one associated subtraction fact.

$8 + 3 = 11 \rightarrow 8 = 11 - 3$ and $3 = 11 - 8$

$7 - 2 = 5 \rightarrow 7 = 2 + 5$ or $5 + 2$

$7 - 2 = 5 \rightarrow 7 - 5 = 2$

These provide an effective background to handle simple equations involving + and - even before formal introduction to equations is made in higher class. Introduction of x or □ could be made to replace any number in the above statements and the question be posed for finding its value.

The simplest is the straight forward one like $8 + 3 = \square$ or $7 - 2 = \square$. The rest involve transformation based on the relational statements.

$\square + 3 = 11 \rightarrow \square = 11 - 3$

$8 + \square = 11 \rightarrow \square = 11 - 8$

$7 - \square = 5 \rightarrow 7 - 5 = \square$

$\square - 2 = 5 \rightarrow \square = 2 + 5$

On similar lines, every multiplication fact yields not more than two division facts and every division fact in turn yields one multiplication fact and one associated division fact

$5 \times 2 = 10 \rightarrow 10 \div 5 = 2$ and $10 \div 5 = 2$

$18 \div 6 = 3 \rightarrow 18 = 6 \times 3$ or 3×6

$18 \div 6 = 3 \rightarrow 18 \div 3 = 6$

These too provide an effective background to handle simple equations involving now multiplication and division.

$5 \times 2 = \square \rightarrow 5 \times 2 = 10$

$5 \times \square = 10 \rightarrow \square = 10 \div 5$

$\square \times 2 = 10 \rightarrow \square = 10 \div 2$

$18 \div \square = 3 \rightarrow 18 \div 3 = \square$

$\square \div 6 = 3 \rightarrow \square = 6 \times 3$

Notion of variable can be introduced at this juncture by means of sentences such as $x + y = 8$ and $xy = 12$, the number of solutions being related to the domain chosen for the variables.

Yet another context for the use of variable and the corresponding letter symbolism presents itself whenever a formula is given in mensuration. For instance, consider rectangles of different dimensions given as positive integers and their areas.

$4 \times 3 = 12$, $6 \times 4 = 24$, $8 \times 2 = 16$, $3 \times 5 = 15$ which on generalisation gives $l \times b = A$ or $e_1 \times e_2 = A$ (l and b representing measures of length and breadth or e_1 and e_2 representing measures of edge 1 and edge 2 and A being the area of the rectangle). By extending the domain to include positive rational numbers, the same formula can be seen to hold good. Finally by extending the domain to include positive real numbers, the same formula gets assumed to be valid. Consideration of perimeters of rectangles and generalising the pattern in 2(4+3)=14; 2(6+4)=20; 2(8+2)=20; 2(3+5)=16, 2(l+b)=P or 2 (e_1+e_2)=P is obtained.

Parameters can also be introduced in the context of perimeters of regular polygons. If the perimeter P=ka is taken as the formula for the perimeter of a regular polygon, k represents the parameter which changes with polygons of each kind. k=3 gives the perimeter formula P=3a for the family of equilateral triangles; k=4 gives the perimeter formula P=4a for the family of squares and so on.

Introduction of integers through postulation of opposites or additive inverses and relation of opposites to the additive identity zero, provides an excellent opportunity to solve certain problems with elegance. If it is asked what should be added to+7 to get say 11, it would be easy to think of the number that should be added to+7 to get zero and then add the required sum to fix the required addend. To solve +7+x=-11, since +7-7=0, +7-7-11=0-11 → +7-18=-11. So the required addend is -18.

Similar situation obtains through the use of reciprocal or multiplicative inverse and multiplicative identity when positive rational numbers or fractional numbers for that matter, are introduced. To find the number by which 3/5 should be multiplied to get, say, 8/13, it is easy to think of the number by which 3/5 should be multiplied to get 1 first and then multiply it by the required product to get the required multiplier or factor. To solve 3/5 x x = 8/13, since 3/5 x 5/3 =1, 3/5 x 5/3 x 8/13 = 1x8/13 →3/5 x 40/39 = 8/13. So the required multiplier is 40/39.

COMPUTATIONAL MATHEMATICS TO AXIOMATICS

Abruptness in passing from computational and manipulative mathematics to propositional mathematics has been one of the major factors for resistance to learning in traditional curriculum and this situation was sought to be changed somewhat in the sixties when modernisation wave was witnessed. Introduction of local axiomatics in lower classes facilitates learning mathematics not only as a body of computational skills and applicable tools but also as a quest for finality or truth in propositions.

Some questions that even children in primary classes can handle successfully with their reasoning powers are:

1) Can there be the last counting number ? Since one more than the given counting number gives the next higher counting number, there cannot be the last counting number.
2) Can there be an even prime number other than 2 ? A number to be prime should have only two distinct factors. Assuming that there could be an even prime number other than 2, a contradiction would occur, as such an even prime would have 1, 2 and itself as factors, that is to say, three factors. So there cannot be more than one even prime number.

Some general propositions can also be considered and proved. Consider the classic proposition about any set of eight persons. At least two of any set of eight persons are born on the same day of the week. If it is a particular set and their days of birth are known, the truth of the proposition can be established by exhaustion or examination of all cases. If it is about any set, then the truth cannot be established by exhaustion. There is need for reasoning by contradiction. There are only seven days in a week. If it is assumed that all the eight are born on different days, that would mean that a week should have 8 days. Since it would contradict the established fact, all the eight cannot be born on different days. If seven of them are born on different days, it would make seven days and the remaining one should be born on any of the seven days. So at least two of them should be born on the same day.

Early experience of disproving a statement through counter example will save learners from commiting the oft-repeated mistake of offering verification as proof. The most appropriate place to introduce disproof by counter example is when even and odd numbers, prime and composite numbers are introduced. A golden opportunity to make conjectures and examine their validity presents itself now. Some of the conjectures are (1) a prime number + a prime number = a prime number. The counter example 5+7=12 disproves it. (2) a prime number + a prime number = a composite number. The counter example 2+11 = 13 disproves it. So the sum of two prime numbers is neither prime nor composite. Similarly, that 1 is neither prime nor composite is also established, given the definition of a prime number as a number having only two distinct factors.

Elements of number theory provide also the opportunity to formulate conjectures and prove them. For instance, consider sets of three consecutive numbers : 1,2,3; 2,3,4; 3,4,5; etc. and their respective sums:6,9,12,etc. The conjecture is that the sum of any three consecutive natural numbers is a multiple of 3.* So the conjecture is proved and hence it becomes a theorem. Again by multiplying the end numbers of each triad of consecutive numbers and comparing the product with the square of the middle number, we get 1 x 3,4; 2 x 4,9; 3 x 5, 16 etc. The conjecture is that the product of end numbers in a trial of three consecutive numbers is one less than the square of the middle number. Since $(n-1)\ (n+1) = n^2 - 1$, the conjecture is proved and it is found to be a theorem.

Some elementary geometrical propositionscan also be proved or disproved. A triangle should have at least two acute angles. This follows from the proposition that a triangle can at most have one right angle or one obtuse angle. To prove this, that the sum of the three angles of a triangle is two right angles is taken as a local axiom.

CONCLUSION

If these readiness inducing links could be well charted and incorporated in text books and teaching, school learners would become predisposed to have a more mature, meaningful and smooth passage from one thought process to another in mathematics.

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* Assuming three consecutive natural numbers to be $n-1, n, n+1$, we find their sum is $3n$ and it is a multiple of 3.

UNMASKING ℕ-DISTRACTORS AS A SOURCE OF FAILURES IN LEARNING FRACTIONS

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Abstract
Natural numbers are serious distractors at the start and during the further course of the process of learning fractions. Indeed, independent namens for fractions are rare in most languages and almost all names for fractions have been met with before only as names for ordinals or are very close to ordinals. What is even worse, is that the symbols for natural numbers reappear as parts of the symbols for fractions with only a horizontal or a slanting stroke added. This means that the arithmetic operations on fractions can easily be confused with the corresponding operations on the numbers figuring in the fraction symbols, and often they are. Failures caused by these circumstances will be called ℕ-distractor failures. These failures proved to be quite refractory as is witnessed by many research results. More often than not attempts at fighting them in the teaching process of fractions have failed. This may be due to the fact that the pupils who make these mistakes rather than being confronted with a cognitive conflict, are left in the state of believing in the correctness of their faulty procedures. Unmasking ℕ-distractors in order to fight failures should be one of the main objectives in teaching and learning fractions. In the present contribution history of mathematics and interview-research results will be the very sources to be drawn on for the solution of this problem. It will be argued that fractions can be taught in a way the students learn both to identify and to unmask the ℕ-distractors by appropriate arguments.

1 *Some historical remarks*

From both a historical and a linguistic point of view it is nonsensical to speak of one fifth, two fifths, ...in order to describe the subsequent number of parts of a whole, which has been partioned into five equal parts. *The* fifth part actually means the *last* part among five. The fifth part was the part, which in addition to the other four parts, completed the whole. That is the way fractional names came into being and acquired their meaning in history. That is why - for instance - Homer made Iliad (K253) say: 'Two parts of the night have passed, the third part remains.'

This historical phenomenon fact leaves us with two conclusions. First fractional names for unit fractions were considered as ordinals and second partitioning was the very concrete source for fractions in the first place.

In olden times the ancient Egyptians already applied fractions. Some important features of the way they dealt with fractions were the following. They distinguished simple fractions like $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, and invented special symbols for them. The eldest fractional system, based on repeated halving was applied in agriculture and economics, for instance, in order to re-allocate the fields after the annual fall back of the Nile within its banks. In a more sophisticated way every non-unit fraction (except $\frac{2}{3}$) was expressed by means of unit fractions, for instance $\frac{5}{6}$ as $\frac{1}{2} + \frac{1}{3}$.

Finally they constructed new relations between unit fractions from previously determined relations by means of substitution (and application of the transitivity of equality).

E.q.

$$\left.\begin{array}{r} \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \\ \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \end{array}\right\} \Longrightarrow \quad \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

These historical facts reflect not only the way in which Western European peoples before the Middle-Age handled simple fractions as their languages show, but also the way young children deal with repeated halving, fractional names and fractions.

(cf. Menninger, 1958; Piaget, 1966; Freudenthal, 1983; Streefland, 1978; 1982)

Exploiting the children's inclinations is necessary at the start and during the further course of the teaching-learning process for fractions. (cf. subsection 3.)

2 *Remarks from research*

A large number of studies display a rich anthology of lapses against rules and procedures for operations with fractions.

(e.g. Pinchback, 1981; Harrison et al., 1980; Hershkowitz et al., 1980; Hasemann, 1980)

Padberg (1983), for instance, presents a survey on mistakes from recent research in Germany regarding 13-14 years old students. Some examples of of the reported mistakes are: $\frac{3}{8} + \frac{2}{5} = \frac{5}{13}$; $\frac{3}{5} - \frac{1}{2} = \frac{2}{3}$; $5 + \frac{2}{3} = \frac{5+2}{3}$; $5 - \frac{2}{3} = \frac{5-2}{3}$ and so on.

It is obvious, that this kind of mistakes can-without any exception-be considered as $\mathbb{N}$-distractor failures. It is also obvious, that the pupils in the cited research walked into this trap because they lacked the means to identify these distractors as such.

Why did they fail? Because their experience with natural numbers distracted them from appropriate rules and procedures. Why could this happen? Because fraction, considered as a new mathematical object, as a brand-new type of (relational)number, as an entity, which obeys its own operational laws, did not acquire this meaning for them. They persisted in regarding two fractions as two pairs of natural numbers, separated by a stroke, and the operation symbol as a signal to operate on them in one way or another, albeit within the context of natural numbers.

What is at the background of this phenomenon? It is plausible to accept the explanation that at the start of the teaching-learningprocess for fractions and also during the further course concrete contexts for fractions have either been disregarded or, were run through in a jiffy as a ceremony. (cf. Freudenthal, 1969). In order not to be accused of arguing on mere speculation I refer the reader to the textbooks for primary and secondary education. Their treatment of fractions betrays the predominant role of rules and procedures at the cost of contexts and applications. The considerations as brought forward in the previous subsections have been taken into account in the research to be reported on in the following subsection.

3 *The present research*

3.1 *Preliminary*

In order to reach a correct interpretation of the research results it is necessary to describe the research context in global terms.

The main aim of the research is to trace the individual learning processes of a group of sixteen fourth-graders (9-10 years old) over a period of two years on the basis of a teaching experiment. Elsewhere I described the basic principles at the background of this experiment and the characteristics of the materials used in it (Streefland, 1984).

Briefly summarized these principles are.

- the constructivistic principle, which means that the pupils themselves construct their operational concept of fraction as a mathematical reality, which will be widely applicable.

This means:

- the construction of fractions from realistic situations, which may serve as a fertile soil to grow them;
- the construction of a language for fractions,
- the construction of rules and procedures for operations with fractions;
- the verbalising construction of a language for these rules and procedures;
- the construction of visual models and schemes as organising tools for the process of mathematisation;
- (partly) the productive construction of problems and exercises.

  The ideas the children already have about simple fractions ans their relations will be both respected and exploited. This means that abstractions and generalisations, rather than being forced upon them, are developed (that is constructed) starting from their own spontaneous informal knowledge and from their own informal strategies in order the learned subject to be made applicable. (cf. subsection 1.)

  The children who cooperated with me in the present research are pupils of a so-called stimulation school, that is a school that is allowed an extra teacher on the ground of a social indication. (underprivileged environment)

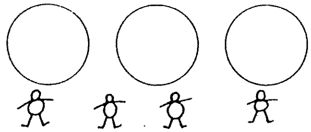
  We chose this kind of school since it may be expected that the difficulties into which many primary school pupils traditionally get with fractions, will show up most clearly with this sample.

3.2 N-distractors produced and unmasked

Since the principle of constructing learning is governing our experiment, some attention shall be payed to it in this brief report. Since children's knowledge of and preference for repeated halving in situations of fair sharing have repaetedly been observed by us and many others, this circumstance shall also be taken into account.

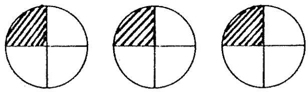
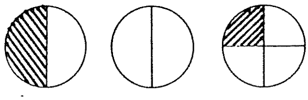
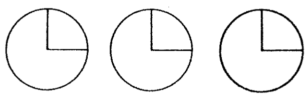
That is why the exploration of distribution situations - as we called them - have played and will play an important part in the teaching experiment.

A situation like that of 4 children equally sharing 3 pancake's

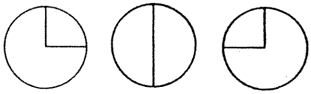


provoked over and over again the construction of a broad spectrum of partitionings and matching descriptions (cf. Streefland, 1982; 1984)

For the sake of brevity results can only globally be summarized.

Observed solutions	*Descriptions given by the pupils*	
1. Successive distribution of one pancake after the other into four parts. 	A quarter and a quarter and a quarter; three times a quarter; three quarters; $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$; $3 \times \frac{1}{4}$; $\frac{3}{4}$	Notice that number of children is the starting point of the distribution
2. Distribution of two pancakes in four halves and of the third in four parts. 	A half and a quarter; a half and half a half; $\frac{1}{2} + \frac{1}{4}$; $\frac{1}{2} + \frac{1}{2} \times \frac{1}{2}$	Notice that again the number of children is the starting point of the distribution. Distribution, however, happens in a more efficient way.
3. Distribution of each pancake into two parts of three quarters and one quarter respectively. 	Three quarters; a whole minus a quarter; a quarter and a quarter and a quarter; three times a quarter; $\frac{3}{4}$; $1 - \frac{1}{4}$ $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$; $3 \times \frac{1}{4}$	Notice that here the result has influenced the way of distributing. Perhaps the clock-dial was the background model.

4. Successive division of the three pancakes directed towards the previously known result.



Three quearters;
a whole minus a quarter;
a quarter and a half;
a half and a quarter;
$\frac{3}{4}$;
$1 - \frac{1}{4}$;
$\frac{1}{4} + \frac{1}{2}$;
$\frac{1}{2} + \frac{1}{4}$

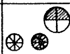
Afterwards the problem was raised to invent arithmetical sentences (in first instance verbal rather than symbolic ones) corresponding to the pictorial ones produced in the previous problems.
This challenge provoked the children as it were to produce a host of diversified answers.
I will confine myself to just one piece of work in order to reconfirm the phenomenon of N-distractor failures.

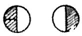
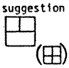
Part of Mike's work.

$$\frac{2}{4} + \frac{1}{8} = \frac{5}{8} \qquad \frac{2}{8} + \frac{2}{8} = \frac{4}{16}$$
$$\frac{1}{4} + \frac{1}{4} = \frac{2}{8} \qquad \frac{2}{8} - \frac{2}{8} = \frac{0}{0}$$
$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{6} \qquad \frac{1}{2} + \frac{2}{8} + \frac{1}{4} = \frac{4}{14}$$
$$\frac{1}{4} - \frac{1}{2} = \frac{0}{2}$$

Besides Mike two other children showed true N-distractor failures.
One girl, Natascha, who made such a mistake, corrected it before she gave in her work.
The strong relation with concrete distribution problems may have prevented the other children from stepping into the N-distractor trap. In order to settle this question we decided to interview all the pupils on N-distractor failures at this stage of their learning processes, which had been traced and described thus far. Nobody will be astonished about the huge

differences between individual pupils that have been observed. That is why the problems to be risen were adapted to the different levels of progression in the learning-process.
Nevertheless three main categories could be identified in the outcomes, which will be described after a brief review of the data. The interviews proceeded according to the following pattern.
The interviewer (I) presented a false arithmetical sentece as taken from the work of some anonymous pupil. The interviewed pupil ($P_{b(oy)}$ or $P_{g(irl)}$) judged the presented sentences either to be correct or false, had to reconstruct the way of reasoning which had led to the faulty outcome and finally was asked to argue the answer, which could be done arithmetically or pictorially. If the final results were in conflict with the first judgement the pupil was made aware of this fact, which - as data wil show - not in every case did the pupil leave with a cognitive conflict.

PUPIL	EXERCISE	FIRST REACTION	PUPIL'S CONSTRUCTION OF FAULTY REASONING	VISUALISATION	(SELF)CORRECTION	ADDITIONAL REMARKS
Sg	$\frac{1}{4}+\frac{1}{4}=\frac{2}{8}$ $\frac{1}{8}+1=\frac{2}{8}$	S. completed with $\frac{2}{8}$ in advance Correct		 (after I.'s suggestion)	$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$ $\frac{1}{8}+\frac{7}{8}=\frac{8}{8}$	S. was very hesitating and acted most of the time at trial and error level.
Mg	$\frac{3}{4}-\frac{1}{2}=\frac{2}{2}$ $\frac{1}{8}>\frac{1}{4}$	Nice way of reasoning, but Incorrect Incorrect	Yes, 3 - 1 = 2 4 - 2 = 2 Yes, 8 > 4		$\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$ $\frac{1}{8}<\frac{1}{4}$	
Og	$\frac{3}{8}-\frac{2}{4}=\frac{1}{4}$ $\frac{2}{4}-\frac{1}{2}=\frac{1}{2}$ $\frac{1}{8}>\frac{1}{4}$	Hesitation. Reproduced faulty reasoning Correct Incorrect $\frac{1}{4}$ is more.	Yes, 3 - 2 = 1 8 - 4 = 4, then second sentence- - Yes, 2 - 1 = 1 4 - 2 = 2 Yes, 8 > 4		$\frac{2}{4}-\frac{1}{2}=0$	
Cg	$\frac{3}{4}-2=\frac{1}{4}$ $\frac{3}{8}-\frac{1}{4}=\frac{2}{4}$	Correct Incorrect (because of the first one)	Yes, 3 - 2 = 1 4 - 0 = 4 Yes, 3 - 1 = 2 8 - 4 = 4		$\frac{3}{4}-2=$ is impossible $\frac{3}{8}-\frac{2}{8}=\frac{1}{8}$	I. invited C. to convert the sentence in one with half quarters. (C. 3 half quarters - 2 half quarters = $\frac{1}{8}$
Mb	$\frac{1}{4}+\frac{1}{4}=\frac{2}{8}$ $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$	Correct Correct	Yes, 1 + 1 = 2 4 + 4 = 8 Yes, 1 + 1 = 2 2 + 4 = 6		$\frac{1}{4}+\frac{1}{4}=$ one half $=\frac{1}{2}$ $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$	M.'s belief in the correctness of his rules was rather persistent.

PUPIL	EXERCISE	FIRST REACTION	PUPIL'S CONSTRUCTION OF FAULTY REASONING	VISUALISATION	(SELF)CORRECTION	ADDITIONAL REMARKS
Sp	$\frac{3}{4} - \frac{1}{2} = \frac{2}{2}$	Hesitation He supposed Incorrect	Argued: $\frac{3}{4} = \frac{6}{8}$, so $\frac{1}{4}$(of $\frac{3}{4}$) and $\frac{1}{2}$ (of $\frac{1}{2}$) will remain and $\frac{1}{4}$ is bigger than $\frac{1}{8}$			By means of converting $\frac{3}{4}$ into 6 half quarters (suggestion of I.) S. was able to show the sentence was obviously incorrect: $\frac{6}{8} - \frac{4}{8} \neq$.
Mb	$\frac{2}{4} - \frac{1}{2} = \frac{1}{2}$	Correct	Yes, 4 - 2 = 2 2 - 1 = 1		$\frac{2}{4} - \frac{2}{4} = 0$, so the given sentence is incorrect	I. had to draw M's. attention on visualisation and number sentences.
Cq	$\frac{1}{4} > \frac{1}{2}$	Incorrect	No, $\frac{1}{2}$ equals $\frac{2}{4}$ and $\frac{1}{4}$ is just one quarter			Cq. was not able to produce the defective reasoning.
Kb	$\frac{3}{4} - \frac{1}{2} = \frac{2}{2} = 1$	Correct	Yes, 3 - 1 = 2 4 - 2 = 2	 (After I.'s suggestion)	K. write: $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$	K. was convinced his first judgement was incorrect.
Fb	$\frac{1}{8} > \frac{1}{4}$ $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{12}$	Incorrect Incorrect	No, $\frac{1}{8}$ is half a quarter and $\frac{2}{8}$ is a quarter $\frac{3}{4}$: 1 + 1 + 1 = 3 4 + 4 + 4 = 12			F. only produced the defective reasoning in second case.
Hb	$1 + \frac{1}{4} = \frac{2}{4}$ $\frac{1}{8} > \frac{1}{4}$	Hesitation Incorrect $\frac{1}{8}$ is half $\frac{1}{4}$	Yes, 1 + 1 = 2, 4 remains just like that Yes, 8 > 4	 $1\frac{1}{4}$($\frac{5}{4}$)	While drawing he stated the sentence was incorrect	
Zg	$\frac{1}{4} + \frac{1}{2} = \frac{2}{6}$ $\frac{1}{4} > \frac{1}{2}$	Correct Incorrect	Yes, 1 + 1 = 2 4 + 2 = 6 Yes, 4 > 2	 (After I.'s suggestion)	$\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ $\frac{1}{2} > \frac{1}{4}$	Z. concluded she was wrong in the first instance.
Ag	$\frac{1}{4} + \frac{1}{4} = \frac{2}{8}$	A. anticipating said $\frac{2}{4}$. I.: $\frac{2}{8}$ Correct after another visualisation of I.	Yes, 1 + 1 = 2 4 + 4 = 8	After suggestion of I.: 	A: $\frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$ A. judged $\frac{2}{8}$ and $\frac{1}{4}$ not to be equel.	A. insisted on $\frac{2}{8}$ being correct either 
Ng	$\frac{1}{4} + 1 = \frac{2}{4}$	Correct	Yes, 1 + 1 = 2 , 4 remains just like that		$\frac{1}{4} + \frac{4}{4} = \frac{5}{4}$	N. concluded her first judgement was wrong.
Ab	$\frac{1}{8} > \frac{1}{4}$	False, because $\frac{1}{8}$ is half $\frac{1}{4}$	Yes, 8 > 4			
Bg	$\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$	Correct	Yes, 1 + 1 = 2 2 + 2 = 4	Distribution situation. 2 bars of liquorice - 4 children. Each child $\frac{1}{2}$	I. The amount of 2 children. B. $\frac{1}{2} + \frac{1}{2} = 1$ but $\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$ is possible too. Other situations.	I. Try pizza's B.  then she decided $\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$ to be a real mistake.

As the data show *six* children did not fall into the trap of ℕ-distractor arithmetical sentences. *Three* pupils reacted correctly in the second instance, after having met a cognitive conflict. *Seven* more children started with an erroneous judgement, but most of them corrected themselves after having faced a cognitive conflict. Only *two* pupils of the last category suggested that both the given outcome and the one they produced themselves, might be correct, because of the different situations. Without going into details I dare to suggest, these judgements show that according to the views of these two children the fractions they had studied thus far depended strongly on the context in which they had been produced. Another phenomenon, which has not been accounted for in the table although it is quite important, is the following. The pupils who accepted the false number sentences as being correct in the first instance were able to reproduce the faulty reasoning substantially faster than the pupils who unmasked the false sentences immediately. Of cource this is due to the excessive resemblance between their own way of reasoning and the rules and procedures that produced the incorrect outcomes.
On the other hand the pupils who did unmask the ℕ-distractors showed to have acquired a solid mental object of fraction, the quality of which was proof against the temptations of ℕ-distractors.

4 *Final remarks*

Pupils passing their long term process of learning fractions need to meet ℕ-distractor conflicts repeatedly. The aim is to develop an attitude of having these distractors both identified and unmasked by appropriate arguments. Since the pupils both are familiar with, and have a preference for, the context of repeated halving and for number relations which describe this process and its outcomes, one can start building up this attitude at the very beginning. If the principle of constructivism is observed as it was in our experiment the pupils will get the opportunity to show frankly and freely their views on fractions and the (incorrect) rules and procedures by which they are governed.
The principle as mentioned not only provokes a challenge for the pupils but also for the theacher (researcher), because the constructions of the children reflect their views on fractions and - as a consequence - bring forward the possibilities to trace the individual learning processes, to diagnose misconceptions and to take remedial measures in order to correct them. The present contribution covers a short period in the learning-

process not only of the pupils but also of the researcher. Even after having collected correct results from all the pupils his job will not yet be finished. Rather than being satisfied with such results he should also mistrust them. Disregard for the decisive influence of the contexts in which problems will be raised and which make the children ascribe specific meanings to fractions and their operational rules and procedures might distract his attention of what is really going on.
Thus identifyning $\mathbb{N}$-distractors with children who learn fractions, means being aware of the distractors, which are already present or can sneak into one's context of discovery.
If history of mathematics is one of them, one will have to his disposal a wealthy source of inspiration and insight, which presages many peculiarities of the informal ways children (will) deal with fractions.

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C. COGNITION/COGNITIVE THEORY

CHILDREN'S CONCEPTIONS AND PROCEDURES IN ARITHMETIC: SOME IMPLICATIONS

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The Strategies and Errors in Secondary Mathematics (SESM) investigation into the causes of secondary school children's errors in elementary algebra revealed misconceptions and alternative approaches to the solution of simple algebraic problems which provided insights into children's understanding of arithmetic as well as algebra. Starting with some particular errors in 'generalised arithmetic' which the earlier Concepts in Secondary Mathematics and Science (CSMS) project had shown were being made by large numbers of children aged 13 to 15 years, the SESM project used individual interviews and short teaching experiments in order to study the causes of those errors (Booth, 1984). The results indicated not only that children had difficulties in algebra due to the way in which they viewed the use of letters, but also that they were often unable to make explicit (and hence to generalise) the procedures they used in arithmetic. Of particular note in this regard was the observation of the use by children in this age group (13 to 15 years) of informal 'child-methods': procedures based mainly on counting and grouping, and which the child could use successfully to solve simple problems involving small whole numbers, but which often did not extend to cases involving large numbers or non-integers, and which rarely lent themselves readily to algebraic representation. In addition, children's attempts to symbolise various procedures showed how widespread were certain erroneous beliefs about the meaning of arithmetical expressions.

Both these issues concerning the kinds of procedure used in handling arithmetical problems and children's views of the meaning of numerical expressions can be illustrated by children's responses to arithmetical word-problem items of the kind illustrated in figure 1, and taken from the

(a) CSMS Number Operations	A gardener has 391 daffodils. These are to be planted in 23 flowerbeds. Each flowerbed is to have the same number of daffodils. How do you work out how many daffodils will be planted in each flowerbed?	391-23 23-391 391+ 23 23x 17	23÷391 391x 23 23+ 23 391÷ 23
(b) CSMS Place Value and Decimals	My car can go 41.8 miles on each gallon of petrol on a motorway. How many miles can I expect to travel on 8.37 gallons?	41.8+8.37 41.8÷8.37 41.8-8.37	8.37÷41.8 8.37-41.8 8.37x41.8

Figure 1. Examples from CSMS Number Operation Test and Place Value and Decimals Test. In each test, children were asked to select the number sentence which expressed wht they would do in order to solve the problem.

CSMS Number Operations and Place Value and Decimals tests respectively. The CSMS data had in fact shown that selecting the correct number expression to represent a given word problem was not a trivial task for many children (Brown, in Hart, 1981). Evidence from the interviews conducted as part of the CSMS research programme further indicated that children were often able to solve the given problem, even if they could not select the appropriate number expression from those presented (Brown & Küchemann, 1976). Clearly these children did not solve the problem by the more formal process of writing an appropriate number expression and then computing the result. Rather, they were observed to proceed by such strategies as 'counting on', repeated addition, or grouping by systematic trial and error. Often the children using these procedures appeared to be unable to relate the procedures they themselves were using to the more formal representations given in the problem.

More recent work with secondary school children conducted as part of the SESM study, as well as work with 10 year-olds in Britain and 12-year olds in Malaysia, has substantiated and elaborated upon these results. In the case of the SESM investigation, an exercise was conducted with a group of 126 children aged from 12 to 15 years (from one school only) in which the children were presented with either a test of nine whole-number word problems of the kind shown in figure 1(a) and taken from the CSMS Number Operations test, or a test of six items of the kind shown in figure 1(b) and taken from the CSMS Place Value and Decimals test. Only five of the 126 children were able to choose the correct number expressions for all the items in the given test, with 105 children making two or more errors. In some cases children were apparently unable to recognise what operation was involved (even though they were subsequently able to solve the problems especially if the numbers involved were smaller whole numbers), while in other cases they chose the correct operation but seemed to have different ideas about the form in which the expression should be written. Such children, for example, would select the form '23÷391' instead of the (correct) form '391÷23', or alternatively would select both expressions as being correct. The same observations were made in the case of the 10 year-olds and 12 year-olds referred to above, and indeed have been noted by other researchers and teachers.

Subsequent investigations into the reasons for children's choice of these incorrect numerical expressions were conducted via both clinical interviews and teaching experiements, and indicated the following explanations:

a) Some children interpret 23÷391 as '23 divided into 391', perhaps confusing the expression with the 23 $\overline{)391}$ form of recording division. An expression such as 4-17 is similarly read as 'take 4 away from 17'.

b) Many children adhere to the rule that 'you always divide the large number by the small number' (similarly for subtraction). These children will choose either or both expressions, since for them the order of recording is irrelevant.

c) Other children interpret the numerical expression in terms of the context of the problem. If they 'know' from the problem which number must be divided by which, they see no difficulty in writing the division in either order, since the sense is defined by the context.

d) Some children consider that all operations are commutative, so that expressions such as 23÷391 and 391÷23, are regarded as equivalent. This appears to be a decision based upon the symmetry of the configuration, and does not take the meaning of the division sign into account. It is likely that these children's acceptance of the commutativity of addition and multiplication is similarly based on symmetrical considerations rather than upon an understanding of the operations involved.

These observations indicate that children's errors in arithmetical recording are not trivial mistakes based upon carelessness or idleness, but rather reflect definite points of view about the meaning of notation, the meaning of operations, and the nature of mathematical representation. The findings also indicate that these beliefs may be as prevalent among 14 and 15 year-olds as among 10 year-olds.

In addition, the same investigations provided support for the suggestion that some children's difficulty in handling mathematical problems derives in part from their use of informal counting procedures. The difficulty here lies not in the 'incorrectness' of these informal methods; indeed, the procedures used are often completely correct. Rather the difficulty stems from the fact that these procedures are often only useful where the numbers involved are relatively small whole numbers which can be dealt with 'intuitively'. The same methods often become extremely cumbersome and difficult to apply when the numbers involved are large, and often are inappropriate to the case of non-integers, being largely based on counting techniques. Children who have not recognised the relationship between their own informal methods and the more formal mathematical representations are unable to change to the latter when their use becomes necessary as a result of the kinds of values involved. Consequently, the range of problems which the child can successfully handle is restricted by virtue of the limited procedures available to the child.

The use by children of informal mathematical procedures at primary school level has been noted by many workers (e.g. Erlwanger, 1975; Ginsburg, 1977; Groen & Resnick, 1977) and is becoming increasingly recognised by others in the field. What has perhaps been less widely recognised is the prevalence of the use of such methods among secondary school children, although the existence of 'alternative conceptions' among children in science has been the subject of considerable interest. The existence of these informal methods in mathematics must have profound consequences for teaching, curriculum development, and research. In the first case, if we wish children to be able to solve the more complicated versions of problems, then we must help children understand the more formal mathematical procedures, since these are the ones that are likely to be needed in order to do this. Using a calculator to overcome problems of computation may be of limited value if the child does not know what buttons to press, and in what order. Helping the child to learn these procedures means recognising the methods that children actually use, and helping them both to understand the relationship between what they are doing and what the teacher is presenting, and to appreciate the value of making this connection, by helping them to recognise the limitations of their own approaches (cf Case, 1978). In addition, attention to the nature of the solution process inevitably requires attention to the forms in which that process might be recorded, since it is often only by attempting to formulate a procedure that the child's attention is directed to the process itself rather than to the answer which is obtained. Children's ideas concerning the question of mathematical representation thus become an important aspect of their work on developing more general procedures. At the same time, the value of children's own informal approaches for the solution of simple problems must not be overlooked. It may well be that for some children, time might more usefully be spent in helping children to develop and extend their own procedures, so that these might be applied to a wider range of problems. This requires a careful analysis of the kinds of methods children do use, and the ways in which these might be extended.

The nature of 'child-methods' used may also have consequences for curriculum development. In some cases, children's informal procedures may be more akin to procedures used by computers or indeed used in 'higher' mathematics (e.g. iterative processes). This may suggest a review of the kinds of procedures taught in school mathematics and the emphasis placed upon them.

The observation of the use by children of alternative methods must also have implications for research in mathematics education. In particular, as Vergnaud has stressed (1979), the notion of 'complexity' in describing problems which are set for children, and from which may be derived assumptions concerning the child's level of cognitive functioning, needs careful attention. A task which the researcher sees as 'complex' may not be so for the child who has an informal procedure for handling that problem. Certainly it may well be true that some apparent contradictions in research findings might be resolved if attention is directed to the procedures by which the tasks were actually performed. The issue may therefore well be one which merits further investigation from the point of view of research and theory as well as pedagogy.

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The mathematical "reversal error" and attempts to correct it

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Introduction

In recent years, there has been interest in the "reversal error", a phenomenon in which many people reverse natural referents for symbols. Rosnick (1980), for example, found that many calculus graduates write the equation 6S = P when asked to produce an algebraic expression for the statement that there are six times as many students as professors in a particular university. Lochhead (1980) collected from university faculty members and high school teachers responses to a request to write an English sentence giving the same information as the equation A = 7S where A is the number of assemblers in a factory and S is the number of solderers. Of 202 faculty members, 35% gave an incorrect response, the proportion being over 50% for those in faculties other than Physical, Natural, Behavioural and Social Sciences. Of the 148 high-school teachers, the proportion incorrect was 47%; for teachers of the physical sciences it was 28%, while for teachers of Natural Behavioural and Social Sciences it was 67%. As a result of tutoring interviews, moreover, carried out with undergraduates, Rosnick and Clement (1980) concluded that "the reversal misconception is a resilient one which is not easily taught away".

In the present study, the reversal error displayed by graduate students enrolled in a Master of Business Administration course was examined and remedial action attempted. The participants, while all aimed at the same eventual qualification, come from a large variety of academic backgrounds.

The study was conducted in two stages: an initial test, and a low-key remedial phase followed by a final test. The first stage involved the whole sample of 104 students; the remedial strategy and final test were administered to a reduced sample of 45 participants.

Method

Stage 1: Investigation of degree of reversal in general.

Of over one hundred full-time Master of Business Administration students at the Australian Graduate School of Management who were invited to participate in the study, about 95% provided usable responses.

Each subject was asked to respond to the following questions:

Question 1

Write *one* sentence in English that gives the same information as the following equation: A = 7S. The letter A represents the number of assemblers in a factory; S is the number of solderers in the factory.

Now, please clarify the answer you gave.

Question 2

Write an equation using the letters C and S to represent the following statement:

> At Mindy's restaurant, for every four people who ordered cheesecake there were five who ordered strudel.
>
> Let C represent the number of cheesecakes ordered and let S represent the number of strudels ordered.

The first of these questions is the one used by Lochhead (1980) in his study with university staff and high-school teachers. It was employed in this study to allow comparisons to be made. The second question was used by Rosnick and Clement (1980) as part of a remedial-teaching unit which comprised a section of their study. The above questions were used in this study in order to provide subjects with different modes of operation: the first question involves interpreting an equation, the second requiring the writing of an equation to represent a situation. We refer to the explanation of an equation in words as the *equation-interpretation mode*; writing an equation corresponding to a particular situation we call the *equation-formation mode*. In addition to the possibility that the reversal phenomenon might be stronger in one mode than the other, the use of both modes allows the examination of tendencies among those who exhibit reversal in one mode but not the other.

As well as being asked to respond to the questions, each participant was asked to state the principal discipline or area of study of his or her undergraduate degree.

Usable responses were received from 104 students. This represents about 95% of those approached.

For each question, a substantial proportion of subjects exhibited reversal. The relevant frequencies and percentages (rounded to the nearest unit) are presented in Table 1.

		Question 2 *(equation-formation mode)* reversal	correct	
Question 1 *(equation-interpretation mode)*	correct	21 (20%)	35 (34%)	56 (54%)
	reversal	39 (38%)	9 (9%)	48 (46%)
		60 (57%)	44 (42%)	104

Table 1: Frequencies and percentages giving correct responses and exhibiting reversal

The results indicate that about 46% of MBA students exhibit reversal when translating from the equation A = 7S, and about 57% reverse when writing an equation corresponding to the situation at Mindy's restaurant. The result for the equation-interpretation mode is similar to that achieved by Lochhead's high-school teachers, 47% of whom displayed reversal. (Of university faculty-members, only 35% exhibited reversal in Lochhead's study.)

About 30% of MBA students display reversal in one mode but not the other; however, the tendency to reverse is much more marked in one direction than the other. In fact, under the hypothesis that the proportion reversing for equation-formation but not equation-interpretation is no different from that reversing in the opposite direction, the probability of the results (and of more extreme results on a two-way test) is about 0.043. It thus appears that among MBA students who exhibit one-way reversal, a far greater proportion reverse in equation-formation than in equation-interpretation.

There are at least two factors which may influence the tendency of Question 2 to invite reversal to a greater extent than Question 1. First, there is the mode-difference already indicated. Secondly, although both (implied) equations may be written in the form $X = aY$, the value of a in Question 1 is a whole number whereas in Question 2 it is a non-integer. It is possible that this factor exerts a substantial influence and that the mode-difference is not important. In a future study, the value of a should perhaps be controlled.

As mentioned previously, graduate students enrolled in MBA programmes came from a variety of undergraduate backgrounds, even though they are ultimately destined for the same graduate qualification. In the sample used in this study, subjects possessed undergraduate majors in Engineering, Science, Mathematics, Computing, Medicine, Architecture, Commerce, Education, Psychology, Social Sciences, Law and Arts. In view of the finding that persons involved in different disciplines have different potential for displaying the reversal phenomenon (Lochhead, 1980), the data were divided according to undergraduate background and re-analysed. The "Science-Technology" grouping comprised subjects who had majored in Engineering, Science, Mathematics, Computing and Medicine. All other subjects were assigned to the grouping labelled "Other". The frequencies and percentages of correct and reversed responses are presented in Table 2.

			Question 2 (*equation-formation mode*) reversal	correct	total
Question 1 (*equation-interpretation mode*)	Science-Technology graduates	correct	12 (22%)	26 (47%)	38 (69%)
		reversal	14 (25%)	3 (5%)	17 (31%)
		total	26 (47%)	29 (53%)	55
	Graduates of other faculties	correct	9 (18%)	9 (18%)	18 (36%)
		reversal	25 (51%)	6 (12%)	31 (63%)
		total	34 (69%)	15 (31%)	49

Table 2: Frequencies and percentages giving correct responses and exhibiting reversal, by undergraduate speciality

Although the percentage of graduates in fields other than science and technology who reverse in Question 2 and not in Question 1 is not much greater than those who exhibit the opposite phenomenon, the same is not true for graduates in the Science-Technology areas. If it is postulated that percentage reversing in one direction is the same as that reversing in the other, the probability of the obtained results (or of any more extreme) is about 0.035. This leads to the conclusion that Science-Technology graduates who reverse in one direction are much more likely to exhibit reversal in equation-formation problems but not in equation-interpretation problems than in the opposite direction.

Stage 2: Remedial action and final test.

The researcher was able to attempt remediation on about half the initial sample, during regular classes in the MBA subject "Quantitative Methods".

The strategy intended to correct the tendency to reverse was deliberately low-key and was introduced during a discussion of proportionality. The students were first introduced to the concept of proportionality between two variables and to the proportionality symbol. It was then demonstrated that a constant of proportionality can be employed in order to transform a statement of proportionality into one of equality:

$$A \propto B$$
$$A = kB$$

A discussion then ensued, with values of k respectively less than or greater than unity, attention being paid to whether A or B was larger for each type of k. An example, $C = 4D$, was discussed, C representing number of cats and D being number of dogs. The earlier "assemblers and solderers" and "Mindy's restaurant" questions were referred to also.

About two weeks after the above episode, the sample was asked to respond to two further questions, as follows:

Question 3

In the equation R = 4G, the letter R stands for the number of Russians in a particular community while G stands for the number of Germans.

Write a simple sentence that conveys the same information.

Are there more Russians or more Germans?

Question 4

For every worker using a Packard machine in a particular office, four workers use Canon machines. Using the letter P to represent the number of Packard machines used in the office and C to represent the number of Canon machines, write a simple equation corresponding to the above statement.

The responses to these two questions were matched with those previously obtained for Questions 1 and 2, the following frequencies being obtained:

Equation interpretation:

		Question 3 (after treatment) reversal	correct	
Question 1 (before treatment)	correct	0 (0%)	23 (51%)	23 (51%)
	reversal	2 (4%)	20 (44%)	22 (49%)
		2 (4%)	43 (96%)	45

Equation formation:

		Question 4 (after treatment) reversal	correct	
Question 2 (before treatment)	correct	0 (0%)	10 (23%)	10 (23%)
	reversal	6 (14%)	28 (64%)	34 (77%)
		6 (14%)	38 (86%)	44

The above results demonstrate that the phenomenon of reversal is not necessarily "a resilient one which is not easily taught away" (Rosnick and Clement, 1980). For the smaller sample treated in stage 2 of this study, only about 4% exhibited reversal in equation interpretation after treatment compared with 49% initially. In the case of equation formation, 14% reversed after treatment compared with 77% initially; in this case, however, the situations were not strictly equivalent because Question 2 involved an equation which had a non-integer constant of proportionality whereas for Question 4 the constant was integral.

Conclusions

The results of this study show that graduate students of business management exhibit equation-interpretation reversal to roughly the same extent as high school teachers; for equation-formation, the reversal rate is slightly greater. Of those who reverse in one case but not the other, a far greater proportion reverse in equation-formation only than in equation-interpretation only.

As would be expected, science-technology graduates reverse to a lesser (but still substantial) degree than graduates in other areas.

A low-key remedial strategy markedly lowered the rate of reversal for this population, in both equation-interpretation and equation-formation modes.

Like other studies, this one has used algebraic symbols which are the initial letters of the objects whose numbers are being represented in the equations. Further investigation might indicate when the onset of reversal occurs in this sort of situation and whether reversal is exhibited if letters other than the initials referred to above are employed. A theoretical explanation of the phenomenon might be useful, also, in helping to eliminate it in the wider populations investigated.

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ACTIONS AS A MEANS FOR ACQUIRING MATHEMATICAL CONCEPTS

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0. PRELIMINARIES

It is a commonplace that concepts in general and mathematical ones likewise are not acquired by rote learning of a verbal definition. But it appears to be a widely open question which could be the means and media by which (mathematical) concepts can be acquired in an as much complete form as possible as to have influence on the practical and mental behavior of the individual (knowledge for action). To give answers to this question it will not suffice to investigate empirically how students or adults know concepts and how they make use of them. This will mainly shed light on a status-quo of learning and teaching. Here I want to analyse mathematical concepts with respect to their genesis and their complex relationships to the individual. This means I do not investigate any given psychological reality but the result of such an analysis will show of which kind (content and form) the cognitive schemata representing (mathematical) concepts in an individual could/should be and how possibly they can be built up and constructed by the individual. A basic assumption underlying this approach is that the epistemological strucutre of concepts reflects the psychological structure of schemata if they are to contain the meaning of the mathematical knowledge condensed in concepts. An argument for this assumption is that the epistemological structure is the result of psychological and social processes which led to the concepts with which the learner now is confronted. The verbal description of a concept often will not convey these processes which on the other side are in a way an integrating and indispensible part of the concepts.

Verbal learning and study of the processes of concept development will not lead to the formation of the corresponding schemata in the learner though knowledge about such processes is important in guiding his or her acquisition of concepts by individual processes. This meta-knowledge will contain information about the mediating means of individual concept formation and it is these mediators which are the topic of the following deliberations. They are based on the thesis that an essential mediator between (mathematical) concepts and the individual are material and mental actions. This position was stimulated by various authors (A. Aebli, 1981, but especially by Soviet psychologists(A.N. Leontev and his school) who have developed activity theory, see Wertsch 1981. A related position is developed in Dörfler 1983a and 1983b.

1. ACTIONS AS MEDIATOR BETWEEN CONCEPTS AND THE INDIVIDUAL

To understand the function of actions mediating between concepts and a learning and developing individual one must have a look at the roles which can be played by concepts in the context of an action. This will show that there is a mutual relationship between actions and concepts. Actions are planned, regulated, controlled, supervised and described by concepts and the latter ones are developed by the corresponding actions as cognitive schemata of the individual. In this sense the individual construct of a concept is developed further, gets more extended, generalized and more specialized and refined by every action in which it is envolved in one way or the other.

Every action specifices several of its components. An action is carried out on certain elements and by the use of certain elements (instruments of the action), it has a goal which often is a kind of result or product, a starting point or state, a way of getting from there to the goal (the process of the action), it realizes the goal and thereby establishes a relationship between the initial configuration of all the elements and the goal. For an action there are certain conditions for it to be executable which usually are expressed as properties of and relations between the elements of the actions.

A concept can stand for each of these components or aspects of an action. This "standing for" means that the concept grasps the schema, the structure, the general form of the respective aspect or component of the action. It is this schema, the pure form of which usually is isolated in the mathematical formulation (definition) of concepts. For the connection of concepts with actions this abstract general form is not sufficient but also their content is needed which consists of concrete, special instances of the general form. From the examples we will see that the same concept (characterized by its form, its abstract structure) can stand for different of the above aspects or components of different actions. In principle there is a process of development with respect to the relationship between actions and concepts by which either part of this system is developed further by making more and more precise the form and simultaneously extending the content. This process is a personal, cognitive one with respect to the learning individual but a social one as well, occurring in the communicative system of a group (of scientists for instance). It is essential to keep in mind the central role of the actions (actual or mental ones) for this process. It is the actions which are the agens which on the other hand makes use of the already built up concepts.

In the following paragraphs examples will be given for concepts standing for each of the aspects of an action as there are:
- elements of an action (which one is acting on)
- instruments of an action (by which one is acting)
- conditions for an action (executability)
- procedure of the action (transformations, transitions, combination of elements and the like)
- goal of the action (result, product)
- relation between inital state and goal (which is established by the procedure of the action).

2. CONCEPTS AS SCHEMA OF THE ELEMENTS OF AN ACTION

Each of the examples in this and in the following paragraphs deserves a detailed analysis which should include theoretical investigations (didactical analysis) and empirical research as well. Thereby the latter one should study how the actually acting individual depends on his or her personal construct of the respective concept and how actions contribute to the building-up of this construct. We give examples where the elements are material and ideal objects as well:
- rectangle as the schema of tiles for tiling walls and floors
- rectangular prism as the schema of bricks for house construction or of objects which easily can be piled up
- geometrical concepts as the schema of building blocks for constructions of a great variety of types and purposes, compare for this the Principle of Operative Concept Formation as formulated by Bender and Schreiber, 1980
- straight lines and planes as the schema of objects which easily can be shifted along each other, compare Krainer 1982

- line segments as the schema of objects which can be assigned a length
- numbers as the schema of quantities for actions like adding two quantities of the same kind
- natural numbers as the schema of elements which can be counted or ordered
- sphere as the schema of elements (objects) which can be rolled easily into all directions
- for practically all mathematical objects we have a variety of operations (mental actions) which we perform with them: we differentiate functions we integrate them, we transform equations, we take the limit of a sequence, we have several algebraic operations for matrices and of course for numbers of any kind. The differentiable function is then the schema of elements for the operation of differentiation and likewise in the other cases. So even pure mathematics abounds with operations (on mathematical objects as elements). Some of these mental actions model material actions whose elements are modeled by the corresponding mathematical elements. This holds for instance for the operations with natural and (positive) rational numbers or for the definite integral as modeling a practical method for determining areas and volumes.

3. CONCEPTS AS SCHEMA OF THE INSTRUMENTS OF AN ACTION

Before citing some examples for this aspect of concepts it should be mentioned that "concept" in the position held here comprises also properties, (logical) consequences of what ususally is used as the definition of a concept. For instance, "rectangular triangle" comprises the Pythagorean theorem and other of its properties. Some examples are:
- rectangular triangle as the schema of means for determining distances in surveying. Similarly the trigonometric functions can be viewd as the schemata of instruments for various actions carried out in surveying.
- proportionality as the schema of means for calculating values of certain quantities (like prize). Similarly this holds for other kinds of functional dependence
- equations of every kind as the schema of an instrument for determining unknown quantities
- matrices as instruments for describing correspondences between quantities
- graphs as schematic instruments for the description of networks; similarly many other mathematical concepts are a means for representing, describing real-life situations whereby the concept schematizes the structure of this instrument.

The concepts of the examples of paragraph 2 can also be viewed from the instrumental point of view. The sphere as a ball is a means for many purposes, the rectangle is the schema of the instrument by which the floor is tiled, the natural numbers are the tool for counting and so on. This shows that usually a concept will reflect many or even all of the aspects considered here. Sometimes even the role of a conept in an action can be viewed to be objective (paragraph 2) and instrumental at the same time.

An other very important instrumental function of concepts is the role which they play for mental actions like planning, anticipating, mental construction, checking alternatives, extrapolation and so on. These are proper actions which are carried out mentally in a schematic way thereby using the schemata of elements, instruments, properties, relations going into the material actions which are planned, anticipated and so forth. I hold the opinion that this specific relation between mathematical concepts and actions of human beings cannot be overemphasized and can be an effective tool for the development of appropriate cognitive schemata of the concepts by the learning individual.

4. CONCEPTS AS SCHEMA FOR THE CONDITION OF EXECUTABILITY OF ACTIONS

Most of the previous examples can be interpreted also in this sense in an immediate way since the conditions for executability are reflected by (the schemata of) the properties of and relations between the elements and instruments of an action). Some new examples are added:
- rationality of a proportion as a condition for measuring one quantity by an other (commensurability)
- additivity (of quantities like length, volume) as a condition for the meaningful execution of actions like sticking together, filling together, combining
- convexity as the schematic condition for reachability along straight paths
- simple connectivity as a condition for shrinking loops or spheres arbitrarily (this is used as an "intuition" in mathematical considerations)
- many mathematical concepts like differentiability describe the schema of the condition for certain operations to make sense. The real numbers can also be understood in this way: condition for taking arbitrary limits; likewise the other kinds of numbers can be viewd as reflecting conditions

for the execution of operations like subtraction (negative numbers), division (rational numbers) or solving (algebraic) equations (complex numbers). The integrating and constituent role of the operations (mental actions) should not be neglected as it is done in the usual mathematical construction of the various number fields.

5. CONCEPTS AS SCHEMA FOR THE PROCEDURE OF AN ACTION

Here again is an abundance of examples:
- algebraic terms as schemata of number calculations, of calculations with quantities, of actions with material objects depending on what the variables and the operation signs (+,x,\) stand for. Here even the symbolic notation points to the reflected schema of the actions. Terms for functions also fit into this pattern as being the schema for calculating (this is an action) the value when given the argument; compare for this Kiesow, Spallek, 1980
- in many cases a concept represents the schema of the construction or production of certain objects (the schemata of which are also an aspect of the concept). The concept of a circle comprises rules for its construction which are the schema of certain actions (for instance construction by using a thread fixed at one of its endpoints). For many people these construction schemata are not part of their personal constructs of the concept though such an operative understanding of the concept is essential not only for many ways of its instrumental use but also for deriving logical consequences from an appropriate definition of the concept.
- place value systems as a schema of counting by bundling. Here the common way of denoting numbers corresponds to the result of the bundling actions. These actions are the genetic source for the place value system.
- fractions as the schema for producing certain parts of a whole; the fraction stands also for the produced part and its relation to the whole (this refers to paragraphs 6 and 7)
- fractions as the schema for measuring a quantity b by a quantity a: find a natural n, such that $\frac{1}{n}a$ fits m times into b; or find m, n such that m.a equals n.b (for instance by joining rods of the respective lengths)
- derivative as the schema for measuring (approximately) instantanteous relative change of quantities.

One central thesis of this approach is that an efficient way for building up the cognitive schema of a concept is to carry out (actually or mentally) the actions (or operations) the schema of which is reflected by the concept. Thereby the concept in a first stage is in fact represented by instructions for the respective actions which then are stepwise formalized and schematized. In a way for the learning individual this is a process of interiorization as it is formulated by Piaget. From this we must discern the change and development of notation for the instructions (for the concept) but these two processes are dependent on each other.

6. CONCEPTS AS SCHEMA FOR THE GOAL OF AN ACTION

In many cases (material) objects of a certain form are the goal of actions and it is then this form (schema) which goes into the adequate concept. We can cite here again many geometrical forms. Other examples are:
- algebraic and function terms considered as standing for the result, the product of the corresponding operations (calculations), i.e. for a certain number
- fractions as standing for the respective part of a (possibly absent) whole
- graphs as the schema of constructed communication networks or of detected social relations
- decimal numbers as the product of counting by bundling ten by ten.

It should be pointed out here that there is a strong relationship between the schema of the goal of an action and the schema of the procedure of an action. The two cannot be separated really in the concrete action but only in an analysis of the structure of actions. Nevertheless the mathematical definition of concepts often is restricted to the static aspect expressed by the goal. Consider for instance rotations in the plane: the procedure itself is not reflected explicitly in the mathematical definition but only the result of it and its relation to the initial position (despite the use of "dynamic" words in the definition).

7. CONCEPTS AS SCHEMA FOR THE RELATIONSHIPS ESTABLISHED BY AN ACTION

We have in mind here relationships (and their schema) between or at elements of the action (including its product) which are intended and realized by the action. These relationships can in fact coincide with the goal of an action which then has no material product but only results into a system of relationships. This again is a static aspect of action which often is reflected by the usual mathematical representations of concepts. For a rotation it is the correspondence between the images and preimages established by the action whereby this correspondence apparently is a mental construct induced by the action. Similarly we have this for all mappings;their mathematical definition as a set of ordered pairs just represents this kind of relationship. A rational number expresses a relationship (ratio) between a part and a whole or between two (commensurable) quantities. If one constructs a quadrangle from rods the quadrangle can be viewed as the established system of relationships between the rods or as the (material) product of the action. It should become clear from the examples how intimately related this last aspect is with other ones especially with the procedure of the action and its rules.

8. CONCLUSION

When holding the position that actions and operations play a constituent role for the epistemological and psychological structure of concepts we don't assert that we can grasp thereby all aspects of mathematical concepts. So we have not considered the descriptive (concepts as models) and the logical aspects (mathematical theory of concepts). But we think that for many basic concepts the corresponding actions can be the genetic way, the mediator for their individual acquisition which should be investigated empirically and exploited didactically. The latter already is being done with considerable success in the lower grades of mathematical education by many kinds of material objects (like Cuisennaire rods).

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SOI-LA IN THE IDENTIFICATION OF MATHEMATICAL TALENT IN CHILDREN

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The authors have for several years been involved with youngsters in the 10-12 years age group who have participated in on-campus mathematics enrichment classes in the second semester each year. These children have been chosen by their teachers as having high ability and potential in the mathematics area, and as being those pupils most likely to benefit from what was being offered at the University.

Our experiences with these pupils to whom we offered a variety of problem solving activities not apparently closely related to "school mathematics", led us to speculate that their teachers might be using somewhat inappropriate criteria in their selection. Indeed, some teachers expressed to us their misgivings over their own capacity to select the pupils with the highest mathematical potential.

Accordingly, a study was undertaken in one large suburban primary school with the grade 6 and 7 children and their teachers. To determine the criteria used by their teachers, and how, using these criteria, the teachers would rank their pupils, a teacher schedule was administered. This information was augmented by data from informal interviews. The pupils completed the ACER profile series operations test and the 13 maths-related tests from the SOI test, Form A. At the grade 7 level we also had available the scores on the "October tests", the T.O.L.A. and MA7X tests.

From these results it was possible to analyse the interrelationships between pupils' scores on the various instruments. Further analyses with respect to process areas of the "October tests" served to identify the process areas of mathematics which are tests, and thus, ultimately, those which are seen by teacher and pupil as "real mathematics", and taught and tested by the teachers. The method of analysis was to categorize items on these tests in terms of the descriptions of the SOI process items.

The Structure of Intellect model was first postulated by J.P. Guilford (1956) and subsequently refined by Guilford and co-workers. It was arrived at by factor analytic studies of many tests of intellect to locate and define special abilities rather than be satisfied with the concept of a global 'g' or general intelligence quotient, which is a pedagogically useless statistic. Guilford conceptualized the structure of the intellect in the form of a three-dimensional cube with 120 three-dimensional components. The three axes of the cube are operations, contents and products. Meeker (1969) provided an interpretation of Guilford's model, and related these to many other tests in current use with school pupils, as well as to teaching-learning situations.

The SOI tests developed by Meeker relate directly to the skills and abilities which must be mastered for learning to proceed to higher levels. By exploding and transforming the three-dimensional Guilford model to a two-dimensional form, Meeker presented a schema depicting the five abilities areas, and the range of processes encompassed by each, for each of which process tests are available.

The five abilities areas, and the range of tests available to test processes within each, are as follows:

SOI-LA TESTS

General Areas

TABLE 1.

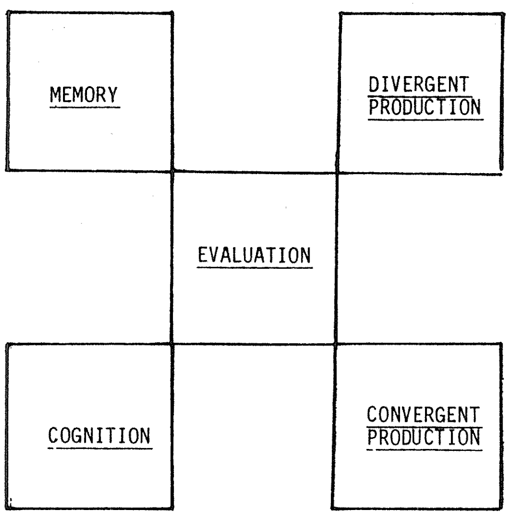

TABLE 2

SOI-LA TESTS

Details of Areas

MEMORY

MFU-memory of figural units

*MSU-V-Visual / A-Auditory — memory of symbolic units

*MSS-V-Visual / A-Auditory — memory of symbolic systems

*MSI-A-Auditory / V-Visual — memory of symbolic implications

MMI-Memory of semantic implications

DIVERGENT PRODUCTION

DFU-Divergent production of figural units

*DSR-Divergent production of symbolic relations

DMU-Divergent production of semantic units

EVALUATION

EFU-Evaluation of figural units

EFC-Evaluation of figural classes

*ESC-Evaluation of symbolic classes

*ESS-Evaluation of symbolic systems

COGNITION

CFU-Cognition of figural units CSS

CFC-Cognition of figural classes

*CMU-Cognition of semantic units

*CFS-Cognition of figural systems

CMR-Cognition of semantic relations

*CFT-Cognition of figural transformations

CMS-Cognition of semantic systems

*CSR-Cognition of symbolic relations

*CSS-Cognition of symbolic systems

CONVERGENT PRODUCTION

NFU-Convergent production on figural units

*NSS-Convergent production of symbolic systems

NST-Convergent production of symbolic transformations

*NSI-Convergent production of symbolic implications

* Mathematics related tests

Researchers such as the Meekers have, since the early sixties, using Guilford's model as a starting point, identified the abilities which are critical for learning. Thus application of tests for these abilities provides a diagnosis of individuals' present learning ability status, and a prescription for developing some or all to an appropriate level for success in tasks being undertaken. Likewise, the information allows presentation of learning opportunities appropriate to individuals' developed learning abilities.

In this study, Form A of the test was administered and the results of 13 mathematics-related tests were used in the analysis. These tests are:

- MSU(a): Auditory memory of symbolic units;
- MSS(a): Auditory memory of symbolic systems;
- MSI(a): Auditory memory of symbolic implications;
- CMU: Cognition of semantic units;
- CFS: Cognition of figural systems;
- CFT: Cognition of figural transformations;
- CSR: Cognition of symbolic relations;
- CSS: Cognition of symbolic systems;
- ESC: Evaluation of symbolic classes;
- ESS: Evaluation of symbolic systems;
- NSS: Convergent production of symbolic systems;
- NSI: Convergent production of symbolic implications;
- DSR: Divergent production of symbolic relations

Thus, for ability to cope successfully with arithmetic, pupils would need to be competent with the vocabulary of maths and verbal concepts (CMU) with the auditory attending (MSU(a)), auditory sequencing (MSS(a)), comprehension of numerical progressions (CSS), judgement of correctness of numeric facts (ESS), application of maths facts (NSS), and judgement of arithmetic similarities (ESC), whereas for success in mathematics, pupils need to have developed their abilities in coping with the constancy of an object in space (CFS), spatial conservation (CFT), inferential memory (MSI), comprehension of abstract relations (CSR) and form reasoning, or logic (NSI).

Bearing in mind that no Australian norms exist for the SOI tests and also the SEC of the school population, it is probable that the translational factor favours our grade 6 and 7 pupils. The whole population means on CMU, CSR, CSS, MSI, ESC, ESS, NSS and NSI were fairly high. However, the scores on CFS and MSU(a) did not show this effect, and the mean scores on

CFT and DSR were extremely poor.

When histograms for each class on each test were inspected, a curious effect was noticed in the tests where the average scores were high. There was an almost bimodal distribution, with a large "gifted" population, a gap where the very able group should be, and another population in the above to below average segment. It looks as if early drills of terms and rules and over-practice has had the effect of accelerating the performance of the upper quartile into the "gifted" zone, thus invalidating the test for identification of the genuine gifted pupils. However, for less mature pupils the teachers' concentration on mechanics appears to have had little effect in hastening the mastery of the learning abilities involved.

Although when the syllabus content is inspected it is not immediately clear why the pupils made such a poor showing on CFT and DSR, the authors have noted over a period of several years that the majority of youngsters attending the on-campus classes for gifted mathematicians have been weak in these very areas to the point where we have adopted the practice of concentrating on geometry and spatial concepts as a matter of urgency early in the classes. We had also found that most children were unable to talk about the way they intended to go about solving problems, at least until they had had some practice at it, which is in all likelihood related to their low MSU(a) scores. Moreover, at first all but a very few appeared not at all confident in employing and testing a <u>variety</u> of problem solving strategies (DSR).

Observations of the usual classroom teaching strategies employed tend to shed some light on the above. Perhaps because early emphasis is placed upon arithmetical accuracy and speed in the system, teachers tend to present "types" of sums with a set unvarying sequence of steps for their correct solution. Pupils practise these, and learn to recognize the "types". They thus are able to achieve results without having necessarily mastered the underlying concepts. This reinforces the tentative explanation of the bimodality of results evident in the high scoring tests. The same thing occurred with these tests except that a very few scored very high, and the second population started around average to low average. Some pupils may well have discovered some of these processes for themselves, and not necessarily within the classroom context, while others were more or less unaware of their existence.

Further evidence to support this thesis came from some of the teachers themselves. Although they did not administer the SOI tests to their pupils in this study, they were present at the scoring of the tests and hence saw and studied the items. Many expressed confusion at first about the spatial tests, and most had great difficulty in understanding the divergent reasoning test. It was undoubtedly a valuable inservice exercise. It must be remembered that the teacher education of these teachers occurred in the same or very similar institutions, and must have reflected the biases, strengths, weaknesses, and emphases of their own mentors, who, as previously noted, would themselves have come out of the same system - a somewhat incestuous sequence. As an example of this, although spatial transformations are referred to at several points in the primary school syllabus, many teachers said all they did about it was "mirrors in grade 4". It was thus not part of their own integral model of mathematics at all, but more in the nature of a one-off somewhat irrelevant and peripheral activity.

The above is not intended as criticism of the teachers at all, but rather to illustrate that gaps exist in our system between syllabus statements and classroom implementation. A Queensland study into this very area (I. Warner, 1981) highlighted some of the same problems mentioned above.

However comprehensive a syllabus, what is actually taught is largely determined by what is tested, and what is tested is what is valued. Teachers prepare their pupils according to the assessment they expect.

The 13 clearly defined SOI operation-content-product processes were used to analyse the content of the items in the T.O.L.A., MA7X and ACER tests used, with the following interesting result.

TABLE 3

GRADE 7 TESTS	SOILA MATHEMATICS RELATED TESTS												
	CMU	CFS	CFT	DSR	CSR	MSUa	MSSa	MSIa	ESC	CSS	ESS	NSS	NSI
1. TOLA - TEST 1 No. of items	35	-	-	-	-	-	-	-	-	-	-	-	-
- TEST 2 No. of items	-	3	2	-	1	-	-	-	2	1	-	4	8
- TEST 3 No. of items	24												
TOTAL TOLA ITEMS	59	3	2	-	1	-	-	-	2	1	-	4	8
2. MA7X	27	2	1	-	-	-	-	-	-	13	1	17	
3. ACER - 60 items or operations:	✓				✓							✓	✓
NOTE: SCOPE OF THIS TEST IS LIMITED BY ITS NATURE													

CMU has to do with the understanding of words and ideas. It is primarily designated as 'vocabulary', and this component comprises approximately half at least of the content of well known tests of achievement and I.Q. It is the vocabulary component in particular which contributes to poorer scores for disadvantaged children of all kinds. CMU items comprise almost 75% of the TOLA test and approximately 45% of the MA7X. CSS, well represented in the MA7X test, is cognition of semantic systems and has to do with understanding classifications and extracting rules. NSI also comprises about 30% of MA7X items. NSI is the convergent production of symbolic implications and has to do with extracting the useful or meaningful information out of a mass of data.

Teachers teach towards what is required for the all-important "October tests", T.O.L.A. and MA7X, and test in much the same way. No doubt their knowledge about what within their context constitutes the "real" mathematics, biases both their teaching and the attitudes they bring to bear upon their pupils in class. The "best" pupils may well be those who pay attention, follow directions and get the right answers. This is borne out when one inspects the criteria teachers reported using in assessing their pupils for possible inclusion in our programme.

TABLE 4

SOME INFORMATION ON PARTICIPATING TEACHERS' BACKGROUNDS AND CRITERIA FOR CHOICE OF GIFTED MATHEMATICIANS

TEACHER	APP. AGE	TRAINING	1.	2.	3.	4.	5. OTHER (SPECIFIED) CRITERIA
1. 6MA	25	Q.T.T.	✓		✓		
2. 6MB	26	Q.T.T.	✓		✓		
3. 6MC	30	Q.T.T.	✓		✓	✓	
4. 6FA	25	Q.T.T.	✓		✓		✓ For one pupil only.
5. 6FB	23	Q.T.T.	✓		✓		
6. 6FC	25	Q.T.T.	✓		✓		✓ Observes speed with which pupils attack a problem.
7. 7MA	30	Q.T.T.	✓		✓		
8. 7MB	50	Q.T.T.	✓		✓	✓	
9. 7MC	33	Q.T.T.	✓		✓	✓	
10. 7FA	22	Q.T.T.	✓	✓	✓	✓	✓ 1. Informal discussions with pupils. 2. Informal observations of pupils.
11. 7FB	33	Q.T.T.	✓		✓	✓	
			11	1	11	5	3

1. CLASS TESTS
2. STANDARDIZED TESTS
3. CLASS PERFORMANCE
4. AFTER DISCUSSION WITH OTHER TEACHERS WHO HAD TAUGHT PUPILS

6)
7)GRADE LEVEL

M = Male   F = Female

Q.T.T. = Queensland Teacher Training - Basic qualification for primary school teaching in Queensland, currently three years' tertiary training in a College of Advanced Education.

The chief criteria were success in class tests and a general impression of the way the pupil worked in class.

Analysis of teacher choices, scores in the 13 mathematics related SOI tests, and scores in TOLA and MA7X showed that teachers were good predictors of pupils' success in the October tests, but fairly poor predictors of mathematical aptitude. Whereas they selected as the most gifted pupils who fell in the top quartile on the SOI battery, they succeeded in selecting only one of the gifted group.

Denton and Postlethwaite (1982) examined the ability of specialist teachers to predict the level of performance of their 13 year old pupils at specific subjects in their 'O' level examinations at Oxfordshire Comprehensive Schools. The subjects considered were English, Mathematics, French and Physics. The English teachers were the best predictors. They were able to specify exactly what skills and processes their pupils should be able to master for success, as well as say where each pupil was at present with respect to their clear-cut criteria. The worst predictors were the Physics teachers. They appeared to overlook the skills and processes approach employed by the English teachers, and to fall back upon describing behaviours of "model pupils" in the classroom setting. Moreover, the Physics teachers lacked the knowledge of individual pupil's performance which the English teachers possessed.

Self-confident individuality and a quest for greater autonomy may well be desirable success characteristics for an aspiring physicist or mathematician, but both the Queensland teachers and the English physics teachers appeared to value cheerful unquestioning compliancy along with demonstrated success in a relatively narrow band of largely convergent skills. Consequently, many potentially able pupils who were individualistic and non-conforming in some ways were eliminated from the reckoning by their teachers, whereas other pupils who could operate successfully within the class framework but who lacked true talent were included.

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A CONSTRUCTIVIST VS A FORMALIST APPROACH IN THE TEACHING OF MATHEMATICS

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A constructivist perspective of the teaching of mathematics focusses on the learner and the question "How can we guide him in the construction of his mathematical schemas on the basis of his existing knowledge?" is at the very heart of this approach. It is this prime concern which leads the teacher in the choice of his pedagogical interventions. Thus, to begin with, he has to determine what kind of knowledge can be used as a foundation for the building of the intended concept and to ascertain that such a basis is present in the student. He must then take care that each step in his proposed construction is accessible to the pupil.

In contrast, a formal approach to the teaching of mathematics concentrates more on the transmission of knowledge than on its reconstruction by the pupil. Rather than favor a long and sometimes arduous construction process, this approach tends to short-circuit it by attempting to transmit a given concept through a "good definition". A formal perspective of the teaching of mathematics is fostered by a formal perception of mathematics which is characterized by an emphasis on the form of mathematical expressions. To a formalist, the mathematical expression is intrinsically endowed with meaning and this can lead him to confuse the mathematical notion with the mathematical form, as in the case of number and numeral.

In view of the failure of those programs developed in the sixties which fostered formalism in the teaching of mathematics, one might believe that this kind of instruction has been relinquished. However, the formal approach is still with us and there are several reasons which could explain it.
First, the need to communicate mathematical ideas requires the use of precise definitions and appropriate symbolism and when these are introduced prematurely they constitute formal teaching. A second reason is related to the pressures imposed by the need to cover a specific program in a given time. Such

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pressures will often prompt the teacher to choose an avenue which may seem to be more efficient in the short term, that of transmitting knowledge rather than having the child construct it. Thirdly, the teacher being an adult, his formal thinking may blind him to the epistemological obstacles that the child must overcome in building his cognition. For example, a teacher may be convinced that a good definition is all that a child needs to understand a given concept. Finally, since most of the practicing teachers were trained in the last twenty years, it is not surprising that they frequently use a formal approach, since quite often, it is the only one they have experienced.

While formalism in the teaching of mathematics can often be avoided by making the teacher aware of some of its causes, it is sometimes inevitable. For instance, the extension of multiplication from N to Z brings about the product of two negative numbers which is difficult to relate to our experience. Similarly, in extending the definition of trigonometric functions from the triangle to the circle, one has to overcome a major epistemological obstacle since the functions no longer correspond to the ratios of the sides of right angle triangles, but rather to ratios of the radius and its projections on the axes, subject to the conventional signs of the four quadrants. Thus, we can see that in the introduction of certain topics a formal presentation cannot always be avoided. And in such cases, the difficulties experienced by most students provide ample evidence of the added hardship brought about by mathematical formalism. However, such topics are practically non-existent in elementary school mathematics. Even for one of the most difficult notions taught at this level, that of division by a fraction, to introduce it by the definition or the rule ($\frac{4}{1/3} = 4 \times \frac{3}{1}$) does not provide meaning or relevance. On the other hand, to introduce it using the usual meaning of division ("If I have 4 pizzas and each child is to receive 1/3, how many friends can I invite?") will allow the pupil to relate division by a fraction to an already acquired action-schema.

At the elementary level, not only is a formal approach unnecessary, but it can be downright harmful. For if the mathematics introduced cannot be related to the child's experience it simply will not make any sense to him and he will be reduced to manipulate meaningless symbols using rules he does not understand. In order to avoid such situations we have looked for ways of making the teacher more conscious of the dangers of formalism and we have tried to find means of fostering a constructivist approach. To some extent, these objectives have been achieved by introducing teachers to the epistemological analysis of the main mathematical topics they teach.

EPISTEMOLOGICAL ANALYSIS

By epistemological analysis we mean answering the question "How does the child construct a given mathematical notion ?" However, since such a question is far too complex for most teachers, a similar end is achieved by asking "What does it mean to understand a given concept?". This brings about many responses and while teachers may recognize in their answers various levels of understanding, they find it difficult to view them as steps in the construction of the given concept. We have tried to cope with this last problem by identifying different levels of comprehension which can be construed as stages in a constructive process and these levels constitute what we have called a Model of Understanding.

The search for criteria which might enable us to characterize these different levels of understanding has not been an easy task. In fact, the last three years have witnessed a gradual refinement and precision in the selection of these criteria (Bergeron & Herscovics, 1981,1982). Of course, this is an ever evolving process and even our last version (Herscovics & Bergeron, 1983) has been improved following a critique by one of our collaborators, Nicole Nantais. In this last version, we had described four levels of understanding (intuitive understanding, procedural understanding, abstraction, formalization) and she had taken issue with our description of the second level. She had quite justly felt that one could hardly talk about procedural understanding by limiting it to the acquisition of procedures while leaving out their appropriate use. In fact, we had considered the ability to choose a suitable procedure for a given mathematical task as a criterion for a first phase of abstraction. But this forced us into the trap of having to consider as procedural understanding a kind of instrumentalism whereby the learner could have acquired procedures as instruments, but without knowing when to use them. These considerations have led us to include both the mastery of procedures and their proper utilization as describing procedural understanding, hence simplifying our third level, that of abstraction, by eliminating the need for two distinct phases. These changes have modified our model which now can be described as follows:

Intuitive understanding refers to an informal mathematical knowledge which is characterized, as the case may be, by pre-concepts (e.g. surface is a pre-concept of area), or a type of thinking based on visual perception (e.g. non-conservation of number), or unquantified action-schemas (e.g. adding to and joining are two action-schemas associated with arithmetic addition), or estimation based on rough approximations (e.g. more, less, few, many).

Procedural understanding refers to the acquisition of mathematical procedures which the learner can relate to his intuitive knowledge and use appropriately. For instance, counting-all and counting-on are two arithmetic procedures which quantify the two action-schemas of adding to and joining. It is only by linking these arithmetic procedures with the two action-schemas that arithmetic addition acquires its meaning. Of course, such linkage will also insure the appropriate use of the procedures.

Mathematical abstraction refers to both abstraction taken in the usual sense as a detachment from any concrete representation and procedures (e.g.when the number 7 exists in the child's mind without requiring the presence of objects or the need for counting), and abstraction taken in the mathematical sense as the construction of invariants (e.g. the conservation of number), or as the reversibility and composition of mathematical transformations (e.g. subtraction viewed as the inverse of addition; strings of additions seen as equivalent to fewer operations),or as generalization (e.g. perceiving commutativity of addition as a property applying to all pairs of natural numbers).

Formalization refers to its usual interpretations, that of axiomatization and formal mathematical proof which, at the elementary level, could be viewed as discovering axioms and finding logical mathematical justifications respectively. But we assign to formalization two additional meanings, that of enclosing a mathematical notion into a formal definition, and that of using mathematical symbolization for notions for which prior abstraction has occurred to some degree. These last two interpretations take into account the particular need for precision and efficient notation in mathematics. Since the reproduction of a formal definition and the correct manipulation of symbols usually cannot by themselves be taken as criteria for understanding, the condition of some prior abstraction is essential for these to be accepted as formalization.

To study a concept with the help of this model does in fact constitute an epistemological analysis since the four levels of understanding do suggest a likely pattern for its construction by the learner. Of course, we are not referring here to a purely spontaneous kind of construction by the child, but rather to constructions taking place in the context of schooling under the influence of instruction. This is quite reasonable, for under the guidance of a competent teacher, learning is more a process of reconstruction than one of re-invention by the student. In fact, it would be totally hopeless to expect each generation to re-invent our mathematical heritage which is the result of thousands of years of reflexion.

The training of teachers in the epistemological analysis of the concepts they teach changes their perception of mathematics by making them aware that this discipline involves much more than rules and procedures. Furthermore, this training can also affect their teaching by making them more aware of the existence and dangers of formalism, as well as encouraging a constructivist approach.

PEDAGOGICAL IMPLICATIONS

Teachers working with our model cannot avoid becoming more aware of formalism in the teaching of mathematics since our fourth level of understanding, that of formalization, deals primarily with the formal aspect of mathematics. Far from discarding the importance of this formal aspect, our model assigns to it a definite role and place, the role being that of the crystallization of existing mathematical ideas and the place being the very end of the construction process. Formalism being the premature introduction of formal mathematics, this approach presents the learner with mathematics as a finished product in a condensed form from which all the steps leading to its initial construction have been eliminated. The underlying assumption here is that the learner will be able to relate the new concept being introduced to his existing cognition and that a lot of time will thus be saved. This may well work when the gap between the new concept and the learner's knowledge is small. However, if the gap is too wide, the student will not have available the steps needed for his reconstruction of this concept, since these are omitted in a formal presentation. This would imply that he would have to re-invent these steps, a task far more difficult than reconstruction. In fact, such re-invention proves to be so difficult for most students that they may have no other choice than to fall back on rote learning.

Since a formalist approach starts from a given mathematical notion which then needs to be related to the student's cognition, for the learner there is inevitably at the outset a cognitive discontinuity between what he knows and what has to be learned. By contrast, a student subjected to a constructivist kind of instruction is spared such cognitive discontinuity since the teacher starts from his existing knowledge and relies on it to help him climb up the different steps of the intended construction. For a student in such a learning situation, each step is an extension of his accrued knowledge and this endows the learning process with continuity.

Using our model as a frame of reference for epistemological analysis leads to a constructivist approach towards teaching. By taking intuitive understanding as a starting point, the teacher has to search for appropriate situations in the pupil's experience within which the learning of mathematics becomes meaningful and relevant. He then has to identify those notions which are prerequisites as well as the levels of understanding at which each one will be needed. For instance, one can start teaching addition to a child who does not yet conserve number, but it would be pointless if he could not count. On the other hand, some conservation of number as well as the ability to count up starting from a given number are needed for the acquisition of the counting-on strategy in addition. Thus, right at the beginning one has to assess what knowledge is needed for the intended construction and whether or not it is present in the learner.

A constructivist approach towards instruction is also brought out by the fact that the different levels of understanding are interlocked. The situations representing intuitive understanding provide the context in which mathematical procedures can be presented as ways of solving classes of problems. And it is when these procedures are reflected upon that the learner will reach a level of abstraction. For instance, familiarity with counting will lead the child to anticipate the result and thus the notion of number can become detached from the procedure which begot it. Abstraction in turn needs to be achieved to some extent since it is a prerequisite for the fourth level of understanding, that of formalization.

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COUNTING VS PIAGET'S INTELLIGENCE STAGES

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According to Piaget, the realization of the concept of number in a child follows the development of his intelligence in three stages:

1) global knowledge;
2) intuitive knowledge;
3) operational knowledge.

How does Piaget consider COUNTING in his learning of the concept of number? Two hundred children have participated in Piaget's experiments on the concept of number. In the course of those experiments, the child sometimes utilizes counting, sometimes not. The question arises:

At what stage does the child utilize counting most naturally?
Starting from measures of central tendency regarding the age of the children at each one the various stages, a further question arises:

Are those measures significant in the case of utilizers and non-utilizers of counting?

According to Piaget, did counting occupy an important place in his experiments on the genesis of the concept of number? In his book: «La genèse du nombre chez l'enfant», in which chapters does he mention the fact that children do count? In actual fact, in what precise circumtances does the child utilize counting and does he not utilize it? To what extent may we suppose that counting is preliminary to the concept of number among children, according to Piaget?

A careful and thorough study of the book: «La genèse du nombre chez l'enfant» will shed light on the search for answers to those questions.

ON THE FORMATION OF THE DIFFICULTY OF NUMBERS,0,1,2,...,9, GRASPED AS CONNOTATIVE MEANING

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INTRODUCTION

It appears that students and adults of Japan have feelings of the difficulty of numbers, 0,1,2,...,9. When they use these numbers for counting or calculating,they feel that numbers 2 and 5 are easier than numbers 3 and 4, number 5 is easier than 6,7,8, and 9, and one of the most difficult numbers is 7,8, or 9.

The difficulty now dealing with may be defined as feelings or impressions about the difficulty of numbers,0,1,2,...,9 when they are used for counting or calculating. Psycholinguistically, a person's feelings or impressions for a concept is called "the connotative meaning of the concept". So the difficulty of numbers,0,1,2,...,9 described above can be called the difficulty of numbers grasped as connotative meaning, abbreviated to DNCM. I already have researched on the DNCM, its measurement and scaling, reliability and scale values of it obtained from groups of elementary school students, prospective teachers, and researchers of mathematics education. I summarized these in the following section.

There are problems not yet clarified concerning to the DNCM. One of these is the process of the formation of the DNCM. In this report, I intend to report the process of the formation of the DNCM of a group of subjects when they were in the 2nd, 3rd, 5th, and 7th grades.

SUMMARY OF PREVIOUS RESEARCHES

I have developed three instruments for measuring the DNCM. One is that of using 10cm continuum. Each subject is requested to write a cross x on the continuum following his feelings of difficulty (or easiness) of a number i which is written in Arabic numeral. His difficulty of the number is represented by the length of the segment between the left side point of the continuum and his mark x, and the scale value L(i) of the DNCM of a group of subjects is the average of the difficulty of the number over the subjects. The second is an instrument using the method of paired comparisons with an analytical procedure for scaling. The scale of the DNCM obtained is an interval scale, and is called G. The third instrument measuring the DNCM is that of using a graphic scale: easy __:__:__:__:__:__:__ difficult. Each scale is scored by assigning 1 to the extremely easy point, and 7 to the extremely difficult point of the seven-point scale, with intermediate values for the point in between. In this method, the scale value F(i) of the DNCM of a number i is the average of the subjects' scores assigning to their responses on the seven-point scale with the number i. When a group of subjects consists of low grade students, five-point scales are suitable and used rather than seven-point scales, because they might be too young to estimate their feelings in such a complicated way as seven-point discrimination.

Minato (1978) examined reliability of the scales L, G and F, and obtained very high coefficients of correlation among them from administering to subjects of prospective elementary school teachers. After these examinations, I have only used the scale F for scaling the DNCM. The scale values of the DNCM obtained from groups of 189 5th and 6th grade students (numbers 2,3,.. ,9), 118 prospective elementary school teachers (numbers 1,2,...,9), and 36 researchers of mathematics education were as follows:

Group	0	1	2	3	4	5	6	7	8	9
5th, 6th students	-	-	1.4	2.1	2.1	1.8	2.6	2.9	2.9	3.0
prospective teachers	-	1.4	1.3	3.0	2.6	2.4	3.9	5.3	4.9	5.9
researchers	5.6	3.5	2.3	3.2	2.8	2.4	3.4	4.7	3.5	4.0

Minato et al. (1979) have studied on the relation between the DNCM and the relative difficulty of the multiplication combinations, abbreviated to RDMC and clarified the existence of significant relations between them, where the RDMC is defined as error rates of the multiplication combinations. One of the operational definitions of the RDMC is so called the scale Sd, which is defined as: for number i, $Sd(i) = \sum_{j=2}^{9} R(i,j)$, where R(i,j) is an error rate of a multiplication combination i x j.

FORMATION OF THE DNCM

For clarifying the process of the formation of the DNCM, I administered the graphic scales to 91 subjects when they were 2nd grade (7,8 years old), 3rd grade (8,9 years old), 5th grade (10,11 years old), and 7th grade (12, 13 years old) students of the Attached Elementary or Junior High School of the College of Education, Akita University, Japan. When they were in the 7th grade, seven-point graphic scales were administered, and when they were in the 2nd and 3rd grades, five-point graphic scales were administered. Moreover when they were in the 5th grade, both of the scales were administered for examining the relation between two scales F and F' of the DNCM from the administration of the seven-point and five-point scales respectively. The coefficient of correlation between these scales of the DNCM was 0.9987, and that the transformation of F' into F fitted well the following theoretically accepted equation: $F = 1.5F' - 0.5$. Then I obtained scales F of the subjects when they were in the 2nd and 3rd grades using above equation, and in the following, the scale F was only used for representing the DNCM. The scales obtained in the study are as follows:

Grade	0	1	2	3	4	5	6	7	8	9
2nd	1.5	1.4	1.6	1.7	2.0	2.4	2.5	2.9	2.8	3.4
3rd	1.6	1.2	1.4	1.7	2.1	1.9	2.3	2.8	3.1	3.1
5th	1.7	1.5	1.6	2.2	2.4	1.9	2.9	3.5	3.5	3.6
7th	2.6	2.0	1.8	2.9	2.7	2.3	3.1	4.3	3.6	4.3

I calculated coefficients of correlation among these four scales, and using a technique of the multivariate analysis I obtained the distances among the nearest neighborhoods, which were as follows: the distance from 2nd to 3rd : 5.5, from 3rd to 5th: 2.3, and from 5th to 7th: 5.1. In spite of a short interval between the 2nd and 3rd grades, the distance was longest of all. The shortest distance which indicated the least change of the DNCM was one between 3rd and 5th grades. Because the scale of the 7th grade students had a close resemblance to the above mentioned scale of the prospective elementary school teachers, it could be concluded that the DNCM had been formed almost completely until the middle of the 7th grade, when the instruments were administered finally in the study.

In the late 2nd grade, students of Japan study construction of multiplication combinations in their school mathematics classes, and are trained to recite, or recall aloud them. Until the middle of the 7th grade, students finish the study on the rational numbers. Although there is not a salient discrepancy between each F and its regression line, there may be something affecting to the DNCM other than the size of the numbers. Then I estimate that the DNCM is formed by two aspects of the difficulty : cognitive and phonetic. The cognitive aspect of the difficulty is the difficulty on representation of numbers: especially on the size, unity, and oddness. On the other hand, the phonetic aspect of the difficulty is one on phonetic distinction of numbers, especially between 4 and 7, which are pronounced as [ʃi] and [ʃitʃi] in Japanese respectively.

A COMPUTER APPLICATION TO MEASURE A MENTAL ABILITY WITH RELEVANCE IN MATHEMATICS LEARNING

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Western researchers have been giving increasing attention to the work of the Russian neuropsychologist, A.R. Luria. In particular, his distinction between simultaneous and successive synthesis of information has generated a considerable amount of research in the psychometric field. Psychological analysis of disturbances of mental processes in patients with local brain lesions revealed that some sections of the brain are responsible for forming simultaneous syntheses, while other sections are responsible for successive synthesis. The parieto-occipital areas are responsible for the simultaneous, and the temporo-frontal regions contribute to the formation of successive syntheses.

FRONTAL LOBE FUNCTIONING

Luria's writings on the part played by the limbic cortex and the frontal lobes in the functional organization of mental activity have received less consideration. Luria studied the mental characteristics of many patients with brain damage in these areas. His work indicates that patients show impaired functioning in certain types of tasks across a variety of forms of intellectual activity. An analysis of Luria's writings enable a list of characteristics to be made concerning the nature of the tasks which caused difficulty for his patients. Tasks requiring preliminary investigation of the conditions and relationships, the formation of plans and careful control of behaviour reveal defects in performances by frontal lobe patients. Tasks requiring high frontal lobe and limbic involvement are those which need sustained and/or carefully controlled performances (under verbal direction), in which certain operations have to be selected and others inhibited, often where a preliminary analysis must be made before processing. Luria demonstrated how the same pattern of defects can be shown in elementary motor functioning, in complex visual perceptions and in the highest levels of intellectual activity. Elementary tasks which reveal disturbances for this type of patient seem to possess certain features:

(i) A rapid and sustained performance is required;

(ii) A degree of selectivity is involved. Some stimuli are to evoke responses, others are to be inhibited;

(iii) A particular kind of "conflict" exists; such as where the appearance of a stimulus conflicts with its meaning as a signal;

(iv) The rule for guiding performance is continually changing (switching). For each characteristic, frontal difficulty increases as the task comes under the control of the experiments and/or requires performances based on verbal instrluctions or under speech control. When the damage is restricted to the limbic region defects are not revealed to such an extent in those tasks where performance can be controlled verbally.

A NEW FACTOR

An analysis of the tasks suggests that limbic and frontal functioning may be reflected in a basic and underlying human ability. Luria defines "attention" in the following way:

> The directivity and selectivity of mental processes, the basis on which they are organized, is usually termed attention in psychology (p.256)

He explains the factor further in the words:

> ... the process which helps a close watch on the precise and organized course of mental activity (p.256)

Thus Luria's work suggests that a selective attention ability in sustained performances may reflect limbic and frontal lobe functioning. This raises the question as to the nature of individual differences in ability to mobilize attentional capacity in tasks involving a sustained and selective performance.

In order to test for the existence of an "elementary" factor, tasks need to be chosen which require minimal involvement from other regions of the brain. Thus tasks with low requirements for the analysis/synthesis of information need to be avoided. The need for "formal" intellectual and higher order perceptual content should be minimized; requirements for memory and learning could confound the target construct.

The search for the factor thus began with the development of several "primitive" tasks with high validity in terms of similarity to those used by Luria with his patients. Luria's tests were only intended to indicate impairment or non-impairment, however with the use of computer based electronics they can be adapted for studies of individual differences amongst non brain damaged individuals. The development of one such test is described below. The test was designed for use with a sample of 129 children aged six years.

THE R2D2 LIGHT TEST

The most compelling test used by Luria for adaptation is his light/motor experiment (1959). The "R2D2" Light Test was designed to parallel Luria's task for simple and automatic use. A model "R2D2" (a character from the "Star Wars" movie) was chosen as an appealing stimulus for the test age group. Two LED (Light emitting diodes) are attached to "R2D2", one coloured red and the other green. During administration of the task a child is able to press responses with the aid of a hand held push button. A Motorolla 6800/D2 Microprocessor kit formed the basis of electronic equipment designed to create an experimental situation closely resembling the one used by Luria. The necessary hardware and software was designed to enable the following functions to be performed:

(i) Allows the lights to be switched "on" or "off", in a programmable sequence under the control of the experimenter;

(ii) The time intervals between lights and the time for which each is "on" can be predetermined;

(iii) All information concerning occurrences at each step in a trial can be transferred for storage on a diskette (in a floppy disc drive). Following experimentation with a group of children the information can be transferred to a computer for subsequent scoring and data analysis.

The equipment has the advantage of being portable and is easily assembled in a suitable quiet location in schools. This avoids the need to bring children into the more usual central laboratory with surroundings strange to the children.

For the test, a child is instructed to press the button as quickly as possible when a red eye winks, but not to press when a green eye winks. Following extensive trials with the age group, it was decided to set the "off" periods fixed at 1.5 seconds and the time for each light to be "on" at 1.0 seconds. Following suitable familiarisation and practice each child in the experimental sample was given the same sequence of 110 lights (60 target reds) which took approximately 280 seconds to complete.

During the Conference presentation detailed information, photographs and diagrams concerning the test equipment and its operation will be displayed. The sequence of lights and the code used for storing information and details of the test administration will also be made available. The appropriate test administration, including the actual words to be spoken by the experimenter were only decided upon following extensive trials with young children of the same age as the experimental sample.

The test itself proved to be extremely popular with the children and they were very eager to meet "R2D2" and try to match wits with him. Typical comments received spontaneously from the children were of the type; "When is R2D2 coming?"; "Can I have a turn with R2D2?"; "R2D2 was great, but he is very tricky!"

In similar tests Luria was able to analyse qualitatively both the patterns of latencies and the types of errors made by patients. With the aid of the coding procedure used for storing information in this experiment, it is a reasonably straightforward procedure to write computer programs for analysing the responses in ways which reflect those used by Luria with his patients.

Subjects are able to make two types of errors in this test; failures to press to the red lights ("misses") and incorrect presses to the green light ("extras"). A program was written to search the data file in order to count the number of times each child made either type of error. These scores define a variable labelled R2TOT. Additionally, individual differences were analysed in terms of the reaction time patterns. Given that 60 red targets occur altogether, it seems appropriate to group the information for each set of 6 red lights in the sequence. A program was written to search the R2D2 data file for each individual, in order to determine the average latencies for each of the 10 clusters of 6 red lights. Whenever "misses" occurred within a group of lights, averages were only based on the correct responses within the set (no non-zero latencies entered into the calculations). The procedure led to 10 measures which are defined across time labelled L1 to L10. Since more than one underlying dimension of variation may exist among these variables, they were subjected to maximum likelihood factor analysis. A two factor solution provided a reasonably interpretable pattern, so factor scores were generated for each individual. The decision implies the existence of two kinds of ability in order to explain individual differences in the reaction time patterns. Thus a total of three scores were generated from the data.

THE R2D2 CONFLICT LIGHT TEST

A second version of the test was also developed and called the "R2D2 Conflict" Light Test.

A task mode switch was attached to the model R2D2. The alternative mode has the effect of presenting the child with a sequence in which either both lights are on together, or only one is on at a time. The child is instructed to press once when two lights are seen and to press twice when only one light is displayed. The task thus presents the child with an experiment in which the appearance of a stimulus conflicts with its meaning as a signal.

Understanding of the task requirements does not appear to present a problem for children of the age group when the test procedures are carefully followed. In general they found the test difficult but challenging and were eager to do well with motivation in evidence throughout the whole test. Children appeared to be well aware of when they made mistakes and were often observed to the chastising themselves for individual failures. When questioned at the conclusion of the test all children agreed that R2D2 had tricked them several times.

The test presents 110 light stimuli. For each child in the sample, the number of mistakes was counted for each group of 11 light stimuli in sequence. This produced 10 scores for each individual and labelled R1 to R10. The program searched the response data file and counted a mistake whenever a subject failed to press once for a two light stimuli, or failed to press twice for a one light stimuli. The sequence LR1 to R10 was analyzed with maximum likelihood factor analysis. The decision was taken that two underlying dimensions were sufficient to explain individual differences in error rates across the test. Factor scores were generated to produce two new variables from this test.

TWO FURTHER TESTS

In similar vein two further tests were developed to parallel the work of Luria. The "Basil" Light Test employed a model clown figure (called "Basil" by the children) in a simple delayed reaction time experiment. A study of reaction time patterns led to the acceptance of a further two variables for use in later analyses.

The "Dalek Voices" Tests ("Daleks" are creatures from the "Dr. Who" television series) employed a synthetic computer voice in an auditory selective attention test. The test provided an auditory equivalent of the R2D2 Light Test. Scores were obtained by counting the total number of errors made by each child. Details of the two tests can be obtained from the author.

THE ATTENTION FACTOR

From the 4 tests, taken as a whole, a set of 8 scores were generated for subsequent analysis.

The hypothesis suggested by Luria's work is that at least one common dimension of variation can be found within a set of measures derived from tasks involving focussed selective attention across time. Later canonical variate analyses involving the 8 scores suggested the existence of a single underlying dimension with the highest correlations coming from variables representing the total errors in the Dalek Voices Test and the R2D2 Light Test, the errors occurring late in the R2D2 Conflict Light Test and performances in the late half of the Basil Light Test. This suggests that the basic trait can be described as an ability to maintain focussed selective attention over time.

When combined with a set of measures for Luria's simultaneous and successive synthesis of information a simple orthogonal three factor solution was obtained.

Subsequently, canonical correlation analysis and multivariate regression analysis demonstrated the effectiveness of the 3 factor model to predict school competence as measured by teacher assessment and school achievement tests in mathematics (and other subjects as well). Canonical correlations 0.7 were obtained (Ransley, 1981).

Whatever the factors are measuring, they seem to be very basic to the kinds of mental functioning required by children to perform well at school. The model has since been shown to be a strong predictor of mathematical competence among low ability 15 year olds (Ransley, 1983).

Currently, software is being written so that the attention ability can be easily measured on a BBC microcomputer. This will allow the experimental procedures to become available to a wider range of researchers and open up the possibility for simple classrom applications.

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ASSESSING COGNITIVE LEVELS IN CLASSROOMS -- PROGRESS REPORT

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"Assessing Cognitive Levels in Classrooms" (ACLIC) is a project sponsored by Alberta Education (Canada) which is designed to assess and compare student cognitive ability levels and curricular cognitive demands in the mathematics topics covered in the six elementary school grades of the recently revised Alberta Elementary Mathematics Program. The specific objectives of ACLIC are (1) to assess the cognitive ability levels demonstrated by students in each grade in topics in mathematics, (2) to identify the cognitive demand levels of objectives, materials, and classroom activities in each strand of the curriculum at each grade, and (3) to assess the extent to which the curricular cognitive demands match the cognitive levels of the students.

The ACLIC project is an extension (to other grades and more topics) of the Calgary Junior High School Mathematics Project (Harrison, Brindley, & Bye, 1981). Two major findings of that project were (1) that there were considerable gaps between student cognitive level and curricular cognitive demand in fraction and ratio tasks in grades seven and eight and (2) that process-oriented teaching materials can ameliorate these gaps and lead to significantly better attitudes and achievement.

In Alberta today much attention is being focussed on the outcomes of schooling and the measurement of student achievement in various subjects at different grades. Unlike these efforts, but complementary to them, the ACLIC project is concerned with qualitative assessment of students' thinking. Following the Piagetian paradigm, students' cognitive ability levels are determined by the ways in which they use cognitive operations in the context of particular mathematical tasks. A student's response is classified as preoperational if no appropriate cognitive operations are used in responding to the task, as concrete operational if appropriate cognitive operations are applied to particular tangible and/or familiar elements in the task, or as formal operational if cognitive operations are correctly applied to generalizations and/or hypotheses. Piagetian constructs and terminology have been used to facilitate communication with researchers and teachers, but some of the testing situations have been extended, modified, or adapted from the original Piagetian tasks.

Recognizing the fact that a child's response to a task in one topic setting (e.g., measurement of length) may be at a cognitive level different from that of a task in another topic setting (e.g., understanding numbers), we have not attempted to generalize across topics or to apply cognitive level labels to students. Rather, we report the cognitive levels of responses to tasks within single topics. In doing this we have been influenced by the "Structure of Observed Learning Outcomes" (SOLO) taxonomy (Biggs & Collis, 1982) and its classification of student responses as pre-structural, uni-structural, multi-structural, relational, or extended abstract. While choosing to use Piagetian terms for cognitive levels, we have borrowed Biggs' and Collis' term "concrete generalization" and applied it to those responses which lie in the transition between concrete operational and formal operational.

On the basis of our understanding of Piagetian cognitive levels and the interpretation of cognitive operations in children's solution of tasks, we have begun to collect and develop testing situations for the assessment. Where possible with older children (i.e., those in grades 3 to 6) we have attempted to present tasks in printed form for large group testing. In many cases we have simply selected from Piaget's writings individual interview situations that have direct relevance to the elementary mathematics curriculum. In other cases we have had to develop new tasks, since Piaget did not address certain topic areas such as the meanings of multi-digit numerals, for example.

Two sources of assessment items in paper and pencil format that we have found very valuable are the Concepts in Secondary Mathematics and Science (CSMS) project (Hart, 1981) and the Australian Council for Educational Research (ACER) Mathematics Profile Series (MAPS) tests and ACER Mathematics (AM Series) tests. The CSMS tests provide a number of examples of paper and pencil presentations of tasks similar to Piagetian interview tasks. The MAPS tests measure "mathematical development" in a variety of different topic areas on a common scale, the MAPS scale. This scale was originally calibrated using items developed by Collis in research on the assessment of Piagetian levels of development in mathematics, and it is possible to relate regions of the MAPS scale to levels of operational thinking. MAPS difficulty values are given for individual items in the MAPS tests and some of the AM Series tests. In selecting and adapting CSMS and ACER items we have carried out limited field testing to determine whether items are of appropriate difficulty and whether the items appear to be valid measures of the use of cognitive operations.

By September, 1984 we will have completed the task of assembling individual interview protocols for grades one, two, and three and paper and pencil tests for grades three, four, five, and six. In October and November, 1984 these will be administered to a provincially representative sample of about 2 000 students in all. As well, we will be assessing the cognitive demand levels of curriculum objectives and of learning activities presented in textbooks and teacher-directed class activities. In assessing cognitive demands the same criteria will be used as are used in assessing cognitive levels of students' responses.

It is expected that, for some topics and some grade levels, the cognitive demand of the curriculum may significantly exceed the cognitive abilities demonstrated by the students for whom the instruction is intended. In such cases, changes in the timing of topics might be recommended, or, more likely, recommendations might be made regarding the need for supplementary teaching approaches designed to help bridge the gap between operational student ability and topic demand. Should such gaps not be found, the soundness of the present curricular structure and its interpretations will have been substantiated. Whatever is found, the paper and pencil assessment tests produced by the project could provide efficent "cognitive check-ups" for use by teachers. The assessment techniques developed in the project could be employed at other grade levels, in other subject areas, and for program evaluations in general.

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LEARNING WITHOUT UNDERSTANDING — HOW DO THEY SUCCEED IN SPITE OF ALL?
(computer based drill and practice)

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§1. LEARNING WITHOUT UNDERSTANDING

In the last decade certain attention was drawn among Mathematics educators to the phenomena of learning without understanding. Two outstanding studies on this theme are Erlwanger, 1973 and Rosnick and Clement, 1980. However, from Skemp's analysis of understanding (Skemp, 1976) follows that an assertion about learning without understanding might be at least ambiguous if not meaningless. This is because the notion of understanding has, at least, two meanings (in later studies, Byres, V. and Herscovics, N., 1977 and Skemp, R., 1979, more meanings were suggested). According to Skemp, there are situations in which the learner thinks that he acts with understanding and the Mathematics educator denies it. Skemp suggested that what some teachers regard as no understanding and some students and some other teachers do regard as understanding will be called instrumental understanding, whereas the "real kind" of understanding, the one for which the mathematicians look, will be called relational understanding.

Essentially, instrumental understanding is knowing how without knowing why. For example: I know that the area of a rectangle is length times width but I do not know why it is so. Unfortunately, a worse situation can occur in which I even do not know that the area of a rectangle is length times width but I can still get the right answers to particular problems by using irrelevant considerations. It seems that this is the case in Erlwanger, 1973 and Rosnick and Clement, 1980. Benny, for instance (Erlwanger 1973), thinks that when one is asked to write

2/10, 5/10 and 4/11 as decimals the answers are 1.2, 1.5, and 1.5 respectively (He adds the numerator and the denominator and then adds a decimal point). On the other hand, Benny managed to get through the IPI (Individualized Paced Instruction) Mathematics course and was considered as above the average pupil. How did he do it? Benny claimed that he found the key to the IPI answers. He answered the questions according to this key and hid his own mathematical views about the correct answers.

Rosnick and Clement dealt with the following question given to a group of first year engineering majors: Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university". Use S for the number of students and P for the number of professors.

Only 63% were able to answer it correctly and only 27% answered correctly a similar but more complicated problem. The common mistake was P = 6 S (the reversal mistake). A remedial teaching was tried but it turned out that "although the surface behavior of the students changed, continuing probing in the interviews revealed that many of the students' misconceptions remained unchanged" (Rosnick and Clement, 1980, p. 3). Again, how could the surface behavior improve without understanding? An interview with a student (p. 7) can give a partial answer to this question. Essentially, the following happens: After reading the students' and the professors' problem the student's impulse is to write P = 6 S but, because of the remediation he got, he knows that this is wrong. Therefore, he changes it and writes: 6 P = S. This is the kind of strategies students use very often to get the right answers. It is not even an instrumental understanding.

Rosnick and Clement claim that "large numbers of students may be slipping through their education with good grades and little learning" (p. 24). They ask: "how can students learn to solve these problems with understanding?" (p. 23). It is of course a very important question. But there is also another question: How can students learn to solve these problems without understanding? Answers to this question might help to find answers to the previous question.

§2. COMPUTER BASED DRILL AND PRACTICE — SOME OBSERVATIONS

The phenomena of learning without understanding has many forms. It exists at all age levels and in all domains. Therefore, it is impossible to explain it generally. It may be different from one situation to another. In this paper we will try to explain it in a specific situation of computer based drill and practice in Arithmetic. It is typical *not only* to a computer situation of learning. There are many situations in the classroom which are quite similar to it. The reason we chose it is that it is easier to observe. An interviewer can stand behind a child solving problems on the computer and he can interfere any moment he thinks something goes wrong. The following observations focus on pupils who have difficulties with the problems posed to them. We will denote the computer by C, the pupil by P and the interviewer by I.

(1) C: *Which number is greater than 4 by 7?* P (third grade): *3.* C: *You are wrong.* I: *What did you do? explain it to me!* P: *I thought what the difference was.* I: *What is the difference between what?* P: *Between 7 and 4.* I: *What was the question?* P: *Which number is greater than 4 . . .* I: *By 7.* P: *I don't understand it.* I: *. . . Do you know, perhaps, which number is greater than 1 by 2?* P: *Of course!* I: *Which?* P: *1.* I: *. . . Again you are calculating the difference. . . . What does it mean, for instance, that 5 is greater than 3 by 2?* P: *. . . 7 plus 4!* I: *. . . Alright, try it!* P: *I make it 11.* C: *This time you are right.*

An analysis of the above conversation shows that the pupil does not understand the phrase: "a number greater than y by x". He even admits it explicitly ("I do not understand it") and shows it again when the interviewer poses to him another example with simpler numbers ("a number greater than 1 by 2"). Probably the pupil knows the phrase by "what the number x is greater than the number y?" He knows also that the answer to this question is given by x-y. This is probably the reason that he calculated 7-4 and 2-1. When he saw that this strategy does not lead him to the right answer, he started to look for another strategy. But how can he look for another strategy if he did not understand the question? It turns out that by the trial and error method it is quite easy. The exercise presented to the pupil two numbers. It is quite clear to the pupil that he has to act on them by means of a certain arithmetical operation. In the first trial he subtracted them. In the second trial he added them. The addition will give him the right answer. The pupil ignored the question: "what does it mean that 5 is greater than 2 by 3?" and also the opportunity to understand it. He is concentrating on finding the arithmetical operation which will lead him to the right answer. The computer encourages him to act like this. In the future he will know that the numbers in similar exercises should be added. One can argue that if somebody does not understand a question he is not able to identify similar questions and to remember how to solve them. The practice, however, does not confirm this view. Pupils identify the structure of the problems and remember their method of solution. The reinforcement they get from the computer makes them even believe that they do understand the subject, although the Mathematics educator is reluctant to call it understanding.

(2) C: $\frac{3}{10} + \frac{1}{5} = \frac{?}{10}$ P (grade 6): . . . I: *How do you add fractions?* P: *I do not know what to write.* I: *Do you know how to add fractions?* P: *I do 3 plus 1, don't I?* I: *Do you want to try (*to type it*) or to discuss it?* P: . . . *7.* I: *How? . . . why? . . . how did you get 7?* P: *I think that 10 is the common denominator.* I: *Alright, and how will you find the numerator?* P: *10 divided by 10, isn't it?* I: *Do you divide 10 by 10 because there is 10 on the right side?* P: *Yes.* I: *So, you divided 10 by 10. What did you get?* P: *It is 1.* I: *Yes.* P: . . . (pointing at the numerator of the first fraction) *it is 3.* I: *Yes.* P: *10 divided by 5 is 2 . . . 3 . . . it is 5, isn't it?* I: *Yes . . . tell me, how did you get 7 a minute ago? Do you remember?* P: *It wasn't my day.*

The situation here is different from the previous one, where two numbers were presented to the pupil and he had to select the operation leading to the right answer. Since multiplication and division are not candidates for such a question the pupil can arrive at the right answer at most at the second step. Here, on the other hand, five numbers are presented to the pupil (we consider the numerator and the denominator of each fraction as two different numbers). It seems that if the child does not know how to add fractions at the first stage he will not be able to reconstruct the addition algorithm for fractions. But also here, as in the previous case, he succeeded in arriving at the right answer with the help of some little hints. In the beginning he made quite a common mistake in adding fractions. He added the numerators and ignored the fact that the denominators were different (usually the mistake is to add a numerator to a numerator and a denominator to a denominator but here the denominator of the result was determined by the computer). After that, the pupil recalled a tiny bit of the addition algorithm for fractions. Probably, he divided 10 (the common denominator) by 5 (the denominator of the second fraction at the left) and added the result to 5 (the denominator). This probably explains how he got 7 (this type of "strange" error is explained in Vinner, Hershkowitz and Bruckheimer, 1981). The next attempt already puts the pupil on the success track. It is hard to tell what would have happened to him in case the interviewer had not helped him. It is possible that he would be able to reconstruct the addition algorithm by himself. It is also possible that he would get some hints from the teacher at the computer room. This way or the other, after reconstructing the algorithm for this particular exercise in this particular set up the pupil would not make an additional effort to learn more about adding fractions in general. The computer will not give him additional reward for it.

Again, unintentionally, we encourage learning without understanding. Note the pupil's reaction to the question: "How did you get 7?" From all possible answers he chose to say: "It wasn't my day" as if it is a matter of chance. This, perhaps, reflects the pupil's point of view about the process of getting the right answer. There are several strategies to deal with such exercises and he has to choose one of them. Unfortunately, he had bad luck and chose a wrong one. It wasn't his day.

(3) C: *Dalia had 11 beads. She bought 5 beads but 7 got lost. How many beads has Dalia now?* P (third grade): *11 minus 7.* I: *Why did you do 11 minus 7?* P: *Because she got the beads and afterwards she lost 7 . . . it is 11 minus 5.* I: *Why minus 5? . . . Let us read the question once more.* P: *Dalia had 11 beads. She bought 5 beads but 7 got lost. How many beads has Dalia now?* I: *You saw there 7 and 5 and what did you do?* P: *It is 6.* I: *How did you get it?* P: *11 minus 5.* I: *Then what about the 7? . . . there are three numbers in the question: 11, 5 and 7. Why did you do 11 minus 5?* P (after a long silence): *Because she lost 7.* I: *So why did you do 11 minus 5?* P: *Because she got 11 beads and afterwards she got 5 more . . . Ah! 11 plus 5 minus 7! (she calculates 11 + 5 − 7 and types 9).* C: *Very good.* P: *How long it took*

me to discover it.

This description is typical of a pupil who has difficulties in word problems. The goal which directs him at the first stage is to find a suitable operation to be carried out on two numbers. The fact that three numbers are given here does not disturb him too much at the beginning (it confuses him a lot later on). The key for the operation selection at the first stage is a verbal one (see for instance Nesher and Tuval, 1975). In this question it is the word "lost". When this word appears it is well known that one should subtract. But to subtract what from What? Of course, one should subtract a small number from a larger number. Since it was said that "7 got lost" the answer "11 minus 7" is quite expected. This is also explicitly expressed by the pupil: "She got the beads and afterwards she lost 7". However, another look at the question show the pupil that the number 5 also appears in it. Therefore comes the trial: 11 minus 5. The pupil systematically responds to the most dominant stimulus at the moment. In the beginning, it was the expression: "7 got lost". Afterwards, the discovery that the number 5 also appears in the question became the most dominant stimulus. The moment the last trial is also questioned by the interviewer the pupil returns to the starting point: "because she lost 7". He does not know to which stimulus he should react now and his answer has no relevance to the question: "why 11 - 5?" Only after the question is repeated does the pupil reconsider the problem and reformulates it in his own words. Only at this stage does he probably realize what the situation is. Now he can see all its components simultaneously. It is a kind of illumination in the sense of the Gestalt Psychology. Note that it was preceded by almost a blind trial and error process. As in the previous example it is worthwhile to note the pupil's last comment ("how long it took me to discover it"). From his point of view looking for the answer is an activity the goal of which is *to discover* the right answer. The word "understanding" is never mentioned.

(4) I: *Please read the question the moment it appears on the computer* (this is done for the sake of recording). P (second grade): *4 children went to the swimming pool in the morning, 3 children went to the swimming pool in the afternoon. Ah! it is . . .* I: *Are you already going to answer it?* P: *7.* I: *But you haven't finished reading the question* ("how many children went to the pool that day?") P: *But I know.* I: *How do you know?* P: *Because I have had*

something like that many times.

The tendency to find out the right answer as quickly as possible is quite clear here. In this question, contrary to the previous one, there are no verbal cues. In spite of that, there is something in the question which leads the pupil to the solution without reading the end of the question. Our claim is that this "something" is the surface structure of the question; structure in the sense of the Gestalt theory. Since we speak about a structure that one grasps globally and not analytically it is hard to characterize it and to describe its components. (We use the word "structure" as a primary notion, in the same sense it has in the Gestalt theory.) Without the above claim it is hard to explain how pupils solve problems which they only partially understand. It is obvious that verbal cues are not enough to explain it.

§3. HOW TO SOLVE IT?

In the last section we claimed that the key to the pupils' strategies in solving verbal problems is the surface structure of the problem and also verbal cues. The surface structure factor can be well demonstrated in the following example:

C: *The mailman delivered 5 letters on Bialik Street. On Herzel Street he delivered 7 letters and on Burla Street he delivered 8 letters. How many letters did he deliver?* P: *20.* I: *How did you get it?* P: *5 plus 7 plus 8.* I (after 15 minutes): *The mailman delivered 4 letters on Bialik Street. On Herzel Street he did not find the address of 3 letters. On Burla Street he delivered 11 letters. How many letters did he deliver?* P: *18.* I: *How did you get it?* P: *4 plus 3 plus 11.*

From the pupil's point of view the two questions have the same surface structure. This surface structure (Gestalt) is grasped globally, without any analysis. This surface structure calls the arithmetical operation which will be carried out on the numbers in the problem.

§4. DO WE ENCOURAGE LEARNING WITHOUT UNDERSTANDING?

There is no need to say that we do not share the behavioristic approach to learning. According to this approach, to understand a subject is to be able to answer a list of questions on that subject. Ausubel, the spokesman of the meaningful learning relates to this problem when he deals with Skinner, one of the most radical behaviorists (Ausubel, 1967, p. 66): "Skinner denies that the concept of meaning is necessary or useful in explaining behavior. He is completely unconcerned with the problem of whether an emitted verbal response represents the stimulus that comes to elicit it. The problem of how meaning is learned is thus solved by the denying that meaning exists."

This paragraph demonstrates a major problem in the question and answer system in the education process and especially in questions and answers by means of computers. As non-behavioristic educators we *are* interested in thought processes that occur in the pupil's mind between our question and his answer (the stimulus and the response). We *do* care about the considerations that lead him to the answer. We *do* expect meaningful communication. The computer based drill and practice cannot encourage all these. On the contrary, it especially encourages two behavioristic principles:

1. Almost blind trial and error processes.

2. Ignoring the processes which occur between the question and the answer.

Thus, in many cases, unintentionally we do encourage learning without understanding. Chasing the right answers became the name of the game in education, especially with computers.

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D. REPRESENTATIONAL MODES

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ASPECTS OF LOGICO-MATHEMATICAL INTUITION IN THE DEVELOPMENT OF YOUNG CHILDREN'S SPONTANEOUS PATTERN PAINTING

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This paper is based on two longitudinal studies (Booth, 1975; 1981)in which aspects of cognitive development and of logicomathematical intuition in 5-to-6 year old children's spontaneous pattern making in painting -- hence 'pattern painting' -- were investigated. Previously young children's spontaneous pattern making in drawing and painting has been related to art learning alone and designated 'design' (for example, Cockrell,1930; D'Amico, 1966; Ellsworth, 1939; Kellogg, 1969, Muth, 1912).

In the first study (Booth,1975,1980) (M=19, F=18) 2105 spontaneous paintings were analysed for structural content and three stages which develop in an invariant order were observed and described. In the first, the Scribble stage, paint marks are piled more or less in the centre of the page leaving the surrounding space blank. In the second, the Topology stage, colours are separated and the whole surface of the paper is covered in irregular shaped patches of colour placed in an irregular order, or covered in dots placed in an irregular order. In the third, the Geometric Pattern stage ('Pattern' for short) congruent shapes which serve as 'elements' in pattern making (lines, dots, circular patches, squares, and others) are placed in regular order covering the whole surface of the paper.

In the second study (M=83, F=73) (Booth 1981) 7838 paintings were examined for structural content and from a content analysis of 1523 of stage three paintings the patterns were categorised into two main classes: Class 1 Patterns, arising from a systematic repetition of an element; and Class 2 Patterns, arising from a division of the plane (see Figs 1 and 2). Further, an invariant developmental order for the three symmetry operations underlying the construction of the patterns was described; thus, 1) translation (slide), 2) reflection (flip), 3) rotation (turn).

The application of Piaget's theory of intellectual development (Ginsberg and Opper, 1969; Inhelder and Piaget, 1964; Piaget, 1953,1972; Piaget and Inhelder, 1956) to the development in pattern painting illuminated its logico-mathematical nature.

Piaget's distinction between empirical knowledge (knowledge of some physical properties of objects) and logicomathematical knowledge (derived from some form of classification and order imposed on the objects) (Piaget, 1972) is applicable to learning experiences in pattern painting.

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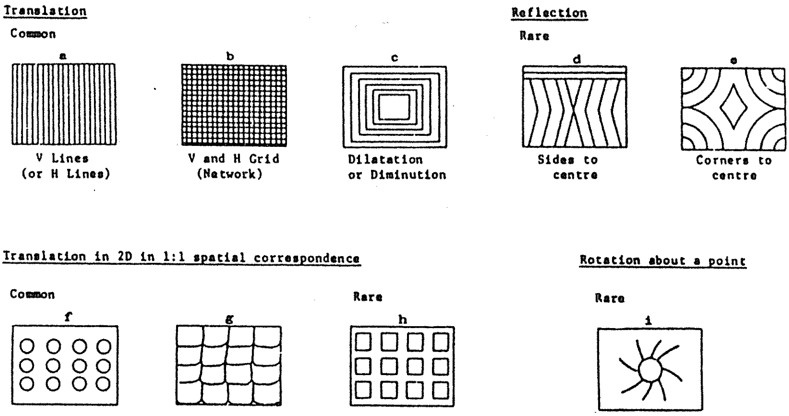

FIG. 1: CLASS 1 PATTERNS: ARISING FROM A SYSTEMATIC REPETITION OF AN ELEMENT
(Source: Booth, 1981)

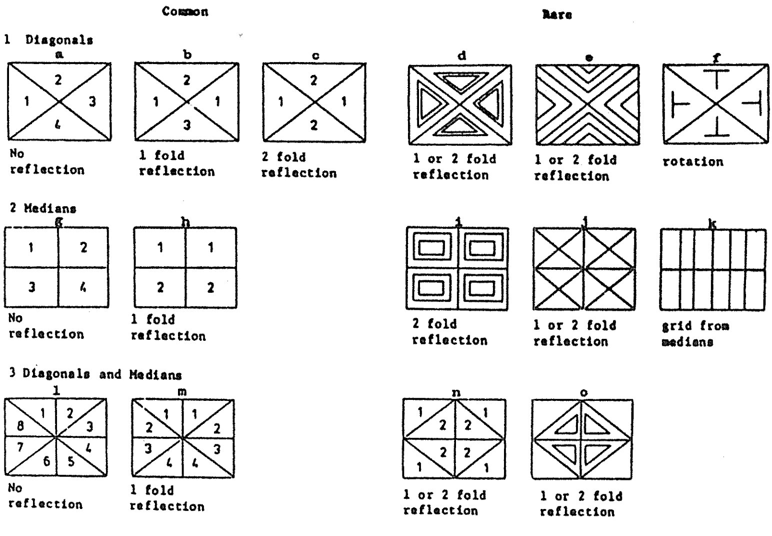

FIG. 2: CLASS 2 PATTERNS: ARISING FROM A DIVISION OF THE PLANE
(Source: Booth, 1981)

Empirical knowledge in pattern painting.-- In spontaneous painting the environment consists of the painting materials: paper, paint, brushes, a water container and water. Each of these has specific physical properties.Thus, paper has a rough and a smooth surface, it has four edges and four corners; brushes come in different sizes, the handle is hard but the bristles are resilient; paint has consistency, and comes in a variety of hues which may be altered by mixing; water is fluid and the amount used affects the consistency of the paint. The degree to which a child by manipulating the materials discovers these properties and learns to control them has a bearing on the final product.

Logico-mathematical knowledge in pattern painting. --Unless raised to a conscious level, logico-mathematical exploration in pattern making remains at the intuitive level. Cognitive skills involved in spontaneous pattern painting are purely at the intuitive level. They encompass intuitive classificatory and spatial behaviour. Classificatory behaviour underlies attention given to colour and to shapes (elements) and their size. Spatial behaviour underlies the organisation imposed on the painting space and the attention given to the position of elements in space and in relation to each other. Logico-mathematical intuition in the topology stage is manifested in the classification of elements (irregular shapes) by colour, and in space filling; and in the geometric pattern stage in the classification of elements by shape, colour, and texture, and in space filling in regular order.

Developmental processes.
The progression through the three stages in pattern painting was interpreted in terms of Piaget's model of adaptation which encompasses assimilation and accommodation as generative processes in cognitive development (Piaget,1953).

The scribble stage. — The scribble stage has four characteristic features: 1) painting is usually started in or near the centre of the page; 2) paint marks are piled on top of each other; 3) the surrounding space is left blank; and 4) an oscillating scribble stroke is used.

Essentially this stage is a manifestation of developing motor skills necessary for handling the painting materials. Primarily four sensorimotor motor skills are developed: gripping the brush handle (assimilating the handle to a hand-grip structure and accommodating the hand grip to the particular shape of the handle), loading the bristles with paint (accommodating to and assimilating the characteristics of the paint and of the bristles to the sensorimotor action of loading the brush with paint), transporting the loaded brush to the appropriate position on the paper, and finally putting a paint mark on the paper. Maximum attention is paid to each of these acts until they are co-ordinated into a temporal sequence forming a primary brush using technique.This technique or skill is practiced by repetition or 'functional assimilation' and forms a scribble action 'scheme'.

The scribble stroke is unconstrained by space, is under minimal sensorimotor control, and initially seemingly executed under little visual guidance. The marks are at first simply perceived as the paint flows off the brush and are later gradually brought under visual control.The scribble scheme is practised for its own sake until control over the tools and materials has been achieved. Attention then shifts from mark making to space and colour.

The topology stage. — Topological attributes are qualitative relations like proximity and separation, boundaries and regions, open and closed curves, which remain invariant under transformation (Hart and Moore, 1973). These attributes are essential features of the topology stage. The characteristics of this stage are: 1) the separation of colour, 2) the filling in of the available painting space, and 3) the random distribution of elements in space.

In the beginning the starting point is still near the centre but towards the end of the stage the starting position tends to be close to a point along an edge of the paper. Starting position, therefore, is not an invariant characteristic feature of topology paintings.

The first indication of classificatory behaviour is attention to separation of colours. And an indication of spatial behaviour is when attention shifts from just making marks to the filling of space. Marks are no longer piled on top of each other but are placed side by side in an oscillating spreading technique. Space filling or colouring is a geometric attribute of pattern (Stevens, 1976; Seymour, 1978). Space filling is thus the earliest manifestation of spatial behaviour and the beginning of geometric activity on the cognitive level in painting development.

With the attention to colour and to space the stroke is brought under greater visual control. Topology painting or non-metric filling of the space is practised for its own sake (functional assimilation) and forms a topology action scheme. When the topology scheme is firmly established the appearance of a regular alignment of a few elements near an edge of the paper foreshadows the geometric pattern stage. As a consequence of the space filling actions and the resulting transformation of space, the motor actions themselves are constrained and modified by space itself. The oscillating painting stroke is transformed into single strokes (lines or dots) at the edges of the paper and at the boundaries of adjacent patches where care is taken not to overlap the colours. The transformation of the painting stroke is thus a function of adaptation processes: accommodation to the environmental constraints, and assimilation of new strokes shaped by the spatial constraints. The transformation of the stroke reflects greater co-ordination between visual and motor actions and improvement in sensorimotor control.

The geometric pattern stage. — The pattern stage arises when single stroke lines (or sometimes dots), which first tend to appear in the topology stage, are assimilated and then consciously used as elements either in repetition patterns or in division of the plane patterns. Characteristic features of this stage are: 1) a 'one-directional' line stroke (Connolly and Elliott, 1972), or dot stroke is used; 2) lines or dots or other shapes are used as basic units or elements in pattern constructions; 3) elements are repeated and aligned in ordered arrays, or a regular division of the plane is carried out; 4) elements are either congruent or similar; 5) patterns are usually initiated at a point on an edge of the paper; 6) standard patterns follow the same rules of construction; 7) in fully formed patterns the whole paper surface is usually utilised.

One rule of construction intuitively followed for a number of patterns is to begin at or close to the upper left corner of the paper. This corresponds to a rule children use in copying geometric shapes observed by Goodnow and Levine (1973) and to the orientation and direction of painting strokes noticed by Connolly and Elliott (1972). Two factors may underlie this phenomenon in painting. Firstly, in the topological exploration of the painting space the edges act as terminal barriers to further action and therefore are brought to the child's attention. Secondly, because the brush stroke, now under visual control, is accommodated to follow the edge it is there that the line becomes most clearly defined. Thus both edge and line are simultaneously brought to the child's attention. The line is then assimilated along an edge which becomes the starting position for most patterns.

Once the line or dot is assimilated attention is centred on shape making which is practised for its own sake (functional assimilation) with a temporary disregard for colour and sometimes also with a temporary disregard for space filling.

The first pattern is usually a translation pattern consisting of congruent elements placed more or less in a linear array. Complex co-ordination processes of hand-motor control and visual control are involved in making each successive shape congruent with the preceding one. The first shape serves as a model. Each successive shape is matched by one-to-one correspondence and is at once a copy of the preceding and a model for the succeeding shape. This involves a process of successive accommodation which differs in its complexity but is essentially similar to successive assimilation thought to underlie linear arrays in graphic collections made with geometric shapes as described by Inhelder and Piaget (1964).

Two-dimensional translation patterns are more complex. In the first row the process of successive accommodation is the same as above. The second and every subsequent row entails successive accommodation not only to the preceding element but also to the element directly above it.

The process of matching shapes by one-to-one correspondence and successive accommodation applies also to reflection patterns. The characteristic features of reflection paintings are: 1) painting proceeds from the edges (or corners) towards the midline or the centre of the page; 2) pairs of matching elements in shape and colour are placed mirror image at the spatial limits defined by the edges of the paper or by the contours of preceding elements. Both shape and colour are integrated forming a reflection symmetry scheme.

A factor that may underlie the initiation of reflection patterns (and diagonal division of the plane) is that in the production of translation patterns the child becomes aware of the terminal edge of the paper because it is a barrier to further actions. Gradually this edge may become a focus of attention and anticipation. At a point in time the anticipation acts as a motivating force to go from the starting point directly to the terminal edge. Typical one-fold reflection patterns arise when a fundamental region (either a vertical line or a column of dots) is painted along one edge and is repeated along the opposite edge in one-to-one correspondence, mirror image in the case of brush-shaped dots. This procedure continues with successive matched pairs of elements, differing from the preceding pair in colour, and sometimes in a modification of the fundamental region, until the elements meet in the centre and the space is filled.

Reflections based on diagonal division of the plane, arise when opposite triangular spaces are filled with elements matched in shape and colour. The starting point for filling may be at the outer edges (as in class 1 type patterns) or it may be at the intersection of the diagonal lines in the centre of the plane.

Which ever way, reflection patterns bring the child's attention back to the centre where the scribble began. This could be a factor that initiates rotation patterns.

Rotation patterns are rare in this age group. The few which have have been observed by Booth (1975 and 1981) are of two kinds. In one, the elements radiate from the centre (see Fig. 1, example 'i'), in another, elements point towards the centre (Fig.2, example 'f').

Throughout the pattern making stage skills of estimating size of elements and distances between elements are developed and refined. The whole process of producing a pattern, whether by repetion of elements or by division of the plane, implies an intuitive metrication of the plane.

Embedded in the patterns are topology structures, Euclidean structures (congruent shapes, parallel lines, equal angles, equal spaces), symmetry structures, and fractions.

Psychological and educational relevance of pattern painting

Of psychological significance are: 1) the invariant developmental order of the three stages (scribble, topology, pattern); 2) the invariant developmental order of the symmetry operations (translation, reflection, rotation); the intuitive mathematical concepts embedded in the paintings, and 3) the use of algorithms or rules of procedure which play an important part in the emergence of the symmetry operations embodied in the painted patterns. Rule based approaches have been observed in other areas of child behaviour, for example, in drawing (Goodnow, 1977, 1978; Goodnow and Levine, 1973), in scientific reasoning (Siegler, 1978), and in counting tasks (Shannon,1978).

Spontaneous pattern painting could play a useful role in teaching since there is general acceptance that pattern is fundamental in mathematics and that it reveals the mathematical order or structure (e.g., Bindon, 1974). Further, painting is intrinsically self-motivating and rewarding (Booth, 1975; Brittain,1979; Connolly and Elliott, 1972). Pattern painting fulfills four requirements considered important for learning: that of 'attention' (Mackworth, 1976) of the inherent interest of the materials, of motivation, and of providing a sense of discovery (Bruner, 1960).Further, patterns are often used as models to highlight certain mathematical ideas. Mathematical models illustrate abstract ideas in a concrete form (Cundy Rollett,1961). Thus children's spontaneous patterns can be used as mathematical model devices in two ways. First, the patterns are named as they arise (Booth, 1981). This is the first step in raising the intuitive thought to a conscious level. Naming a pattern also endows it with a symbolic significance that could provide a foundation for better mathematical understanding. Second, children's patterns could be used as 'models' in the teaching of mathematical concepts (Booth, 1981).

Finally, the teaching of symmetry and other geometry concepts should have their starting point in children's spontaneous activities and follow the natural developmental order: two-dimensional concepts from two-dimensional creations (painting, drawing, collage, etc.) and three-dimensional concepts from three-dimensional creations (for example, clay, and block play). The spontaneous symmetry patterns could be an appropriate beginning in developing in children an understanding of the unity of the sciences and their interaction with other subjects.

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AN EMPIRICAL VALIDATION METHOD FOR PRIMARY SCHOOL MATHEMATICS TEACHING

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In the primary school dual emphases focus on both the student's cumulative knowledge and on their maturing ability to learn. In both these areas what is known about learning and concept development is fundamental to teachers and curriculum planners, who must be concerned with successful learning experiences of all students, and not merely with those who seem to be gifted.

A central problem in teaching mathematics, especially in the primary school, is how to combine epistemological issues with those drawn from the insights of learning theorists. As a proposed solution to this problem, a model of teaching mathematics in the primary school has been developed and is being trialled in a Grade 2 classroom and forms the basis of a method course for pre-service primary teachers. This model combines active experience in the deductive process of mathematics in problem solving with the psychological recommendations of Bruner and Dienes.

Mathematics is a distinctive form of knowledge on account of both its abstract nature and its deductive process (Hirst & Peters, 1970.) There is however, opposition on psychological grounds to the use of the deductive process in primary schools.

> "This point of view [learning by logical reasoning] does not work for the majority of children from ages 5 to 10 years, and indeed can interfere with real learning. Mathematical ability at this age does not consist of being able to deduce conclusions from long chains of reasoning." (Fehr & Phillips, 1972, p.17)

Bruner (1965) on the other hand considered that education must take into account what he calls "the psychology of a subject matter", wherein a discipline's way of thinking is taken as a central focus of a discipline and children are provided at the earliest opportunity with access to this way of thinking. Since deduction is the generally accepted method and approach for discovering truth in mathematics, then even primary school children should be encouraged to participate in this particular process of establishing knowledge. In fact justification rests on this participation.

> "It is important to justify a good mathematics course by the intellectual discipline it provides or the honesty it promotes as by the mathematics it transmits." (Bruner, 1965, p.1012.)

Piaget's developmental stages in learning Bruner accepted, but modified the principle of readiness to include not simply the child, but also the subject matter. The presentation of subject matter was also stage dependent. The same concept could be presented at a manipulative or enactive level, at a pictorial or ikonic level and finally at a symbolic or formal level.

It is generally accepted that primary school children are typically at the pre-operational and concrete operational stages of concept development, and therefore rely heavily on their physical environment. Their learning experiences need to be based on observation of the real world, and then move to more and more abstract mathematical representations.

The teaching method outlined in this paper combines real world exploration that allows both deductive and inductive learning through discovery methods, together with progressive abstraction towards symbols and their manipulation. A disciplined form of mathematical enquiry is thereby combined with a recognised psychological basis of concept development for young children.

The mathematical knowledge developed by this empirical approach is consciously made available to all children to prevent its process becoming esoteric. Validation of the processes and solutions are accessible to public scrutiny thereby encouraging honesty in manipulations. Mastery of the mathematical process is assisted by taking successive stages in introducing symbols and manipulating them in algorithms. Empirical observations and validations are available at each stage.

Trialling of this method with a Year 2 primary class is encouraged by the ready acceptance of this type of activity. There is also heard, quite frequently, from pre-service teachers, remarks about having suddenly discovered meanings and understandings of primary mathematics that had previously mystified them.

METHOD DESCRIPTION

From a problem given in words or symbols a mathematical form of the problem is stated in either enactive or ikonic representation. A solution is then arrived at by deduction in accordance with the mathematical meanings of the problem's description. This solution can be seen as a tautology of the original problem stated in another form, usually simpler. The final mathematical statement can be merely a concluding sentence describing the problem and its solution in symbolic language.

The principal element in the application of the deductive process is the validation required to accept the solution as correct. Instead of relying on the teacher or the text book as the authority for validating the solution, the process of empirical validation makes the solution acceptable from the honesty of the obvious re-arranging of the materials. The method allows for individual variations in the solution process in accord with the perceptual and maturation capacities of individual students.

METHOD EXEMPLAR

Consider a verbal problem has been reduced to the mathematical statement

$$\frac{3}{4} + \frac{1}{2}$$

Both past and present teaching, in the main, describes the solution technique that must be followed as:

rename the fractions with like denominators : $\frac{2}{4} + \frac{3}{4}$

perform the arithmetic operation - add the numerators but do not add the denominators : $\frac{2+3}{4}$ [not $\frac{2+3}{4+4}$]

rename : $\frac{5}{4}$ [not $\frac{5}{8}$]

rename : $1\frac{1}{4}$

To questions such as why the numerators were added but not the denominators and how we know the answer is correct we can appeal to the teacher as the expert or the text book as authorities.

Later students are taught there is a different method when multiplying fractions. We do not rename in equivalent fractions, and we multiply both the numerators and the denominators:

$$\frac{1}{2} \times \frac{3}{4} \Longrightarrow \frac{1 \times 3}{2 \times 4} \Longrightarrow \frac{3}{8}$$

There is later another method taught for division by a fraction, in that we invert the fraction and proceed to multiply:

$$\frac{3}{4} \div \frac{1}{2} \Longrightarrow \frac{3}{4} \times \frac{2}{1} \Longrightarrow \frac{3 \times 2}{4 \times 1} \Longrightarrow \frac{6}{4} \Longrightarrow 1\frac{1}{2}$$

To the student the apparent arbitrariness of the several techniques presents difficulties in remembering both the order of the steps within each technique and which technique to apply to a particular problem. Competency in these techniques is important as they are used in succeeding grade levels right up through the secondary levels.

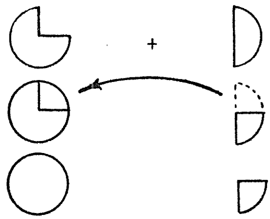
The alternative empirical validation method can be illustrated in solving the problem by manipulating objects and recording pictorially as follows:

problem : $\frac{3}{4}$ + $\frac{1}{2}$

meanings : +

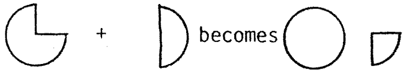
addition operation :

which are the same quantities as :



description in symbols : 1 $\frac{1}{4}$

Thus the problem :



Conclusion of a symbols sentence : $\frac{3}{4} + \frac{1}{2} = 1\frac{1}{4}$

The process is deductive: meanings are established and the material is re-arranged so that each re-arrangement is a tautology of the previous statement. The student is then doing mathematics in a way similar to the mathematician. Operating empirically with materials allows the meanings and re-ordering to be seen by others and the truth thereby checked. Recording pictorially allows scrutiny of the process after the manipulations are completed. The process is thereby self-validating.

If an algorithm is to be developed for manipulation of symbols alone, then its solution can be validated against the empirically validated solution already obtained. Hence why only the numerators were added and not the denominators can be resolved by comparing the solutions arrived at against the solution obtained by the empirical validation method.

CONCLUSION

The method of mathematics education developed through empirical validation for primary school children is deliberately based on Bruner's ideas of the psychology of the subject matter and the psychology of the learner. The deductive way of thinking of mathematics is encouraged in young children by the successive use of the enactive, ikonic and symbolic levels of operation. The empirical validation of each statement of the deductive process makes the truth of the new knowledge open to the scrutiny of all and prevents any manipulation becoming esoteric. The positive reactions of both Grade 2 students and pre-service teachers suggest the empirical validation method could be a viable method in mathematics education.

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A STUDY ON THE REPRESENTATIONAL SYSTEMS IN MATHEMATICS TEACHING

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1. INTRODUCTION

It goes without saying that in lessons of mathematics (including arithmetic), many mathematical concepts, principles and rules are taught. How these concepts are formed and understood by children depends largely on what method of representation has been employed to represent them. Adequacy of a method of representation constitutes a great element that determines the effectiveness of the lessons.

This study examines methods of representation used by teachers and students in mathematical lessons on the basis of such views and from pedagogic, semiotic and psychological viewpoints, and through such examination, intend to induce some applicable principles of the effective, suitable methods of representation in lessons.

The study proposes a classification of the modes of representation, analyzes methods of representation in arithmetic textbooks based on such classification, and investigates through the experimental lessons the roles of manipulative mode, one of the modes of representation in lessons.

2. ON THE PREVIOUS STUDIES

The representational systems have been studied by J.S.Bruner, R.Lesh and others.

It is well known that Bruner (1966) has classified the representational modes into three modes,that is, 'enactive mode','iconic mode' and 'symbolic mode '. Then, he indicates that the nature of intellectual development seems to run the course of these three modes of representation until the human being is able to command all three.

According to Bruner's theory, enactive representation is easier to understand for children than iconic representation and the latter is easier than symbolic representation. These merits are put to practical uses in mathematical lessons. But Bruner's representational system is not proposed as the representation of mathematical knowledge. So, that system has been improved through subsequent studies. Among others, Lesh's studies are important.

Lesh and his group have examined and extended Bruner's three modes and proposed an interactive model for using representational systems as follows.

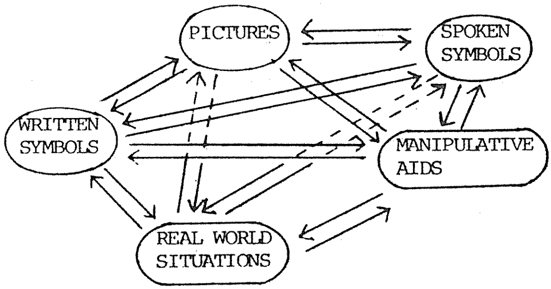


Fig.1. An Interactive Model for using Representational Systems by Lesh.

Lesh reconceptualized Bruner's enactive mode, partitioned iconic mode into manipulative aids and static figural models (i.e., pictures), and partitioned symbolic mode into spoken language and written symbols. Furthermore, he indicated that these system of representation were interpreted as interactive rather than linear. A major hypothesis of his project is that an interplay or translation among and between these several modes of representation enhances meaningful learning,retention, and transfer of mathematical ideas.

Lesh's system is surely better than Bruner's for mathematics teaching. However, it still open to some improvements. As Lesh's spoken language is usually accompanied with each of the other representational modes, it is better that spoken language is not treated as one independent mode. Moreover, there are many differences between mathematical symbols and written language, so Bruner's symbolic representation should be partitioned into these two representational modes.

3. CLASSIFICATION OF MODES OF REPRESENTATION

The study classifies hypothetically the modes of representation used in mathematics teaching into five modes as follows. Each example is the representation of 2 + 3.

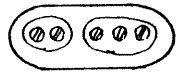
(A) Realistic mode : the representation with realistic or real world situations.

(Ex.) To gather two real apples and three real apples.

(B) Manipulative mode : The representation with manipulative aids.

(Ex.) To gather two marbles and three marbles.

(C) Pictorial mode : the representation with static figures or pictures.

(Ex.) 

(D) Language mode : the representation with written everyday language

(Ex.) " To add two and three "

(E) Symbolic mode : the representation with mathematical symbols

(Ex.) " 2 + 3 "

Then, this study shows their mutual relationships in Fig.2.

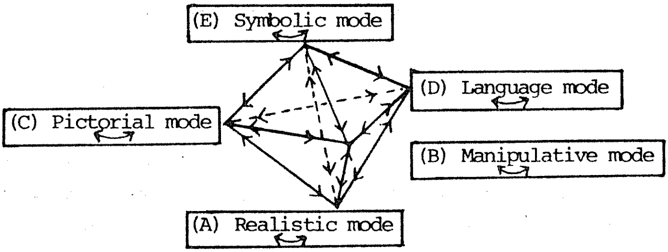


Fig.2. The Representational System by Ishida.

In many cases mathematical lessons ultimately aim at making the representation withmathematical symbols understood. For this reason, how the other four modes of representation used to make the representation understood becomes a major subject to be examined to conduct a better lesson.

Among other things, the representation with manipulative aids and the representation with pictures or figures visualize and embody abstract mathematical concepts, so that they can help make such concepts easier to learn. Therefore scrupulous examination must be exercised, when lessons are composed, as to what manipulative aids and what pictures or figures to be used.

Furthermore, expressing the representation by mathematical symbols in other mode of representation and making children understand their mutual relationships result in children'understanding of various aspects of the concept and thereby leading to a rich and clear formation of the concept. For this reason, Fig.2 can serve as an important viewpoint in thinking about the composition of mathematical lessons and in examining lessons.

4. EXAMINATION OF TEXTBOOKS BASED ON REPRESENTATIONAL SYSTEM

Next, let us examine the mode of representation in Japanese arithmetic textbooks, based on the representational system in the foregoing Fig.2.

As is generally known, the Ministry of Education stipulates the contents of teaching on arithmetic following the Course of Study. Since textbooks are written on the basis of the Course of Study, the current arithmetic textbooks published by the six publishers are comparatively alike in contents. Japanese arithmetic textbooks are not mere collections of questions nor workbooks but are written in such forms and contents with which lessons may be proceeded. In fact , many teachers proceed with their lessons following the development and statements of textbooks. Therefore, examination of the mode of representation in textbooks is of equal value to the examination of the mode of representation in lessons.

Now, currently used arithmetic textbooks contain, for example, an introductory scene of addition as shown in Fig.3. Here, all of the five modes of representation shown in Fig.2 are employed. Although it is a typical example, four modes (A),(B),(D),(E) or (A),(C),(D),(E) are very often used in introducing other concepts, principles and rules. Thus it is from the textbooks published in 1980's that at least four of the five modes of representation came to be used in many cases. Most textbooks before that time used only three modes (A),(D) and (E) or (C), (D) and (E).

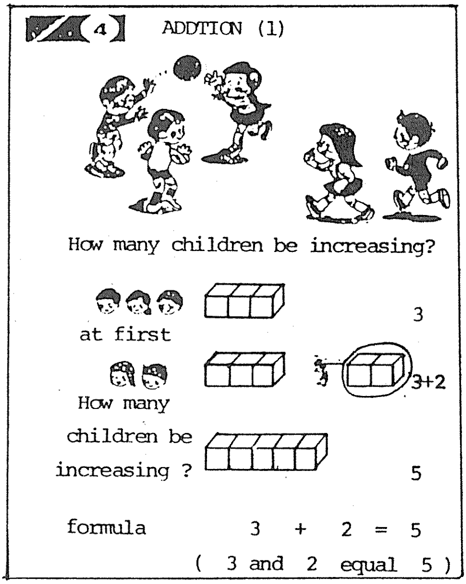


Fig.3.

The facts that a variety of modes of representation are used in a textbook for a certain concept and rule is ideal, considering the viewpoint that clear, rich concepts could be formed. Therefore it can be said that textbooks of 1980's have been improved, as compared to those before that time in term of representational system.

The reason why the modes of representation in textbooks have been improved as such, especially the reason why the modes of representation (A),(B) and (C) have come to be used is that studies on the mode of representation have been promoted, which can be evidenced by the following two points:

(1) Manipulative activity was emphasized in the Course of Study for

arithmetic that was revised in 1977.

(2) In 1970's it was pointed out that children who were poor in arithmetic increased, and efforts have been exercised to make arithmetic lessons more understandable.

5. A STUDY ON THE ROLES OF MANIPULATIVE ACTIVITY

Following are the results of a study made through experimental teaching to see how representational activities with manipulative aids functions to bring up children's mathematical concepts and thinkings in lessons. Experimental teaching was performed on the " areas of rectangles and squares " and was applied to 45 pupils each in classes (I) and (II) in the fourth grade of Tomo elementary school in Hiroshima city.

A pretest made before the experimental teaching revealed that many children thought they could compare the areas of rectangles by comparing the sums of the sides of rectangles. It is pointed out that the conventional teaching method on the matter would end up ares's formulas of rectangles and squares by rote, without understanding what the formulas meant.

In view of such problems we worked out lessons incorporating the following manipulative activities.

(1) Make a variety of shapes using strings of the same length and compare their areas.

(2) Compare the areas of two rectangles by overlapping one over another.

(3) Prepare a number of rectangles 1cm long and 1cm wide and compare the areas of two rectangles by spreading the prepared squares over those two rectangles to be compared and by counting the number of prepared squares that could be spread over.

In order to investigate the difference between the manipulative activity performed by individual children and that demonstrated by the teacher, the former was applied to class (I) while the latter was applied to class (II).

Thereafter a common posttest was conducted on the children of both classes. The result was that the children of both classes gave good marks as a whole and that the manipulative activities from (1) through (3) above proved effective in teaching area ideas.

Aside from the above, the following differences were observed between classes (I) and (II).

(i) Children who attempted to compare the areas of rectangles by the sums of the sides were more in class (II) than in class (I).

(ii) Children who formed a habit of comparing the areas of rectangles numerically were more in class (I) than in class (II).

(iii) Children who came to know that difference shapes of the same area could be made were more in class (I) than in class (II).

The above (i), (ii) and (iii) indicate that the children in class (I) gave more ideal results than those in class (II). In conclusion, it could be said that in teaching mathematical concepts and ideas, manipulative activities by children themselves play an important and effective role.

6. TASKS TO BE SOLVED FROM NOW

Studies on representational systems may be made from a variety of viewpoints and for different objectives. This study is intended, as has been mentioned under introduction, for creating better mathematical lessons. This study, which is a basis for such objective, has proposed a classification of the modes of representation used in mathematical lessons into five modes and proved partly the adequacy and effectiveness of the representational systems based on such classification.

The first thing I must do from now is to define through school teaching the roles of each of the proposed representation modes in lessons, particularly the roles of and the differences between the realistic mode, manipulative mode and pictorial mode, taking into account the personal differences of children and the contents of teaching materials .

I would also like to study how these modes of representation function as a means of curing and teaching those children who have become stuck with mathematics.

Thus the study ultimately aims at establishing some principles on which five modes of representation can be effectively applied.

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ALGORITHM LEADING TO ABSURDITY, LEADING TO CONFLICT, LEADING TO ALGORITHM REVIEW

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Failure to solve a mathematical problem may have many different causes. A serious one is the application of an incorrect algorithm. Students, especially at the elementary level, are trained to use a specific algorithm to solve a given problem. In many cases the textbook and/or the teacher does not suggest a few alternative algorithms which may be used in the problem solution. In more extreme situations, the student will be discouraged from employing an algorithm other than that given in the text or presented by the teacher, even though his algorithm may be correct. As a consequence, many students will be satisfied if they have chosen what seems to them *the* algorithm, and will employ it, paying little if any attention to the result it gives. By presenting problems which, if solved by routine algorithms, give rise to "nonsense", we can create a conflict situation for some of the students. There is a possibility that students who experience such conflict, will be more aware of the necessity of checking their results, and may be thus encouraged not only to check their result, but also, eventually, to question the chosen algorithm itself. The naive assumption of the existence and universitality of a unique algorithm, may thus be weakened.

In this paper we discuss three such tasks, and the results obtained when they were given to students.

TREES AND CHILDREN

The following two tasks, which differ in context only, were given to students in grades 5, 6, 7, and 9. We wished to see how age and context influence student reaction in the conflict situation.

Task 1 (Trees)

236 trees were planted in 6 rows. How many trees do you think there were in each row?

Task 2 (Children)

147 children went on an outing in 4 buses. How many children do you think there were in each bus?

The students were not used to such questions - the usual problems which they had met, could be answered by the standard division algorithm, to give a unique sensible answer. Here the algorithm gives an unreasonable answer, and without the algorithm, the answer is not unique. The analysis of student responses led us to the following categories.

Category I - unreasonable answer

The students in this category used the standard division algorithm (236:6 or 147 : 4), obtained a fractional answer and stopped. This is the only category in which there was no conflict at all, and as can be seen from Table 1, the percentage of students in this category decreases markedly with age.

Category II - no solution

The attitude of students in this category was that, since the division algorithm leads to an impossible result, the problem itself is impossible. Thus, for example, Dina (grade 6)

> "It can't be done - 236 can't be divided by 6."

Very few students were satisfied with such an answer (see Table 1), and this is encouraging.

Category III - reasonable but incorrect or incomplete answer

For example, Uri (grade 6)

> " 147 : 4 = 36 3/4. This is mathematically possible - but in reality it's impossible, because it's impossible to cut up children."

The use of the division algorithm leads to a conflict between the answer obtained and the physical fact. The student either, as in the example, does not know how to continue, or continues but does not take all the facts into account. Thus a typical conclusion was 40 trees in each row, which is not in agreement with the data.
A relatively large number of students fell into this category, which is again a good sign: The conflict seems to exist naturally. Probably, with a little attention, they can be made aware of the necessity to fit the answer to the data.

Category IV - correct answer

For example, Noa (grade 7), after obtaining 36.75 by the division algorithm, wrote

"In my opinion, there will be 37 children in three buses and 36 in the fourth".

or, Maya (grade 7) after trying division, found an allocation of children to buses by "trial and error"

"Bus 1 - 39, Bus 2 - 26, Bus 3 - 41, Bus 4 - 41",

where the first 41 was a correction of 40, to obtain the correct total.

The three latter categories exemplify ascending levels of effectiveness of the conflict situation. In category II the conflict merely prevents an absurd answer, in category III causes further activity if not entirely successful, whereas in the final category it achieves maximum effect. Table 1 shows the percentages of students in each category according to grade and task.

	Category	5 - 6		7		9	
		"trees" (n=130)	"children" (n=127)	"trees" (n=59)	"children" (n=59)	"trees" (n=53)	"children" (n=63)
	I Unreasonable answer	32%	34%	19%	8%	19%	2%
Conflict effect	II No solution	6%	4%	2%	-	-	2%
	III Reasonable but incorrect answer	39%	34%	57%	24%	45%	23%
	IV Correct answer	23%	28%	22%	68%	36%	73%
	Total	68%	66%	81%	92%	81%	98%

Table 1 : Student responses to tasks 1 and 2

It is clear from the figures in the table, that performance improves with age, suggesting that the value of the conflict situation may be connected with general maturity. It is also noticeable that the context of the conflict seems to make a difference. Thus in grades 7 and 9 "fractional trees" seem to be more reasonable than "fractional children"(category I) and there are more correct responses for "children" than for "trees" (category IV).

This suggests that the choice of the conflict, if not crucial, can play an important role - in some sense, it seems that "trees" are more remote from the children's imagination.

BACTERIA

The following contextually more sophiticated task was given to 145 grade 9 students, after they had studied the regular curriculum dealing with linear and quadratic functions. In previous studies we had found that students' responses to tasks involving functions and graphs were essentially "linear". Here, the assumption that only the "linear" algorithm exists, leads to an eventual conflict.

Task 3 (bacteria)

A scientist conducted experiments using cultures, containing different types of bacteria. The number of bacteria in a culture depends on temperature and the type of bacteria.

The figure shows the number of bacteria found at 10°C and 25°C.

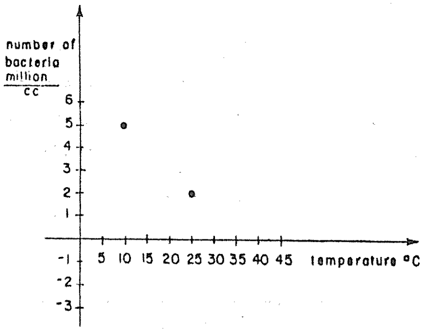

Figure 1

a) *The scientist wants to know the number of bacteria at 20°C. What can you say about the number of bacteria at this temperature? Explain your answer.*

b) *What can you say about the number of bacteria at 30°C? Explain your answer.*

c) *What can you say about the number of bacteria at 45°C? Explain your answer.*

The "linear" algorithm (linear interpolation in section a and extrapolation in sections b and c leads the students to an unreasonable (negative) answer in section c, designed to create a conflict between the result of the algorithm and reality. The fact that the algorithm leads to reasonable answers in the first two sections would seem to make the situation more subtle than that in the first two tasks. Further, in this task, the algorithm itself is essentially incorrect, which is not the case in the first two tasks, where the algorithm can be used to indicate a reasonable answer.
Student responses to sections a and b can be divided into three categories.

Category I - linear

The students in this category used linear interpolation in section a and linear extrapolation in b.

Category II - quasi-linear

The students gave a range for the number of bacteria, lying between 5 and 2 in section a, and 2 and 1 in b.

Category III - non linear

Those in this category wrote that it is impossible to determine the number of bacteria from the given data. Thus Amir, for example,

> "I cannot tell, because the number of bacteria does not have to be in a fixed relation to the temperature."

Table 2 gives the percentage of students in each category.

Category	Percentage of students (n=145)
I Linearity	86%
II Quasi-linear	5%
III Non-linear	9%

Table 2 : Student responses to sections a and b.

There were students who continued to use linear extrapolation in section c, but a large percentage did have a conflict and resolved it by saying that the number of bacteria is zero. Thus, Ronit, for example, wrote

> "From the graph we see that for x = 45, y = -2. But there cannot be a negative number of bacteria - so there can't be any at all."

Table 3 gives the percentages of students according to types of response.

	Category	Percentage of students (n=145)
	<u>Linear</u> $-2\ \frac{\text{million}}{\text{cc}}$	20%
Conflict effect	zero	54%
Conflict effect	<u>Non-linear</u> one cannot know	11%
Conflict effect	Total	65%
	Something else, incorrect	15%

<u>Table 3</u> : Student responses to part c.

As can be seen, about half the students resolved the conflict by concluding that there are zero bacteria at 45°C. Although the "linear algorithm leads to a ridiculous result, *they did not question the algorithm*, but chose the "nearest" meaningful result. However, there were the relatively few who, as a result of the conflict, thought again about the algorithm itself. Thus, for example, Judith wrote

> "The number is -2 (sic) if the function is linear.
> But I don't understand how it can be (-).
> If the function is not linear, then the number depends on the graph you draw."

Of course, a single task is not going to correct a phenomenon as pervasive as the "linear dependence" of students, which has been noticed by many researchers (Karplus, 1979; Markovits, 1982; Markovits et al, 1983). But there is here a hint that we might be able to use the deliberate conflict situation, not only to make students more aware of the necessity of checking the reasonableness of the results, but also to remove methodological misconceptions (Swan, 1983).

CONCLUSION

It should be remembered that the three tasks described, were given to students who had not previously, to our knowledge, been put into such conflict situations. Problems on children, trees, etc. in the curriculum are of the well-known standard type, for which there is an algorithm leading to a unique answer. The fact that so many students, in the first two tasks, were affected by the confrontation with the conflict, suggests that the judicious and consistent use of such problems over the whole curriculum, may well encourage attention to the reasonableness of results, and as a positive influence on other student mathematical behaviour.

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AN ANALYSIS INTO STRUCTURES AND MECHANISMS OF 5TH GRADE CHILDREN'S MATHEMATICAL KNOWLEDGE

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INTRODUCTION

From analyses of children's behavior when learning mathematics emerges the crucial role of the mental representation of their mathematical knowledge. Instead of being coherently organized, relevant knowledge structures which need to be coordinated for success with a broad range of mathematical situations appear to develop initially as isolated "packets" which are more or less restricted to the situational context in which they were acquired.

The postulate that knowledge is stored in memory in discrete and self-contained units has been raised by several authors and has been captured in notions like "frames" (Davis, 1980), "microworlds" (Lawler, 1981), "Subjektive Erfahrungsbereiche" (Bauersfeld, 1983), and others not mentioned here. Besides this postulate, we will take into account that organizational structures of the memory system are given by the kind of the connections established between such memory units. That is, we view the structures of the knowledge in memory to be constituted by at least the following two things:

(i) self-contained memory units ("packets of knowledge");
(ii) connections between these ("organizing path networks").

Both are the result of the individual's interaction with the outside world and of internal reflective processes, and both can be subject to pedagogical interventions. In such structures is constituted the personal knowledge and beliefs of an individual, including all misconceptions and inaccuracies.

Another postulate we submit to in this paper is that in learning takes place accretion, that is, everything committed to memory is stored permanently. Access structures may change but representations are not actually deleted from memory (e.g., Davis, 1980). This postulate becomes critical when memory units concern the same topic of knowledge, for it will support the fact that competing knowledge may come to coexist in memory.

Out of the context of a two-year clinical study on the acquisition of rational number concepts with 4th/5th grade children (see acknowledgement), interview material was chosen for an analysis of characteristical features of mental representation structures of children's knowledge, and of cognitive mechanisms acting on such representation structures. The piece of work presented here will focus on one child's performance on a complex problem solving task to give a detailed example in support of such hypothetical mechanisms. Interviews with other subjects on the same task are available and will be used to back-up, and further expand on, these hypotheses in future work. Cognitive phenomena which may interfere with successful performance will be discussed, and questions will be posed on the nature of pedagogical activities taking account of such mechanisms.

CONTEXT

The "Gray levels study" was done as part of the assessments in the Rational Number Project after completing 30 weeks of experimental instruction, and was reported on earlier (Wachsmuth, Behr, and Post, 1983; a detailed report is in progress). In videotaped one-on-one clinical interviews, sixteen 5th-grade subjects were presented with a scale of 11 distinct gray levels increasing in darkness from 0% (white) to 100% (black) in stages of 10%, and with a set of twelve fraction cards: 0/20, 1/5, 2/7, 6/20, 2/5, 4/10, 6/15, 2/4, 4/8, 4/6, 6/9, 12/15. Interpreted as representing ink mixtures with *a* parts black ink in *b* parts solution, these cards were to be ordered and attached to the scale according to their "darkness". Requiring the coordinated application of a broad scale of relevant skills, this task was expected to elicit particular features of children's rational number knowledge.

One subject, called Terri in this paper, arrived at the following solution:

0	10	20	30	40	50	60	70	80	90	100%	
$\frac{0}{20}$	$\frac{1}{5}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{7}$	$\frac{4}{6}$	$\frac{4}{8}$	$\frac{4}{10}$	$\frac{6}{9}$	$\frac{6}{15}$	$\frac{6}{20}$	$\frac{12}{15}$

Terri (age 11;6;24) was a low-achieving subject who was observed to have severe difficulties impeding successful learning. Rather than building coherent knowledge structures guided by the instruction, Terri was likely to invent her own, often flawed "theories" and procedures. Interventions in classroom instruction sometimes made her arrive at an "insight" which could turn out to have been an ephemeral one the very next day. The interview following her solution of the gray levels task reveals some deep insights into her thinking styles, which can serve to raise some hypotheses about the structures and mechanisms of this child's mathematical knowledge. At the time of the interview, the interviewer had known Terri from daily classroom contacts and other interviews for more than one year and was quite familiar with her idiosyncratic styles of thinking.

COMMENTED INTERVIEW TRANSCRIPT

In the following transcript is displayed some of the inconsistencies and misconceptions in Terri's knowledge of fraction equivalence and rational number ordering. In her behavior there is found indication for several conflicting "frames" or "theories" serving as bases for her decisions on comparisons of the fractions she was presented with. From earlier observations, Terri was known to persistingly call two fractions - when presented to her as written or spoken symbols - equivalent if they had the same denominator.

In the present problem situation, Terri had attached the fractions 6/15 and 12/15 to different gray levels (90% and right of 100%), apparently following some kind of "lexical" ordering bearing on the whole number symbols in the 12 fractions. This fact raises doubts over whether Terri had understood at all the interpreation of fraction symbols by means of gray levels. However, at least in the beginning it seemed to have been clear to her as will be seen from the following dialog. After she has placed all fraction cards at the gray level scale, Terri is asked at first why she has put 0/20 at the beginning of the scale (white, i.e. 0%). Terri explains:

0. TERRI: Because there'd be no black ink, no black ink so it would be clear water.

After a short dialog about 4/8 and 4/6 which Terri calls about equally dark, but 4/8 still a little bit darker than 4/6, Terri is asked about the two fractions 6/15 and 12/15.

1. INTERVIEWER: Now, Terri, what about 6/15 and 12/15?
2. TERRI: They're equal, like [laughs].
3. INTERVIEWER: OK, but you put them in different positions, though, why did you do that?

By this question, Terri's attention is called back to the task situation where she, not necessarily through an interpretation in terms of gray levels but presumably through her strategy of "lexical ordering", has rated 6/15 and 12/15 as being different. This is in contradiction to her momentary opinion that these two fractions are equal. She responds:

4. TERRI: Because! That's the way I thought I should do it! [moves and messes up chart].

That is, confrontation of her momentary opinion with her previous one results in a cognitive conflict which Terri apparently tries to escape from by destroying the solution she constructed. After a short dialog (Terri should have been asked more questions to other cards) the interviewer continues (without Terri's solution being further displayed):

5. INTERVIEWER: I would still like to know: you say six-fifteenths and twelve-fifteenths are equal?

The interviewer returns to this question to find out which was Terri's reason for calling the fractions equal before; besides, he is interested *now in which of her opinions will persist after the conflict.*

6. TERRI: Right.

7. INTERVIEWER: But you put them on different parts ...

The conflict is evoked another time; Terri's response confirms that she had in mind, without making any connection to the gray levels, an ordering strategy guided by the whole number relationsships in the fraction symbols:

8. TERRI: 'Cause six comes before twelve so I thought that's the way you do it ...

Her thinking was that this was the way to make sense of the task. Now the interviewer wants to find out whether gray levels played any part at all in her doing (Terri's explanation on placing the 0/20 card had actually made reference to gray levels).

9. INTERVIEWER: OK, did you think in terms of darkness when you did that?

10. TERRI: Yeah, sorta like ...

Terri's answer does not sound convincing. Even when gray levels have played a part in the beginning, one is tempted to assume that later she has focused on a whole number ordering strategy. The next question is to find out whether Terri, in the situational context of gray levels, realizes that 12/15 represents a darker mixture than 6/15 does.

11. INTERVIEWER: Which would be darker? Six-fifteenths or twelve-fifteenths?

12. TERRI: Twelve-fifteenths.

She does rate 12/15 as darker than 6/15, but can she reach a conclusion on the ordering of the fractions 6/15 and 12/15 from this?

13. INTERVIEWER: OK, and which fraction would be bigger?

14. TERRI: Twelve-fifteenths.

Terri apparently infers that 12/15 should be the greater fraction of the two. This inference is based on an interpretation of the fraction symbols which grounds its meaning on gray levels, but it already states a "greater" (and no longer "darker") relationship between the two fractions. The inferred statement, however, continues to be in conflict with Terri's earlier opinion about the relationship between 6/15 and 12/15 which apparently resulted from her flawed "theory" of when two fractions should be equivalent (i.e., when presented to her in a purely symbolical context, Terri calls same denominator fractions equivalent).

The interviewer's next question is to find out whether Terri's opinion inferred meaningfully (12/15 > 6/15) outweighs her earlier opinion which was based on her "theory" of symbolical fraction equivalence. A critical section in the interview begins here. Through careful wording (namely, as it had been used in a stereotypical fashion in repeated interviews presenting fraction comparisons in a symbolical setting[1]), the interviewer on purpose attempts to trigger Terri's "theory" of symbolical fraction equivalence.

1) The original wording in these interviews was like "One-fifth and one-sixth, are they equal or is one less? - Which one is less? - Tell me how you know!"

15. INTERVIEWER: And if I ask you, six-fifteenths, twelve-fifteenths, are they equal or is one less?

If Terri's "theory" were activated by this, she should reply 'They are equal'.

16. TERRI: It's less.

I.e. one is less, that is, Terri does not call them equal which is (surprisingly at the moment) not the answer anticipated from her "theory". Should the conclusion she inferred on the basis of gray levels have ultimately affected Terri's belief? The interviewer's next question ('which one is less') is posed even though Terri had previously named 12/15 as the greater fraction. This question corresponds - in wording and in the sequence of events - to the stereotypical situation of the interviews on symbolical fraction comparisons and thus again addresses (as is the interviewer's hypothesis at this point) Terri's "theory" on equivalence of fractions presented to her symbolically.

17. INTERVIEWER: Which one is less?

18. TERRI: Six... um... fifteenths.

And now the interviewer wants to know which of Terri's theories her opinion, after all, is based upon.

19. INTERVIEWER: And why did you say it's less?

20. TERRI: 'Cause it ...oh! [puts head in hand and sighs] No, they're equal. Because they have the same denominator.

DISCUSSION

We observe several instances of the crucial part which cognitive structures and mechanisms play in Terri's behavior.

1. Relevant knowledge can remain latent in a task situation, i.e. the subject "knows" but does not access, at any time, particular facts which apply to the situation. That is, the "momentary opinion" of the subject does not resemble the global knowledge she has acquired which is relevant to the task (were it so, the subject would have to become aware herself of the inconsistencies existing in her knowledge). Rather, we can conclude that her "momentary opinion" is based on a local subset of her knowledge being determined by her actual focus, i.e. the knowledge she has "in sight", and by the part of knowledge which she can access from this point.

Another point which has been observed is the possible lack of mutual accessibility of relevant knowledge units, e.g., in the context of one the subject may be able to access another one but the opposite may not be true. This is yet another instance in support of the fact that knowledge tends to develop in discrete units and that attention has to be given to the development of a proper access framework. Still more evidence for this point is found in other interview dialogs not presented here. Future work will elaborate on this.

2. We mention the crucial role of language cues (and of other cues possibly generated from a situation). As was shown in the dialog, certain lanquage can serve to shift the subject's focus to access knowledge contained in other memory units while loosing sight of the factual situation focussed on previously. A striking instance of this is documented in lines 15 - 20 of the above transcript. The momentary opinion of Terri (lines 16 and 18) was obtained by the chain of inferences she had undergone before (lines 11 - 14) and was supported by the meaning constructed from the situational context of gray levels. Presumably, the resulting conclusion ($6/15 < 12/15$) is still present in Terri's short-term memory while the chain of inferences which made her arrive at this conclusion is no longer present in her short-term memory. But then the interviewer, again, calls for reasons while cueing her knowledge on symbolical fraction equivalence (line 19). Indeed, Terri's focus turns out to have shifted back to this realm: In order to give a reason, Terri has to make a new inference, based on her current focus. And - no way out of there - she comes up (line 20) with an according opinion (changed again!), together with an appropriate reason.

3. Cognitive restrictions can limit the use of relevant knowledge a subject possesses and can possibly intercept the change of incorrect beliefs. One is tempted to resign on the usefulness of a Socratic style of dialog and on whether incorrect beliefs of a subject can be changed through such a dialog. Admittedly, the example discussed is an extreme one and probably requires further analysis in terms of attitudinal patterns in the interaction of interviewer and subject. It shows, however, that a single intervention does not necessarily lead to an "insight" which becomes persistent instead of being a momentary one. Apparently, the access structures calling on Terri's flawed knowledge on symbolic fraction equivalence are much stronger than the connection made on the basis of a several-step inference which the subject is not likely to achieve all by herself. A point can be made that a more global consistency check and revision of acquired knowledge structures requires cognitive capabilities this child has not developed so far. A momentary and single "insight" is not sufficient for a long-term change in the cognitive structures manifested in Terri; it will need more than that.

CONCLUSION

At the beginning, we made a distinction between the memory units which knowledge is locally organized in, and the global organizational structure given by the connections established between such units. It is one thing that a subject can have acquired incorrect knowledge (e.g., Terri's "theory" on fraction equivalence); it is yet another thing that relevant knowledge, whether or not it is correct, does not become activated in a situation when it should be. Moreover, the fact that incorrect as well as correct knowledge on the same topic can coexist in memory calls for particular attention on how instruction can help to improve access to the right piece of knowledge at the right time. Future work will be devoted to the critical part which mental representation plays in the learning of mathematics.

ACKNOWLEDGEMENTS

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2. The mental structures postulated of Terri in this paper have been modeled in a knowledge representation system implemented in a PROLOG machine. Parts of the dialog have been reproduced by the system. The author is grateful to Helmar Gust of Osnabrück University who has made available his PROLOG system MLOG and has helped implementing the model.

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E. PROBLEM SOLVING

SEQUENTIAL versus CONCEPTUAL
TWO MODES IN ALGORITHMIC THINKING

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INTRODUCTION

Since 1981, the group "Foundations of Mathematics and Mathematical Education" at the University of Osnabrück has been studying the thinking processes of pupils as they deal with algorithmic problems in the field of computer science. In the beginning, we developed an experimental design to represent an algorithm on different levels of concept formation (COHORS-FRESENBORG 1982). Our design enables us to represent an algorithm (on natural numbers) as a sequence of elementary actions with matchsticks, as a kind of working flow-chart (constructed as a network with the bricks of the didactical material DYNAMIC MAZES) or as a program, written in a formal language for the model computer REGISTERMACHINE. In the first studies, we looked for differences in the ability to construct or to analyse an algorithm. In our experiments we found examples for all four possible categories:

kind of task \ categories	1	2	3	4
analytic	+	+	-	-
constructive	+	-	+	-

Pupils which belong to category 2, we called analytic pupils; those belonging to category 3 constructive pupils. Because there exist examples for both categories 2 and 3 one cannot assume that the two kinds of given tasks are of different difficulty . Further experiments with a more sophisticated design (COHORS-FRESENBORG 1983) verified our previous results. The existence of these two categories of competence (beside the two trivial ones, 1 and 4) was proved by a behavioristic design. We have measured the success in two well-defined classes of tasks. Still, there was no theoretical explanation beside the common sense one, which led us to the two names "analytic" and "constructive" for the two categories of competence.

THEORETICAL ANALYSIS ON THE BASIS OF COGNITIVE THEORIES IN MATHEMATICAL EDUCATION

In order to understand why we had found the two different competences in algorithmic thinking of our pupils, we started an analysis of the thinking and learning process, which had been documented on videotape during our investigation. First, we tried to explain the different competences by well-established theories in mathematical education. HASEMANN (1984, p. 235-260) has shown, that neither the theory of van HIELE nor that of HERSCOVICS/ BERGERON or VERGNAUD could explain the different behaviour and the different success of our pupils. But he did show that in the theory of DAVIS/McNIGHT, the concepts of *frame* and *visual moderated sequence* (VMS) can be used as a suitable framework to explain some of the children's behaviour.

A deeper and more precise analysis of the children's behaviour(i.e. an analysis of words with which they described the working of the Registermachine) led us to the hypothesis, that there exist at least two different cognitive strategies in dealing with our tasks of constructing and analysing algorithms.

DIFFERENT COGNITIVE STRATEGIES: SEQUENTIAL VERSUS CONCEPTUAL

We started with a very detailed analysis of the videotapes (including the transcripts) of the sessions of two of our pupils, who behaved quite differently and who were classified by their competence to be an analytic and a constructive type. KAUNE (1984) has established two concepts: *sequential* and *conceptual* cognitive style in dealing with our algorithmic tasks. Because the first analysis was made on observing two pupils with different behaviour and competence, we were led to assume that it is usefull to use the concept *cognitive style*. However, deeper analysis led us to introduce the concept of *cognitive strategies*. (It is clearer to classify the cognitive behaviour in a given situation than to postulate that a child (nearly) always behaves in a certain manner).

In the following we describe the two different strategies as they occured in our experiment:

sequential	*strategy*	*conceptual*
	given a constructive task	
transformation of a (spontaneous?) idea, which has the goal in mind but is not yet final ↓		developement of a global idea, using concepts to describe the problem ↓
→ invention of a concrete algorithm ↓		decomposition of the problem ↓
analysis of an interim solution		invention of a concrete algorithm or modification of a known one ↓
If it is wrong, step by step correction of the solution until the problem is solved (→ back to invention of a concrete algorithm)		analysis of the solution ↓
		if wrong, modifying the modular decomposition (→ back to developement of a global idea)

sequential	conceptual
preference for the use of match-sticks or the working with the constructed network, because by this procedure it is possible to analyse the sequential working of the program	preference for the use of the program language, because in this representation it is easier to decompose the problem and to structure the program. The use of match-sticks even hinder an abstraction.
Quicker reaction to a given task, because the sequence of actions may be started only on the basis of a (vague) idea. The further ideas for solving the problem come during the execution of the actions. This behavior may be described by a VMS in the sense of DAVIS/McKNIGHT.	Needs a longer time before a reaction is seen. A reason for this behaviour could be that he needs time to structure the problem and to form concepts.
This strategy may be preferred by impulsive pupils. The use of correct mathematical language is not important. Even wrong names for mathematical concepts don't disturb the sequential and interactive way of problem solving, because the thinking process is more orientated on the actions than on the labels.	This strategy may be preferred by reflective pupils. The correct use of the mathematical language is important, because the names (and the labels for them) represent the concept. The use of wrong names would in the next step induce wrong ideas for solving the problem.

DISCUSSION

Our hypothesis (that there exist the two different strategies *sequential* and *conceptual*) has to be compared with other propositions for classifying cognitive strategies. We will compare our approach with that of PASK (1972). He established the categories *serial* and *holistic*. We have not yet had the opportunity to test our pupils with his materials. Still, we expect the following differences between our two approaches:
First, there is a big difference between the two kinds of tasks. The tasks of PASK are - from a fundamental mathematical point of view - dealing with a set-theoretical or logical way of concept formation. The hierarchy of the concepts is *static*. It has to be *learned*. Our (constructive) tasks are dealing with the invention of a sequence of *actions*. From the mathematical point of view, the basis is a concept of *functions*. These mathematical differences between a logical and a functional point of view will certainly be verified as a psychological difference in the thinking process. We assume that the important difference is between the concepts *serial* and *sequential*. The sequential way of organizing needs a specific kind of holistic orientation on a meta-level. Because our tasks offer an actively sequential structure, there exists an action-orientated way of inventing and analysing the algorithm. The strategy is guided by ongoing analysis of the interim solutions.

The work of PASK has influenced an investigation on individual differences in cognitive style and their effect on the learning of a programming language (van der VEER 1983). Van der VEER/van de WOLDE (1982) "did not find any interaction between cognitive style and feasibility of programming language". We are not very astonished by their results because of their specific kinds of tasks. Their problems are given as a hierarchy of conditions. Writing a program in their case means only to formalize the logical structure in a computer language (instead of propositional calculus). We could not find any hint that the centre of the problems were to *organize* a complex sequence of actions (instead of analysing a sophisticated network of conditions). If one represents the problem Cook II (van der VEER 1983, p. 261) as a structured tree and compares it with the Gaudlemuller taxonomy in PASK (1982, p. 231), it is easy to see that both belong to the same kind of analytic problems. The complexity is determined by the complexity of the hierarchy, which is given in a (natural) language.

One kind of analytic task in our investigation is given in the following way:
The pupils have to analyse the semantics of an algorithm (which is represented as a computational network of DYNAMIC MAZES or as a program word in the language of REGISTERMACHINE). A second kind of analytic task asks for the computational complexity of given programs. Because all programs contain loops, they reach quite a high complexity for pupils at the age of 13. For both kinds of analytic tasks exists a sequential strategy (and not only the obvious conceptual one), which means, the pupils solve the problem by imagining, how (by the sequence of elementary actions) the REGISTERMACHINE will execute the program. From this idea they form the desired formula by a generalisation of their procedure and not by an abstraction.

Further problems

The investigation of thinking processes in computer programming is still in the beginning. In our conceptual framework there are quite a lot of open problems. Here we list some of them:
How can the introduction of the concept of the cognitive strategies "*sequential*" and "*conceptual*" explain the differences between constructive and analytic types, which we have found (COHORS-FRESENBORG 1982, 1983)?
What is the role of visualization for sequential strategies? In the study of SUWARSONO (1982) some of the problems in his mathematical processing test (i.e. problem 13 page 292) are of the kind, that they can be solved by a strategy, which we would call a sequential one. However, *he* points out, that the main difference in the strategies is the use of visualization. In this review "how spatial ability is defined", CLEMENTS (1983) deals with the problem, if "some people consistently prefer to use *visual* strategies, and others more *analytic* strategies". On the other hand he remarks, that "a person's spatial ability, as measured by traditional paper-and-pencil tests, is largely independent of his/her preference for using visual or non-visual thought processes". We believe that it would be fruitfull to investigate, whether these differences between spatial abilities and visual strategies can partially be explained by the fact that most of the visualization procedures are sequentially acting procedures (and not a kind of spatial understanding).

Holistic versus *serial*, or *conceptual* versus *sequential strategies* may be used by problem solvers with a consistent preference, so that one may define *cognitive styles*. Are there any connections to the dominant use of one of the hemispheres? Connected with our studies we made a small test using the program of FIDELMAN (1982). There was a tendency, that right-handed pupils, who prefer a conceptual strategy are dominant on the left hemisphere, while the sequential ones are dominant on the right hemisphere. This needs further research.

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SOME COGNITIVE ABILITIES AND PROBLEM SOLVING BEHAVIOUR: THE ROLE OF GENERALISED IMAGES AND/OR SIMULTANEOUS PROCESSING

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This paper reports some results of a wider study exploring the role of some socio-cultural factors and cognitive abilities as predictive variables for school acheivement and problem solving behaviour. Factor scores, hypothesised to represent abilities in simultaneous information processing, successive information processing and 'attention' according to Luria's model of brain function were used as criterion variables for the selection of 64 fifth grade subjects in a manner that allowed a factorial research design for the analysis of variance of a number of problem solving behaviours. The data on problem solving behaviour was collected using a co-operative interview technique for a series of problems with a range of difficulty. The results obtained suggest that ability in simultaneous information processing is significant in the comprehension and analysis of problem data. The results indicate significant sex differences and some socio-economic differences for this ability.

It is the intention of this paper to report on one aspect of the results of a study described in detail in a previous paper by the author (1983). That is, those research results that provide information about the role of simultaneous synthesis/analysis as described in Luria's (1973) model of brain function and later research by Das (1973) and others, in problem solving ability.

According to Luria's model, simultaneous synthesis/analysis occurs in the occipital/parietal region of the cerebral cortex and in 'the zones of overlapping' with the frontal lobes. At a perceptual level the ability is associated with visual/spatial information processing. However, particularly in higher order cognitive activities, processing is not restricted to visual spatial sensory input. Rather, simultaneous synthesis/analysis appears to involve processes where surveyability of a number of pieces of information and the relationships between them is important. In language comprehension Luria suggests that simultaneous processing abilities are associated with the comprehension of logico-grammatical expressions of relationships. Recent research by Das (1973) and others has confirmed the presence with individual variations of a distinct factor corresponding to Luria's description in normal school populations. A study by Hunt et al (1976) suggests that the ability appears to be associated with the comprehension and analysis of relationships. Cummins and Das (1978) have confirmed the role of simultaneous processing in language comprehension. The factor has been defined in Western research by tests of visual/spatial information processing and memory including Raven's Progressive Matrices.

This ability is of particular interest to mathematics educators. Since the early influence of Piaget there has been considerable interest in the role of sensory input in concept development. Skemp (1981) has discussed the limitations of verbal definitions of more abstract concepts as meaningful communication without prior concept development based on sensory input. The detailed analysis of semantic categories for addition and subtraction by Nesher et al (1982) and the research by Carpenter et al (1983) suggest that with current teaching procedures in mathematics education, young students experience difficulty in translating semantic categories of problem information into mathematical symbolism. Carpenter et al suggest that there is:

> "a shift from direct modeling of problem structure as a result of instruction in sentence writing ... Prior to instruction, the general strategy that most children use to solve addition and subtraction problems is to model directly the action or relationships described in the problem". (p. 68)

The author's previous research (1981) suggests that prior to instruction simultaneous processes are used to interpret problem information. There appears to be an educational gap in the development, through curriculum experience, of strategies for recording surveyable information about relationships from specific contexts to more abstract forms in either logico-grammatical terms or graphic symbols.

In contrast to Western educational literature, there have been a number of articles in Soviet literature describing teaching procedures for encouraging the use of the "structure" of mathematical problem solving processes in pupil instruction. For example, Yaroshchuk (1969) suggests that the ability to correctly identify problem 'structure' is an important factor in correct and independent solution. Chetverukhin (1969) writes about the importance of developing spatial imagination. More recently, Ivanova (1980) in an article entitled: "The Acquisition of Generalized Images and Their Use by Pupils in Problem Solving Situations" suggests that pupil experience in actively constructing 'generalized images' of a problem assists in the application of knowledge and transfer of techniques.

In Australia, a recent report by Lawson and Kirby (1981) suggests that teaching procedures are influential in the extent to which simultaneous information processing is used to complete school tasks. That is, teaching methods placing an emphasis on visual/spatial information as well as verbal and symbolic means of communication appear to facilitate the use of simultaneous processes. The research reported below is an attempt at a detailed analysis of the role of simultaneous processes in school problem solving.

METHOD

A sample of 64 fifth grade students were selected from a wider sample using factor scores for simultaneous, successive and attention processing abilities (generated using PA 2 version of SPSS Factor Programme) as criterion variables. The sorting procedure allowed analysis of results using a 2 x 2 factorial design with 16 subjects in each cell for any two factors while controlling for a third. The subjects were interviewed using a detailed protocol derived from Luria's (1973) model of problem solving. Each subject was asked to solve four problems of differing semantic complexity. The first three might be classified in Nesher et al's

(1982) terms as 'combine', 'compare' and 'equalize' types. The fourth problem involved a more complex multiplication relationship. The problem order was randomized to minimise error. The questions used to elicit information about problem solving processes were structured as far as possible to probe rather than impose differences in cognitive style and range of known strategies. The student behaviour at each point in the protocol was scored according to a three point scale (Yes, No, Partial) for each behaviour for each problem. The total scores for each behaviour give an indication of the range of behaviours available for each student over a range of problems. A modified version of the ANUI occupational classification scale (Broom et al, 1965) was used to code father's occupation for each subject.

RESULTS

The problem solving process scores were analysed using a factorial design as illustrated in figure 1 below.

	Simultaneous		
	Low	High	Attention
	n = 16	n = 16	High
	n = 16	n = 16	Low

figure 1

It should be noted that in each cell, half the subjects also belonged to the 'High' group and half to the 'Low' group for scores in the third factor representing successive synthesis.

Using S.P.S.S. analysis of variance programme ANOVA and the classic experimental approach, results suggest that ability in simultaneous processing is a significant source of variance for a number of problem solving behaviours. There was little evidence found in the analysis of significant association between factor scores for simultaneous processing and either of the other sets of factor scores. The results are summarized below in Table 1.

Table 1

Problem Solving Behaviour/Competence	Significance of Simultaneous Processing Ability as a Source of Variance	
	F	Significance
Adequate Understanding of problem	11.336	.001
Ability to express problem in own words	8.565	.005
Able to distinguish relevant information	7.764	.007
Adequate language competence to discuss relationships	17.594	.000
Able to select appropriate tactics for obtaining a solution	15.689	.000
Inclusion of relevant information during implementation of tactics	9.643	.003
Flexible selection of appropriate materials for implementation of tactics	9.118	.004
Reasoned sequence of steps taken in achieving solution	7.786	.007
Presentation of 'correct' answer	6.129	.016

Results suggested that sex and father's occupational status were significant as predictors of simultaneous processing ability but not for successive processing or attention. The relevant information from the output of the appropriate SPSS regression programme is shown below in Table 2.

Table 2

Analysis of Variance	Df	Sum of Squares	Mean Square	F
Regression	2	1229232385.66	614616192.83090	11.97001
Residual	88	3132126584.33	51346337 .44817	

	Variable	B	St. Error B	F
Multiple R .53089	Sex	.5966293 (+004)	1719.66786	12.037
R square .28185	Father's Occupation	-.1432118 (+004)	445.36190	10.340
Adjusted R square .25830	Constant	-.2438876 (+004)		
Standard Error 7165.63587				

There were also significant differences between the sexes in the ability to use visual representation to organise information.

CONCLUSION

During the individual interviews that were conducted in obtaining the results described above, there was little evidence of known strategies for using visual means of obtaining a generalized abstract representation of relationships. In addition only five of the 64 subjects used any method of checking their answers other than a repeat of the algorithmic steps when prompted by the interviewer. This possibly indicates a lack of experience in these processes. In view of the significance of simultaneous processing abilities in many important aspects of problem solving and the well documented difficulties in the education of students to competently solve verbal problems, this seems a serious omission in the curriculum. The significance of sex and father's occupational status as predictors of this ability suggests that life experience as well as inherent individual differences may be important in the development of simultaneous processing abilities.

In particular, the development of language competence in the appropriate register seems likely to be associated with adequate experience in the use of language to discuss logical and comparative relationships. Models for, and experience in, the use of such language are notably absent from most primary school curricula.

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THE INFLUENCE OF REWORDING VERBAL PROBLEMS ON CHILDREN'S PROBLEM REPRESENTATIONS AND SOLUTIONS

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Recent studies on simple addition and subtraction word problems have produced convincing evidence that the semantic structure of verbal problems strongly influences the relative difficulty of such problems and the strategies used by first and second graders to solve them. Typical of this kind of research is the work by Riley, Greeno & Heller (1983), and by Carpenter & Moser (1982). The results of studies in our center over the past few years are generally consistent with the findings of the American investigators. However, this work suggests at the same time that, in addition to the semantic structure, some other task characteristics also have an important effect on children's problem-solving processes, namely, the sequence of the known elements in the problem text and the degree in which the semantic relations between the given and the unknown quantities of the problem are made explicit in the verbal text. In the present paper we will report a recent study in which the influence of the latter characteristic of word problems on children's problem representations and solutions was systematically verified. Before giving an overview of the design and the results of this study, we first outline its theoretical background.

THEORETICAL FRAMEWORK

The study of the influence of rewording verbal problems on children's problem representations and solutions was designed within the framework of our competent problem-solving model (De Corte & Verschaffel, 1985), which is based on work done by Greeno and associates in which semantic processing is considered to be a crucial component in skilled problem solving (Greeno, 1982; Riley et al., 1983). The first phase of this model is conceived as a complex, goal-oriented text-processing activity : starting from the verbal text the pupil constructs a global, abstract, mental representation of the problem in terms of sets and set relations. This mental representation is considered as the result of a complex interaction of bottom-up and top-down analysis, i.e. the processing of the verbal input as well as the activity of the competent problem solver's word problem schema (De Corte & Verschaffel, 1985) and semantic schemata (change, combine and compare schema) contribute to the construction of the representation.

The verbal problems that are usually given to children in schools are most often stated very briefly and sometimes even ambiguously, unless one knows and takes into account various textual presuppositions (see also Nesher & Katriel, 1977; Kintsch & Greeno, in preparation). As an illustration, let us consider the following problem : "Tom and Joe have 8 marbles altogether; Tom has 5 marbles; how many marbles does Joe have ?" In this problem text it is not stated explicitly that Tom's five marbles mentioned in the second sentence also form at the same time part of the eight marbles that Tom and Joe have altogether. This combine-subset unknown problem can be reworded in such a way that the semantic relations between the given and the unknown components of the problem are stated more explicitly : "Tom and Joe have 8 marbles altogether; 5 of these marbles belong to Tom and the rest belongs to Joe; how many marbles does Joe have ?"

Experienced problem solvers have no difficulties in overcoming the indistinctness of the usual word problems and in constructing an appropriate representation because they process the verbal text largely in a top-down way, i.e. the processing is conceptually-driven using the semantic schemata mentioned above. Competent problem solvers' well-developed semantic schemata enable them to compensate for omissions and ambiguities in the problem statement. In less able and inexperienced children, however, the semantic schemata are not yet very well-developed and, therefore, these children depend more on bottom-up text-driven processing to construct an appropriate problem representation. Therefore, we would suggest that, especially for those children, rewording verbal problems in such a way that the semantic relations are made more explicit will facilitate the construction of a proper problem representation and, by extension, of finding the correct solution.

METHOD

Materials. Two series of six rather difficult word problems were constructed : Series A and Series B. Each series consisted of two change problems in which the start set was unknown, two combine problems in which one of the subsets was unknown, and two compare problems in which the difference between the referent set and the compared set was unknown. In Series A the problems were stated in the traditional form. In Series B the same kinds of problems were reformulated in such a way that the semantic relations between the sets were stated more explicitly, so that they would be clearer to young children. Table 1 gives examples of both series of word problems.

Table 1. Traditional (Series A) and reworded (Series B) verbal problems

Type of problem	Series A	Series B
Change/start set unknown (change 5)	Joe won 3 marbles. Now he has 5 marbles. How many marbles did Joe have in the beginning?	Joe had some marbles. He won 3 more marbles. Now he has 5 marbles. How many marbles did Joe have in the beginning?
Combine/subset unknown (combine 2)	Tom and Ann have 9 nuts altogether. Tom has 3 nuts. How many nuts does Ann have?	Tom and Ann have 9 nuts altogether. Three of these nuts belong to Tom. The rest belongs to Ann. How many nuts does Ann have?
Compare/difference unknown (compare 1)	Pete has 8 apples. Ann has 3 apples. How many apples does Pete have more than Ann?	There are 8 riders, but there are only 3 horses. How many riders won't get a horse?

Subject and procedure. Both series of word problems were collectively adminstered near the end of the school year to four first-grade classes (6-7 year olds) and four second-grade classes (7-8 year olds), with a total number of 89 and 84 children respectively. In both grades, half of the pupils were given Series A first and Series B one week later; for the other half of the children the order was reversed. The data collected were subjected to quantitative analysis as well as to error analysis.

RESULTS

In both grades and in the total group the reworded problems of Series B were solved significantly better (t-test; $p < .01$) than the standard verbal problems of Series A. For each of the three problem types (change, combine, and compare) the following null hypothesis was tested, using the X^2-test :

the proportions of correct and wrong answers are equal for Series A and Series B. This null hypothesis was rejected in all cases ($p < .01$). This implies that the general finding that Series B is solved significantly better than Series A holds also for each problem type separately in each grade.

To obtain a more detailed analysis of children's responses we have classified their answer on each word problem in the following five categories :

(1) correct answer (CA);
(2) adding error (AE), i.e., adding the two given numbers in the problem instead of subtracting the smaller number from the larger one (as all problems in our study were subtraction problems, adding always yielded an incorrect answer);
(3) given number error (GNE), i.e., answering with one of the given numbers in the problem, either the first (FGNE) or the second (SGNE);
(4) a miscellaneous category containing low frequency errors, either technical errors or errors for which we have no ready explanation (MC);
(5) no answer (NA).

Table 2 gives the distribution of the answers over these categories, separately for the three types of problems. Inspection of Table 2 shows (1) that the great majority of errors belongs to two categories, namely, adding error (AE) and "first given number" error (FGNE), and (2) that the rewording lead to a substantial decrease in the number of errors in those two categories.

To get a better insight into the origins of children's errors, we asked them to write down on their answer sheet how they obtained the solutions of the problems. This technique did not yield much interesting data. However, on the basis of work by others and the results of a longitudinal study in our center, we assume that the AE and especially the FGNE are mainly due to shortcomings in the children's understanding of the problems that can be ascribed either to a lack of understanding of semantic relations (Riley et al., 1983), or to misunderstanding isolated words and/or sentences in the verbal text (e.g. misinterpreting a sentence like "Person A and Person B have x objects altogether" as follows : "Person A has x objects, and Person B also has x objects") (De Corte & Verschaffel, 1985).
Another source of errors, and especially AE, could be that children process the verbal text only superficially : instead of trying to construct a mental

Table 2 . Distribution (in %) of the answers over the different answer categories for each type of problems

Problem type	Answer categories	Group of pupils					
		First grade		Second grade		Total group	
		Series A	Series B	Series A	Series B	Series A	Series B
Change 5	CA*	13	33	61	79	36	55
	AE	30	17	14	8	22	12
	FGNE	46	36	14	5	30	21
	SGNE	2	5	5	3	4	4
	MC	4	8	3	5	4	7
	NA	5	2	3	0	4	1
Combine 2	CA	43	57	71	83	56	70
	AE	23	14	11	3	17	9
	FGNE	15	14	9	4	13	9
	SGNE	5	2	4	3	4	2
	MC	11	10	4	7	7	9
	NA	3	3	1	0	3	1
Compare 1	CA	47	70	76	90	61	80
	AE	34	14	16	4	25	9
	FGNE	11	6	3	1	7	4
	SGNE	2	2	1	1	1	2
	MC	5	7	4	4	5	5
	NA	1	1	0	0	0	0

* CA = correct answer
AE = adding error
FGNE = "first given number" error
SGNE = "second given number" error
MC = miscellaneous category
NA = no answer

representation of the problem as a whole, they focus on a key word that is associated with a certain operation (e.g. "altogether" is associated with adding). The facilitation effect of the rewording can then be attributed either to the circumvention of the need for a semantic schema by making the relations more explicit in the text (Riley et al., 1983), or to the elimination of possible misunderstandings of words or sentences in the problem, or to the breaking of the association between a key word and an aithmetic operation.

DISCUSSION

The results of the present study support the hypothesis that rewording verbal problems in such a way that the semantic relations are made more explicit without affecting the underlying semantic and mathematical structure facilitates the understanding of word problems for, and the solution of these problems by young elementary school children.
Over the past few years a considerable body of research has yielded evidence

that the semantic structure of word problems significantly influences the difficulty level of the problems and children's strategies applied to solve them. The findings of the present study are not in conflict with this well-documented finding but rather complement it. Indeed, our data show that, with respect to young problem solvers, considerable differences in the level of difficulty can occur within a given problem type, depending on the degree to which the semantic relations between the sets in the problem are made explicit, obvious and unambiguous in the surface structure of the verbal text.

Kintsch & Greeno (in preparation; see also Van Dijk & Kintsch, 1983) have recently developed a model that can account appropriately for our findings and that, in so doing, provides a refinement of our competent problem-solving model. In the Kintsch & Greeno model, the initial stage of the problem-solving process, namely, the construction of a mental problem representation, is divided in two substages : in the first phase, the problem solver transforms the verbal input into a propositional text base; in the second phase, starting from those propositions, he constructs the internal representation of the problem situation. This model implies that modifications in the usual problem text (e.g. adding or changing words, expressions or a sentence) will give rise to a different text base. More specifically, our rewordings, which consist mainly in rendering the semantic relations between the sets in the problem statement more explicit, will result in a more elaborated text base. As a consequence, the construction of an appropriate mental representation of the problem situation starting from this more elaborated text base will be facilitated.

The present study is also relevant in the perspective of educational practice. An important implication relates to the formulation of verbal problems in textbooks for elementary mathematics education. Usually, textbook writers pay more attention to the purely arithmetic aspects of word problems than to the wording of those tasks. Our investigation demonstrates clearly that children are often given problems that they fail to solve, not because they lack the necessary arithmetic skills but because they do not succeed in constructing an appropriate problem representation due to their inability to understand correctly the condensed and sometimes ambiguous statement of the problem. The present study also contains suggestions concerning the direction in which one can search for rewordings, that are helpful in overcoming some of the difficulties that children experience.

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ABOUT RECURSIVE THINKING

- ON THE MODELLING - DOWN HEURISTIC -

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The heuristics of mathematical problem-solving may be defined as being those broad underlying principles, techniques and skills which enables one to construct valid solutions to problems. Indeed, teaching the heuristics of problem-solving is one of the major goals of education. But students seem to have great difficulties in applying these broad underlying ideas, or even recognizing that they are applicable. Although literally thousands of books, articles and speeches have been given heralding paradigms and strategies on heuristics, their overall effect for enhancing problem-solving ability is dismally discouraging. Students seem to approach even the simplest of tasks as independent exercises void of common reasoning patterns. Wagner (1981) illustrated this beautifully in her work on variables ; when subjects were asked to evaluate various values of a function f(n) for specific values of n, which were listed in table form, they did so without difficulty. The letter n was then replaced with the letter x, and with the table they had just constructed in front of them, they were asked to compute the values of f(x) for the same values they had used for n. Clearly one-third of the students who had already studied algebra constructed the new table by re-evaluating f(x) for each "new" value of x. The ability to transfer these broad concepts seems to be illusory at best. Siegler (1983) commented on this as follows :

> "A persistent issue in the investigation of children's reasoning has been whether they approach each task as a new problem or whether they reason in similar ways on different tasks. Much research has been conducted to try to identify common reasoning patterns, but the results have been disappointing. If broad unities exist, they are extremely difficult to identify."

Not using problem-solving heuristics seems to affect not only novice problem solvers, but expert ones as well. Mager and associates in a series of studies designed to predict behavior as a function of training and experience level summarized their work (Mager et. al 1961a, b, 1963) :

> "The ego-deflating results have uniformly demonstrated that those naïve organisms commonly referred to as 'college sophomores' and 'underpaid secretaries' were essentially as accurate as judgments made by highly trained and experienced clinical psychologists. The

> expert, in other words, did not make judgments any better than the novice."

The Mager studies showed that when subjects were presented with novel problems, they floundered and groped for direction. They did not rationally analyze an approach, nor did they reflect on it. Mager was essentially interested in the initial paths the subjects took in constructing their solutions, but overall, his subjects demonstrated a dearth of problem-solving heuristics. Indeed, Lesh and Akerstrom (1982) contend that content-independent heuristics are both unteachable and of dubious value ! Although they were speaking of heuristics for solving "real" problems, there is abundant evidence that in spite of our efforts, their criticism to this point in time also applies to classroom exercises.

We, however, retain an unswerving belief that heuristics can be taught to and internalized by students. Our failure to do so heretofore is more a polemic against our methods than the heuristics themselves. Moreover, most of the research on problem-solving lumps together the various techniques and approaches needed to effect a solution. Tests which are labeled problem-solving abound, but they are a smörgasbord of problems encompassing many different types of strategies which are needed to effect a solution. Problem-solving heuristics have been studied heretofore in this larger setting... and as a result, everything seems to get jumbled together. See for example the researches of Schoenfeld (1982), Lester (1980) and Lucas (1982). It is nearly impossible to make a definitive statement on specific heuristics from these studies.

No paradigm for teaching the heuristics of problem-solving has received more attention than Polya's four-step model. Presented therein are types of questions one should ask himself when faced with a mathematical problem for the first time. His four-step program (How to solve it : first, understanding the problem ; second, devising a plan ; third, carrying out the plan ; fourth, looking back) crystallizes the type of analysis we would love for our students to internalize, and it ranks as one of the most elegant of heuristic models (see Fig. 1). Let us focus our attention on his second step and in particular on the series of questions wich ask the learner if he can "model-down" the problem to a simpler one, solve the simpler problem, and then generalize the solution to fit the original statement. This technique is fundamental to mathematics, but the problems themselves often trigger its use. Indeed, here is where mathematical experience and maturity enter the picture. When confronted with counting the total number

HOW TO SOLVE IT

UNDERSTANDING THE PROBLEM

First. You have *to understand* the problem.	*What is the unknown ? What are the data ? What is the condition ?* Is it possible to satisfy the condition ? Is the condition sufficient to determine the unknown ? Or is it insufficient ? Or redundant ? Or contradictory ? Draw a figure. Introduce suitable notation. Separate the various parts of the condition. Can you write them down ?

DEVISING A PLAN

Second.	Have you seen it before ? Or have you seen the same problem in a slightly different form ?
Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a plan of the solution.	*Do you know a related problem? Do you know a theorem that could be useful ?* *Look at the unknown !* And try to think of a familiar problem having the same or a similar unknown. *Here is a problem related to yours and solved before.* *Could you use it ?* Could you use its result ? Could you use its method ? Should you introduce some auxiliary element in order to make its use possible ? Could you restate the problem ? Could you restate it still differently ? Go back to definitions. If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem ? A more general problem ? A more special problem ? An analogous problem ? Could you solve a part of the problem ? Keep only a part of the condition, drop the other part ; how far is the unknown then determined, how can it vary ? Could you derive something useful from the data ? Could you think of other data appropriate to determine the unknown ? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other ? Did you use all the data ? Did you use the whole condition ? Have you taken into account all essential notions involved in the problem ?

CARRYING OUT THE PLAN

Third. *Carry out* your plan.	Carrying out your plan of the solution, *check each step.* Can you see clearly that the step is correct ? Can you prove that it is correct ?

LOOKING BACK

Fourth. *Examine* the solution obtained.	Can you *check the result ?* Can you check the argument ? Can you derive the result differently ? Can you see it at a glance ? Can you use the result, or the method, for some other problem ?

Figure 1 : Second step of the Polya four step model.

of squares in a 10 x 10 grid, would one not sooner or later fall upon this modeling-down technique ? How natural is it for one to adapt this approach ? Would it be used for a 5 x 5 grid ? Do the problems themselves really dictate its use or is it always one of the alternatives one thinks of consciously or otherwise when first seeing a problem... and if so, what types of problems ? Is its development a function of age and experience, or simply intelligence ?

The purpose of this research is to study this particular "modeling-down" heuristic in isolation.

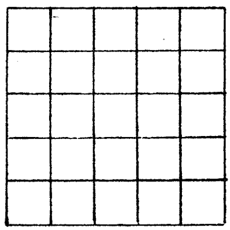
THE STUDY

Problems were chosen, all of which could be solved... at least in two different ways (see Table 2). It was felt however, with the exception of # 8 that the modeling-down technique would be a more natural way to approach them. The problems were given in both verbal and written format to a variety of populations (children, students, and teachers) in both group and individual settings. Two problems were given to each subject. The subjects were told that we were more interested in their methods of solution rather than the solutions themselves. The methods of approach, the analyses they used and strategies they employed in deciding upon a route to pursue were classified as to their closeness to the modeling-down stratagem.

The initial results of this study will be presented and discussed.

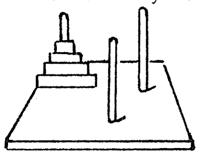
Table 2 : Example of such problems.

1. How many squares are there in the following configuration (the answer is more than 26)



2. The tower of Hanoï.

   You want to transfer the rings to one of the other two poles. However, you are not allowed to place a larger ring on top of a smaller ring. Each time you move a ring, it counts as one move. How many moves are necessary to accomplish this transfer.



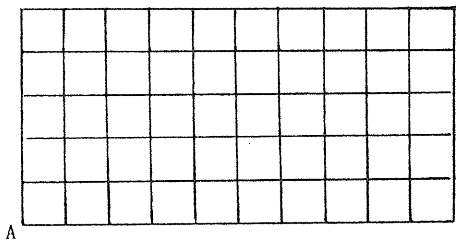
3*. If p, q, r and s are strictly positive real numbers, prove that

$$\frac{(p^2 + 1) . (q^2 + 1) . (r^2 + 1) . (s^2 + 1)}{p . q . r . s} \geqslant 16$$

4*. Define $\sigma(S)$ to be the number of subsets of S which have an <u>odd</u> number of elements. E.g.
If $S = \{a, b, c\}$ then $\sigma(S) = 4$. Determine $\sigma(S)$ if S is a set with 26 elements.

5. Find the sun of the coefficients when we expand $(x + 1)^{30}$.

6. Taxicab geometry.
A cab going from A to B must follow the lines representing the streets and avenues of this part of New York. Determine the number of paths of minimal length, from A to B. B



7. On a chess board, a tower controls exactly one horizontal and one vertical. Determine the number of configurations (on a 8 x 8 chess board) with 8 towers, such that none of them threatens any of the 7 other ones.

8. Ten people are in a room. Each must shake hands with each other (exactly once). How many handshakes take place ?

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SOME INCOMPATABILITIES BETWEEN CONDITIONS FAVOURING LEARNING AND PROBLEM SOLVING IN MATHEMATICS

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Problem solving plays a central role in most mathematics curricula. Mathematical concepts are taught normally by presenting some information, demonstrating the use of that information with a minimal number of worked examples and then presenting a relatively large number of exercises or problems to solve. In the last few years the nature of what needs to be learned in complex cognitive domains and the conditions that facilitate learning have become clearer. In light of this knowledge, there may be some doubt whether current procedures are optimally efficient.

Work on expert-novice distinctions has thrown light on the cognitive characteristics of expertise. De Groot (1966) found that chess masters and novices did not differ on expected dimensions such as width or depth of search. In contrast, major differences were found in memory of realistic chess positions. After 5 seconds exposure a board configuration extracted from a real game could be reproduced with a high degree of accuracy by masters. In contrast, novices were poor at this task. The difference was due to previously acquired knowledge rather than short term memory because both groups were equally poor at reproducing random board configurations. This finding suggests that a major aspect of expertise is the ability, during a game, to recognise each configuration encountered and know which move is most appropriate for that configuration. Schemas may be used to describe this knowledge where a schema is defined as a mental construct which allows patterns or configurations to be recognised as belonging to a previously learned category and which specify which moves are appropriate for that category.

Chi, Glaser and Rees (1982) using physics problems also found that a differential ability to categorise problems distinguished experts from novices. Experts categorised problems according to solution mode, novices according to surface structures. These findings can be used to explain the different strategies used by experts and novices. Larkin, McDermott, Simon and Simon (1980) and Simon and Simon (1978) found that on physics problems, novices tended to work backward from the goal while experts worked forward from the givens. Novices were using general search strategies such as means-ends analysis which involve attempting to reduce differences between each problem state encountered and the goal state. Experts were using previously acquired schemas which allowed them to recognise a problem as belonging to a particular category and choose moves appropriate to that category.

What are the conditions that might facilitate schema acquisition? There is considerable evidence that conventional problem solving is not a suitable vehicle for learning. Mawer and Sweller (1982), Sweller (1983), Sweller and Levine (1982) and Sweller, Mawer and Howe (1982), using puzzle problems, found that conventional problem solving activity massively retarded simple rule induction. Sweller, Mawer and Ward (1983) found that if the use of means-ends analysis was prevented on physics or geometry problems by eliminating the conventional problem goal and instead substituting the statement "calculate the value of as many variables as you can", then the development of expertise was considerably enhanced.

We can account for these findings by assuming that conventional problem solving search, especially when it involves means-ends analysis, firstly directs the attention of problem solvers away from those aspects of a problem conducive to schema acquisition, and secondly that search activity requires a large amount of cognitive processing capacity. There are good theoretical grounds for both of these assumptions. A person engaged in problem solving search, for example means-ends analysis, must simultaneously consider the problem goal, the current problem state, the relation between the goal state and the current problem state, the relation between the latter relation and potential problem solving operators and lastly, a sub-goal stack must be maintained. This activity can be expected to require considerable cognitive processing capacity which may be consequently unavailable for schema acquisition because problem solving search is largely unrelated to schema acquisition. To acquire an appropriate schema, a problem solver must note which previously attained problem states have been successfully combined with the use of particular operators. Problem solving search activity for the most part ignores previous moves.

The above concepts have been tested over the last few years in a long series of experiments involving puzzle problems, physics, geometry, trigonometry and algebra problems. Only the algebra experiments will be summarised here. (A complete description may be found in Sweller and Cooper, in press.) Two distinct types of experiments were run. The first experiment was designed to test the hypothesis that expertise in algebra requires the acquisition of a large number of limited, domain specific schemas. Essentially, we were attempting to duplicate the results of de Groot (1966) using algebra transformation problems rather than chess. The remaining experiments were designed to indicate that schema acquisition is retarded by conventional problem solving activity. In this case the comparison was with worked examples.

If experts have acquired large numbers of domain specific schemas this should manifest itself in a superior memory for realistic algebraic equations compared to novices. This difference should disappear if strings of random algebraic symbols are used rather than equations. The subjects were a group of Year 9 students, a group of Year 11 students and a group of mathematics teacher education students. Equations such as (c + d)(a + e) = (c + d)(b) were presented for 5 seconds followed by 30 seconds in which subjects had to reproduce the statement as accurately as possible. The equations were interspersed with random symbols such as b-=)dc(cb()e+a for which a similar memory test procedure was followed.

The longest correctly recalled string length was used as a dependent variable. Results indicated a differential ability to reproduce the equations between the Year 9 students and the remaining two groups which did not differ. This difference was not apparent when random symbols were used rather than equations.

The implication of this result is that by the time a mathematically competent student has reached senior secondary school levels, he or she has learned to recognise large numbers of individual equation configurations. This knowledge may be an important, even major determinant of expertise. I hypothesised previously that the prior acquisition of appropriate schemas which allow problem solvers to recognise problems as belonging to a particular category and provide an appropriate move for that category, is an important ingredient of expertise. The results of this experiment provide some evidence for this suggestion.

The remaining experiments I wish to summarise tested the hypothesis that schema acquisition is retarded by conventional problem solving search activity when compared to a heavy use of worked examples. These experiments used Year 8 and 9 students with a limited prior exposure to algebra manipulation problems. The basic experimental design presented the conventional problem groups with a series of eight problems during an acquisition period. An example of one such problem is $(af+e)/b = c$, solve for a. The eight problems were divided into four different categories of two problems each. The two problems in each category had an identical structure. They differed only in terms of the variables used. Thus the alternate problem belonging to the same category as the above problem could be $(aw+x)/y = z$, solve for a. Problems in different categories had different configurations. The two problems belonging to the same category were presented in immediate sequence. The worked example groups were presented identical problems except that the first of each pair belonging to a particular category consisted of a worked example rather than a problem to solve. Subjects could study this worked example for as long as they pleased up to a maximum of five minutes. Following these acquisition problems, four test problems were presented. These were similar in structure to the acquisition problems. They differed only in that different variables were used.

The two major dependent variables were time to solution and number of algebraic errors. The first finding to note was that the conventional problem groups required four to five times as much time to solve a problem during the acquisition period as the worked example groups needed to study the equivalent worked example. The major question is whether this reduction in time spent on acquisition had deleterious effects on the subsequent test problems. In fact, the worked example groups required significantly less time and made significantly fewer errors on the test problems than the conventional problem groups. The use of worked examples was of clear benefit both during initial acquisition and on the test problems.

Given the reduction in errors on the test problems, it is possible that the worked example groups have gained general proficiency in the use of algebraic rules. If this knowledge is independent of domain specific schemas, we might expect it to transfer to different problems requiring similar algebraic manipulations. Alternatively, if as suggested by the results of the memory experiment, knowledge acquisition is heavily tied to highly specific schemas, we might expect minimal transfer. Learning on one problem may not heavily influence skill on a different problem. These hypotheses were tested by using identical acquisition formats to those described above but substituting structurally different test problems. The same algebraic manipulations to those used on the acquisition problems were required but since the initial equations differed, the order in which particular algebraic rules were used varied. Large differences were again obtained during the acquisition period favouring the worked example groups but there were no differences on test problems using either time to solution or number of errors.

In sum, these results suggest that the acquisition of narrow, highly specific schemas are an important component of expertise and that the acquisition of these schemas is comparatively retarded by conventional problem solving activity. There are at least two objections that can be made to this conclusion. Firstly, if expertise requires us to study virtually every type of problem we are ever likely to come across, expertise would develop extremely slowly. In fact, expertise does develop slowly and this may be because of the large number of schemas we need to acquire. Some personal evidence that expertise has the characteristics I have described may be obtained by solving the equation $(a+b)/c = d$ for a. Most people reading this paper will be able to solve it immediately without engaging in problem solving search activity. We will not for example, engage in the initial activity of attempting to subtract b from both sides followed by a rejection of that move as being inappropriate as a first move. We will not even consider it as a first move because we recognise the equation as being of a type that requires multiplying out the denominator first. We acquired the appropriate schema long ago.

It might also be objected that if an emphasis on problem solving is reduced, something valuable may be lost. In fact, our results do not suggest that problem solving should be eliminated from curricula. They do suggest that its use should be modified. Less emphasis should be placed on problem solving during learning and more on its role as a motivating factor and as a means of establishing what has been learned and understood and what still needs to be learned, understood and studied further. It may be desirable to use alternative techniques during actual learning.

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AN ANALYSIS OF INTERVENTION PROCEDURE IN MATHEMATICAL PROBLEM SOLVING

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An intervention procedure was proposed to improve children's ability to solven word problems. It was assumed that there are two main processes in solving word problems: Problem understanding and problem-solving skills. Children were grouped according to their problem solving abilities. These abilities utilize the above two processes resulting in four types of problem solvers: The first type is children who have difficulty in either process. The second is children who have difficulty in the first process (U Group). The third is the opposite of the seconnd (S Group). The last is children who have difficulty in both processes (U-S Group).

Method.

147 sixth graders were administered the pretest. In the pretest, both calculation problems and word problems were given to all children. The calculation problems were 28 fraction. The word problems were 14 fraction. Each word problem had three sub-questions. First, subjects were requested to select one of four rules which they thought was necessary for solving the problem. Next, they were asked to select from a list of 12 the correct equation. Last, they were asked to compute the problem using the equation they had selected. All children were divided into one of the four ability types based on both the score of the computational problems and total score of the first and second sub-questions of the word problems.

In the 76 children the U, S and U-S groups were assigned to one of three intervention conditions: Individual intervention, group intervention, and control (i.e., no intervention).

Our intervention programs had four aims: First, to generate hypotheses bbout children's incorrect strategies from responses to the computational problems. Second, to understand such basic schema as part-shole relation. Third, to eliminate children's incorrect strategies and to teach correct rules. Fourth, to consolidate skills through two days of drill practice.

The number of days cenessary for finishing the programs varies from child to child.

In the group intervention condition children were taught all together the second and third steps of the intervention program for seven days.

Results and Discussion.

The gains form pretest to posttest in calculation problems are shown in Table 1. Although we did not try to improve children's ability of problem solving skills, the U and U-S groups indicated significant gains except for ones of the control condition. In the posttest, children in individual intervention condition showed no incorrect strategies. However, children in group intervention condition remained some incorrect strategies even in posttest. The gains from pretest to posttest in word problems of addition are shown in Figure 1. The gains of the U and U-S group on individual intervention condition were superior to the S group. However, although the group intervention condition had great advantage for the S group, there were less benefits for those who less understood on goal or structure of the problem.

Table 1

Mean number of correct responses in both pre- and post-test for calculation.

Group	Individual		Group		control	
	pre	post	pre	post	pre	post
U	17.7	22.7	15.7	19.5	16.9	19.8
S	26.6	27.5	23.4	23.8	25.9	20.1
U-S	9.7	17.8	9.7	15.8	7.8	9.4

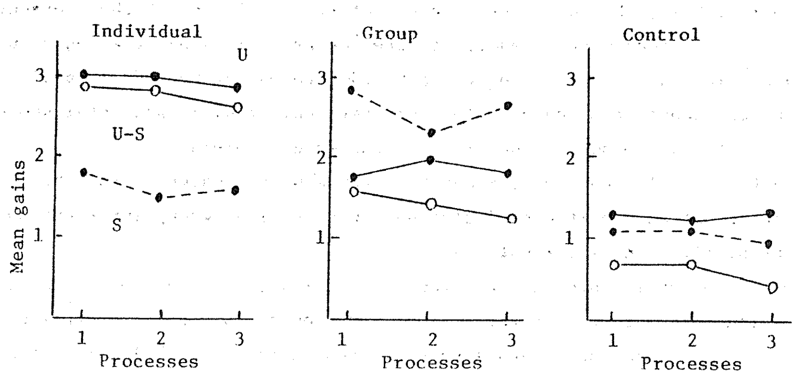

Fig.1. Mean Gains from pre- to post-test of all the groups in each intervention condition for addition problems.

F. OPERATIONS

THE TRANSFER BETWEEN TWO KINDS OF SUBTRACTION PROCEDURES

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This study aimed at examining whether the computation procedure with abacus would influence that with paper and pencil when third grade children had the experience to solve the same problems alternately by abacus and written computational procedures. These third graders had been able to solve the problems correctly by abacus, but had shown bugs in written computation.

METHOD

1. Subjects   16 third grade children who could solve correctly by abacus 22 or more of the 30 3-digit subtraction problems, including one-step or two-step borrowing, showed some distinctive errors in written computation, which have characteristic features such as Borrow-From-Zero, Borrow-Across-Zero, Smaller-From-Larger etc.

2. Procedures   This study consisted of Pretest, Transfer experiment and Posttest.

1) Pretest   The experimenter gave the subjects 10 3-digit subtraction problems including one-step or two-step borrowing, and asked them to solve the problems in writing. If a subject showed no error here, the study was terminated.

2) Transfer experiment   Subjects were given Transfer experiment comprising of the following five Phases.

Phase A: Subjects who had shown any wrong answer at Pretest were asked to copy the target problem on another sheet of paper, and also to solve it in writing again and explain his/her solution process. If a subject showed the correct answer, no later phases were given.

Phase B: Subjects were asked to copy the problem to which they had shown the wrong answer at Phase A, and to solve it by abacus, and explain the process verbally.

Phase C: Subjects were asked to copy the target problem and to solve it in writing again. If the subject showed the correct answer, the Experiment was terminated.

Phase D: Subjects were asked to copy the target problem and to solve it by abacus in an unusual order, i.e., from ones to hundreds, just like the order used in writing. If a subject showed a wrong answer on this Phase, experimenter was to show the same subtraction procedure by abacus from ones to tens, tens to hundreds repeatedly and led him/her to be able to make the correct answer.

Phase E: Subjects were asked to copy the target problem and to solve it in writing.

In this experiment, unlike Resnick (1983), the subjects were not required to alternate between the two procedures in every substep of computation.

3) Posttest 10 3-digit subtraction problems in writing, which included similar computational components to those in Pretest, were given to the subjects.

RESULTS AND DISCUSSION

Transfer esperiment was given to 11 subjects who did not get 10 points on Pretest. 8 of the 11 unexpectedly showed the correct answer on the target item(s) in Phase A. The remaining 3 subjects who had shown the wrong answer in this Phase could not give the correct answer in writing until Phase E. Namely, they, showing the correct answer on Phase B and D by abacus, repeated the same wrong answer in Phase A, C and E when they calculated in writing.

As for these 3 subjects, it was clear that getting the correct answer by abacus did not influence directly computation in writing and the experience of solving the target problem by abacus in an unusual order (same order as in writing) did not affect computation in writing either.

For example, answers of subject SS in Phases were as follows. For $\begin{array}{r}804\\ \underline{-436}\end{array}$, the answer in Phase A was 388. Her verbal explanation was "As I cannot remove 6 from 4, I borrow 10 from the next column, and removing 6 results in remaining 4, --- 4 and 4 become 8, --- I cannot remove 2 from 0, and borrow 100 from the next column, and ----- it gives 8, if 2 is removed ------ and subtracting 4 from 7, I get 3."

SS understands that if the number in ones cannot be reduced, she has to borrow from the next column, but since in this case, the top numeral of tens is zero, she borrows ten from the bottom numeral, namely 3. She slways showed the same type of error when the top number of tens was smaller than the bottom number.

Her answer on Phase B was 368. It was correct. The verbal explanation was "I place 804 on the abacus, reduce 4 from 8, and it becomes 4 --- and I cannot reduce 3 from 0, --- borrow from the next column, and reduce --- number in hundreds becomes 3, and as I cannot reduce 6 from 4, borrow 10 from the next column of 30 --- and borrow 10 from 70, and reduce 6, it becomes 4, and 4 and 4 become 8."

After that, the answer in Phase C was 388, 368 in Phase D, and 388 in Phase E.

In case of subject MH, the answer in Phase A to $\begin{array}{r} 836 \\ -\ 448 \\ \hline \end{array}$ was 308. The verbal explanation was "I cannot reduce 8 from 6, and borrow 10 from tens, 16 minus 8 is 8, the answer is 8, --- I cannot reduce 4 from 3 at tens, and as the column of tens lends 10 to ones, I cannot reduce 4 from 2 --- borrow 10 from hundreds and 13 minus 4, Oh! I made mistake." T: "Write down the correct answer under the former answer." C: "(The answer turns 398) 13 minus 4 is 9 --- it is possible to reduce 4 from 8 at hundreds, but the column of hundreds lends 10 to the next column of tens because I cannot reduce the number at tens, and 7 munus 4 is 3." T: "Why did you write 0 here? (pointing the zero of 308)" C: "I could not reduce 4 from 3 and I wrote 0 since I forgot to borrow the number from the next column of hundreds."

She answered 308 because she could not remove 4 from 3. When the top number was smaller than the bottom number, MH sometimes dealed with this situation by making the answer zero. The answer 398 was obtained by 13 minus 4 at tens, namely forgetting to reduce the number in tens by one for the borrowing necessary for ones.

The answer on Phase B was 388. It was correct. The verbal explanation was "At first, place 836 on the abacus, 800 minus 400 is 4, and borrow 10 because I cannot reduce 4 from 3 at tens, 4 on hundreds turns 3 because the column of hundreds lends 10 to tens, --- and 13 minus 4 becomes 8 ------ and ------ correct number is 9 but since I cannot reduce 8 from 6 at ones, and borrow 10 from tens, it becomes 8 ------ and 16 minus 8 equals 8." After that, the answer of Phase C was 398, 388 in Phase D, and 398 in Phase E. Above result indicates that even when they had the experience of solving the same problems alternately by abacus and by written procedure, it was difficult for them to modify spontaneously the latter procedure by referring to the former, as they followed procedures which they believed correct in both cases.

Next was the parts of an informal interview with subject SS, asking how she would think about her obtaining different answers by computation in writing and by abacus.

T: "Does the answer of computation in writing and by abacus always coincide with? or not?" C: "I suppose they are the same ------ but ------ I don't know exactly." (omission) T: "Is the correct answer always only one? Or is there the case where there are two correct answers?" C: "One." T: "Then when the answers of the computation in writing and by abacus are different, one of them must be wrong?" C: "No, --- Yes."

T: "Both are correct answer?" C: " ------- (shakes her head)" (omission) T: "Either of them is wrong answer isn't it?" C: " ------ (nod)" T: "Is that right?" C: "I don't know."

She is inclined to think that there is only one correct answer and answers by abacus and written procedure should be the same, but she was not convinced of these. She did not recognize clearly either the reason why she got different answers.

REMOVING, A SORT OF SUBTRACTION, IS EASIER THAN ADDITION. WHY?

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INTRODUCTION

Which is easier for children, addition or subtraction? School textbooks and most of the literature in Japan usually introduce addition first. What is the reason for that?

You will easily see that the reason cannot be found in the courses of study of kindergarten and elementary school.

Belgian psychologist, Docroly, said that subtraction seems to be much easier than addition. This made Mr. Kanji Hatano say; "The results of Docroly's experiments seem to show that children learn subtration earlier than addition. If all the children discover subtraction first and then addition, this means that subtraction is easier than addition, and that it is wrong for us to start Arithmetic education with addition."

Therefore in this paper;

(1) The author wants to confirm the Docroly's theory that subtraction is easier than addition through experimental research.

(2) We want to consider the way that the results of this experimental research are to be put into the practical use in Arithmetic education for kindergarteners and for lower graders at elementary school.

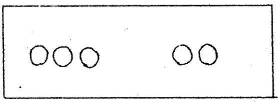
CONCERNING OF THE RESEARCH

1. The date of the research was November in 1981.
2. The examinees;

From 4-to 6-year-old children, each age group having 15 boys and 15 girls; subtotal of 30 children. The total of 90 children. And each examinee grasps certainly the concepts of number by 5. (N.B) The children were selected at random.

Experiment 1; Union

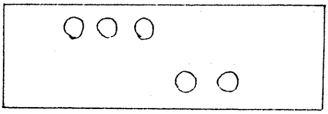
Place the marbles in front of the examinee as in the picture (the distance between the two sets is 10 cm),



and ask him the question; "Here are so many marbles(three), and here are so many marbles (two) which you can put together. How many marbles are there in all?"

Experiment 2; Increasing

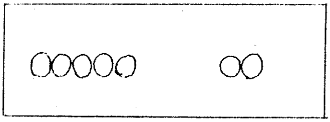
Put three marbles in fornt of the examinee and take two marbles out of the box and present them as in the picture (the distance of two sets is 15 cm), and ask the question;



"Here are so many marbles (three). If these marbles (two) are added to them, how many marbles are there?"

Experiment 3; <u>Difference</u>

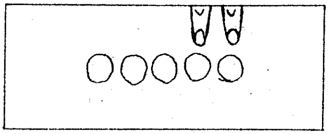
Put marbles in front of the examinee as in the picture (the distance of two sets is 10cm), and ask the question;



"Here are so many marbles (five) and here are so many marbles (two). Which set has more marbles? and by how many?"

Experiment 4; <u>Removing</u>

Put marbles in front of the examinee as in the picture, and ask the question;



"Here are so many marbles (five). If these marbles (two) are put back into the box, how many marbles are left here?"

THE EXPERIMENTAL RESULTS AND CONSIDERATION ON THEM

(1) Consideration by the percentage of correct answers Table 1. The numbers of those who have given the correct answers

age / exp	4	5	6
uni.	19 (63.3)	29 (96.7)	30 (100)
inc.	19 (63.3)	29 (96.7)	30 (100)
dif.	3 (10.0)	8 (26.6)	12 (40.0)
rem.	23 (76.7)	29 (96.7)	30 (100)

The number in the parentheses means percentage

When the four experiments are put in order by the correct-answer-percentage from the high to the low is the following;

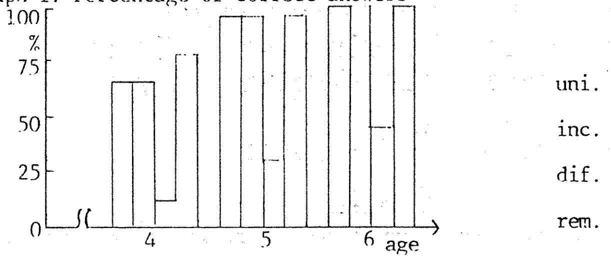
4-year-old: Removing —— Union = Increasing —— Difference

5 & 6-year-old: Union = Increasing = Removing —— Difference

This shows that the correct-answer-percentage of difference in each age is low. Even in 6-year-old children, while the other three experiments are proud of 100% correct answers, only difference has the low percentage with no more than 40%.

Now, the content of the Table 1 is put into a graphic chart so that you can easily get an intuitive grasp of the content with the aid of your vision. It is shown below.

Graph 1. Percentage of correct answers



Paying attention to the growth of correct-answer-percentage, that of union, increasing & removing is remarkable from 4-to 5-year-old, and the correct-answer-percentage is reaching about 100% already at the age of 5. But such phenomenon can not be found with respect to difference, and the growth takes place only gradually from 4-to 6-year-old children, and correct-answer-percentage is remaining at 40% at the age of 6.

By the t-test for the 0.05 level of significance, there is a significant difference between 4-and 5-year-old children regarding union, increasing and removing.

So, it can be said that those concepts develop rapidly from 4-to 5-year-old children.

And there is a significant difference in each age between union, increasing & removing, and difference. It follows that for kindergarteners difference is more difficult than the other three.

(2) Consideration by the time

Here the data, processed with the addition of a new factor of the required time added to the correct-answer-percentage in Table 1, is classified in the next three stages.

Those who answering promptly (0 to 3 seconds)

Those who needed time to answer (4 to 10 seconds)

Those who required some time to answer(more than 11 seconds)

Table 2. The number of children giving the correct answers by the time required

time / age / exp	0 - 3sec.			4 - 10sec.			11 - sec.			incorrect a.		
	4	5	6	4	5	6	4	5	6	4	5	6
uni.	10 (33.3)	22 (73.4)	25 (83.3)	6 (20.0)	6 (20.0)	5 (16.7)	3 (10.0)	1 (3.3)	0 (0)	11 (36.7)	1 (3.3)	0 (0)
inc.	12 (39.9)	20 (66.7)	25 (83.3)	5 (16.7)	8 (26.7)	5 (16.7)	2 (6.7)	1 (3.3)	0 (0)	11 (36.7)	1 (3.3)	0 (0)
dif.	2 (6.7)	4 (13.3)	8 (26.7)	0 (0)	4 (13.3)	4 (13.3)	1 (3.3)	0 (0)	0 (0)	27 (90.0)	22 (73.4)	18 (60.0)
rem.	22 (73.4)	26 (86.7)	27 (90.0)	1 (3.3)	3 (10.0)	3 (10.0)	1 (3.3)	0 (0)	0 (0)	6 (20.0)	1 (3.3)	0 (0)

By the way, focussing on the prompt-answering-stage which has the large distributional number of the children, the four experiments are placed like below in order from the one with high prompt-answering-percentage to the one with low prompt-answering-percentage;

4-year-old: removing — increasing — union — difference

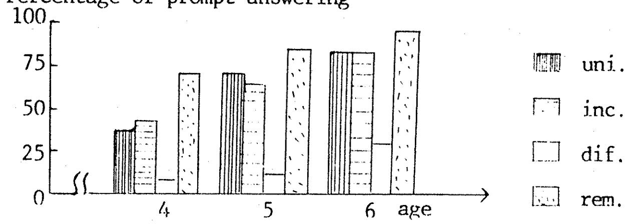
5-year-old: removing — union — increasing — difference

6-year-old: removing — union ‖ increasing — difference

It shows that over every age groups prompt-answering-percentage of difference is low.

Again the content of prompt-answering-stages in Table 2 is put into a graphic chart so that you can easily get an intuitive grasp of the content. Now, here is the Graph 2.

Graph 2. Percentage of prompt-answering



As in the case of the correct-answering-percentage in Graph 1, by the t-test for the 0.05 level of significance, it is clear that in 4-year-old children, there is a significant difference between removing & union, and between removing & increasing. Therefore, it is possible to say that for 4-year-old children removing is easier than union and increasing.

You can also find a significant difference between union, increasing & removing, and difference in each year group. This means that, compared with the other three, difference is more difficult to kindergarteners.

(3) Consideration by sexes

In general, some examinations show a clear difference by sexes and others do not.

By the way, the conceptional development of addition and subtraction, being very closely related items to Arithmetic, is a matter of good concern and interest.

If the result shows a significant difference only among junior and senior high school students, it is not so surprising. But if the significant difference can be obtained among kindergarteners, this causes greater interest since it touches on the foundation of Arithmetic education in elementary school.

Table 3. Correct-answer-percentage by sexes.

age / sex / exp.		4	5	6
uni.	B	10 (66.7)	15 (100)	15 (100)
	G	9 (60.0)	14 (93.3)	15 (100)
inc.	B	9 (60.0)	15 (100)	15 (100)
	G	10 (66.7)	14 (93.3)	15 (100)
dif.	B	3 (20.0)	4 (26.7)	5 (33.3)
	G	0 (0)	4 (26.7)	7 (46.7)
rem.	B	12 (80.0)	15 (100)	15 (100)
	G	12 (80.0)	14 (93.3)	15 (100)

(N.B.) Here, B and G are the abbreviation of boys and girls respectively.

Table 4. Correct-answer-percentage by sexes and times

time / sex / age / exp.		0 − 3sec.			4 − 10sec.			11 − sec.			incorrect a-		
		4	5	6	4	5	6	4	5	6	4	5	6
uni.	B	5 (33.3)	11 (73.3)	13 (86.7)	2 (13.3)	3 (20.0)	2 (13.3)	3 (20.0)	1 (6.7)	0 (0)	5 (33.3)	0 (0)	0 (0)
	G	5 (33.3)	11 (73.3)	12 (80.0)	4 (26.7)	3 (20.0)	3 (20.0)	0 (0)	0 (0)	0 (0)	6 (40.0)	1 (6.7)	0 (0)
inc.	B	4 (26.7)	10 (66.7)	11 (73.3)	4 (26.7)	4 (26.7)	4 (26.7)	1 (6.7)	1 (6.7)	0 (0)	6 (40.0)	0 (0)	0 (0)
	G	8 (53.3)	10 (66.7)	14 (93.3)	1 (6.7)	4 (26.7)	1 (6.7)	1 (6.7)	0 (0)	0 (0)	5 (33.3)	1 (6.7)	0 (0)
dif.	B	2 (13.3)	2 (13.3)	3 (20.0)	0 (0)	2 (13.3)	2 (13.3)	1 (6.7)	0 (0)	0 (0)	12 (80.0)	11 (73.3)	10 (66.7)
	G	0 (0)	2 (13.3)	5 (33.3)	0 (0)	2 (13.3)	2 (13.3)	0 (0)	0 (0)	0 (0)	15 (100)	11 (73.3)	8 (53.3)
rem.	B	11 (73.3)	13 (86.7)	13 (86.7)	1 (6.7)	2 (13.3)	2 (13.3)	0 (0)	0 (0)	0 (0)	3 (20.0)	0 (0)	0 (0)
	G	11 (73.3)	13 (86.7)	14 (93.3)	0 (0)	1 (6.7)	1 (6.7)	1 (6.7)	0 (0)	0 (0)	3 (20.0)	1 (6.7)	0 (0)

In order to see whether the result of the experimental research by the author shows a significant difference by sexes, the t-test is done for the 0.05 level of significance, there is no significant difference against each age. It is also the same even if a new factor of the required time is added to that.

SUMMARY

The results obtained through these four experiments are summerized in connection with the experimental objectives in the following.

1. The older kindergarteners grow, the higher the correct-answer-percentage gets, means the development of the concept.

2. The correct-answer-percentage shows no significant difference between union, increasing & removing in each age group.
However, difference develops more slowly than the others, so it is the most difficult concept.

3. The prompt-answering-percentage shows that in 4-year-old children, there is a significant difference in development between removing & union, and between removing & increasing, indicating that removing is the easiest of them.
At the age of 5 to 6, a significant difference disappears between union, increasing & removing.
Difference has a significant difference from the other three in each age, and develops more slowly, meaning that it is the most difficult concept.

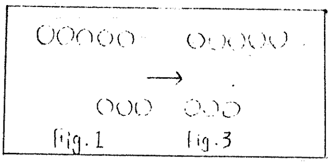
4. It is granted that in scientific thinking boys are superior to girls, but there is no significant difference between sexes with respect to union, increasing, difference and removing.

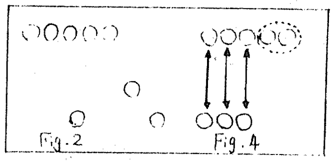
As mentioned above, the author has considered the development of addition & subtraction concepts, through the four experiments using tangiable objects, and has reached the conclusion that he has not been able to approve of Piaget's assertion;
"Subtraction is innately the reverse of addition in operational systems. If it is to be emphasized as an individual operation, it must be taught with addition simultaneously as can be seen in the reversibility of concept."
Neither can be easily accepted Docroly's theory that subtraction is easier than addition.
The reason for this is that, while the concepts of addition & subtraction have always been subdivided into many concepts, as in these experimental researches by the author, the concept of addition is divided into union &

increasing; and the concept of subtraciton is divided into difference & removing, and then they are placed in order from the easy one on the left to the difficult one on the right;

		union	
4-year-old:	removing	‖	difference
		increasing	
	removing		
5-& 6-year-old:	union		difference
	increasing		

This shows the difficulty of difference.
If we present this content by using such terms as addition & subtraction
4-year-old: subtraction (removing) --- addition (union, increasing) ------ subtraction (difference) with these it is impossible to say unconditionally that subtraction is easier than addition.
What has enabled Docroly to say that subtraction is easier than addition is that, to be more precise, removing, only one of the concepts of subtraction happens to be so.
Finally, when the above mentioned results are to be practically used in Arithmetic education at kindergarten and in the lower grades of elementary school, the following conclusion is deduced.
A. At kindergarten removing should be introduced first, and after that union, & increasing are to be introduced. Difference is not to be taught.
B. Now in the first grade addition (union & increasing) and subtraction (difference & removing) are introduced. But the correct-answer-percentage of difference in 6-year-old children being at 40%, it must be avoided and passed on to the second grade.
C. In order for children to solve the question of difference, they have to be taught to acquire the ability which enables them to recognize that it is of the same nature as the question of removing.
In other words, they have to acquire the ability to change view points, which make it possible for them to see that difference is, from a different angle, an isomorphism of removing.
To achieve these,
(1) They must acquire the conservation of numbers.
Three marbles in Fig. 1 are moved into the new position in Fig. 2.





At this time it is not certain that the number of the marbles is still the same as before.

Three marbles may have the possibility to change their position, and even if the marbles are changed their position, children must observe closely what is invariant.

They don't proceed directly from Fig.1 to Fig.3, but they go through the process of Fig.2 just the author has mentioned above.

And so, they acquire the conservation of numbers.

(2) They must understand one-to-one correspondence is nothing but the act of removing.

Now, one-to-one correspondence has done in Fig.4. And at this time, they must be able to set up a new stand point, and from this visual point, they must understand that one-to-one correspondence is actually nothing but the act of removing.

If they proceed like this, the problem of <u>difference</u> will turn into the problem of <u>removing</u>. We call this generative understanding.

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CHILDREN'S PRENUMERICAL ADDING SCHEMES*

Leslie P. Steffe

The paradigm that I use to understand children's problem solving behavior differs from my early studies in this area (Steffe, 1970; Steffe & Johnson, 1971). Then, it was my intention to perform correlational studies based on the performance of groups of children in an attempt to establish significant relationships between "psychological" variables and certain other variables that were apparent to me in the problematical situations. It is now my intention to perform interpretative studies of the problem solving behavior of children in teaching episodes. Teaching episodes are the basic means of observation in teaching experiments that are conducted with individual children over extended periods of time (Steffe, 1983b; Cobb & Steffe, 1983). The problematical situations that are used are designed on the basis of an historical interpretation of the child's problem solving behavior. These interpretations, while theory laden, are revised or elaborated to fit observations made in further teaching episodes. The interpretations are never taken as absolutes and are superseded or changed as new interpretations are made. Thus, there is never a "right" or "correct" model of the child's knowledge. Rather, there are possibilities for understanding the child's knowledge and possibilities for how that knowledge might change under the guidance of a knowing adult.

Upon the conclusion of a teaching experiment, retrospective analyses of the teaching episodes are conducted using video taped records. The retrospective analyses provide time for reflection -- time for a study of the schemes that are only indicated by the child's behavior (Piaget, 1980; von Glasersfeld, 1980). Isolating itineraries of the construction of these schemes by several children is of central importance because observable regularities across these itineraries can lead to the formulation of a developmental model of the particular schemes.

Stages in the Construction of the Counting Scheme

Retrospective analysis of the construction of adding schemes by three children who were subjects in a two-year teaching experiment has led to specifying adding schemes within observed stages of the construction of the counting scheme. Over the duration of the teaching experiment (October 1980 to June 1982), there were stretches of time where the three children could count only when perceptual collections were in their visual field (except in special circumstances). These perceptual periods lasted until at least March of 1981-- at least 7 months into the school year. A motor period followed for each of these three children that lasted until at least December of 1981. In this period, the children could count their movements as substitutes for the items of perceptual collections. All three children had entered a verbal period by February of 1982 and remained in that period until June of 1982 (Steffe & Cobb, 1982). In this period, the children had curtailed motor activity in counting and their number words signified countable items. None of these three children constructed abstract unit items as countable items over the two year duration of the school year. In other words, they were yet to construct number (cf. Steffe, von Glasersfeld, Richards, & Cobb, 1983, for a model of children's counting types).

Two distinct stages were observed in the construction of the counting scheme by the three children -- the perceptual stage and the figural stage. The perceptual stage corresponded to their perceptual periods and the figural stage to their motor and verbal periods. The justification for positing these stages is twofold. First, there was a distinction in the composite wholes that the children could create and count (cf. von Glasersfeld, 1981 for a discussion of the composite wholes that children can construct) . In the perceptual stage, the children could count the items of collections in their visual and auditory fields but not re-presentations of these items. In the figural stage, they could count substitutes for re-presentations of the items of collections that were not in their visual field. Second, there was a reorganization of the counting scheme that corresponded to the transition to the figural stage. The children could now count substitutes for the items of the second of two figural collections by a continuation of counting substitutes for the items of the first. It was essential, however, that the number word that referred to the second of the two collections refer to a spatial pattern or to a finger pattern that the children could use in the decision of when to stop counting (cf. von Glasersfeld, 1982 for a discussion of figural patterns). There were no reorganizations of counting that were observed in the verbal period that would justify distinguishing stages.

Three other children who were subjects of the teaching experiment achieved an operational stage in their construction of the counting scheme. There was a distinct advancement in the composite wholes these children could create and count when they were in this stage as well as a reorganization of counting. The children could count the units of a number, whereas, in the preceding stages, they could at most count substitutes for the figural items of a hidden collection. Moreover, in the operational stage, the children could itentionally keep track of a continuation of counting when the numerosity of the counting acts of the continuation exceeded the specific plurality of a spatial or finger pattern. In the sequel, I do not discuss the adding schemes of these three children. Such adding schemes have been suggested elsewhere (cf. Woods, Resnick, & Groen, 1975; Steffe, Herstein, & Spikes, 1976; Davydov & Andronov, 1981; Eshun, 1982; Cobb, 1983; Steffe et al. 1983; Carpenter & Moser, 1984; and Fuson, 1984).

The Perceptual Stage

I focus on the attempts of the three children who started the experiment in the perceptual stage to solve tasks that involved partially hidden collections, for these attempts reveal their adaptations as well as their limitations (cf. Steffe, et al. 1983 for a discussion of the attempts of counters of perceptual unit items to solve additive tasks when collections were not hidden.).

BRENDA. Brenda was shown three squares and was told that four squares were covered by a cloth. She was then asked to find how many there were in all. First, she attempted to raise the cloth but was thwarted by the teacher. So, she counted the three visible squares. The remainder of her solution is presented in protocol form.

B: 1-2-3 (touches each visible square in turn).
T: There are four under here (taps the cloth).
B: (Lifts the cloth, revealing two squares) 4-5. (She touches each of these squares and puts the cloth back.)
T: O.K., I'll show you two of them (folds back the cloth to reveal the two she had counted). There's four here. You count them.
B: 1-2-3-4-5 (touches each visible square in turn).
T: There are two more here (taps the cloth).
B: (Attempts to lift the cloth but does not count).

Brenda's attempt to lift the cloth indicates that she was aware that some squares were hidden. But this awareness of a hidden collection did not trigger counting. She was yet to coordinate the utterance of a number word with the production of items in visualized imagination.

Brenda could, however, make perceptual replacements for hidden items. On one occasion, the teacher covered six of nine marbles with his hand and asked Brenda to count all of the marbles. She first counted the teacher's fingers and then continued, counting the three visible marbles. The teacher pointed out that he had six marbles beneath his hand and Brenda replied "I don't see no six!" Brenda's creation of perceptual replacements was a result of her search for perceptual items to count. That she took the teacher's fingers as countable item was fortuitous -- they just happened to be there.

Upon suggestion by her teacher, Brenda began using a finger pattern scheme. For example, in the context of finding how many marbles were hidden beneath two cloths, one hiding six and one hiding three, she first simultaneously put up five fingers of her right hand and one of her left hand. She then put up two more fingers of her left hand, forming a finger pattern for "three". She failed to make a separation between the finger pattern for "six" and the finger pattern for "three". Finally, she counted her eight extended fingers. The solution is typical of those which involved creating a pattern of more than five fingers and then another finger pattern. Moreover, Brenda gave no indication that her answer "eight" referred to the marbles. This suggests that once she established a finger pattern, the pattern served no substitutive function but simply superseded the collection of marbles.

TARUS. The first major adaptation that a child named Tarus made in the perceptual stage was to construct what I call spatial-motor patterns for the number words "two" through "five". He could use these spatio-motor patterns when solving tasks involving two hidden collections of five or fewer items. For example, Tarus uttered "1-2-3" in synchrony with tapping on a cloth hiding three items, his points of contact forming a triangular pattern. He then continued, tapping on another cloth hiding four items in synchrony with uttering "4-5-6-7", his points of contact forming a square pattern. The number words referred to spatial patterns that Tarus could re-present in visualized imagination. That he substituted these figural patterns for the hidden items is indicated by his touching the cloth covering the items. His enactment of a figural pattern is what I refer to by the phrase "spatio-motor pattern".

Tarus' second major adaptation in the perceptual stage consisted of substituting finger patterns for hidden perceptual collections. In contrast to Brenda, his substitutions were made independently of any directives. He tapped on a carpet on which he was sitting as his teacher dropped four marbles into each of two cups. To my surprise Tarus introduced the novelty of finger patterns.

> T: (After Tarus had counted dropping four marbles in each cup) how many altogether?
>
> Ta: (Sequentially puts up four fingers on his left hand) 1-2-3-4. (Pauses,then puts up his thumb) 5. (Sequentially puts up four fingers on his right hand) 6-7-8-9. Nine! (He intently looked at his fingers as he put them up).

Tarus substituted finger patters for the hidden collections. This is confirmed in subsequent tasks where he oscillated between looking at his fingers and looking into the cups. Even though he sequentially put up fingers, Tarus counted his fingers as perceptual items to establish finger patterns. This is indicated by his failure to make a separation between establishing a finger pattern for "six" and a finger pattern for "two" as did Brenda. Nevertheless, putting up fingers provided him with an occasion to isolate that motor act as an experiential item. He was soon to enter his motor period.

JAMES. James used a scheme for adding that is quite typical for some children. He was shown five marbles arranged in a domino five pattern which he immediately recognized as "five". He was then shown two other marbles which he recognized as "two". The teacher covered the marbles with his hands and asked James how many marbles were altogether.

> J: (Sequentially touches his lips with the fingers of his open right hand) 1-2-3-4-5. (Continues, touching his lips with two extended fingers of his left hand) 6-7.

James characteristically touched his lips in synchrony with uttering number words. I believe that what he counted were tactual perceptual items. However, his intention seemed to be to count his fingers because he first established finger patterns and then counted them.

Discussion of adding schemes in the perceptual stage

Brenda could not re-present spatial patterns. As a consequence, she did not develop spatio-motor patterns for number words. She compensated by developing finger patterns which she used as perceptual replacements for hidden perceptual collections. When Brenda counted, she took her fingers as direct perceptual unit items that had no representative function. Counting was triggered by the perceptual collection of fingers and a number word such as "eight" referred to this collection after she counted. Her use of finger patterns was on a par with the perceptual replacements that she made when counting the teacher's fingers.

Although I have not provided a protocol to document it, Brenda's scheme for counting two hidden collections soon changed. She could imagine the finger patterns to which number words referred and coordinated uttering number words with the sequential instantiation of the elements of those patterns (which consisted of sequentially putting up fingers). This adaptation led to a reorganization of counting in that she could establish two finger patterns by performing a single sequence of counting acts. After establishing a finger pattern by counting, she could continue counting to establish a second finger pattern. Brenda's re-presentation of a finger pattern reflected her fingers as perceptual items in the optical sense that Piaget (1980) explained. In other words, a re-presentation of a finger pattern for "four" was active throughout sequentially putting up four fingers in synchrony with uttering "four", "five", "six", "seven". Likewise, Tarus did not first put up fingers and then count them. Instead, he put up fingers as he went along, indicating that he anticipated counting a plurality of fingers. The manner which he failed to separate two finger patterns suggests that he was yet to enter his motor period. His adding scheme, however, was emerging as a figurative scheme (as was James') because there was good reason to believe that their finger patterns were substitutes for the hidden collections in re-presentation.

The prominence of finger patterns in the emergence of figurative adding schemes can be partially explained by their accessibility to the children. Another reason is that the fingers of a pattern can appear to co-occur in a unitary whole. These figurative or perceptrual complexes served as content, or material, of number words or numerals. The semantic connections that the children had established between finger patterns and number words were the results of counting. As counted collections, the finger patterns had a "quantitative" aspect that was not accounted for by their status as collections. They embodied the counting activity that was used in their establishment.

The Figural Stage

The motor period.

There were two primary advances in the adding schemes of the children while they were in their motor periods. First was the emergence of the motor act as a countable item and second was the coordination of patterns with the counting scheme.

JAMES. In a task that involved fourteen blocks hidden by a cover and four more hidden by the teacher's hand, James sequentially put up all ten fingers in synchrony with uttering "1-2-3-4-5-6-7-8-9-10" and then simultaneously moved both open hands while uttering "11-12-13-14". To complete counting the blocks, he continued by touching the teacher's hand in four places while uttering "15-16-17-18".

Simultanously moving both hands was a modificatioon of putting up fingers and constituted prime indication that he was counting his motor acts. That he did not put up fingers a second time when counting beyond "ten" suggests that he still experienced the perceptual constraint of the finger pattern for "ten" -- he had used all of his fingers. Touching the teacher's hand four times does indicate that it was his intention to count the hidden blocks and that his motor acts were substitute countable items.

TARUS. Tarus overcame the perceptual constraints implicit in his use of finger patterns while he was in his motor period. The following solution is exemplary.

> Ta: (Puts eight and then four more marbles in a cup. To find how many were in the cup, he sequentially put up eight fingers) 1-2-3-4-5-6-7-8. (He then held the three extended fingers on his left hand with his right hand while putting up his thumb and little finger. He then put up two more fingers of his right hand.) 9-10-11-12. Twelve.

There are three significant observations in his solution. First, Tarus held the three fingers of his left hand with his right hand, separating the first eight counting acts from those he was going to perform. Second, he bridged his hands when he made the pattern to which "four" referred. Third, Tarus put up two fingers for a second time when he counted "11-12". This and the two previous observations lead to the inference that he took the motor acts as countable items. Tarus' ability to bridge his hands indicated that his finger patterns embodied action patterns.

BRENDA. While in her motor period, Brenda still experienced the perceptual constraints of finger patterns. Nevertheless, she provided strong indication that she took the motor acts of putting up fingers as countable items. For example, to solve a word problem involving "13" and '4', Brenda asked if she could count and then sequentially put up all ten fingers, closed one hand and sequentially put up three fingers for a second time, all while synchronously uttering "1-2-3- ... - 13". At this point, the teacher intervened and told her to close her fingers before continuing because he was aware of her reliance on finger patterns. Brenda did so and then sequentially put up four fingers while synchronously uttering "14-15-16-17".

Discussion of adding schemes in the motor period

The capacity of the three children to count beyond an already completed counting activity was based on a coordination of their counting scheme with patterns. In the case of James and Tarus, the number words could refer to spatial patterns which they could re-present prior to a continuation of counting, and continuing comprised an enactment of these patterns. Previously, enacting spatial patterns took the form of spatio-motor patterns. Now, enacting spatial patterns took the form of action patterns, e.g. moving the hand in a spatial location so many times or sequentially putting up any four fingers. These action patterns were also abstractions from the activity of putting up fingers. The patterns that Brenda re-presented prior to the continuation of counting were finger patterns, which explains why she continued to experience the perceptual constraints of particular finger patterns.

The extensions of counting the children made were intuitive because they involved patterns -- figurative schemes that triggered a continuation of counting (Steffe, 1983a). There seemed to be partial anticipations of making an intuitive extension prior to continuing to count, but there was no apparent over-all plan of action prior to starting to count from "one".

The verbal period.

There were several adaptations of the adding schemes of the children when they were in their verbal periods. First, they curtailed the coordination of producing a countable unit item in an act of counting and the number word uttered came to signify the curtailed unit item. This curtailment was apparent especially in re-enactments of counting. Second, all three children, under the guidance of the teacher, learned to count-on. Third, there were contextual counting solutions of missing addend tasks.

JAMES. James curtailed the coordination of pointing acts with the utterance of number words when counting the first of two hidden collections. The protocol following was extracted from a teaching episode where James first made the curtailment. This teaching episode occured one and one-half months after the preceding protocol where I documented James counting his motor acts.

T: There are 15 here (pointing) and five here (pointing). How many altogether?
J: (Shuts his eyes and subvocally utters number words) 1-2-3-4-5- ... -15. How many you got here (pointing to the other cloth)?
T: Five.
J: (Touches the table where the points of contact form a domino five pattern) 1-2-3-4-5. (Slaps the other cloth) fifteen right here. (Subvocally utters, tapping on the table in synchrony with the first two utterances) 1-2-3-4-5- ... -15. (Pauses, and then touches the table where the points of contact form a domino five pattern while subvocally uttering) 16-17-18-19-20. Twenty.

After counting to "five", James re-enacted counting. But it was still a requirement that he start with "one". "Fifteen" was not an index of the number word sequence "1-2-3-4-5- ... -15". Nevertheless, he curtailed his pointing acts when counting to "fifteen". On the very next task, where collections of 19 and 5 items were hidden, James shut his eyes and subvocally uttered "1-2-3-4-5- ... -19", paused, and then continued, deliberately uttering "20-21-22-23-24" without pointing or tapping acts.

Over the next one and one-half months, several unsuccessful attempts were made to lead James to a realization that he could count-on. He was finally placed into a situation where he could recursively count so many more. First, he was to start at 3 and count three more as indicated by a sequence of circles with arrows drawn from each circle to its successor. The numeral "3" was placed in the first circle.

J: (Simultaneously puts up three fingers of his left hand.) Three. (And then three more on his right hand) three more. (Immediately) Six.
J: (Writes "6" in the next circle. Puts up six fingers and then sequentially puts up three more.) 7-8-9. Nine! (Writes "9" in the next circle.)
J: N-i-n-e (puts up nine fingers. He then puts up the one left and sits quietly. He finally utters) ten!
T: Put those fingers in your head!
J: (Closes his eyes and presses all nine fingers to his forehead.) Nine (Sequentially puts up two

fingers) 10-11--12.

J: (Writes "12" in the next circle. He then presses his open hands to his forehead) T-w-e-l-v-e (Sequentially puts up three fingers in synchrony with subvocal utterances) Fifteen!

J: (Continues in this manner until reaching "27" but pressing only one hand to his forehead and sequentially putting up three fingers on the other.)

The major adaptation that James made in his adding scheme was to take a number word as signifying counting activity that was embodied in a re-presented collection of fingers. This is indicated by his focusing inwardly on a figural collection of nine fingers. I interpret the number word that he drawled as signifying the activity of counting that was implicit in the figural collection. Having "registered" counting to "nine", he could then continue to count three more. His awareness of saying number words was a prominent feature of his adaptation. In a subsequent task (where he started with "4" and went four more), after using two finger patterns for "four" to start at four and go four more, he hit his head with both hands and then continued counting! He then went on quickly, first touching his forehead and then putting up three fingers but uttering four number words to show that he was smart. He finally curtailed putting up fingers and just uttered number words as if scanning a linear four. The whole activity became a play on his number word sequence. The same teaching procedure was used in the cases of Brenda and Tarus while they were in their verbal periods.

TARUS. Re-enactments of counting perceptual unit items by counting verbal unit items were quite typical for the children in this period. Such re-enactments led to contextual adaptations of their adding scheme. In one such instance, Tarus' teacher showed him seven blocks and asked him how many he had.

Ta: (Counts the blocks) Seven.

T: (Covers the blocks and places some more with them) Now we have eleven. How many more did I put under there?

Ta: (After guessing twice, he sits up straight after his teacher said "eleven" when reposing the problem. After a 20 second pause when Tarus was obviously deep in thought) Four!

T: How did you do that?

Ta: I count!

T: How did you count?

Ta: I go "1-2-3-4-5-6-7" (rapidly) "8--9--10--11".

T: Did you use your fingers?

Ta: No.

Two aspects of Tarus' solution were crucial. First, he actually counted the seven blocks before they were hidden. Second, he viewed the blocks as belonging to one collection that he intended to count. His re-enactment of his solution confirms this. Rapidly uttering "1-2-3-4-5-6-7" indicates a re-enactment of an immediate past experience. The separation of this counting activity from its continuation is also quite significant and is taken as an unarticulated awareness that he had to complete counting the blocks.

Tarus' solution to two tasks that immediately followed the one above shows how important it was that he count the initial collection prior to its being covered. In the first of these two tasks, he counted eleven blocks prior to their being covered. After some extra blocks were added to the eleven covered and he was told that there were now 16 covered, it took Tarus 17 seconds to answer that "five" were added. He said that he counted when asked how he knew that five were covered and re-enacted his solution by uttering "1-2-3-4- ... -16" without pause. This is solid indication that he recognized a temporal pattern for "five" rather than guessed.

When he solved the second of the two tasks, he did not count the initial twelve blocks of a collection of 17. On this occasion, he guessed "ten". It wasn't enough that "twelve" and "seventeen" refer to hidden collections. He had to actually re-enact counting for there to be a solution process other that guessing. I emphasize that Tarus did not independently use counting to solve missing addend tasks. Even in those cases where he did count part of the collection, had it not been for the interventions of the teacher, Tarus would not have counted to solve the task.

The partial anticipations that are characteristic of intuitive extensions were present for Tarus. He was aware that he was going to continue to count the collection which he started to count, but that was only after he re-enacted counting. He could not separate, in thought, counting the two parts unless he actually counted the first. Moreover, the mechanism of the separation was a re-enactment of counting rather than the mental operations of uniting and separating the contents of a collection.

BRENDA. Brenda frequently counted subvocally once she had entered her verbal period and often focused "inwardly" on her counting activity. Eventually, she substituted a re-presentation of counting for the activity. This occured on 31 May 1982 -- four months after she had entered her verbal period.

T: (Makes the sentence "40 + = 50" using felt numerals.) Can you do that one?
B: 41-42-43-44-45-46-47-48-49-50 (sequentially closes all ten fingers).
T: What did you find?
B: (No response).
T: How many times did you count?
B: (Sequentially wiggles each of her fingers) ten.

Brenda re-presented counting to "40", but her task seemed to be to count from "40" to "50". This is confirmed by her subsequent failure to solve a story problem corresponding to "13 + = 21". She did not even attempt to count beyond "13". However, after being directed by the teacher to find how many times she counted in the protocol, she "solved" "40 + = 53" as follows.

> B: 41-42-43- ... -50 (sequentially closes all ten fingers) -51-52-53 (sequentially re-closes three fingers) 13.

Brenda's "solutions" were only contextual. She did not independently anticipate that she could keep track of her own counting acts before counting. A counting act in the continuation of counting did not serve the double function of being a counting act of counting from "one" as well as from the first "addend". Her lack of assimilation of the story problem using her adding scheme confirms that the observed changes were contextual accomodations. She cound not "run through" counting in thought before carrying it out and anticipate finding how many times she would count when counting from one number to another.

Discussion of adding schemes in the verbal period

Uttering number words carried the significance of a curtailed production of motor unit items for the children while they were in their verbal periods. However, a curtailment of motor activity did not yield the capacity to substitute a re-presentation of counting for the activity until more than a month after the on-set of the verbal period. This was the case even though the children were taught twice a week in one-half hour sessions where they were given repeated opportunity to curtail counting activity. In these sessions, the first "addend" was varied while the second was held constant in repeated trials. These attempts only led to momentary curtailments of starting to count from "one". There were no re-presentations like those made by James when he recursively counted three more. I should point out, however, that while he took counting as a given and proceeded, the unitary whole to which "nine" referred was not a number. It was a re-presented collection that embodied counting -- a specific plurality.

The ability to substitute a re-presentation of counting for the activity (or count-on) emerged after the children could re-enact counting perceptual unit items by counting verbal unit items. These substitutions allowed all three children to appear to be able to solve missing addend sentences but their solutions were contextual and guided by the teacher. There were no indications that the children could run through the solution in thought before carrying it out. Their adding schemes were yet to become operational.

FINAL COMMENTS

There were no indications of addition as a joining operation in the prenumerical adding schemes of the children over the duration of the teaching experiment. The first possibility of a joining action was Brenda's finger pattern scheme. However, after she established two finger patterns, the elements of the patterns were in her visual field as an experientially bounded whole -- a collection -- that triggered counting. There was no necessity that she "join" the two patterns into one pattern before counting. In fact, I think of her finger pattern scheme as two subschemes. The first consisted of successively establishing two finger patterns and the second of counting the collection formed by the results of the first. Addition consisted of carrying out an action (establish a finger pattern) followed by carrying out a like action. The results of these two actions were not joined prior to counting but were juxtaposed in experience and triggered counting, the only possible joining action. I emphasize that there were no indications that Brenda was aware of the differing functions of establishing finger patterns and counting the fingers of those patterns. These were activities she carried out, but they did not refer to anything outside of themselves.

The spatio-motor pattern scheme of Tarus must be interpreted differently from the finger pattern scheme of Brenda. A number word such as "three" referred to a spatial pattern that he could re-present in visualized imagination. When these number words referred to a hidden collection, he could re-present the elements of the collection in a constellation -- e.g. a triangular three or a square four. His adding scheme consisted of, first, re-presenting a collection of three hidden marbles in a constellation, and then counting the elements of the constellation, completing a spatio-motor pattern for "three". Completing this activity triggered the next activity -- re-presenting the remaining part of the collection of marbles to which "four" referred in a constellation and continuing to count the collection, completing a spatio-motor pattern "4-5-6-7". The notion "counting the marbles" was the "glue" that held the additive activity together.

The finger pattern scheme Tarus used while he was in the perceptual stage must be interpreted differently from the finger pattern scheme of Brenda in the same stage. For Brenda, a number word referred to a particular finger pattern that she could not re-present. Thus, she first had to actually establish two finger patterns and then count the elements of the result of the activity. Tarus, on the other hand, could re-present finger patterns in visualized imagination just as he could re-present the elements of a hidden collection as a constellation. Consequently, he could substitute a finger pattern for a hidden collection. His finger pattern scheme for adding consisted of first, substituting a finger pattern for part of a collection hidden in a particular spatial location and then counting the elements of that pattern. This triggered substituting a finger pattern for the second part of the collection that was hidden in another spatial location and continuing to count the collection by counting the elements of that pattern. Counting the elements of the collection hidden in two spatial locations was the only possible joining action. James finger pattern scheme also differed from Brenda's. Like Tarus, he substituted finger patterns for hidden collections even though he simultaneously rather than sequentially put up fingers to establish finger patterns. But, like Brenda, the finger patterns were only juxtaposed in experience and there were no indications of a joining operation prior to counting.

The intuitive extensions that were observed in the motor period differed from the spatio-motor scheme of Tarus and the finger pattern schemes of Tarus and James in two important ways. First, the children had isolated motor acts as countable items. Consequently, it was not required that the number words that referred to the two visually separated but hidden parts of a collection each refer to some pattern. A number word such as "fourteen" could now refer to actually counting to "fourteen" by counting motor unit items. This enactive meaning of a number word was a definite advance because, in the perceptual stage, the meaning of number words consisted primarily of patterns. They could refer also to perceptual collections, counted or uncounted, but not to the activity of counting them. The intuitive extension scheme, then, consisted of establishing enactive meaning for a number word that referred to one part of a collection hidden in a particular location by counting up to the number word. Completing this counting activity triggered substituting a pattern for the other part of the collection hidden in a different location, and enacting that pattern. Again, it was the activity of counting the particular elements of the collection that held the activity together. There was no indication of a joining operation.

There were two distinct advancements in the the intuitive extension scheme in the verbal period. First, the children could establish meaning for a number word by simply uttering number words up to and including that number word. Moreover, these number words could signify curtailed productions of countable items. Second, the children could count the elements of a pattern by simply uttering number words. This latter activity often took the form of nodding the head, say, four times as if scanning a spatial four. The major advance, then, consisted of a change in the countable items the children could create and count, rather than a structural reorganization of the scheme. There were contextual adaptations of the intuitive extension scheme where the children appeared to solve missing addend problems. However, these apparent adaptations were not made independently by the children. The only possible structural reorganization occurred when the children learned to count-on. However, this was only a substitution of a number word for the activity of uttering number words up to and including that number word. I do not trivialize this substitution because of the great difficulty that the children had in making it. The substitution required that the children re-present the activity of counting a collection without enacting the counting activity. That occurred in the context of re-presenting a finger pattern that signified the counting activity that it embodied. Counting activity itself could not be re-presented -- only re-enacted. Consequently, it was crucial that the items of a collection be re-presented that co-occurred in a unitary whole and that embodied counting activity. While this could be considered a structural re-organization of intuitive extension, I consider it as a curtailment even though the mechanism of the curtailment was rather complicated.

The question of whether these children learned their addition facts is of practical interest. For the most part, by March of the second year of the experiment, they had learned those facts that corresponded to use of the finger pattern scheme -- where each addend was five or less. I must point out that these children received regular school instruction as well as instruction from us. In March of 1982, documentation was made of the children as they attempted to learn the other facts. They could remember them for only short periods of time, often for only one day. They had difficulty estimating a sum and their extimations were usually wildly inaccurate -- they were only guesses.

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G. LANGUAGE

LANGUAGE AND MATHEMATICS IN PAPUA NEW GUINEA: A LAND OF 720 LANGUAGES

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In a country of three and a half million people, where 16% of the world's languages are spoken, it is not surprising to find that the effect of language on learning has been a research topic for a number of years. English is the official language of communication in schools. However 'Pidgin English' and to a lesser extent 'Police Motu' are the common lingua franca of the people. In the villages where most people live, the village or clan language (Tok Ples) is spoken. There are in excess of 720 different Tok Ples; that is 720 distinct languages, not dialects.

The interplay between language learning and concept formation has always been of interest to educational psychologists. The difficulties of learning mathematics in your second or third language makes the whole process that much harder. Since the mid seventies a number of studies have looked specifically at the problems Papua New Guinean (PNG) students have in learning mathematics. Most have either centred on the language issue or at least have taken this factor into account.

Austin and Howson (1979) in an authorative essay on the interaction between mathematics and language used three major categories to give a framework for their discussion. They were 'the language of the learner', 'the language of the teacher (and author)' and 'the language(s) of mathematics'. They further subdivided their second category into two sections; 'readability' and the verbal 'language in the classroom'. The studies reviewed in the present article have all been carried out in PNG and can all be classified in Austin and Howson's framework under the subsection of 'readability'.

THE EXTENT OF THE PROBLEM

The readability of written mathematical problems is a concern from primary school level through to university.

Clements and Lean (1980) investigated various cultural, memory, spatial and mathematical variables in a cross cultural study on years 2, 4 and 6 students. They found no educationally significant differences between the PNG students and the English speaking students on memory tasks, some of the mathematical tasks and on Piagetian conservation tasks. However on pencil-and-paper tests of mathematics and on a mathematical language test there were differences. In particular the authors noted that the PNG students handled word-free computational problems quite well, but with verbal arithmetic problems they had great difficulty.

Edwards and Bajpai (1979) gave 2604 post year 10 students a mathematics test which included symbol type problems and word problems. Analysing the results the authors were appalled at how weak the PNG students were. Sullivan (1981) reanalysed the results and showed that with the non-verbal problems the PNG students were as good, and in some cases better, than comparable groups of Australian and United States students. However with the verbal problems, the PNG students were in fact very weak.

Lean and Clements (1981) investigated the relationship between first year university students' spatial ability and their mathematical performance. They found that only 22% of the variance was explained by the variables they used. They suggested in their conclusion that language competence may be a variable that would probably be especially important in the PNG context.

These three studies suggest that language is seen as an important variable in the learning of mathematics at all levels of the education system in PNG. In particular the readability of mathematical problems is a real cause of difficulty for many students.

THE DETAILED STUDIES

A number of specific studies have investigated the interrelationship between readability and mathematic's learning.

Two hundred and ten year 10 students were tested late in the year using a test/retest design (Clarkson, 1981). One test was a set of written questions. The second test consisted of the same questions but with diagrams relevant to the questions appended. Although the study was dealing with spatial ability, an analysis of the results revealed among other things that the mere presence of the diagrams did not make the questions any easier. But further, it was clear on interviewing students that they did not bother with the diagrams at all. The questions this raises is just how often do students analyse questions and understand them? Do they in fact just go straight to trying to do a calculation?

In 1978 Meek and Feril constructed an instrument to test the comprehension of eleven key words such as solid, construct, edge and pyramid, found in a year five geometry unit of work. Three contexts were devised for each word: a diagram recognition item, a gap filling item and a sentence completion item. Thus the test had a top score of 33. One hundred and seventy one year 6 students and 240 Primary School Teachers' College students were tested. The year 6 students only had an average of 44% the Teacher College students a low 68%. More worrying still was the fact that for only one of the eleven words did more than a third of the year 6 students obtain a correct answer in all three contexts. This occurred for six of the eleven words for the Teachers' College students. It appears that students did not fully understand the words that they were supposed to use.

Jones (1982) studied other specific words important to mathematic's learning. He investigated the acquisition of the relational concept more-less from years 2-10. This concept has implications for a wide area of mathematics including measurement, geometry and arithmetic. The instrument constructed consisted of questions in three contexts:

Which is more 10 or 13? (termed this 'comparative')

What number is 1 more than 5? (termed this 'direct')

The number 8 is 2 more than what number? (termed this 'indirect')

It has been suggested that for students to have mastery of this concept, the three contexts must be understood. Jones' results indicated that although both the PNG students and English mother tongue students learnt the three contextural meanings of this concept in the same order, that shown above, the PNG students were some four years slower. This meant that many of the PNG students still had not mastered the 'indirect' context by year 10. What is worrying is that the syllabi in use in the schools assumes that students have mastered more-less by years 3 to 4 and also the follow on ideas of bigger-smaller, shorter-longer in all their complex forms.

Clarkson (1983) used the Newman technique of error analysis to investigate the types of errors PNG year 6 students made on written mathematical problems. This technique gives categories of reading, comprehension, transformation, process skill, encoding and careless errors. An analysis of 95 student responses on a 36 item test indicated that these students made a significant proportion of reading (12%) and comprehension (21%) errors in their attempts to solve the problems. Reading errors were of the type 'spaces' read for 'shapes', 'meaning' for 'missing' and 'volume' for 'value'. Some words like 'daughter' were simply unknown. Key words which were read correctly but which students did not understand were of the type 'value', 'simplest form'; logical terms such as 'and', 'or', 'not'; and confirming Jones' findings 'more than' and 'less than' caused problems. In trying to sort out the characteristics of students who made these types of errors,the frequencies of the error categories were correlated with a number of variables. One salient fact arose from this analysis; there were quite different patterns of correlation for the two types of errors suggesting that different types of remediation techniques are called for.

The Indigenous Mathematics Project began in 1976. The first phase of this project ran until 1979 and documented indigenous counting, classification and measuring systems. It also produced a number of cross cultural cognitive studies. The second phase turned to the classroom. Studies were carried out which undertook detailed observations of classroom teaching, investigations of mathematic's learning by primary school students and development of learning materials. One result from this project indicated that the most difficult section of the course was that of Problem Solving. This strand of the syllabus is often associated with story or word problems and hence is closely tied with language skills. As part of the investigation of mathematic's learning Souviney (1983) has reported on the development of tests based on the objectives of years 2, 4 and 6. These were given to various groups of students as well as tests of visual memory, memory space, language and a number of conservation tasks. The language tests comprised a reading-sentence completion test and a dictation test. The analysis of results suggest that the language and cognitive variables play an increasingly important role in mathematic's achievement as the years of schooling increase and hence the complexity of the task increase.

The last study to be reported on was carried out with second year university students by Reed (1981 and 1982). He started with a model for mathematical activity:

A. Language comprehension

B. Concept formation

C. Mathematical symbolism

It was suggested that in solving a mathematic's problem students often progress through stage A and then oscillate between stages B and C in the actual solving process. In early years of schooling Reed suggests that most effort is needed on stage A with the other two stages being relatively simple. By the time university is reached most effort is devoted to stage C.

Reed devised five instruments which dealt with word questions, geometric diagrams, producing diagrams given a textual description, on proof and a test composed of mathematical symbols (equations to be differentiated). The intercorrelations between these tests tended to group the first three together and the last two together. He suggested that the first three relate more to stage A whereas the last two deal with Stage C. The results from these tests were also correlated with the students' first year results in language and mathematics. There was no significant correlation with the language results and only a significant (positive) correlation between the mathematics examination results and the test of symbols. The non correlation of the language mark with the mathematical tests is not surprising. Such global scores are not fine enough at this level to show relationships. Reed suggested that the correlations between the first year mathematics results and the test of symbols perhaps indicated that the dependence on language, which the first three tests tended to measure was fading fast as his model predicted. But were the end of year mathematic's examinations only measuring symbol manipulation? Should they be testing other aspects of mathematical learning?

Since English is the official language and the language of instruction there is merit in this. However no studies have ascertained whether students' competency in their mother tongue and/or in 'Pidgin English' has an effect on mathematical learning. This is an area which needs urgent investigation.

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To further this study Reed requested students to fill in a questionnaire which asked them to list their preferences of working with numbers, symbols, diagrams, words and class discussion, and the extent to which they thought in English when solving mathematical problems. The interesting result was that the extent to which students used English in their thinking did not correlate significantly with the mathematical tests given except for that on proof. The question arises as to whether there is a relationship between proof and the underlying structure of English.

Reed is really rather blunt in his analysis. However his work does indicate the likelihood of there being a different type of relationship between mathematics and language at university level compared with that which exists at levels lower in the education system.

SUMMARY

There are some generalisations which can be drawn from the studies reported:

1. There is a concern at all levels of the education system with the quality of mathematical learning and a suspicion that language difficulties are partly the cause of problems in mathematics learning.

2. There is evidence to show that students and probably a number of primary school teachers have difficulty with many important mathematical words and the underlying concepts.

3. There is some evidence which indicates that students do not analyse questions carefully.

4. There is a suggestion that the type of relationship between mathematical learning and language learning may well change as students progress through the system.

All the studies reported have focussed on competency in English and related this to mathematical learning.

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H. PROPORTIONAL REASONING

ADOLESCENTS' STRATEGIES ON AN INVERSE PROPORTIONALITY TASK

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The understanding of proportion in a functional relationship is tacitly assumed to be a major conceptual strand in secondary school mathematics and science. Yet researchers, such as Lovell (1961), Hart (1978) and Karplus and colleagues (1972, 1977a, 1977b) have studied the difficulty of this schema while contributing to an understanding of its complexities. In a recent text in the psychology of mathematics education (Lesh & Landau, 1983), both Vergnaud and Karplus authored chapters relating to proportionality. Both emphasized the need for further study of ratio-concept development. This study is significant for its investigation of the understanding of inverse proportionality as a functional relationship by adolescents who were successful on a measure of direct proportions.

The inverse proportionality measure was a version of Inhelder and Piaget's (1958) projection of shadows task which emphasized feedback and allowed subjects a second trial on each subtask. The equipment design, interview outline and scoring procedures were developed for use in an earlier study (Farmer, Farrell, Clark & McDonald, 1982). In that study, Farmer et al. (1982) incorporated a repeated trial procedure and systematic feedback to encourage optimum performance by subjects. Yet these task characteristics appeared to have little effect on the difficulty level of the task for that sample (see Table 1).

TABLE 1
Classification by Three Inhelder and Piaget Tasks

	2A		2B		3A		3B		
Task	#	%	#	%	#	%	#	%	Total
Solution	5	(7.2)	48	(69.6)	16	(23.2)	0	(0.0)	69
Rods	20	(30.0)	24	(34.8)	22	(31.9)	3	(4.3)	69
Shadows	7	(10.1)	59	(85.5)	1	(1.4)	2	(2.9)	69

Tapes of the interviewers' comments revealed some common error patterns as well as some unusual reactions to feedback. It was hypothesized that a more homogeneous, high-achieving group who were able to succeed on a prerequisite task would be distributed more normally across the four classifications. Moreover, it was expected that their reasoning strategies would be more structured and that such subjects would display more ability to learn from feedback. In view of previous research (e.g., Maccoby & Jacklin, 1974) showing gender differences on spatial tasks, it was also expected that males would outperform females on the shadows task.

METHOD

The subjects were all of the 1,200 students enrolled in tenth, eleventh or twelfth grade college-bound mathematics and/or science classes in five school districts in upstate New York, U.S.A. The subjects included 44 ninth, 475 tenth, 229 eleventh and 153 twelfth graders. The ninth graders were enrolled in tenth grade classes. There were 576 males and 624 females in the total sample. The Tall-Short task (Karplus, Lawson, Wollman, Appel, Bernoff, Howe, Rusch & Sullivan, 1977b) was administered as a first order direct proportion ($y/x = k$) measure of prerequisite ability. As a result of some test administration irregularities in some class sections of one school, data from only 901 subjects (433 males and 468 females) were analyzed. From the 474 subjects classified as successful (stage 3), a random subsample of 128 subjects (75 males and 53 females) was tested on the projection of shadows task.

INSTRUMENTS

Tall-Short Task. Tall-Short is a measure developed by Karplus (Karplus et al., 1977b), which has been used by researchers such as Goodstein and

Boelke (1980) and Hart (1978), as well as in multiple studies by Karplus and colleagues (1970, 1972, 1977a). The version administered in this study was essentially that pictured in Karplus et al. (1977b). The scoring on Tall-Short was modeled after that reported in Karplus (1977a). The subject was classified as 3 only if a proportional method had been used successfully. An additive response was classified as 2A while a response using one ratio or some concrete approach toward equating the two units was classified as 2B. Each response sheet was scored independently by two trained scorers. The interscorer agreement attained in the training period was 98%. Researchers checked a random sample of all papers and resolved the small number of scoring questions posed by scorers.

Projection of Shadows Task. The projection of shadows task progressively tests first order direct ($y/x = k$) and first order inverse ($yx = k$) proportion in a spatial, geometric context. Unlike Inhelder and Piaget's balance beam task, which also involves first order inverse proportion, the shadows task requires an understanding of proportion as a function. In fact, Piaget, Grize, Szeminska and Bang (1977) found that subjects often succeeded on the balance beam by working with the complement of the inverse, rather than by setting up the inverse proportion itself. It appears to be impossible to succeed on the shadows task by any similar circumvention.

The apparatus, script and scoring designed for an earlier validity study (Farmer et al., 1982) were used with no modification. The team of clinical interviewers viewed a researcher-produced videotape of a variety of concrete and formal operational subjects being administered the projection of shadows task. After discussion and analysis of the videotape, each interviewer studied relevant transcripts in the original Inhelder and Piaget reference and role-played one of the described subjects or acted as interviewer. Finally, junior and senior high students (from schools not to be included in the research) were interviewed by team members. Videotapes of these interviews were analyzed to assist team members in details of the interview procedures. The data from these same tapes were also used to obtain a measure of interscorer agreement on classification. An interscorer agreement of .95 was obtained during the training period.

All interviews were conducted by two members of the research team with one team member acting as a "checker." Both kept written records on the prepared script sheets. An audiotape recording was made of each interview and any written work produced by subjects was attached to the completed script sheet. Interscorer agreement was reached on classification immediately after the subject left the room.

RESULTS

Performance on Measure of Direct Proportions. Of the 901 papers in the reduced sample, 474 (53%) were classified as 3, 189 (21%) as 2B and 238 (26%) as 2A. Correlation coefficients between scores on Tall-Short and age, grade, sex, number of previous science courses and number of previous mathematics courses were .103, .213, -.167, .258 and .349, respectively--all statistically significant with $p < .01$. Thus, stage increased slightly with age, grade, number of previous science courses and number of previous mathematics courses. The negative correlation with sex indicated that females tended to be classified at lower stages and males at higher stages.

A stepwise multiple regression was performed to determine the relative importance of each of these variables in stage classification according to scores on Tall-Short (see Table 2).

TABLE 2

Multiple Regression on Tall/Short

Variable	Multiple R	R^2	R^2 Change
Grade	.396	.157	.014
Number of previous science courses	.403	.162	.005
Age	.411	.168	.006
Number of previous math courses	.349	.122	.122
Sex	.378	.143	.021

The strongest predictor was number of previous mathematics courses. When the effect of this variable was removed, the others contributed very little to the prediction. Sex and grade were second and third respectively, but as can be seen from the change in R^2, their contribution was small. Number of previous science courses and age contributed almost nothing.

Performance on the Projection of Shadows. Of the sample interviewed on the shadows task ($\underline{n}$ = 128), 24.3% were classified as formal (22 subjects as 3B and 9 subjects as 3A) and 75.7% were classified as concrete (92 subjects as 2B and 5 subjects as 2A).

Correlation coefficients were calculated between classification on the shadows task and age, grade, sex, number of previous science courses and number of previous mathematics courses. These correlations were .013, -.029, -.096, -.011 and .186, respectively. Only the correlation of .186, between classification and number of previous mathematics courses, was significantly different from zero ($\underline{\alpha} < .05$). However, this correlation is very low. What is perhaps of more interest is the lack of gender difference found for this sample on this task. This result is considered in a later section.

The transcripts of subjects' responses were analyzed to explore (1) the relationship between the successful use of a second trial on an early subtask and subsequent responses to later subtasks and (2) the potential inhibiting effect of a strategy based on direct proportion.

Subtask two required that the student correctly predict whether two equal (diameter) rings at different locations produce equal or unequal shadows; and if unequal, correctly identify the ring which produces the larger (smaller) shadow. As can be seen in Figure 1, 73% (91) of those succeeding on subtask two did so on one trial. This same group of subjects eventually yielded 19 of the 23 subjects classified as 3B.

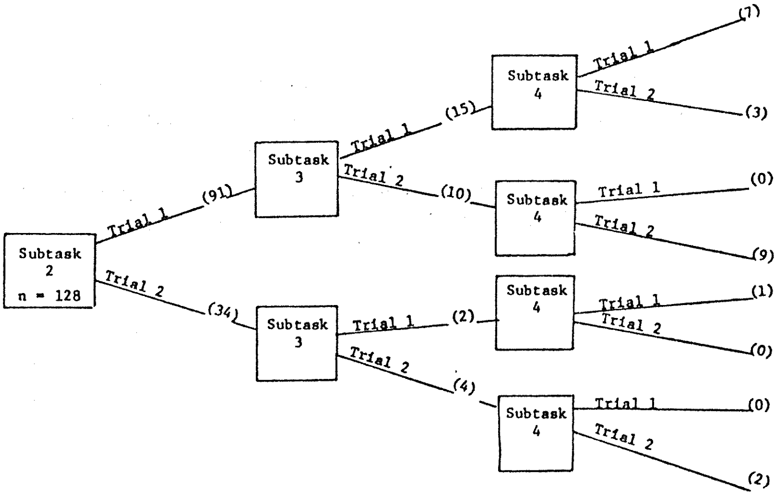

FIGURE 1: Number of Successes per Number of Trials on Subtasks 2-4.

Next, the analysis of the data focused on the role of the number of trials needed for success. Only 7 of the 22 subjects classified as 3B succeeded on the first trial for each subtask. Half of the 3B subjects required two trials on the last two subtasks. Since subjects who had two trials got visual feedback on errors and successes, one might conjecture that these additional contrasting prompts are critical for eventual classification as 3B for many subjects. However, only 14 of 108 subjects (13%) succeeded on the second trial of subtask three. For these subjects, the additional feedback was of little help. They either had no idea how to construct a metrical relationship or struggled with inappropriate rules. Such students knew that the 3 cm ring should be between the light and the 9 cm ring in order to produce coincident shadows but they were unable to predict locations based on a metric. A discussion of some of the stumbling blocks faced by high and low 2B subjects may be found in Farmer, Farrell, and Rumsey (1981).

The literature on the projection of shadows task frequently alludes to the function of reasons associated with various subtasks. The protocols in

Inhelder and Piaget (1958) refer to the subject's focus on the light versus the screen in subtask two and multiplicative versus additive reasons in subtask three. Students who attend to distance from the screen have been characterized as those searching for a direct proportion--further from screen, larger shadow. Since visual feedback on this subtask does not contradict this strategy, such subjects, it might be assumed, would try to find a metrical relationship for subtask three based on the same relationship. Figure 2 illustrates a pattern of reasons for a subsample of subjects.

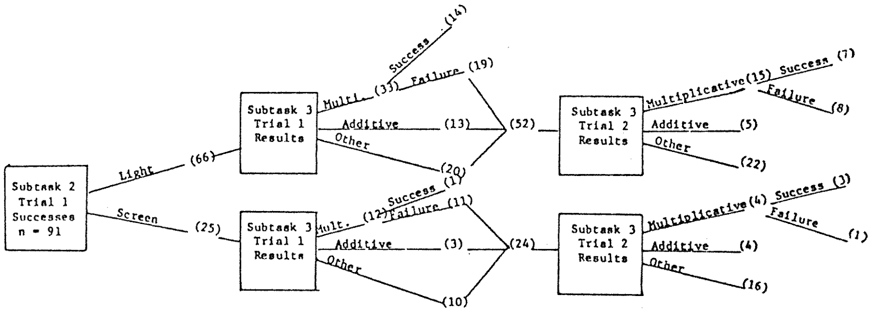

FIGURE 2: Pattern of Reasons on Subtask 3 for Light versus Screen Respondents.

Seventy-three percent (73%) of these subjects attended to the light in the successful completion of subtask two. This is probably a higher percentage than would appear in the original secondary school population tested on Tall-Short. Notice that the additive reason is an unlikely one for both groups, again a probable result of using the screened sample. The reason designated as "other" included diverse mixed rules, guessing and "eyeballing" the distances. For example, a subject might have attempted to recall the size of the shadow displayed in the feedback given on subtask two so that he/she could "cut-and-try" a likely location. The multiplicative reason was used by half of the "light" subjects and half of the "screen" subjects on trial #1 of subtask three. However, on trial

#2, although both groups shifted from the multiplicative reason, the "screen" subjects were more likely to choose another reason. An example of the use of a multiplicative reason by screen respondents follows: "The 3 cm ring has to be three times further from the screen than the 9 cm ring to get a larger shadow." The feedback following this prediction appeared to undermine the confidence of these subjects. Thus, rather than seek other variables to relate in a multiplicative way, they often abandoned multiplication and either tried a different metrical relationship or "eyeballed" the distance. Such subjects gave no indication of having considered the possibility of an inverse proportion. The data in Figure 2 appear to highlight the pervasive inhibiting effect of a strategy based on a direct proportion regardless of the feedback and repeated trial characteristics of this task.

AN ANALYSIS OF THE PROJECTION OF SHADOWS TASK

The results on the projection of shadows task have major implications for both curriculum and instruction in mathematics and science, as well as suggesting areas of further study. The fact that over 76% of these highly selected students were unable to recognize inverse proportions in this task signals a serious lack of understanding of this functional relationship. Is the task difficult because it involves a spatial, geometric component? What exactly is the nature of the mathematical relationship embodied in the projection of shadows? Figure 3 is a schematic of the problem components where m represents the distance from

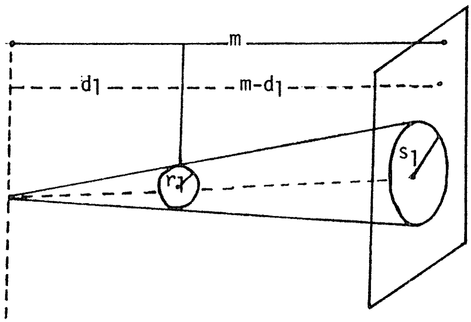

FIGURE 3: Schematic of the Shadows Problem

the light to the screen; r_1, the radius of ring #1; s_1, the radius of the shadow produced by ring #1; d_1, the distance from the light to ring #1. If a ring #2 were shown, then r_2, d_2 and s_2 would denote similar lengths. For two such rings, the following relationships hold:

$$(1) \quad \frac{r_1}{s_1} = \frac{d_1}{m} \text{ or } \frac{s_1 d_1}{r_1} = m$$

and

$$(2) \quad \frac{r_2}{s_2} = \frac{d_2}{m} \text{ or } \frac{s_2 d_2}{r_2} = m$$

Thus,

$$(3) \quad \frac{s_1 d_1}{r_1} = \frac{s_2 d_2}{r_2}$$

This final relationship illustrates the several variables and the direct and inverse proportions between them, taken two at a time. Each of the first three hierarchical subtasks in the projection of shadows is based on this relationship but in each equal values are assigned to a different pair of corresponding variables. In subtask one, the unequal rings are placed on the same hanger; $d_1 = d_2$--a first order direct proportion. In subtask two, the rings are the same size; $r_1 = r_2$--a first order inverse proportion. In subtask three, the subject is told to place unequal rings <u>so that</u> their shadows coincide; $s_1 = s_2$. Mathematically, the reduced relationship now becomes nothing more than a first order direct proportion. Yet the subject cannot simply ignore the shadow size. The subject must be aware of a functional relationship relating ring size to its shadow size. How does the successful subject conceptualize the problem? What is the nature of the obstacle for the unsuccessful subject who <u>does</u> realize that if $r_2 = 3r_1$ that shadows must relate to corresponding rings by a factor of 3 and who knows enough to arrange the rings in the order: light, r_1, r_2, screen?

Why do subjects who attend to a distance relation remain fixed on the same pair of distances? In this case the screen distance plus the light distance (to the object) is equal to the constant 100 cm. An analogous situation may shed some light on the problem. A student may be told that the sum of two numbers is 14. "If one of the numbers is represented by x, how can we represent the other in terms of x?" The question, which requires the student to represent one part in terms of the whole and the other part, is always a difficult one for algebra students. In the shadows task this problem is compounded by the need to analyze a functional relationship and to coordinate the two features of subtasks one and two.

Finally, the shadows task is even more spatially-oriented than Tall-Short. Why are the sex differences present in the scores on Tall-Short not evident in the shadows data? In a recent study of adolescent reasoning, Linn and Pulos (1983) found that males performed slightly better than females on the proportional reasoning measures but that these sex differences were not accounted for by various measures of aptitude, including spatial ability. In fact, Linn and Pulos found a lack of relationship among tasks that individually yielded sex differences. However, as in the studies reported in this paper, the researchers drew attention to the generally low attainment of proportional reasoning by both sexes. In this study, it seems that whatever factors account for sex differences on Tall-Short, the effect of these in the more complex proportion tasks is neutralized by success on the first order direct proportions task offered by Tall-Short.

SUMMARY

The exploratory work in this study goes beyond the investigation of proportional reasoning in terms of a linear functional relationship. The inverse proportion is representative of a class of curvilinear relationships repeatedly found in scientific data patterns. Yet this select sample generally used a multiplicative, rather than an additive strategy and still could not overcome the inhibiting effect of focusing on direct proportions. The results also showed that feedback and second trials were essential for success for most (68%) of the 3B subjects. Is the dimensional analysis strategy emphasized by Goodstein (1983) an appropriate instructional response to these psychological obstacles?

Is that strategy useful only for a limited type of proportionality problem? To what extent has teaching which stresses "cross-multiplication" actually inhibited development of the understanding of proportion (Karplus, Pulos & Stage, 1983)? If tasks were couched in different language and posed in a two-dimensional framework, would response patterns improve? Fajemidagba's study (1983) of proportionality among Nigerian adolescents showed some evidence of success given concrete referents and real world examples. These are a few of the issues which remain in the study of proportional reasoning.

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Developmental Research on teaching and Learning in Mathematics Education.

---A View of Doubling and Halving Based on Ratio Concept.----

N. NOHDA THE UNVERSITY OF TSUKUBA. JAPAN.

1.Preface

In the elementary mathematics we experienced great difficulties in the teaching-learning process of rate, ratio and proportion. The research project "Strategies and Errors in Secondary Mathematics"(SESM) at Chelsea College has investigated the methods used by students in solving ratio and proportion problems (K.M. Hart). The main error student committed in these topics is the use of the incorrect addition strategy(identified by R.and E.Karplus, B.Inhelder, J.Piaget).

Recently, the IEA survey shows the following percentages of student of 13 years old in the performance of some items of ratio and proportion questions.

For example,

P.1 In a school of 800 pupils,300 are boys. The ratio of the number of boys to girls is,

A. 3:8 B. 5:8 C. 3:11 D. 5:3 E.*3:5

P.2 A painter is to mix green and yellow paints in the ratio of 4:7 to obtain the colour he wants . If he has 28 liters of green paint, how many liters of yellow paint shuold be added ?

A. 11 B. 16 C. 28 D.*49 E. 196

The Table 1 shows the percentages of correct answer in these problems above.

Problem No	Pre	Post	Both
1	33.6	31.5	11.9
2	62.8	60.6	41.7

Large numbers of Japanese students failed to answer problem of proportion such as the Problem, because they are obligated to learn how to use many mathem atics expressions about ratio or proportion and less to the meanings of these concepts. This tendency is clearly shown in the following types of problem.

For example. My father's height is 180cm(b,given). The ratio of son's height (a,unknown) to father is 3:4.

Can you find the height of the son?

Which is the correct sentence for solving this problem?

a:b = 3:4 , a=b×3/4 ,

a=b×3 ÷4 , a=b÷4 ×3 ,

In our research, almost same results are obtained as IEA survey.

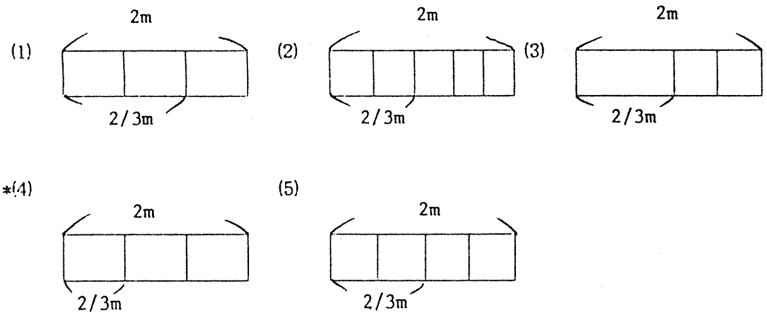
For example, "Halving and Rate" problems and results are the followings.

P.3 What is the number of 1/2 of 'half' ?

Can you find the correct answer?

(1) 1 (2) 2 (3) 4 (4) 1/2 *(5) 1/4

P.4 Can you find the correct item which shows the length of 2/3 m?



The Table 2 shows the percentages of correct answers of these problems.

Prob./ Grade	1st	2nd	3rd	4th	5th
3	32	44	46	----	-----
4	---	3	9	10	11

In the elementary grades, ratio problem that cannot be solved by a routine operation cause a great deal of difficulty. Most of the studies for determine student's difficulties in such problems use broad categories for characterizing the problems. Understanding student's behavior in such problems requires a clear description of the types of problems with respect to the operations which are commonly used by student to solve the problems(K.M. Hart).

For example, Dr. Hart pointed out the types of problems as followings.

(1) Doubling and Halving

(2) Multiplication by an integer

(3) Given a rate per unit, apply this rate

(4) Find a rate per unit and then apply it and so on.

Student's difficulties while solving ratio problems could be due to the difficulties in understanding the basic concept and concepts related to ratio. Several years ago, the characteristics of these students who could recognize fraction as part of a whole and multiplication of fractions have been studied in our research. The present study proves the cognitive role of Doubling and Halving in student's performance in solving some simple problems. (Problems will be shown in detail, see. Appendix)

2. Purpose

Some contents could be understood by pupils without teaching. In order to facilliate pupils' learning, it is not always for all content to be taught. So, the content that was needed to be taught and the content that pupils could learn by themselves had been identified.

As mentioned before, pupils of lower and middle achievement in elementary schools failed to answer the ratio and proportion problem. In order to promote their understanding of "ratio", that is the relative magnitude of a value to another value, they were to be taught with intension to ensure thier understanding of these ideas. Then, as mentioned just before, the fact that pupils could learn some contents by themselves existed.We assumed that the doubling and halving would be this type of content in ratio, and not all grades could learn these concept as the basis for understanding of multiplication and division, which might depend on pupils' intellectual developement. So, the following questions was examined in accordance with different grades.

If pupils have been taught "the doubling and halving" adequately,
can they understand the "ratio" concept?

And, which grade can learn these concepts by themselves?
Which grade requires teaching in order to understand these concepts?

3. Method

(1)Setting and Procedure of Experimental Teachin

This experimental teaching was perfomed on pupils from first grader to fourth grader,each grade with two classes,in an elementary school of the Tsukuba Research Academic City near by Tokyo. At each grade, one class was taught and the other was presented with material only,and in each of the classes was given subjects material with good presentation and the rest were with poor presentation. They were divided into two groups each consisting of equal number of boys and girls with homogenious grouping. In the class with teaching, the author and a co-worker gave 10 minutes'"mini-teaching".For the teaching content and presentation conditions, the attached sheets were referred.One class of each grade was given explanation, and half of this class was given adequate presentaion. This group was referred to as group P. The remaining pupils to be given inadequate presentation was referred to as group Q. For the other class of each grade with out explanation, the group of pupils to be given adequate sheets was referred to as group R,while the rest be group S. Thus,these first to fourth grade pupils were devided into the P,Q,R and S group.

(2) 'Mini'-Teaching

That experimental teaching was called 'mini' teaching above. In the P and Q groups, about 10 minutes'demonstration is carried out with concrete materials ,and contents were given in advance to the pupils. That demonstration activity for the first grader was a preparatory teaching on doubling/halving concept, because they had not been taught about the contents .For the second and third grade, pupils had been taught roughly about the doubling/halving concept, this demonstration would help in their understanding. As the fourth grader, they seem to have been taught and had understood commpletely, this demonstration helped them to recall. Thus, in the P and Q groups from the first to the fourth grader, each demonstration was carried out in a short time and to a small number of pupils.That was the reason why it was named 'mini-teaching'.

Present./ Teach.	Conducted	Not Conducted
Adequate	P	R
Inadequate	Q	S

Fig.1
Groups divided under conditions of teaching and presentation.

(3) Used Problem Items.

Twenty problems were used in this research.The set of problems from tenth to nineteenth was inadequate presentation. The set of problems from twentieth totwentyninceth was adequate presentation.Problems for Q and S groups were having more choices than P and R groups.

(a) Problem Category.

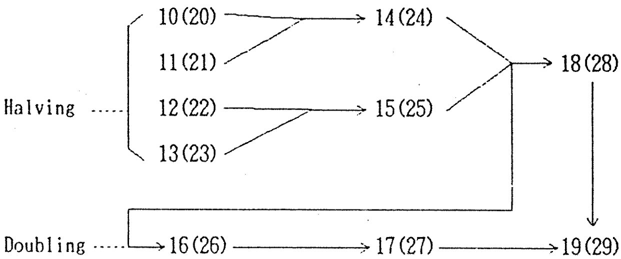
1, Meaning of one half   I= (10,11,14) or (20,21,24)

2, Number of 1/2   II= (12,13,15) or (22,23,25)

3, Two-fold or Four-fold   III= (16,17) or (26,27)

4, Relation between halving and doubling.   IV= (18,19) or (28,29)

(b) Problem Relation.



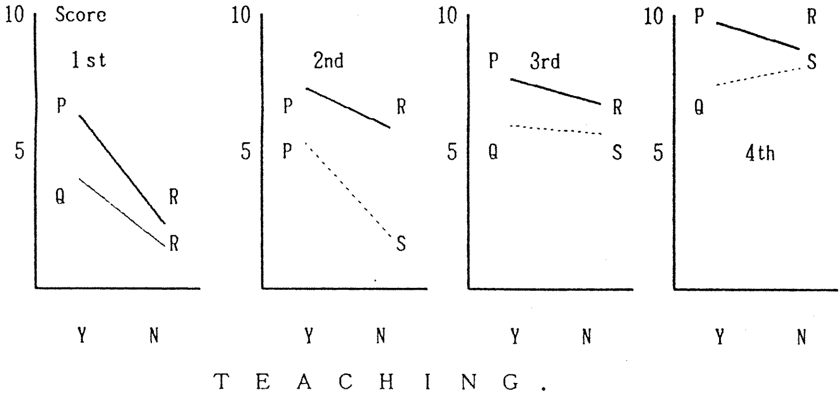


Fig.2 Differences by The Conditions of Teaching and Presentation.

4. Results and Discussions.

(1) Influence of Factors of Teaching and Presentation on Pupils' Performance.

Fig.2 shows the differences in the P, Q, R and S groups by teaching and presentation conditions from the first to fourth grader.

For detailed values,refer to the final page. Difference in each grade is analysed and then discussed below.

Analysis of Result

(1)In the first grader, because the research problem was the contents not taught in advance, there was a great influence on whether teaching to be carried out or not (significant of 1% level).Therefore, teaching contributes effectively to pupils'learning.And the adequacy of subject matter presentation had influenced too (significant of 5% level).So, in the case of the first grader, teaching and subject matter presentation functioned efffectively. However, there was no significant difference in the interactions of these factors.

(2) For the second grader, nearly a half of the subject matter were not taught before so that the factor of teaching is nearly as effective as the first grader. However, it was not so much as the first grader. The second grader was characterized by the marked effect of subject matter presentation factor. Interactions of factors of teaching and presentation were observed from the first grader, but no significant difference was found.

(3) For the third grader, most of subject matter had already been taught so that the factor of teaching scarcely worked,with the factor of presentation influenced to some degree was also not so much as for the second grader. There was no interaction between the factors of teaching and presentation.

(4) For the forth grader,all of subject matter had already been taught, so the factor of teaching scarecely worked. But the factor of presentation was more influence than the third grader.And there was significant difference in the interaction of these factors.

Discussion.

The factors of teaching and presentaion from the first to the forth graders have been examined. What is common to all grades is that the factor of teaching effectively influenced the pupils' learning. The role of this function differs before and after teaching. Before teaching, the factor of presentation works more effectively on the learning volition than the factor of teaching. After teaching, it works effectively on recall. From these results, the following is suggested as the teaching strategies we proposed. First, if the teaching contents are those with pupils' readiness not formed before learning, the factor of teaching works more effectively than the factor of presentation. Next, at the stage in which the pupils' readiness has been formed, and just before learning, the factor of presentation works effectively on the pupils' learning rather than the factor of teaching. If the contents are those that have been taught sometime before, the factor of presentaion works effectively on the pupils' recollection and recall of the learning contents rather than the factor of teaching.

In summary, when it is found that the pupils are not ready to learn, generally teaching should be stopped. If the contents are important, it is recommended that the pupils' readiness be promoted by translating the contents into words that the pupils may understand easily. When pupils' readiness has already been achieved, teacher should devise adequate factor of presentation to aid their learning and then make adequate environment to promote their learning volition. Teachers should attempt to develop pupils' volition for self-learning by helping them to enjoy in discovery by their own.

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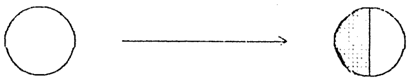
DATA Demonstration Teaching.

This sheet indicates how to teach the halving and doubling.

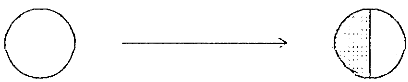
〔1 〕Divide a circle into two equal parts.Shade one of them,read the size of it as 'one half' and write "1/2".



〔2 〕Fold the semi-circle once more into two equal parts. Shade one of them, read it as '1/2 of one half' and write "1/4".

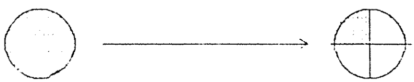
〔3 〕Divide another circle of the same size into 4 equal parts.Shade one of them,read the size of it as 'one fourth'and write "1/4".

〔4 〕On the other hand,a circle is two-fold of a semicircle,and a semicircle is two-fold of a quadrant.



Left circle is double the size of the right'shaded part.

Left shaded part of the left circle is double the size of the right shaded part.



So, the left circle is four times the size of the right shaded part.

group ----- grade	P		Q		R		S		
	M	SD	M	SD	M	SD	M	SD	
1	5.59	2.44	3.86	3.29	1.40	1.71	0.67	1.04	85
2	7.41	2.47	5.88	2.27	6.28	2.18	2.47	2.57	69
3	7.84	2.08	6.11	2.71	6.88	2.18	5.44	2.65	72
4	9.88	0.32	8.06	1.76	9.00	1.19	8.53	1.23	70

1st grade

SV	SS	df	MS	F
A	288.57	1	288.57	** 51.79
B	33.09	1	33.09	* 5.94
A X B	5.29	1	5.29	0.94
Residual	451.38	81	5.57	
Total	778.2	84		

2nd grade

SV	SS	df	MS	F
A	87.82	1	87.82	14.65
B	124.25	1	124.25	20.72
A X B	22.36	1	22.36	3.73
Residual	389.73	65	5.99	
Total	624.16	68		

A : Teaching Factor, B : Presentation Fctor. ** p < o.ol, * p < o.o5

3rd grade

SV	SS	df	MS	F
A	11.74	1	11.74	1.18
B	45.57	1	45.57	** 7.30
A X B	0.42	1	0.42	0.07

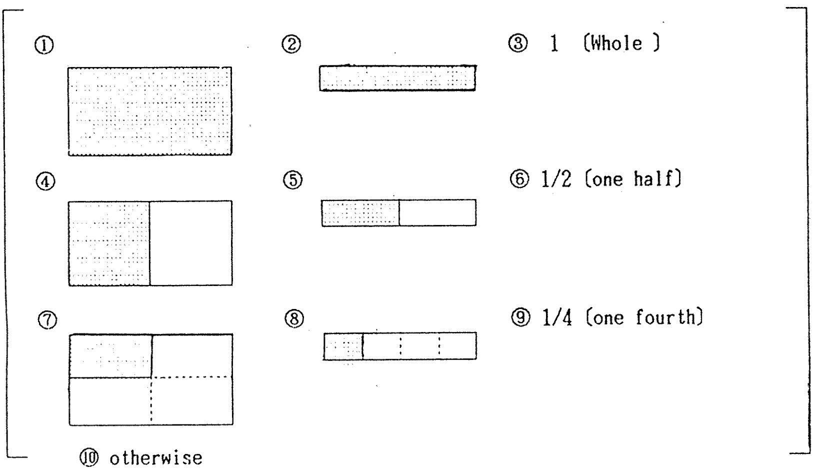
Residual	424.51	68	6.24	
Total	482.24	71		

4th grade

SV	SS	df	MS	F
A	0.63	1	0.63	0.38
B	22.33	1	22.33	** 13.71
A X B	7.95	1	7.95	* 4.89
Residual	107.44	66	107.44	
Total	138.35	69		

Table, Mean, Standard Deviation and Analysis of Variance 1-4 Grade.

A. Inadequite Presentation Style



Selecct an appropriate number from the follwing figures or numbers, and fill in blank.

10, ☐ is 1/2 of①.
11, ☐ is 1/2 of②.
12, ☐ is 1/2 of④.
13, ☐ is 1/2 of⑤.
14, ☐ of ① is ④.
15, ☐ of ② is ⑧.
16, ② is two-fold of ☐.
17, ② is four-fold of☐.
18, ☐ is 1/2 of ⑥.
19, ③ is double of ☐.

B. Adequite Presentation Style

Select an appropriate number from the suitable figure or numeral.

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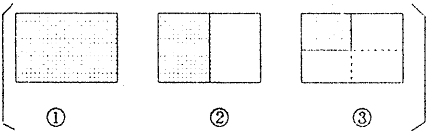

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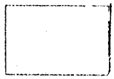

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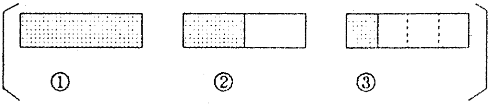

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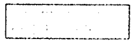

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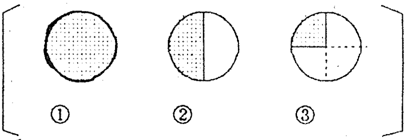

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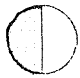

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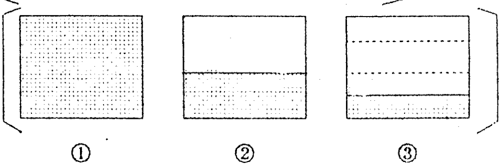

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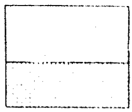

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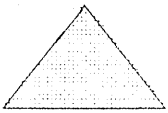
①, 1
②, 2
③, 4
④, 1/2
⑤, 1/4

of



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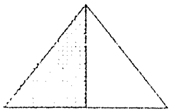

and so on.

P / T	Condu.	Non Co	Row.M
ade.	5.59	1.40	3.60
inade.	3.86	0.67	2.30
Col.M	4.73	1.04	

1st grader

P / T	Condu.	Non Co	Row.M
ade.	7.84	6.88	7.40
inade.	6.11	5.44	5.78
Col.M	6.98	6.16	

3rd grader

P / T	Condu.	Non Co	Row.M
ade.	7.41	6.28	6.83
inade.	5.88	2.47	4.18
Col.M	6.65	4.38	

2nd grader

P / T	Condu.	Non Co	Row.M
ade.	9.88	9.00	9.44
inade.	8.06	8.53	8.32
Col.M	8.97	8.74	

4th grader

Table, The Results of Experiment.

VISUALISATION AND OPERATIVITY IN PROPORTIONAL REASONING

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The present contribution tries to apply some notions about the process-structure of learning activity and human activity in general to the teaching of proportional reasoning. An empirical study following these theoretical orientations was conducted in order to produce results that may give rise to possible modifications, elaborations and confirmations of the underlying concepts.

The main theme of the contribution is the role of visualisations, graphic displays, and diagrams in the teaching (and didactics) of proportional reasoning. Empirical work that has been done so far by Karplus and co-workers (see for instance Karplus & Karplus 1972; Karplus, Pulos & Stage 1981, 1983a, 1983b), by Hart (1981), Suarez (1977), and Noelting (1980) has produced beneath other important results growing interest in the question how visualisation of mathematical facts influences the solution of proportion tasks. There is no simple or clear cut answer to the question, whether visualisations promote the acquisition of mathematical knowledge: sometimes they do, sometimes they block or hamper the acquisition process (cf. Otte 1983, Jahnke 1984). The empirical study of school reality and the problem solving or task-oriented behavior of students does obviously not by itself structure the knowledge continuum in this domain. Therefore a theoretical structure appears to be necessary that gives a description of the interaction between tools and media of human (learning) activity and its process quality.

I will start with an outline of the situation that can be found when a learning system interacts in certain ways with a variety of other aggregated systems, that can be called "learning environment" or "learning ecology". Between these two system aggregates there exist different forms of interrelations and exchange: fusion, symbiosis, communication, and interaction according to the degree of autonomy that the respective systems maintain (cf. Jantsch 1979 following Maturana).

Looking at the tools and media we find them organizing the exchange and communication between the learning system and its environment. They are powered by the engine of living activity. The processes running between the learning system and its environment can be described from the point of view of interiorisation vs. exteriorisation. The psychological significance of these concepts has been elucidated by Wygotski. Piaget introduced the concepts of assimilation vs. accomodation for the description of the process-structure of activity. Analyzing the role of tools and media for the interiorisation vs. exteriorisation of structures between communicating systems we find a reversal of direction of process-structures, if compared with the relation of assimilation and accomodation analyzed by Piaget.

Exteriorisation as a mediated relation between structures can be said to be structurally determined or algorithmic: the learning system exports a sequence of processes fixed by the tool or medium to the outside of its system boundaries. The machine is the incorporation of cristallyzed internal processes driven to a maximum. It can be said that rigidity and reproducibility are the outstanding characteristics of the exteriorisation of structures by the learning system.

Interiorisation as a mediated process is more oriented towards the surprise or news value of an imported structure. The tools and media of interiorisation organize the process quality of the exchange between the systems. Flexibility and creativity are the outstanding characteristics of interiorisation, that can best be expressed in the ambiguity of pictures and visual symbols.

Interiorisation and exteriorisation are complementary processes. In his theory of pragmatic communication has E. von Weizsäcker analyzed the interactions between systems from the point of view of initiality vs. confirmation. Jantsch (1979) has applied theses concepts to a general system theory of evolution. Using their ideas it can be said, that interiorisation continuously transforms initiality into confirmation, whereas exteriorisation is dominated by confirmation. The interaction between the systems comes close to a balance, if confirmation or exteriorisation prevails. The interaction is far from balance, if initiality is transformed into confirmation during interiorisation.

The application of these concepts to the situation of students in today's school gives the following picture. The system can only fulfill its central function, namely learning, if it is organizing the transformation of initiality into confirmation by itself securing for the future the occurence of initiality, a state far from balance giving rise to learning. Learning in schools, teacher-student-interaction and the use of visualisations, however, always aim at maximal confirmation and a balanced relation between learning system and its environment. Teachers very often find those teaching efforts especially excellent that have succeeded to teach something new to the students without them becoming aware of the fact that it was something new. This reduction of new knowledge to old one is a common feature of school mathematics (cf. Brousseau 1983), whereas the system can only learn, if existing knowledge is projected onto a "zone of proximal development" (Wygotski) that is based on the explanation of existing knowledge with the help of new one.

Coming back to visualisation we can give a tentative summary: according to whether the analysis predominantly adheres to the exteriorisation or interiorisation of structures visualisations are viewed either as "machines" or as "symbols".

The notion of the process-structure of learning activity is different depending on an interiorisation or exteriorisation point of view. Two models can be distinguished:

- the emphasis on exteriorisation leads to a model of learning activity that gives a hierachic-sequential picture of the organization of action, where lower levels of action are completely controlled by higher levels. The idea of an algorithmic nature of activity processes prevails and the concept of "the one best way" elaborated by Taylor for the organization of industrial work lies very close to it. The models of teaching that are based on a hierachic-sequential model of the process-structure focus on the solution of tasks and on task analysis (cf. Resnick & Ford 1981). The reproductive characteristics of process-structures are heavily stressed.

- The emphasis on interiorisation leads to a model of heterarchical organization of the process-structure of activity, that is not directed towards the vertical control executed by higher levels on lower ones. Instead it stresses the self-organisation of interacting living systems having a relationship of co-evolution. The models of teaching that reflect this view aim at meaning and understanding of a mathematical fact, that result if a variety of interrelations can be established between this fact and intended applications. In this view the productive and creative characteristics are heavily stressed.

The following table tries to summarize the statements presented so far:

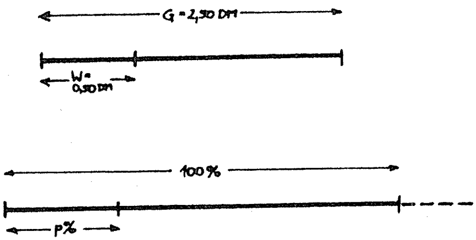
	interiorisation	exteriorisation
process-structure of activity	heterarchy of living systems	hierarchic-sequential control
models of teaching	meaning and understanding: productive/ creative	solution of tasks: reproductive
evaluation	multiple presentation	right/wrong
visualisation as tool and medium	"symbol"	"machine"

The empirical study reported here used percent word problems from lower secondary level (class 7 and 8). Visualisations of the "symbol"- and "machine"-type were selected from a number of different visualisations. In two different test-versions these visualisations were used as introductory tasks to a test consisting of six percent word problems. A third test-version (control) received no introductory task. The two test-versions are described below.

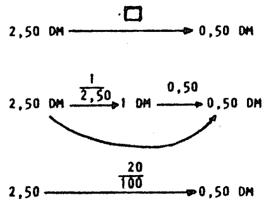
The introductory task read as follows:

"Thomas holds a weekly allowance of DM 2.50. Regularly he saves DM 0.50. His friend Peter tells him: 'I save 25% of my allowance'. Who saves the greater amount of money in percent?"

Test-version G received the following visualisation (G = base, W = percentage, and p = rate):



Test-version O received the following visualisation:



The two test-versions G and O represent two different approaches to the use of visualisations in the teaching of mathematics. Approach G considers them more as "symbols" and emphasizes the geomentrical nature of visualisations, whereas approach O bases the effect of visualisation on the concept of the "operator". Unequivocally directed arrows are established between magnitudes using operator-arrows, thus trying to build up a sequence of operative steps. The operator-approach and with it the operator-diagram is the conception that has received the widest dissemination in the teaching of ratio and proportion in the secondary school in West- Germany. The test was administered to 600 students of classes 8 and 9 from different types of schools. The students were 13 to 17 years of age. It was recorded which textbook was used by the teachers when they introduced percent. In the mathematics curriculum percent is usually at first taught in class 7 and repeated in class 8 in a more shortened version as "application of percent". The classes 8 and 9 were choosen to give to the test problems a definite reproductive character.

Empirical findings. The table below shows the total results for the six test problems divided into "No answer", "Wrong", "Right". It may give a first impression how task-difficulty was distributed.

	Task 1	2	3	4	5	6
no answer	2,7%	8,6%	18,3%	9,7%	48,6%	28,7%
wrong	10,8%	10,5%	43,7%	18,5%	19,6%	37,0%
right	86,5%	80,9%	38,0%	71,8%	31,8%	34,3%

Task 1: find percentage; Task 2: find rate; Task 3: find 100% - rate; Task 4: find base; Task 5: find base plus percentage; Task 6: find base minus percentage

There is an obvious difference in difficulty between tasks 1,2,and 4 on the one side and tasks 3,5, and 6 on the other. With tasks 3,5, and 6 the greater difficulty results from the fact that they require more than a simple routine. There is an additional operation required that analyzes the relation between written context of the word problem and the procedures at hand. The question is, whether differences can be found that can be seen as a result of visualisations. Looking at the effects of the different test-versions C, G, and O no significant differences can be found that are exclusively due to the effects of the introductory tasks. It would have been a surprise if you compare the rather modest effect of the introductory tasks with the powerful variables "textbook", "type of school", and "age". Especially in interaction with "textbook", however, the introductory tasks did have a significant effect with certain task strategies.

Ten different task-strategies could be identified, characterized by a different use of visualisation. Basically this were four different approaches: (1) simple calculation, (2) operator diagrams, (3) percent equa tion/formula, (4) rule of three. Simple calculation prevailed, very often with the help of hand held calculators. Five different textbooks could be identified: each of them used operator-diagrams, but they differed in the way they did that. Textbooks TA and TE introduced percent with the help of the operator diagram, whereas textbooks TB, TC, and TD put it at the end of an introduction with various visualisations and approaches. The table below shows the differences between textbooks cross-tabulated with the task-strategies for the tasks 1 to 6.

	Tasks 1	2	3	4	5	6
calculation	n.s.	v.s.	v.s.	n.s.	v.s.	v.s.
operator diagram	n.s.	n.s.	v.s.	n.s.	n.s.	n.s.
equation	n.s.	n.s.	s.	n.s.	n.s.	v.s.
rule of three	n.s.	n.s.	n.s.	n.s.	s.	n.s.

n.s.: no significant difference
s.: difference at 5%
v.s.: difference at 1%

Selecting the task strategy "calculation" for a closer analysis and cross-tabulating test-versions C, G, and O iwth the different textbooks (for tasks 2,3,5, and 6) we get the table shown below (percent of "right" answers).

	Task 2			Task 3			Task 5			Task 6		
Textbook	C	G	O	C	G	O	C	G	O	C	G	O
TA	74%	87%	83%	19%	20%	33%	25%	38%	17%	20%	25%	14%
TB	90%	90%	94%	42%	58%	53%	77%	55%	83%	13%	36%	60%
TC	86%	78%	71%	86%	45%	57%	43%	29%	50%	17%	12%	66%
TD	95%	93%	69%	47%	30%	47%	89%	57%	80%	80%	58%	70%
TE	48%	58%	62%	9%	11%	9%	14%	36%	31%	25%	23%	28%

The results in the above table show that students taught according to textbooks TB, TC, and TD generally produce more "right" answers than students taught according to TA and TE - the latter being the textbooks that favored introduction of percent with the help of the operator-diagram. The table shows that for the TA and TE - students the introductory tasks have produced better results in the test-groups when compared to the control-group. The data suggest that visualisations may in the teaching of percent (and proportion) play such an important role that further theoretical and empirical analysis seems appropriate and desirable. The operator diagram so vastly used in the textbooks is possibly not so adequate as has been supposed. The reason may be that the visualisation in the operator diagram is heavily influenced by the machine - metaphor.

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I. RATIONAL NUMBER

LEARNING FRACTIONS IN DISCRETE AND CONTINUOUS QUANTITY CONTEXTS

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Previous research (Hunting, 1983b) has demonstrated the potential of children for solving fraction problems cast in discrete quantity contexts using intellectual processes for defining and relating units based upon sharing and partitioning strategies. It was concluded that schemes children possess for dealing with units and unit relationships far exceed the mental mechanisms needed for solving fraction problems if fractions are conceived of as flexible, contextually determined units and their multiples. Much school instruction about fractions has traditionally relied upon experiences with continuous quantities and their subdivisions. It is possible that children may have additional strategies for constructing and defining units based on processes of estimation and measurement which may be applied in continuous quantity contexts. But which, if any, such estimation and measurement processes can be used to solve fraction problems involving discrete quantities? It was conjectured that a knowledge of fractions based on discrete quantities would constitute a superior basis for solving fractional number problems compared to a knowledge of fractions based on continuous quantities.

To investigate this question six children were interviewed using a comprehensive set of problems involving fraction representations, comparisons, equivalence, and operations, at the conclusion of their participation in a teaching experiment (Hunting & Korbosky, 1984). Three children participated in 14 instructional meetings of approximately 30 minutes each, in which activities, problems, and games emphasised discrete quantity contexts using popsticks. These children were identified from a pool of 26 children as having relatively weak fraction knowledge and relatively strong strategies for defining, constructing, and relating units in discrete quantity non-fraction contexts. The other three children received 14 lessons in which activities, problems, and games were based on continuous quantities such as paper strips and regions. These children also had been identified as having weak fraction knowledge and had exhibited diverse strategies for dealing with units and unit relationships in continuous quantity, non-fraction contexts.

At the conclusion of the teaching sessions an interview was given to all six children individually to assess dimensions of fraction knowledge established previously. Problems were included which were similar to those presented in the initial fraction interview. Other problems were adapted from tasks, games, or activities developed for the teaching episodes thought to highlight the potency of a child's control over some aspect of fractions. Still other problems, such as the operations problems, were included to test the generality and transferability of the child's knowledge in a novel situation.

The interview problems were cast in both discrete and continuous contexts, or allowed solutions to be developed using either discrete or continuous quantity materials. Each of the six children was given a standard set of problems. However some children attempted more problems than other children, due to differences in competence.

METHOD

An adaptation of the clinical method (Hunting 1983a) was used whose features included (a) careful trialling and scripting of interview plans to anticipate as many contingencies as possible; (b) structuring of interview plans so that every opportunity was given for success on each problem given, including problems of graded difficulty within most settings; (c) choice of problem settings that would arouse the child's interest and provide motivation for solving a problem (where appropriate, physical materials were made available); and (d) provision of appropriate assistance by the interviewer if the child appeared to encounter difficulty with a problem.

Each interview was recorded on videotape. Preliminary analyses were carried out for each child across all problems attempted. Data recorded included the outcome of each problem given, together with notes of the strategy used to reach that outcome. Strategies used were either directly observed as the child interacted with physical elements of the problem, or self reports elicited by the interviewer from a child during or after the completion of a problem. Such reports were useful in cases when there was no overt activity by a child while attempting a solution, or when observed actions were ambiguous. Analyses of solution processes were then carried out for all children problem by problem.

Three classes of problems were presented to each child following the teaching episodes: representation problems, problems of order and equivalence, and operations problems. Data is presented here for Tasks 1 and 2, and Tasks 5 and 6 of the representation problems only. In Tasks 1 and 2 a number of swapcards are placed on the table. The child is asked to imagine having collected a fraction of the given set of cards, and to find the number of cards representing that fraction. In Task 1 the child is asked to find one-third of 12 swapcards; in Task 2 three-fifths of 10 swapcards. Supplementary problems are available if necessary. For example, in Task 1, if the child cannot find one-third of 12 cards, the interviewer repeats the problem using first three cards, then six cards. If the child succeeds with these simpler problems, the original question is repeated.

Tasks 5 and 6 match Tasks 1 and 2, except these problems are set in the context of finding the right length of ribbon to wear to a football game. In Tasks 5 and 6 the child is asked to find one-third and three-fifths of given lengths of ribbon, respectively.

RESULTS

All children in the Discrete group satisfactorily determined four swapcards as one-third of 12 for Task 1 (see Table 1). No physical interaction with the available material was observed.

EXPERIMENTAL GROUP	SOLUTION OUTCOME	SOLUTION PROCESS
Discrete		
Donna (10;6)	4	"I went 3 fours are 12". "I was pre-
Rachel (9;10)	4	tending that I had them in 3 piles."
Ricky (10;4)	3, 4	"Its gotta be 3 groups and 4 goes into 12 3 times"
Continuous		
Kerrie (10;8) (one-third of 3 also 3)	3	"Because one-third means that you've just made um.. got 3, you collected 3..."
Neil (10;3)	4	"I worked out threes into 12 goes 4."
Sean (10;10) (one-third of 3 also 3)	3	Organised cards into 4 piles of 3

TABLE 1 : One-third of 12 swapcards (Task 1)

Solution processes reported included the use of the whole number relationship 3x4=12 and visualisation of three piles of cards. Neil was the only successful member of the Continuous group. Sean and Kerrie both said one-third of 12 cards was three. Their solution processes and responses to the follow-up problem of finding one-third of three swapcards indicate an interpretation of one-third as a set of three objects.

Noone in the Continuous group was successful with Task 2 -- finding three-fifths of 10 swapcards (see Table 2). Kerrie said that

EXPERIMENTAL GROUP	SOLUTION OUTCOME	SOLUTION PROCESS
Discrete		
Donna (10;6)	6	Organised cards into 5 piles of 2; collected 3 piles together
Rachel (9;10)	6	"Well I said, um, put them in 5 piles and there'd be 2 in each pile. Then I'd say 2 threes are six."
Ricky (10;4)	2, 6	Placed cards together in pairs, then grouped 3 pairs together.
Continuous		
Kerrie (10;8)	5	Guess. Three-fifths of 5 was 3. Three-fiths of 15 was 3.
Neil (10;3)	3	"Just a guess". Maintained 3/5 of 5 wouldn't go evenly.
Sean (10;10)	5	Organised cards with 2 groups of 5. After discussion covers 3 cards from a group of 5.

TABLE 2 : Three-fifths of 10 swapcards (Task 2)

three-fifths of five cards was three in the follow-up problem. Sean's action of grouping the 10 cards into two piles of five indicated an incorrect understanding of the meaning of fifths. In contrast, the children in the Discrete group gave correct solutions supported by appropriate processes. Two children, Donna and Ricky, chose to physically partition the swapcards into five piles of two followed by actions of grouping three piles together. Rachel, though not interacting with the material, reported a similar internal process.

Tasks 5 and 6 involved finding a fraction of a piece of ribbon to wear to a football game. No apparent pattern of behaviour was evident for either group of children in Task 5 (see Table 3). Only Donna of the Discrete group marked up the ribbon using visual unit estimates alone. All other children employed supplementary strategies for ensuring that the desired fractional unit was as accurate as possible. Rachel and Neil folded their material first before marking. Ricky requested a ruler which he used to determine equivalent segments. Kerrie noticed that her folding technique resulted in a shorter final segment, so she adjusted the position of her first fold to correct the discrepancy. Apart from Donna, there was evidence that the children knew that the goal of their endeavours was to show three congruent segments with their paper strips.

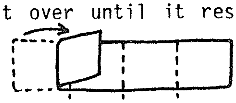
EXPERIMENTAL GROUP	SOLUTION PROCESS
Discrete	
Donna (10;6)	Marked her material using visual unit estimates without folding.
Rachel (9;10)	Folded, then marked
Ricky (10;4)	Requested a ruler. Laid ruler over ribbon and carefully marked with a pen.
Continuous	
Kerrie (10;8)	Folded over twice. Final segment was too short, so refolded after adjustment.
Neil (10;3)	Could use a ruler. Folded and marked after being asked for an alternative method.
Sean (10;10)	Began to mark, then decided to fold. After marking observed segments were uneven.

TABLE 3 : One-third of a given length of ribbon (Task 5).

Representing three-fifths of a length of ribbon in Table 6 elicited more exploratory behaviour by children from both Discrete and Continuous groups, typified by the unsuccessful application of several different strategies before attaining a satisfactory result (see Table 4). Such behaviour could be expected from children in the Discrete group, whose use of continuous quantity materials for representing fractions in the teaching episodes was negligible. But it was of interest to observe Sean and Kerrie, who had engaged in similar kinds of activities in the teaching episodes, attempt a number of different procedures. For

EXPERIMENTAL GROUP	SOLUTION PROCESS
Discrete	
Donna (10;6)	Marked. Two segments are noticeably larger than others.
Rachel (9;10)	Folded 4 segments using linear method. Found initial estimate unsatisfactory. Began again with another strip.
Ricky (10;4)	Used ruler. Attempts mental computation but concluded "doesn't go evenly." Suggested folding : 1. Folded into 4 2. New paper: halved, halved, halved. 3. New paper: Linear method which left final segment too large. 4. New paper: Rolled and flattened into 6 segments. 5. Made first segment larger.
Continuous	
Kerrie (10; 8)	1. Used finger spans to mark strip into 3 segments. 2. Halved, halved again. 3. Halved, thirded; counted 6 segments. 4. New paper: Rolled and flattened into 6 segments. 5. Made first segment larger.
Neil (10;3)	Requested ruler, which was placed alongside ribbon. Decided to fold. Marked the third fold.
Sean (10;10)	1. Attempted linear fold with initial segment estimate. 2. Made fourths using halving. Folded over one end to midway between first and second fold. Subdividedby visual estimation. 3. Made fourths, unfolded and made shorter segments. Shaded 3 segments

TABLE 4 : Three-fifths of a given length of ribbon (Task 6)

example, Sean first attempted an initial segment estimate followed by a linear fold. He discarded this strategy in favour of halving, then halving again to make four segments. After making small marks on the folds he took one end of the strip and folded it over until it rested mid-way between the first and second fold thus: 

He then unfolded the strip, and visually determined the remaining segments. After expressing dissatisfaction with this result Sean took a fresh strip of paper, made four segments by halving, then refolded the strip using shorter segments. He appeared to be satisfied with the result so obtained and proceeded to shade three of the segments.

DISCUSSION

The behaviour of the Continuous group children on the representation problems showed them to be generally less successful in their efforts to solve problems cast in discrete quantity contexts compared with the Discrete group children. It would seem that the Continuous group children were not able to readily restructure their conceptions of fractions to these contexts. Children in the Continuous group had encountered thirds and fifths extensively during the teaching episodes. In contrast, on correspondong problems with continuous settings, such as those involving the football game ribbons, there was action by all children from both groups targeted at producing accurate units. The Continuous group children displayed unstable strategies not expected for children who had been exposed to 14 teaching sessions using this material. It was not unusual for these children to discard two or even three distinctly different strategies for determining fractional units before they became satisfied with their results.

Actions used to subdivide continuous quantities are different to those used to subdivide discrete quantities. The goals may be identical but the means are not. Children's conceptions of fraction may therefore be quite different if restricted to either contextual material as a consequence. Children in the Discrete group were observed to select discrete material to solve problems cast in continuous quantity contexts (for example, Ricky used a ruler several times). Children in the Continuous group were observed to choose discrete material to solve problems cast in continuous quantity contexts. Children of both groups then, indicated a preference for discrete material to solve problems cast in either discrete or continuous contexts. It is possible that a continuous quantity meaning base for fractions, while not sufficiently durable to allow children to solve problems cast in discrete settings, may be prerequisite knowledge for being able to interpret and solve discrete quantity fraction problems.

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AN ANALYSIS OF THE DEVELOPMENT OF SEVERAL SUBCONSTRUCTS OF RATIONAL NUMBER

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This study, carried out in 1983, followed the line of investigation begun several years earlier into the development of rational number concepts in children and adolescents. The study by Kieren and Southwell (1978) resulted in the following conclusions in relation to the operator subconstruct:

i) A subject who mastered the direct fractional number task was likely to master the related inverse task. It appears that it is the nature of the operator that determines performance rather than the aspect of inverse within an operator.

ii) A pattern of growth over time is indicated in the operator subconstruct of rational number. Three major levels of development seem to occur, the " " operator level, the unit operator level and the general operator level.

iii) The difference between male and female performance on the machine mode was found to be significant at the 0.01 level.

iv) Children in the elementary school appear to be able to handle composition of two functions provided they already have mastery of the two original component parts.

v) There appears to be a delay of two years between mastery of the tasks using the two different representations, machine and pattern, the latter being a more abstract approach.

A further study by Southwell with a sample of Papua New Guinean students, in which the machine representation was again used, as well as the orange juice test (Noelting, 1978) resulted in similar conclusions in relation to inverse tasks, a general pattern of achievement by age but with a greater delay than with the Canadian sample, and sex differences.

At the same time the data analysed indicate that the stages of proportional reasoning identified by Noelting (1978) do hold with the Papua New Guinean sample with some variations. The similarity between the two samples appears to increase as the stages develop. Differentiation in these stages does not appear to be determined by grade.

The strategies adopted by Papua New Guineans are similar to those of Canadian subjects at the pre-operational and concrete operational levels. No clear statement of strategies within the formal operational level is possible.

A study (Southwell, 1982) using an Australian sample indicates that Australian students develop the operator subconstruct of rational number in three general stages with a delay of approximately one year on Canadian students. Findings regarding inverse functions and composition reflect the Canadian results.

PURPOSE OF THE STUDY

The purpose of this further study was:

a) to further investigate whether there exists a pattern of growth by grade in the five subconstructs of rational number;

b) to compare the development of mastery for each subconstruct;

c) to examine the processes used by the subjects in completing rational number tasks, and

d) to discuss the implications for the teaching of rational numbers.

SAMPLE

The sample consisted of 522 subjects in Grades 6, 8 and 10 in community (primary) and provincial high schools in Papua New Guinea. The sample distribution is given in Table 1.

TABLE 1 - SAMPLE DISTRIBUTION

GRADES

SEX	6	8	10	TOTALS
FEMALE	76	115	67	258
MALE	81	101	82	264
TOTALS	157	216	149	522

INSTRUMENTS AND PROCEDURES

Two tests were administered. The first and most extensive was the Rational Number Thinking Test developed by Kieren. The other was a much shorter test developed by the writer. In the two tests, eight subtests tested various aspects of rational number. The tests are described more fully elsewhere (Southwell, 1983).

RESULTS

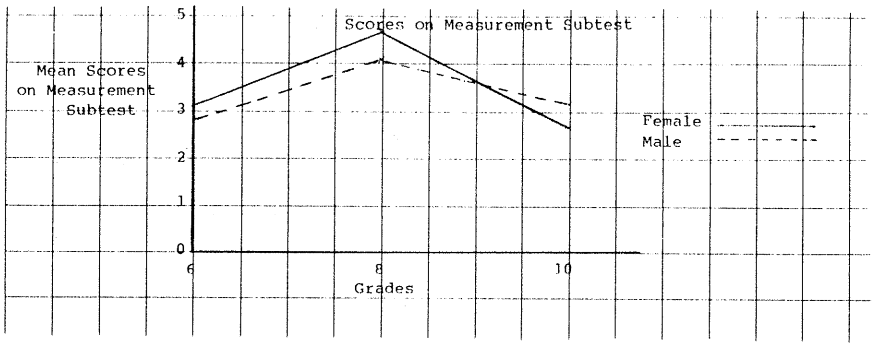
a) Previous Results

Previous results reported by Southwell (1983) indicate that there is an increase by grade in performance on the rational number tasks as a whole. There also appears to be a strong correlation between performance in the subconstructs measured by the tests administered.

b) Further Results on Achievement by Grade

Each of the subtests were analysed to see if there is an increase by grade in achievement. As reported previously (Southwell, 1983) there appears to be a gain in achievement by grade for both males and females on the Drinks, Share, Operator, Inverse and Scale subtests. On the Measurement subtest, however, neither males nor females showed gains. This phenomena is shown in Figure 1.

FIGURE 1



The other subtests on which there was not a gain in achievement by grade were the Problems, and Inequalities. These are shown in Figures 2 and 3.

FIGURE 2

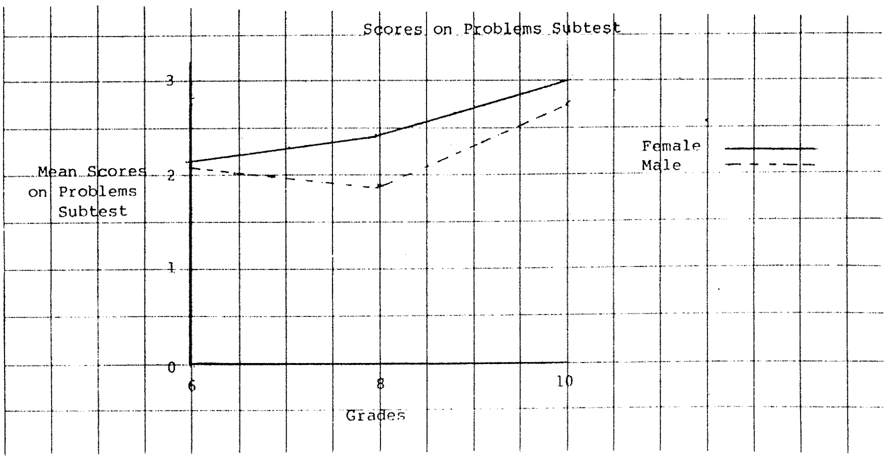
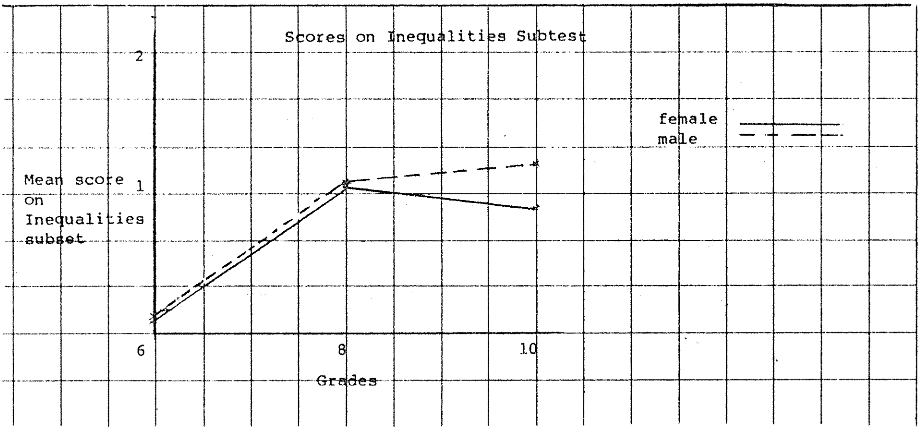
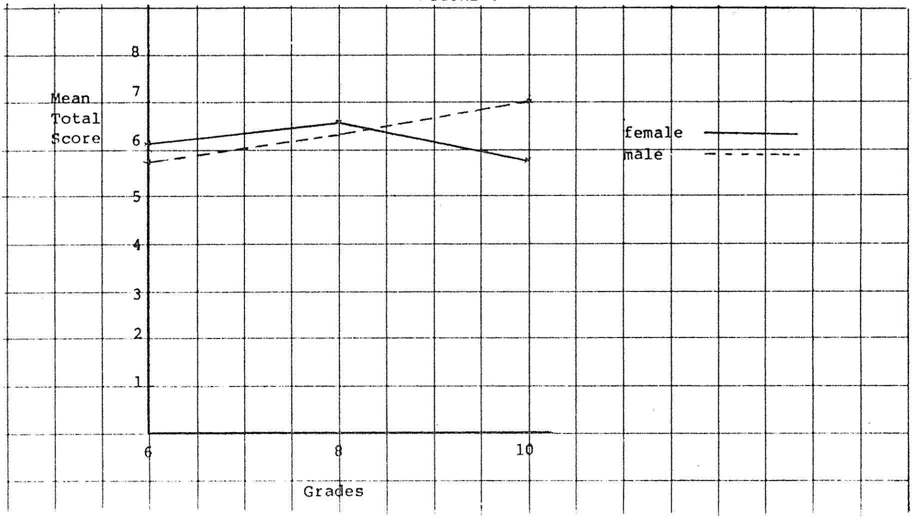


FIGURE 3



The overall pattern for total scores on all subtests is given in Figure 4. While the male subjects showed an increase by grade, the females gained between Grades 6 and 8 but then declined in Grade 10 to a level below that of the Grade 6 subjects.

FIGURE 4



The differences between performance on each of the subtests at each of the three grades were tested for statistical significance. With the exception of the following, all other differences were significant.

For females between Grades 8 and 10	: Drinks, Operator, Inverse and Inequalities
For females between Grades 6 and 8	: Problems
For females between Grades 6 and 10	: Measurement
For males between Grades 8 and 10	: Inequalities, Scale
For males between Grades 6 and 8	: Drinks, Problems
For males between Grades 6 and 10	: Measurement

c) Further Result on the Comparison of Performance on Different Modes of Representation

Correlation analyses (SPSS Correlational Analysis) were carried out to measure the strengths of the relationship between the eight subtests for the male and female subjects separately. The strongest correlations for each grade and sex are given in Table 2.

TABLE 2 - CORRELATIONS BETWEEN SUBTESTS

GRADE	SEX			
	MALE		FEMALE	
6	*Operator/Inverse	(.001)	Measurement/Inverse	(.001)
	Problems/Measurement	(.001)	*Operator/Inverse	(.006)
8	Drinks/Share	(.001)	Drinks/Share	(.001)
	Inverse/Inequalities	(.001)	Drinks/Operator	(.001)
	Problems/Measurement	(.001)	*Operator/Inverse	(.001)
	Measurement/Inequalities	(.001)	Operator/Problems	(.001)
	Share/Operator	(.001)		
	Share/Inverse	(.001)		
	*Operator/Inverse	(.001)		
10	Drinks/Share	(.001)	Share/Operator	(.001)
	*Operator/Inverse	(.001)	*Operator/Inverse	(.001)
			Measurement/Inequalities	(.001)
			Measurement/Scale	(.001)
			Inequalities/Scale	(.001)

d) Factor Analysis

A factor analysis was carried out to explore the patterns which may exist between the subtests. The method used was Principal Factor with iterations and produced five factors as shown in Table 3.

TABLE 3

VARIMAX ROTATED FACTOR MATRIX

TESTS	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4	FACTOR 5
Drinks	0.15552	0.46980	0.04717	0.01095	0.23789
Share	0.20543	0.64037	0.01204	0.11702	0.03324
Operator	0.73773	0.22761	0.02661	0.19088	0.26262
Inverse	0.75525	0.23853	0.07340	0.15184	-0.04532
Problems	0.06681	0.14950	0.17326	0.10305	0.49156
Measurement	0.03544	-0.01378	0.54782	0.11917	0.17994
Inequalities	0.07387	0.22833	0.42112	0.47225	-0.01852
Scale	0.15345	0.01954	0.08155	0.40568	0.10698

DISCUSSION

a) Achievement by Grade

On the subtests used, there does not appear to be a general or overall increase in achievement by grade there is, however, an increase in achievement from Grade 6 to Grade 8 on the part of both male and female subjects, and from Grade 8 to Grade 10 on the part of the male subjects.

The overall achievement of the female subjects is better than that of the males, as is also their achievement at Grades 6 and 8, while Grade 10 males did better than Grade 10 females. The only subtest, however, in which the female students were consistently superior to the males was the Problems subtest.

The relatively weaker performance of the Grade 10 females needs further consideration. It is a fairly recent phenomenon in Papua New Guinea for girls to be given the same access to education as boys. The societal pressure for girls to keep to the traditional roles of wife, mother and gardener may be more real to the Grade 10 students than to the younger ones.

(b) Comparison of Performance on Different Modes of Representation

The results of the factor analysis support the links between the following pairs of subtests:

Operator and Inverse
Drinks and Share
Measurement and Inequalities
Inequalities and Scale.

The Problems subtest stands relatively alone, though it does have weak links with the Drinks and Operator subtests. Compared with the five rational number subconstructs posited by Kieren (1978), those subtests relate in the following way:

FIGURE 5

SUBCONSTRUCT	SUBTEST	FACTOR
Part/Whole Division	Problems	5
Ratio	Drinks and Share	2
Operator	Operator and Inverse	1
Measurement	Measurement, Inequalities and Scale	3, 4

Further investigation of the percentage achievement of the total subjects on each of the subtests indicates the following order:

TABLE 4 - PERCENTAGE MEAN SCORES

ORDER	SUBTEST	MEAN SCORE (%)
1	Measurement	44.625
2	Share	41.75
3	Inequalities	40
4	Problems	39.17
5	Drinks	38.17
6	Operator	17.25
7	Scale	7.5
8	Inverse	6.77

Consideration of the relative performance of male and female subjects between grades indicates that improvement is good for the Share, Operator, Measurement and Inequalities subtests. The greatest increase on the Problems Subtest took place between Grades 8 and 10. It may be reasonable to assume, therefore, that students from Grade 6 are able to benefit from instruction in the subconstructs exemplified in the four subtests Share, Operator, Measurement and Inequalities, with Problems being better managed after Grade 8.

CONCLUSION

In terms of the stated purpose of this present study a development by grade of the ratio and operator subconstruct has been established. Nothing definite can be said about the part/whole and division subconstructs. Partial development only is indicated on the measurement subconstruct.

From the results stated above, it would appear that students perform better on some forms of the measurement subconstruct and on the ratio subconstruct than on the operator subconstruct.

Further investigation needs to be completed to ascertain the relative position of the part/whole and division subconstructs.

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J. GIRLS AND MATHEMATICS

GIRLS AND MATHEMATICS :

Affective variables associated with the selection of courses in senior secondary school.

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It has been well documented that in western societies females have not participated in mathematics courses to the same extent as males (Keeves 1973, Fennema & Sherman 1977). In 1967 Husen reported that in the twelve countries studied in the International Study of Achievement the ratio of males to females enrolled in mathematics at the end of secondary education ranged from 1.73 to 7.13 with an average ratio of 3.70 (p.234). The data for Australian states show that when the study of mathematics becomes optional girls are less likely to participate in these courses (Widdup 1981). Furthermore when girls do take mathematics they tend to do at a lower level (Widdup 1980; Leder 1980; Atweh 1980).

Whether high school leavers intend to seek employment or engage in further study, the trend is for them to need better and more extensive mathematics training (Atweh 1980). The Mathematical Association of Victoria found that three quarters of tertiary courses require a pass in H.S.C. (Year 12) mathematics and two-third of certificate of technology courses require a pass at Year 11 level (Costello 1978). Even in those courses that do not have stated mathematics prerequisites, students find their way barred to extended study because of heavy dependence on quantitative treatment of data (Carss 1981). Mathematics performance is also used as an entry sorting device for trade courses, apprenticeships and other employment opportunities (Carss 1981; Atweh 1980). Consequently Sells (1973) has labelled mathematics as the "critical filter" which terminates the opportunities of many females.

A common explanation for female avoidance of mathematics is that females have innately inferior mathematical abilities when compared to males. However Fennema (1978) has observed that several studies which reported female inferiority in mathematics ability and performance at secondary school were effectively comparing males who spent more time studying maths, typically 4 years, against females who typically only took 2 years of mathematics. Recent studies have shown that when the amount of previous mathematics training is controlled, sex differences disappear (Fennema & Sherman 1977; Fennema , 1978; Wise 1978). The performance data in the N.S.W. 1980 H.S.C. examination shows that females equalled and were occasionally superior to males at each level of mathematics taken (Mack 1981). Several researchers have shown that the magnitude of sex differences in mathematical ability vary with cultural (Husen 1967) school (Fennema 1977) and socioeconomic factors (Mackay and Fary 1976).

It seems evident that female avoidance of mathematics cannot be attributed to sex differences in mathematical ability. However the affective variables which are defined as those dealing with feelings, interests, attitudes and values (Kruthwohl et al 1964) may play an important role in the selection of mathematics courses. There has been an increasing amount of literature dealing with specific affective variables and their relationship to sex related differences in mathematics study (Fox et al 1979; Fennema 1977).

Confidence in mathematics is the belief that one has the ability to learn new mathematics and to perform well on mathematical tasks. Sherman & Fennema (1977) reported a strong relationship between confidence and intent to take advanced courses in secondary school. Some studies also report that males are more confident in their ability to deal with mathematics than females despite the fact that there was no sex related difference in mathematics achievement (Fennema & Sherman 1977; Anderson et al 1975).

Although it is commonly accepted that mathematics is stereotyped as an activity more appropriate for males than for females (Keeves 1973; Fox 1976) the evidence suggests that this stereotyping does not occur until the adolescent years (Stein 1971; Morgan 1979). Fennema & Sherman (1977) have reported that there are some sex differences in stereotyping with more males than females rating mathematics as a male domain. It is suggested that such a masculine view of mathematics is undoubtedly communicated to females and contributes to their avoidance of mathematics courses.

These stereotypic views of mathematics has implications for the fear of success theory regarding females' achievement motivation. Horner (1968) has hypothesized "that fear of success is more strongly aroused in competitive achievement situations and particularly when the competition was against males and when the tasks involved (were) generally considered masculine such as tasks of mathematical, logical, spatial etc. ability" (p.24). Both Leder (1980) and Sherman & Fennema (1979) have reported that females who opt out of intensive mathematical training are characterized by higher fear of success scores.

Another effective variable which helps explain why females do not elect to take mathematics is the perceived usefulness to them of mathematics (Fennema 1978). Among senior secondary school students more males than females consider mathematics useful for future study and career prospects (Morgan 1979; Fennema & Sherman 1977). There is an abundance of evidence that females are less oriented towards careers (Scanlon & Barnes 1981) restrict their career choices to a narrow range of lower status occupation which are traditionally feminine (Leder & White 1981) and envisage conflicts between working and homemaking roles (Hawley 1971; Astin 1974). There is also some evidence that girls who are interested in careers in nursing, education and the social sciences, are likely to be unaware of the relevance of mathematics and science to these professions (Fox et al 1979).

Support and encouragement from parents are crucial for girls in their decision to elect or decline extensive mathematics course in secondary school (Haven 1971; Luchins 1976). Studies by Fennema & Sherman (1977) and Fox (1975) reported that females perceived that both parents encouraged them less and valued their mathematical talent less than males. The perceived support or lack of support of teachers appears to be significantly related to student participation in mathematics courses beyond the required level (Sherman & Fennema 1977). However many teachers have different expectations for high school (Ernest 1976; Levine 1976). Frazier and Sedaker (1973) referred to teachers as "hidden carriers" of society's sex role stereotyping where they interact more with males than females especially in mathematics classes (Spender & Sarah 1980). Although peers seem to play an important role in mathematics participation, most of the evidence to support this claim has been annecdotal (Angel 1974; Widdup 1980).

Although each of the affective variables has been linked with mathematical course participation their relationship to each other has received less attention. Two recent studies by Brush (1980) and Armstrong (1980) have studied the relationship between these variables. Utilising factor analysis both studies classified these variables into three factors:

- Firstly, those variables relating to the feelings and attitudes of the student of mathematics toward the subject such as confidence, anxiety, stereotyping of mathematics, success in mathematics etc. (SELF)
- Secondly, those variables relating to the usefulness of mathematics to the student's educational and career aspirations (USEFULNESS)
- Thirdly, those variables relating tothe perceived attitudes of significant others - parents, teachers, peers, towards the learning of mathematics (SIGNIFICANT OTHER)

Armstrong (1980) has demonstrated that each of these three factors were strong predictors ofmathematics course participation. However Brush (1980) has indicated that only the self factor is a strong predictor of course participation. However both studies did not determine the unique contribution that each of the factors makes to the decision to engage in optional mathematical courses. Since these factors are logically related to each other it seems important to determine their independent contribution to mathematics course participation.

METHOD

Subjects were 339 females Year 10 students of three Sydney state schools. The age of students was between 15-16 years. All Year 10 students present on the day of testing participated in the study. The results of 5 subjects were discarded because of incomplete data.

The basic instrument of data collection was a questionnaire which included both attitude scales and additional questions. Most of the questionnaire was precoded with forced choice items but a few questions invited free responses.

Attitudes towards mathematics has been measured by the Fennema-Sherman Mathematics Attitudes (FSM) Scales (1976, 1977) which comprise nine scales with a total of 108 items. A pilot study conducted by the present author revealed that a major fault of the FSM scales was their excessive length which contributed to some loss of data. Love (1980) revised the FSM scales (RFSM) which were derived from factor analysis of FSM items. The RFSM scales consisted of six scales each containing eight items, four worded positively and four negatively, in a five point Likert-type format. A small number of items were reworded for an Australian student population.

The six attitude scales of RFSM are:

1. Attitudes toward success in mathematics (Success)
2. Perceived attitude of teacher towards self as a learner of mathematics (Teacher)
3. Confdience in learning mathematics (Confidence)
4. Perceived attitude of mother towards self as a learner of mathematics (Mother)
5. Perceived attitude of father towards self as a learner of mathematics (Father)
6. Mathematics as a male domain (Male D)

In addition to the six RFSM scales the 12 item Perceived usefulness of mathematics scale (usefulness) from the original FSM was administered. Since the FSM scales did not address the issue of peer influence directly a 16 item scale, 8 representing the girl friend scale and 8 representing the male friend scale, was developed and administered in the present study. Additional questions were also administered in order to obtain data on the subjects' educational and career aspirations, preference for mathematics compared with other courses and demographic variables. After the questionnaire was completed by subjects some additional information regarding the level of mathematics achievement and the sex of the teacher was also collected.

RESULTS

The percentage of subjects who intended to enrol in each Year 11 mathematics course and terminate their education is presented in Table 1.

Courses	Number	%
4 Unit	6	1.8
3 Unit	92	27.5
2 Unit	123	36.8
2 Unit A	36	10.8
No mathematics	12	3.6
Not continuing to Year 11	65	19.5
	334	100.0

TABLE 1: Mathematics course taking and Year 11 enrolment

The percentage of subjects not continuing to Year 11 (19.5%) were eliminated from the subsequent analyses. Consequently only 269 subjects were included in the mathematics participation variable.

The Pearson product moment correlation coefficient is used to measure the relationship between mathematics participation and several other variables. The correlation coefficient for each variable is presented in Table 2. Some categorical variables are recoded to obtain this measure of correlation.

Variables			r
Level of mathematics achievement	..		.536 **
Students' attitudes towards maths			
Difficulty	..		.557 **
Anxiety	..		-.070
Enjoyment	..		.445 **
Confidence (RFSM Scale)	..		.550 **
Maths Favourite Subject	..	#	-.412 **
Maths Least Favourite Subject	..	#	.446 **
Stereotyping Maths as a Male Domain:			
Male domain (RFSM Scale)	..		.219 **
Attitudes towards Success in Maths (RFSM Scale			.208 **
Mathematical activity:			
Hours a week on mathematics homework	..		-.139 **
Extra curricular maths activity	..	#	-.023
Usefulness of maths to future -			
Usefulness : (FSM Scale	..		.526 **
Education:			
Intend going to university	..	#	-.326 **
Occupation:			
Necessity of maths for job qualifications		#	-.361 **
Home Vs. Career :			
Intend to havechildren	..	#	-.005
Work immediately after children	..	#	-.074
Work after child school age	..	#	.084
Work full-time at 30 years	..	#	.108 *
Work part-time at 30 years	..	#	.007
Siginficant Others:			
Mother : (RFSM Scale)	..		.407 **
Father : (RFSM Scale)	..		.416 **
Teacher : (RFSM Scale)	..		.348 **
Sex	..	#	.060
Peers :			
Importance of having best friend in maths class			.142 **
Male Peers (Male friend scale)			-.027
Female Peers (Girl friend scale)			.095 *
Demographic :			
Father : Education	..		-.120 *
Occupation	..		.169 **
Mother : Education	..		-.148 **
Occupation	..		.129 **

TABLE 2 : Correlations of all variables with participation in Year 11 mathematics
* $p < .05$ ** $p < .01$ # Recoded categorical variables

Table 2 shows that the levels of Mathematics Achievement and the Demographic variables are correlated with participation. The higher the level of achievement and the higher the educational and occupational level of parents the more likely the participation in higher level mathematics courses.

Most of the variables measuring students' attitude towards mathematics have a high correlation with participation. Subjects who participate in higher levels of mathematics are significantly more likely to rate mathematics as less difficult, more enjoyable, consider it a favourite subject and are more confident in their mathematical ability. Participation at higher levels is also significantly correlated with less stereotyping of mathematics and a more positive attitude towards mathematical success.

The more useful mathematics was rated the higher the level of participation. Similarly the intention of going to university and the belief that mathematics is necessary for job qualifications are correlated with higher levels of participation. The orientation towards a career rather than a family seems to increase the likelihood of participation at higher levels but fails to reach significance.

The positive and supportive attitude of both parents and teachers seem to highly correlate with participation at higher levels. Whereas mathematics participation is significantly correlated with positive attitude of female peers the more negative attitudes of male peers seems to encourage higher participation but not to a significant level.

Since all the variables in the present study are correlated both with mathematics participation and with each other multiple regression is used to evaluate the contribution of a particular variable when the influence of other variables is controlled. The dependent variable in the regression analyses is participation in mathematics. The background factor which comprised the level of achievement in mathematics and parental occupational and educational variables contributed to about 30% of the variation in mathematics participation. To control for the effect of these variables the background factor was entered first into the regression equation. The regression analysis of the variables in each of the categories of Self, Usefulness and Significant Other are presented in the following tables.

Table 3 shows that whereas confidence in one's mathematical ability contributes a significant amount of variation in participation when the other self variables are controlled, the variables including difficulty, enjoyment, anxiety, stereotyping maths as a male domain and mathematical activity variables fail to contribute any significant variation in participation. Attitudes towards mathematical success and the preference for mathematics do contribute a significant amount of variation in participation when controlling background factors and mathematical activity.

	R^2	R^2 Change	F
Background Factor	.308	.308	
Time spent on maths homework	.323	.014	.24
Extra curricular maths	.323	.000	.08
Maths least favourite subject	.421	.098	1.52
Maths favourite subject	.466	.044	3.43 **
Success (RFSM Scale)	.484	.017	3.07 **
Male Domain (RFSM Scale)	.492	.001	.05
Difficulty	.500	.008	.11
Enjoyment	.505	.005	1.23
Anxiety	.507	.001	1.19
Confidence (RFSM Scale)	.529	.021	11.68 **

TABLE 3 : Self Variables controlling for Background Factor

** $p < .01$

The regression analysis presented in Table 4 shows that the Usefulness scale and the intention to go to university accounts for a significant amount of variation in participation when the other usefulness variables are controlled.

	R^2	R^2 Change	F
Background Factor	.308	.308	
Intend to have children	.309	.000	1.12
Work immediately after child	.309	.000	.03
Work after child school age	.309	.000	.94
Work full-time at 30	.320	.006	2.84 **
Work part-time at 30	.320	.000	.86
Maths necessary job qualification	.397	.073	.79
Intend to do university	.421	.024	7.51 **
Usefulness (FSM Scale)	.518	.096	51.40 **

TABLE 4 : Usefulness variables controlling for Background Factor

** $p < .01$

However most variables which measure an orientation to home or career do not significantly contribute to variation on participation when other usefulness and background factors are controlled. One exception is the involvement in full-time employment at 30 which does account for some variation in participation.

The regression analysis of the variables presented in Table 5 shows that attitudes of parents and teachers account for significant amounts of variation in participation when the other variables are controlled. The sex of teacher also contributes to some variation in mathematics participation when the peer variables are controlled.

	R^2	R^2 Change	F
Background Factor	.308	.308	
Importance of having best friend in maths class	.312	.004	3.44 **
Male friend Scale	.314	.001	2.19 *
Female Friend Scale	.320	.006	.02
Sex of Teacher	.337	.017	12.49 **
Teacher (RFSM Scale)	.396	.059	10.81 **
Mother (RFSM Scale)	.444	.047	3.99 **
Father (RFSM Scale)	.467	.022	11.00 **

TABLE 5 : Significant Other Variables controlling for Background Factor
* $p < .05$ ** $p < .01$

Although the attitudes of both male and female peers do account for some variation in participation only the male peer scale reached significance when importance for having a best friend in maths class is controlled.

Despite their high correlation to mathematics participation some variables did not reach a significant level as predictors when other variables are controlled. These variables included difficulty, enjoyment, a dislike of maths and a necessity of mathematics for job qualifications. Similarly variables which obtained significant correlation with matheamtics participation such as the Male Domain scale, the hours spent on maths homework and the female peer scale, do not reach a significant level as predictors when other variables are controlled.

It is apparent that only some of the affective variables within each category account for most of the variation in mathematics participation. Consequently subsets of these variables were chosen for the affective factors. The relationship between affective factors and mathematics participation when the background factor is controlled is presented in Table 6.

	R^2 Total	R^2 Change	F
Self	.503	.194	66.90 **
Significant Other	.451	.143	36.01**
Usefulness	.505	.196	67.48 **

TABLE 6 : Affective Factors controlling for Background Factor
** $p < .01$

The table shows that both the self and usefulness factors each account for approximately 19% of the variance in participation and the significant other factor accounts for 14% of the variation.

However since these factors are undoubtedly correlated their contribution to variance in participation when all other factors are controlled are presented in Table 7.

	R^2 Total	R^2 Change	F
Self	.600	.073	38.80 **
Significant Other	.600	.034	38.80 **
Usefulness	.600	.036	38.80 **

TABLE 7 : Affective Factors controlling for All other Factors
** $p < .01$

The results show that the self factor uniquely contributes 7.3% to the variance in participation. The unique contribution of both the significant other and usefulness factors are approximately 3%.

These results have important ramifications for intervention programs designed to increase female participation in mathematics. Firstly, the strong predictive capabilities of the affective factors suggest that programs designed only to improve mathematical achievement will only be partially successful. Secondly, the interdependence of these factors suggest that programs aimed at one affective variable such as overcoming anxiety will only have partial success. Consequently the most effective programs will have components to not only change the attitudes of the learner of mathematics but the attitudes of significant others and to increase the awareness of the usefulness of mathematics. However limited resources often dictate that the most comprehensive programs are never implemented. The results of the present study indicate that if a choice must be made between the factors, the most effective programs will be those aimed at the self factor. But it must be emphasised that without the support of significant others and the belief that mathematics is useful, intervention programs directed merely at changing the attitude of the student will only have limited success in increasing participation.

Although stereotyping mathematics as a male domain and the home Vs. career variables failed to significantly predict mathematics participation it is possible that their effect was to act as a mediator for other affective variables such as confidence, usefulness and perceived parental attitude.

Consequently intervention strategies to reduce stereotyping and increase the career options of women should be directed not only to mathematics students but to their teachers, parents and peers. However it is also apparent that there needs to be systematic evaluation of such intervention strategies to assess effects on both students and significant others.

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SEX DIFFERENCES IN MATHEMATICS PERFORMANCE BY COGNITIVE LEVEL

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ABSTRACT

The purpose of this study was to look more closely at the possibility of sex-differences in cognitive functioning affecting the respective mathematics performances of males and females. A large-scale study of students within cognitive levels was made possible by the development of a group test of cognitive reasoning in mathematics (A.C.E.R., 1977). Using this test, subjects, both male and female, were categorised into various levels of cognitive functioning. A test of general mathematics performance was then used to determine sex-differences which occur within any of these levels.

Analyses of the data revealed that although there was no significant interaction between cognitive level and sex, there was a uniform difference in favour of boys across the four cognitive levels. This prompted a closer investigation of possible causes relating to this sex difference. The middle concrete operations stage was selected as a section of the population to be thus investigated, being the stage at which the difference between mean scores of females and males was largest. Analysis of the achievement of 50 females and 50 males at this level revealed a number of topics in which males tended to do better than females. Individual interviews which addressed the reasons for these differences showed that the greater difference was in attitude - the females displaying less motivation to persevere with a problem.

INTRODUCTION

When comparisons have been made between the sexes, it has been found that proportionately fewer females than males succeed in mathematics (Fennema, 1974, 1974a; Fennema and Sherman, 1977a; Burton, 1978). If it is accepted that a mathematics education is a valuable asset for both effective living and to give flexibility to one's vocational opportunities, then it is important that the potential of everyone be maximised. The evidence suggests that only a few are achieving this fulfilment and that of those who do only a small proportion are women. Thus many people, especially women, deprive themselves, or are deprived of, careers involving mathematics as well as the developments in logical reasoning and aesthetic awareness to be derived from its study.

A number of factors have been suggested to account for this apparent trend for girls to 'drop out' of mathematics study. Some writers suggest that this is derived primarily from traditional environmental and cultural expectations - stereotypes enforced by society, influenced by home, school and the wider community - the 'person-in-his-total-life-situation' (Smedslund, 1977, p.5). Others suggest that it may be due, at least partly, to sex-differences in personality traits (Aiken, 1972; Gardner, 1974; Currie, 1979). Others indicate that sex differences in various abilities may be contributed to by differences in physiological factors (Guilford, 1967; Fennema, 1974a; Poteat, 1977; Widdup, 1980). However, little consideration has been given to the possibility that girls may be more inclined to fail to achieve because of sex differences in cognitive functioning rather than in social stereotyping, personality traits or other psychological factors.

There is research which indicates that cognitive developmental factors are crucial in mathematics achievement (Collis, 1976a, 1976c, 1978; Halford, 1978) and some research which suggests that boys have the advantage in certain developmental factors related to mathematics (Goldschmid, 1967). Moreover, a number of studies (Bell et al., 1974; Burton, 1977; Case, 1978a, 1978b; Tamburrini, 1975) suggest that lack of cognitive readiness for a topic at the time of presentation is one of the major contributors to the failure to maximise potential of mathemtics performance, and to the overall tendency for students to avoid later study in the area. The purpose of the study reported here was to look more closely at the possibility of sex-differences in cognitive functioning affecting the respective mathematics performances of males and females.

THE MAIN STUDY

Overview of Research Design:

The aim of the study was to compare the mathematics performances, as measured by a specific achievement test, of males and females in the four sub-stages of cognitive development commonly found in the primary and secondary schools. All students in a secondary school and its three feeder schools, and a selection of Higher School Certificate mathematics and non-mathematics students were tested. The schools were selected because of their situation in a relatively homogeneous socio-economic area, this being predominantly middle-class.

Two testing instruments were employed to investigate the effects of the independent variables, cognitive level and sex, upon the dependent variable, mathematics achievement.

To indicate the stages of cognitive development of the large number of subjects in the study, an instrument was required which could test the population on a large-scale basis. The Mathematics Profile Series Operations Test (A.C.E.R., 1977) was selected as it was a group test which used mathematical items to define stages of cognitive development within the required range namely, early concrete operations to formal operations.

To test the mathematical achievement of subjects of mixed abilities, ages and grades, it was necessary to use a testing instrument which would discriminate between students at all levels, as well as testing mathematical ability rather than dependence on specific knowledge. A number of instruments was considered and the Sequential Tests of Educational Progress (STEP) series selected as the set which could best be adapted to fulfil the needs of the project.

Sample Characteristics:

Because socio-economic background has been established as a variable affecting mathematics performance (Keeves, 1975; Lansdown, 1978) the population was limited to one high school and its feeder primary schools in a homogeneously middle class socio-economic area. The Higher School Certificate classes were drawn from a wider area, but still within a predominantly middle class background. Altogether 1071 students (566 males and 505 females) were tested for categorization into the following cognitive levels: early concrete (circa 8 to 10 years), middle concrete (circa 11 to 12 years), concrete generalisation (circa 13 to 15 years) and formal (16+ years). Of these 543 subjects were dropped from the sample because their scores indicated that they were likely to be in transition between two of the stages. The final sample consisted of 528 subjects (262 males and 266 females).

The Tests:

ACER Mathematics Profile Series, Operations Test. This test is based on a set of items used in research by Collis (1972, 1975) to define, by denotation, different levels of cognitive functioning. The scoring scale, developed using the Rasch model (Wright, 1968), is marked in equal units called 'brytes'. The relationship, used in this study, between bryte scores and Piagetian stages of cognitive development is shown in Table 1 along with examples of the item types used at each stage level.

Table 1: Bryte Scores, Cognitive Stages and Item Types

Cognitive Stage	Brytes	Item Types (Find Δ)
Early Concrete Operational	30 to 40 Brytes	$3+4 = 4+\Delta$
Middle Concrete Operational	45 to 50 Brytes	$4283+517-517 = \Delta$
Concrete Generalisation	55 to 60 Brytes	$7-4 = \Delta-7$
Formal Operational	65 to 70 Brytes	$a \div b = \Delta \div a$

STEP Mathematics Test (as adapted). Four levels of the STEP Mathematics Test were available. They ranged from one suitable for mid-primary school through to one suitable for the top grade in high school. Sixty items were selected across the levels by a stratified sampling technique which took into account both the number of students in the various school grades and the content coverage required to test achievement over the range of grades involved. The items and coverage were checked for validity against both class teacher views and the State Education Department guidelines. A pre-test was conducted with a sample of 100 students drawn from grades four, five, six and ten; these students were not included in the population for the major study. From the results of this testing, items were rated according to difficulty indexes and estimates of validity and reliability calculated. As the test was satisfactory on these technical measures the 60 multiple choice items, each with three distractors, were arranged in booklets for administration in the schools.

Administration of Tests:

The test battery was administered to the students in the schools they attended by the regular mathematics teacher of each class who had been thoroughly briefed by the investigator and provided with a set of standard instructions to be followed in the testing sessions. The ACER Operations Test was given first (time allowed: 40 minutes including instructions) and the adapted STEP Mathematics Test as soon after as possible (time allowed: 50 minutes including instructions). The tests were scored and the results recorded by the investigator.

Data Analyses:

The mean achievement scores and standard deviations of all subjects at each cognitive level were calculated and are presented in Table 2. The results in this table show that the mean scores

Table 2: Means and standard deviations of of achievement scores (sex-by-cognitive level)

	Male Ss			Female Ss			Total		
Cognitive Level	Mean	SD	N	Mean	SD	N	Mean	SD	N
Early Concrete Operational	9.51	5.46	37	7.73	4.36	37	8.62	4.94	74
Middle Concrete Operational	23.58	7.53	79	19.88	7.37	76	21.73	7.45	155
Concrete Generalis-ation	34.54	7.21	91	33.97	7.02	94	34.25	7.12	185
Formal Operational	46.13	9.61	55	43.63	8.09	59	44.88	8.86	114

are in the predicted order - formal operational reasoners having the highest total and the early concrete operational the lowest.

It was decided to use multiple regression with interaction (Ahlgreen & Walberg, 1975) to analyse the sex x cognitive level hypotheses. The hypotheses were three in number:

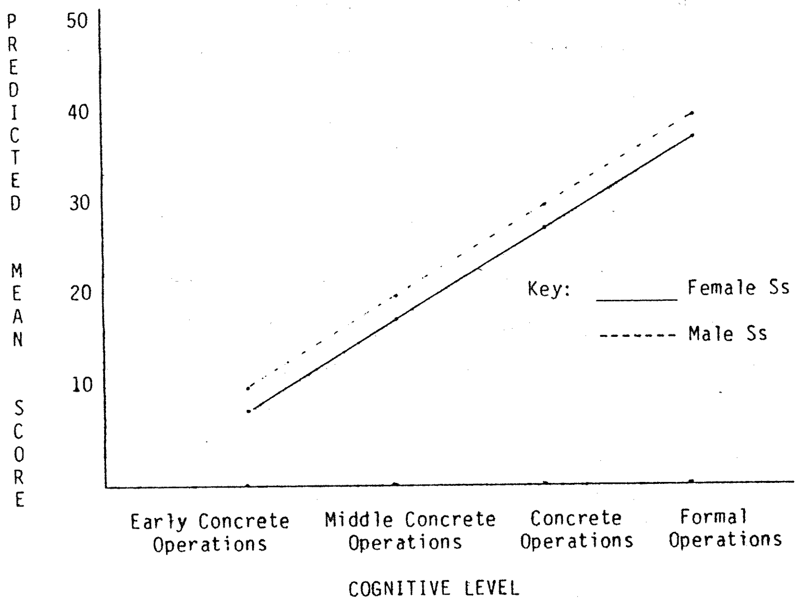
(1) no significant interaction between cognitive level and sex in their effect on mathematics achievement;

(2) no significant effect of cognitive level on mathematics achievement;

(3) no significant difference in mathematics achievement between male and female students within each stage of cognitive development.

The results showed that hypothesis 1 must be accepted (F = 0.17) but that hypotheses 2 and 3 may be rejected ($p < 0.01$ in each case). An examination of predicted mean achievement scores for females and males at different cognitive levels, Figure 1, confirmed a uniform sex difference in favour of males across the four cognitive levels.

Figure 1: Predicted means for female and male Ss at four cognitive levels (modified STEP achievement test)



Discussion:

From this analysis of the data three factors become apparent:

STEP mathematics achievement scores increase with cognitive level; cognitive level does not explain sex differences in scores on the achievement test; there is a significant sex difference in achievement scores in favour of males across cognitive levels.

Although no significant sex x cognitive level interaction occurred the study confirmed previous work which showed an overall sex difference in favour of males (Husen, 1967). It was decided to pursue these differences in more detail by investigating further the nature of the differences between the males and females at the middle-concrete level. This level was chosen because it is the one where the greatest differences occurred in this study (see Table 2) and the one which work by one of the authors (Collis, 1976; Biggs & Collis, 1982) suggests is critical in the child's development in the area of cognitive functioning.

MINOR STUDY *

Investigation of Sex Differences in Middle Concrete Stage

Two procedures were adopted in this part of the study:

(1) selection of a random sample of the achievement tests of 100 students (50 males, 50 females) in the middle concrete operations stage for detailed examination of their responses to items on the test;

(2) selection of a sample of 12 children at the middle concrete operations stage (6 males and 6 females) from one school to be

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* Because of limitations on space only a basic outline of the procedures involved in this part of the study will be presented here. Full details may be obtained by consulting the original thesis, Taplin, M.L. 'An Investigation of Sex Differences in Mathematics Performance at Four Stages of Cognitive Development', obtainable via University of Tasmania Library Service.

interviewed individually by one of the authors with the aim of getting more detailed information on the approaches adopted for solving the problems.

Sample of 100:

The first step taken in this analysis was to ascertain the items which had a greater number of one sex correct than the other. Out of the 60 items in the test, 54 favoured one sex or the other - 41 favoured males and 13 favoured females. However, of these 54 only 8 had a difference large enough to indicate that it was unlikely to have been caused by chance and all of these differences favoured the males. These 8 questions formed the basis for the interview aspect of the study discussed later. In the meantime all 54 items, in which a difference was discerned, were examined to see what possibilities existed in the items for sex related differences.

The 54 items were classified by a panel of 4 experienced mathematics teachers in the following ways and the achievement of both sexes noted:

(i) the sex-biases of the questions, i.e. whether the item dealt with matters of predominantly female or male interest or set the scene with male or female participants;

(ii) the type of answer required for each question (verbal, numerical or symbolic);

(iii) the type of skill strategy required (i.e. problem-solving strategies as opposed to strategies taught at school or related to specific experiences);

(iv) the six categories (Number and Operations, Measurement and Geometry, Probability and Statistics, Function and Relation, Proof-Deductive and Inferential Reasoning and Symbolism) into which the modified STEP achievement test items were classified;

(v) the specific operations and elements appropriate to the modified STEP achievement test.

Differences between the mean scores of the sexes were tested for significance by using a one-tailed t-test in each case. A summary of the results is given below:

(i) Sex Bias of Question: males' results > females' results (.01 level), regardless of whether item dealt with male, female or neutral elements;

(ii) Type of Answer Required:
verbal: non-significant difference;
numerical: males' results > females' results (.01 level);
symbolic: non-significant difference;

(iii) Skill Strategy Required:
non-school taught problem-solving strategies: males' results > females' results (.01 level);

school taught strategies: males' results > females' results (.01 level);

specific non-school experience required: non-significant difference.

(iv) STEP Achievement Categories:

number and operations: males' results > females' results (.01 level);

measurement and geometry: males' results > females' results (.01 level);

probability and statistics: not calculated - insufficient items;

function and relation: males' results > females' results (.01 level);

proof-deductive and inferential reasoning: males' results > females' results (.05 level);

symbolism: non-significant difference.

(v) Operations and Elements in STEP Test:

problem-solving processes: males' results > females' results (.01 level);

measurement: males' results > females' results (.01 level);

operations on whole numbers: males' results > females' results (.01 level);

operations on decimals and fractions: males' results >

females' results (.01 level);

operations on money and time: non-significant difference;

interpretation of graphs: non-significant difference.

In general summary then it appears that females on this test perform equally with males in the verbal/symbolic area and when non-school experience is called upon (money, time). Males appear to perform better in most areas but especially in school taught areas.

Sample of 12:

It was decided in this section of the study to begin by concentrating on the 8 items which had shown a statistically significant difference between the sexes in the original testing. The sample of students chosen mirrored this result. In addition there was no statistically significant difference between the male and female subjects chosen for the interview and their counter-parts (i.e. middle concrete operational groups) in the main sample with respect to age, grade, bryte score or STEP achievement score.

The interviews were conducted individually by one of the authors and were tape-recorded for later analysis. Each subject completed two interviews of up to 30 minutes each - the first concentrating on the 8 items where males had performed significantly better than females, the second on 14 randomly selected items where the differences had not been significant.

The interviews were structured. When the child was given an item he or she was asked to respond to the following statements/questions as he or she worked:

(i) Please read the question to me. If you don't know a word leave it out.

(ii) Tell me how you are going to find the answer.

(iii) Tell me what the question is asking you to do.

(iv) Show me what to do to get the answer. Tell me what you are doing as you work.

(v) What is the answer: A, B, C or D?

If the response to the second request indicated that subjects did not understand how to solve the problem they were asked, "What is it about the question that you do not understand?"

In analysing the data thus obtained the investigators looked for the following: the number of items correctly answered; the indication of an understanding of an item regardless of the correctness of the answer; the strategy employed to find the solution; the desire to persevere with a difficult or unfamiliar problem; the comprehension of verbal explanations; and the tendency to become confused during explanations.

It is clear that no generalisations could be made from such a small sample but to test the patterns which emerged with respect to the sexes the Mann-Whitney U-Test was used.

A summary of the results of these analyses is given below:

(i) Number of items correctly answered: males' scores > females' scores (.05 level);

(ii) Understanding of item: non-significant result;

(iii) Solution strategy employed, i.e.

a) working out problem and selecting alternative;

b) selecting one criterion relevant to answer and checking against given alternatives;

c) going through alternatives and eliminating.

Non-significant result although males tended to favour alternative (a) and females (b);

(iv) Perseverance with unfamiliar problem or when difficulties arose: males' results > females' results (.05 level);

(v) Comprehension of verbal explanations: non-significant result;

(vi) Becoming confused during explanations: non-significant result.

In summary the only basic difference between the males' and females' method of working on the problems in this study appeared to be one of tenacity (the tendency to select procedure (a) by the males and (b) by the females in (iii) above is perhaps a reflection of this phenomenon). It may be of significance to note that, whereas boys who did not complete items gave typical 'opting out' reasons, such as 'we have not been taught that', girls were more likely to respond with 'I do not understand what to do'. This may reflect a difference in approach, that boys tend to rely on experience and what they have been taught whilst girls seem to attempt to understand what they are doing and lack confidence in they cannot understand. This finding is linked to the earlier result that boys performed better than girls on items related to strategies taught at school.

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K. ATTITUDES

BEHAVIOURAL OUTCOMES OF COMMUNICATION APPREHENSION AS A CO-VARITE OF EXPERIENCE AMONG MATHEMATICS TEACHERS

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INTRODUCTION

Research on mathematics teaching has tended to concentrate on :

(a) philosophy of learning;
(b) curriculum - e.g. "New Maths" of the sixties;
(c) individual differences and self paced learning;
(d) educational media; and
(e) the affective domain.

The work of Page (1963) is rare amongst studies of communication in mathematics classrooms, although more recently Dekkers and Shield have attempted to devise a valid and reliable instrument for recording events and behaviours in mathematics classrooms in Australia.

Mathematics remains a compulsory subject until Year 10 in New South Wales and is virtually mandatory in senior years. Any hindrance to communication would only harden negative feelings towards mathematics and exascerbate the phenomenon.known as 'mathematics anxiety'.

The field of speech communication has been concerned with the measurement of fear and/or anxieties relating to oral communication since Gilikson (1942) devised an index of speaker confidence.

THE INDEPENDENT VARIABLE

Communication Apprehension

Definitions

James McCroskey (1976, p.1) has defined communication apprehension as a "broad based fear or anxiety associated with either real or anticipated communication with another person or persons".

The highly communicative apprehensive individual has negative experiences across the broad communication continuum,whereas a "normal" person may occasionally experience stage fright in public settings.

Importantly, McCroskey's estimates suggest that 10-20% of the American population suffer from extreme communication apprehension and up to 20% more experience moderately high levels; research outside the U.S. is very much in its infancy, but Hansford and Hattie (1981, p.1.) reported a "considerable number" of mild to extreme apprehensives and no differences in mean levels between U.S. and Australian samples (n = 1784).

McCroskey and Shehan (1976) reported that low apprehensives were twice as likely to sit in the high interaction zone of classrooms (20% of total seats); high apprehensives were four times more likely to sit outside this area.

A replication of the study for small group, classroom and lecture theatre situations in a N.S.W. setting was attempted in a pilot and final study. The author fears that classroom interaction is less pronounced than in the American setting and that findings may be culturally incongruent, but generalizability will be argued at a later stage.

The classroom teacher must remember that forcing apprehensive children to participate orally will only aggravate their problem and that such children deserve other avenues of demonstrating competence for grading purposes. Experienced teachers will probably identify high apprehensives from cues and patterns, including those related to personality, peer-acceptance, and maybe seating position, although the latter requires successful Australian replication.

DEPENDENT VARIABLES

A Kinesic Correlate (Non-verbal variable)

A study by Comadena and Anderson (1978) investigated the relationship between communication apprehension and hand-movements of 74 undergraduates in two communication courses at a major Eastern American University.

The researchers defined (p.8) illustrators as "socially learned movements which are directly tied to speech and illustrate or amplify what is being conveyed verbally" and found partial support (curvilinear relationship) for the hypothesised link between illustrators and communication apprehension.

Comadena and Anderson, then, argued that illustrators are used more frequently by dominant individuals, or as Ekman and Friesen (1972, 1974) put it "those in dominant positions" and that high apprehension is rarely linked with dominance. They requested (p. 11) "continuing research in this area . . . (to) hopefully produce characteristic behaviours of communication apprehension".

A Verbal Behavioural A Verbal Behavioural Outcome

Rhetorical Interrogatives (Defined as "non-ah" types of speech disturbance, such as "you know?", "see ?", "okay?", "right ?").

Powers (1977), argued that rhetorical interrogatives are unique speech disturbances in that they consist "ironically, of complete sentences, adequate articulation, with denotative semantic meaning" (p. 44). Denotatively, the individual is asking the other person to respond to the quality of the communication, but responses are neither sought nor desired. Connotatively, however, the utterance is meaningless. Powers theorized higher rhetorical interrogative ratios amongst high communication apprehension subjects, arguing that speech disturbance ratios are a function of the level of (trait) anxiety experienced by the communicator.

Apprehension scores of 430 subjects in communication courses were correlated with usage number of rhetorical interrogatives : significant differences ($\underline{F}(1,56) = .004$, $\underline{p} > .1$) were not found, neither were interaction effects. Powers requested further research to possibly strengthen what he described as only partial support for the theorized explanations. (N.B. Significant differences were found for ratios of rhetorical interrogatives).

Outcomes With Both Verbal & Non-Verbal Components

Disconfirming Responses

As teaching requires considerable interaction (Flanders, 1970), then the quality of interpersonal responses would seem to strongly influence communication effectiveness in classrooms. Buber (1957) and Laing (1961) considered such responses as either confirming or dosconfirming in terms of their quality, and an instrument was developed for the identification of disconfirming responses. Hansford and Aveyard (1981) found greatest positive correlation between high communication apprehension and categories of disconfirming responses labelled interrupting and impervious.

Wiemann and Knapp (1975) defined interruptions as "attempts to assume the speaking role before it has been relinquished by the current speaker". Impervious disconfirming responses occur in classrooms when teachers ignore or disregard communicative attempts . Hansford and Aveyard speculated on "impervious and interrupting responses being major contributions to dysfunctional classroom behavior" (p. 9).

A CO-VARIATE

Teacher Experience

Despite the multitude of conflicting empirical reports relating to communication and teacher effectiveness, the relationship between the quality of interpersonal responses and degree of experience and effectiveness seem, in the words of Hansford and Aveyard (1981, p.2), "Intuitively strong". Indeed, their plea for further research on the teaching process suggests that "an interesting project would be to compare responses of trainees that impair communication with response behavior of experienced teachers" (p. 9). They see establishment of effectiveness criteria and communicatiom courses at pre- and in-service level as corollaries of findings from such a project.

Sandefur (1969) identified major changes during the first year of in-service teaching as the use of indirect methods and display of greater fairness, understanding, poise, responsiveness, and confidence. The latter two characteristics are, of course, directly related to the problem in hand, as they embody both the independent variable (confident teachers should have low apprehension levels) and one dependent variable, responsiveness.

METHODOLOGY AND GENERALIZABILITY

McCroskey (1975) developed a variety of instruments designed to measure Communication apprehension via personal report (Personal Report of Communication Apprehension or PRCA). One of these, PRCA-College as a reliable, (concurrently) valid instrument with high loading items, was selected to measure the independent variable in the study, despite some shortcomings and criticisms regarding construct validity. McCroskey's self reports on seating preferences were also used in an attempt to replicate his findings on apprehension and seating preference correlation.

For the parts of the study involving classroom instruction, a 15 minute video-tape of lesson segments, where high teacher pupil interaction had been a stated objective, was utilized. Timed, edited video-tape was coded according to the cited definitions of illustrators, rhetorical interrogatives, impervious responses, and interrupting responses. For each category, a coder other than the researcher was chosen to test reliability of sample tape.

The author's involvement as a secondary mathematics teacher meant that, in terms of practicalities, a sample of colleagues and trainees was the most accessible given the difficulties involved in researching Australian Schools. Colleagues (experienced sample, n = 10) had between 5 and 22 years experience in secondary mathematics teaching; trainees (inexperienced sample, n = 10) were in their final year of a seocndary mathematics training course.

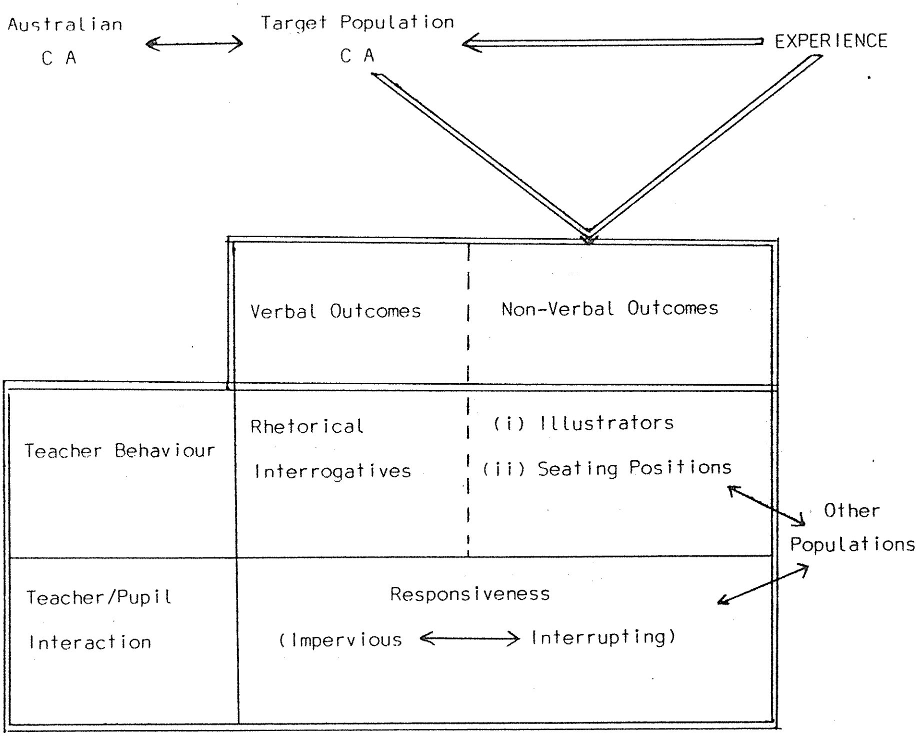
Because the investigator had been forced to "cluster" sample, caution was paid about generalizing beyond the constraints embodied by the sample limitations:

(a) all subjects were mathematics teachers;
(b) all trained or practised in a rural setting;
(c) the trainee subjects were all in their final semester;
(d) the experienced subjects were all volunteers; and
(e) the sample was small and with a predominance of males.

Despite these limitations, it still seemed feasible from the pilot study to hypothesise for a target population of "inexperienced and experienced mathematics teachers in New South Wales".

RESEARCH HYPOTHESES AND FINDINGS

HYPOTHESIS SCHEMA



Key: ⟷ Comparison   C A = Communication Apprehension

Independent variable ⟹(effect) Dependent Variable

Mathematics Teachers' Communication Apprehension

Hypothesis 1

That there is no significant difference between the communication apprehension levels of mathematics teachers and other Australians. Hypothesis 1 was accepted.

Experience and Apprehension

Hypothesis 2

That experienced mathematics teachers possess significantly lower levels of communication apprehension than inexperienced mathematics teachers.

Hypothesis 2 was rejected and it seemed that experienced mathematics teachers did not possess significant lower levels of communication apprehension than inexperienced ones (rather that there was no significant difference).

Seating Positions : Non-Verbal Behaviour

Hypothesis 3

That low apprehensive mathematics teachers prefer small group seating positions preceived as highly interactive and vice-versa.

Hypothesis 3 was accepted and it seemed as in the pilot study and McCroskey and Shehan's (1976) findings that low apprehensive mathematics teachers preferred small-group seating positions perceived as highly interactive and vice-versa.

Hypothesis 4

That mathematics teachers in Australia exhibit significantly different seating preferences to Americans in different situations and that Hypothesis 3, therefore, does not carry over to classrooms and lecture theatres.

Given limitations noted, there was highly significant difference between the American and Australian samples for small group discussion situations on the mean selection rate for highly interactive seats. This is despite the consistencies within each culture verified by Hypothesis 3.

Illustrators : Kinesic Teacher Behaviour

Hypothesis 5

That highly apprehensive mathematics teachers exhibit significantly fewer illustrators in their teaching than those with low apprehension.

Hypothesis 5 was rejected, and it seemed, as in the pilot study and to some extent in Comadena and Andersen's (1978) study, that highly apprehensive mathematics teachers did not use significantly fewer illustrators than those with low apprehension (rather, that there was no significant difference).

Hypothesis 6

That inexperienced mathematics teachers exhibit significantly fewer illustrators in their teaching than experienced mathematics teachers.

Hypothesis 6 was confidently accepted.

Rhetorical Interrogatives : Verbal Teacher Behaviour

Hypothesis 7

That highly apprehensive mathematics teachers exhibit a significantly fewer number of rhetorical interrogatives in their teaching than those with low apprehension.

As in the pilot study, and in Powers' (1977) and Aveyard's (1979) studies, there seemed to be no siginificant difference between high and low apprehensive mathematics teachers in the frequency use of rhetorical interrogatives.

Hypothesis 8

That experienced mathematics teachers exhibit significantly more rhetorical interrogatives in their teaching than inexperienced mathematics teachers. Experienced mathematics teachers did not use significantly more rhetorical interrogatives than inexperienced ones (rather, that there was no significant difference).

Impervious Responses

Hypothesis 9

That highly apprehensive mathematics teachers will exhibit a significantly greater number of impervious responses in their interaction than those with low apprehension.

As in the pilot study, and that of Hansford and Aveyard (1981), it seemed that highly apprehensive mathematics teachers did not exhibit significantly more impervious responses than those with low apprehension (rather, that there was no significant difference).

Hypothesis 10

That experienced mathematics will exhibit significantly fewer impervious responses in their interaction than inexperienced mathematics teachers. Hypothesis 10 was accepted.

Interrupting Responses

Hypothesis 11

That highly apprehensive mathematics teachers will exhibit a significantly greater number of interrupting responses in their interaction than those with low apprehension.

Highly apprehensive mathematics teachers did not exhibit significantly more interrupting responses than those with low apprehension (rather, that there was no significant difference). The results do not match those of Hansford and Aveyard (1981).

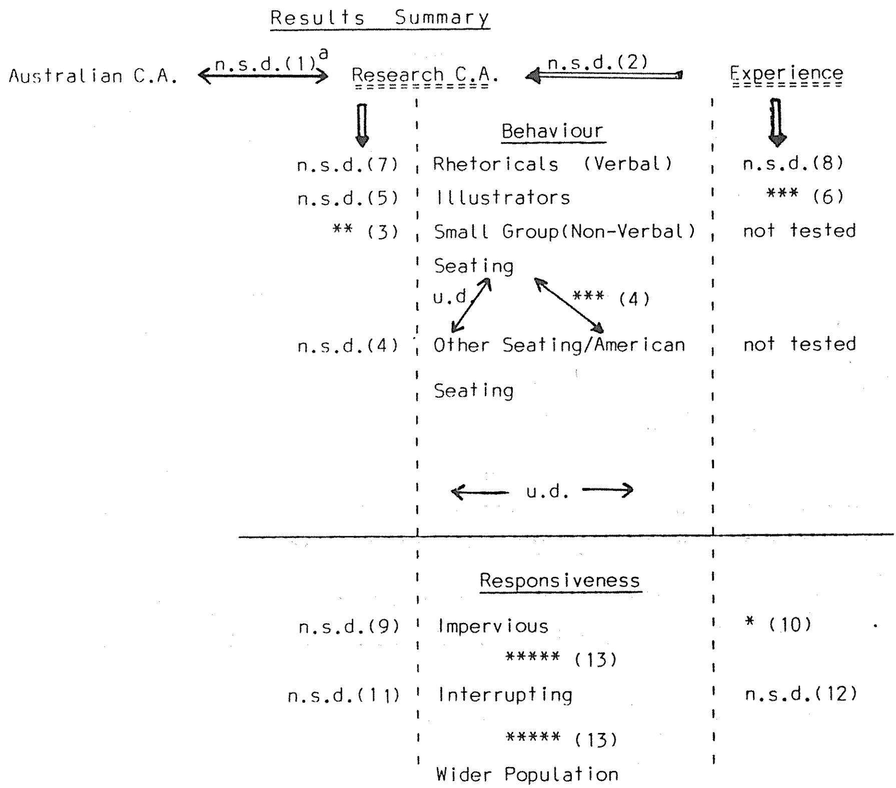
Hypothesis 12

That experienced mathematics teachers will exhibit significantly fewer interrupting responses in their interaction than inexperienced mathematics teachers. Experienced mathematics teachers did not exhibit significantly fewer interrupting responses than inexperienced ones (rather, that there was no significant difference).

Hypothesis 13

That in mathematics classrooms fewer interrupting responses occur either than in other classrooms or compared with impervious responses.

Hypothesis 13 was accepted.



Note: Key: ↔ Comparison   Independent Variable (Effect) ⇒ Dependent Variable

C.A. = Communication Apprehension

n.s.d. = No significant difference

u.d. = Untested difference

a Hypothesis number parenthesised

* p < .05

** p < .01

*** p < .005

**** p < .001

***** p < .0005

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CLASSROOM ATMOSPHERE AND THE ATTITUDES OF CHILDREN AND THEIR TEACHERS TOWARDS MATHEMATICS

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INTRODUCTION

The term 'classroom atmosphere' is frequently used synonymously with the terms 'classroom climate' and 'classroom environment', while social psychologists tend to use the more general expression 'psychosocial environment' when referring to human social environments of which 'classroom atmosphere' is but one example. It has been suggested (Insel and Moos, 1974) that just as people have different personalities, so too will social environments have different social climates and, in general, the social climate within which an individual functions is likely to have a direct influence on his attitudes. In particular, the importance of a proper classroom atmosphere for the learning of mathematics, together with the belief that this is largely dependent on the teacher, has been noted by several writers (Butler, Wren and Banks, 1970; Dienes and Golding, 1971; Johnson and Rising, 1972; Sobel, 1975).

The importance of attitudes towards mathematics is shown in the increasing number of research articles and dissertations pertaining to this topic, together with the deeper concern that is reflected in recent trends in curriculum development in mathematics education (Howson, 1976). The results of investigations into a possible relationship between teachers' and children's attitudes towards mathematics are equivocal, with some studies suggesting that a positive relationship exists and others concluding that there is no relationship (Anttonen and Deighan, 1971; Garner, 1966; Gilbert and Cooper, 1976).

Several research studies (Anderson, 1970; Anderson and Walberg, 1968; Walberg, 1971; Walberg and Anderson, 1972) have shown that the relationship between classroom atmosphere and attitudes towards the subject matter of the class is significantly positive, although Lawrenz's (1975) study did not support this finding.

MEASURING CLASSROOM ATMOSPHERE

Much of the pioneering work related to the measurement of classroom atmosphere has been carried out by Moos and his associates (Insel and Moos, op.

cit.; Moos, 1974;1975; Trickett and Moos, 1973). Building on the theoretical formulations of Murray (1938) and Lewin (1951) and the empirical work of Moos (1974), Silbergeld, Koenig and Manderscheid (1975;1976) developed the Class Atmosphere Scale (CAS). The scale was intended to assess systematically and quantitatively the perceptions held by high school teachers and their pupils about their common class milieu. Classroom atmosphere is measured by the CAS on 12 dimensions : Spontaneity, Support, Practicality, Affiliation, Order, Insight, Involvement, Aggression, Variety, Clarity, Submission and Autonomy. Results from the initial study showed the CAS to have satisfactory psychometric properties (Silbergeld et al., 1975) and provided support for the three global factors: Relationship, System Maintenance and Personal Development, proposed by Moos(1975).

MEASURING ATTITUDES TOWARDS MATHEMATICS

Many attempts at measuring attitudes towards mathematics have been reported in the literature, some based on the premise that the construct is uni-dimensional and others that it is multi-dimensional (Aiken, 1963;1974; Anttonen, 1969; Dutton, 1954; McCallon and Brown, 1971; Sandman, 1974; Scharf, 1971; Schofield, 1981), while a wide variety of attitude measurement techniques are in existence, the two most commonly used being Likert's (1932) method of summated ratings and Osgood's semantic differential (Osgood, Suci and Tannenbaum, 1957).

The view adopted for this investigation was that an attitude is (i) a unidimensional construct and (ii) a learned predisposition to respond in a consistently favourable or unfavourable manner with respect to a given object (Fishbein and Ajzen, 1975). The choice of measurement technique, determined in part by the above rationale, was influenced by the research of McCallon and Brown (op.cit.) and Schofield (op.cit.), which, together with the views of Shannon (1979), points strongly towards the use of a semantic differential instrument, an instrument that has been used successfully for measuring the attitudes of both teachers and pupils towards mathematics.

METHOD

The study was carried out with teachers and pupils in 10 high schools in Harare, Zimbabwe. The sample contained 27 high school mathematics teachers, together with a random selection of one class taught by each teacher. This produced a sample of 27 classes from Form 1 to Form 4 containing a total of 968 pupils. The mean class size was 36 pupils.

The Class Atmosphere Scale was used to assess the classroom atmospheres perceived by the pupils and their teachers, while the attitudes of pupils and teachers towards the concept MATHEMATICS were assessed using a 15-scale semantic differential : Attitude towards MATHEMATICS (ATTM). Teachers' attitudes towards the concept MYSELF AS A TEACHER OF MATHEMATICS were measured using a similar 15-scale semantic differential : Attitude towards MYSELF AS A TEACHER OF MATHEMATICS (AMTM). The teachers also provided information about themselves.

RESULTS

Each CAS dimension contained 10 true-false statements concerning specific classroom behaviours, with dimension scores ranging from a low of 0 to a high of 10. Percentile norms were calculated for each of the 12 dimensions and compared with the American norms obtained by Silbergeld et al. (1975). Table 1 shows the 10th, 25th, 50th, 75th and 90th percentiles for each of the 12 CAS dimensions, grouped according to the three global factors of Moos(1975), which facilitate interpretation of dimension scores. It should be noted that the American norms were obtained from a sample (N=509) of Art, English, Music, Science and Mathematics classes whereas the Zimbabwean sample (N=968) contained Mathematics classes only.

When compared with the American sample in terms of the Relationship factor, the Zimbabwean children perceived the class atmosphere as being lower in emotional support (Affiliation, Support), more restrictive (Spontaneity), but more interesting (Involvement). System Maintenance dimensions suggest that the Zimbabwean classes were more structured and controlled (Order, Submission) than their American counterparts and that defined expectations were greater (Clarity). With respect to Personal Development, the Zimbabwean classes were perceived to be marginally more effective learning situations (Insight, Practicality), but less effective in promoting self-sufficiency (Autonomy).

The percentile norms were used as standards for assessing the profiles of mean dimension scores for each class in addition to making comparisons of teacher-class profiles. Scores lying outside the range of the 10th and 90th percentiles were interpreted as deviant (Silbergeld et al.,1975).

Figure 1, for example, compares the profile of Class 6 with that of Teacher 6 as well as with the 10th, 50th and 90th percentiles. For this teacher

Table 1

Zimbabwean percentile norms for the Class Atmosphere Scale

Relationship factor

Percentile	Spontaneity	Affiliation	Support	Involvement
10	1,6 (2,0)	3,6 (4,0)	3,5 (4,0)	4,6 (2,0)
25	2,4 (3,0)	4,6 (5,0)	4,7 (5,0)	6,0 (4,0)
50	3,5 (4,0)	5,8 (6,4)	6,0 (6,0)	7,4 (5,0)
75	4,7 (5,0)	7,0 (8,0)	7,2 (7,0)	8,7 (7,0)
90	5,7 (7,0)	8,1 (9,0)	8,2 (9,0)	9,4 (8,0)

System Maintenance factor

Percentile	Aggression	Order	Submission	Clarity
10	2,7 (4,0)	3,9 (2,0)	5,5 (4,0)	5,4 (3,0)
25	4,0 (5,0)	5,7 (4,0)	6,5 (5,0)	6,8 (4,0)
50	5,4 (6,0)	7,6 (5,7)	7,4 (6,0)	8,0 (5,5)
75	7,0 (7,0)	8,8 (8,0)	8,4 (7,0)	8,6 (7,0)
90	8,1 (8,0)	9,7 (9,0)	9,2 (8,0)	10,0 (8,0)

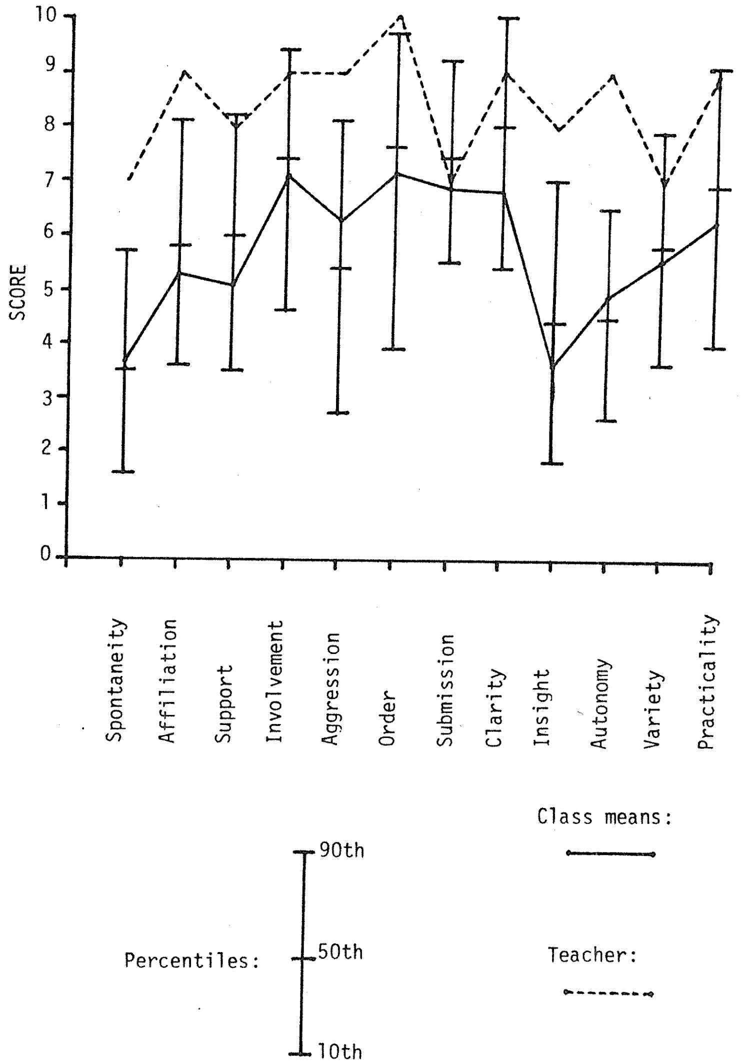
Personal Development factor

Percentile	Insight	Autonomy	Variety	Practicality
10	1,8 (1,0)	2,6 (3,0)	3,6 (3,0)	4,0 (3,0)
25	3,0 (2,0)	3,5 (4,0)	4,6 (5,0)	5,4 (4,0)
50	4,4 (3,5)	4,5 (5,4)	5,8 (5,6)	6,9 (5,6)
75	5,8 (5,0)	5,5 (6,0)	6,9 (7,0)	8,2 (7,0)
90	7,0 (6,0)	6,5 (7,0)	7,9 (8,0)	9,1 (8,0)

(American norms are given in parenthesis.)

Figure 1

Comparison of CAS profiles for Class 6($N = 37$) and Teacher 6 with the 10th, 50th and 90th percentile norms.



and his class, there are pronounced differences between their scores on all dimensions except Submission. This suggests very strongly that Teacher 6 perceives an idealised classroom atmosphere compared with that perceived by his pupils, which is lacking in many facets of the Relationship, System Maintenance and Personal Development factors.

Appraisal of the 120 items making up the 12 CAS dimensions was carried out using the criteria suggested by Moos and Houts (1968) and which were adopted by Silbergeld et al. (1976) in their analysis. The obtained results compared favourably with those of the American study and showed that the Class Atmosphere Scale, which was designed for and tested in high school classrooms in America, functioned satisfactorily in the different cultural setting of high school mathematics classrooms in Zimbabwe.

For the ATTM questionnaire, an oblique principal component cluster analysis was used to determine which of the 15 bipolar adjectival scales were representative of Osgood's Evaluative dimension (Osgood et al., op. cit.). This resulted in six evaluative scales from the teachers' data and using the Whitney (1978) t statistic, which was calculated by means of a FORTRAN program (Glencross, 1981), the attitudes of the teachers towards MATHEMATICS were found to be significantly positive. It was concluded that for the teachers in the sample, MATHEMATICS was seen to be Enjoyable, Good, Valuable, Pleasant, Active and Varied.

A similar analysis was carried out for the AMTM semantic differential, which resulted in five evaluative scales. The attitudes of the teachers towards the concept MYSELF AS A TEACHER OF MATHEMATICS were found to be significantly positive and it was concluded that the teachers saw themselves as Active, Comfortable, Good, Secure and Strong.

The analysis of the ATTM data for the pupils in the sample resulted in eight evaluative scales with significantly positive attitudes being found on all eight scales. For the Zimbabwean children in the sample, MATHEMATICS was seen to be Good, Enjoyable, Nice, Loveable, Active, Pleasant, Comfortable and Varied.

The correlation coefficients between class mean attitude scores on the eight-scale ATTM instrument and the corresponding teacher's total attitude scores on (i) the six-scale ATTM instrument and (ii) the five-scale AMTM instrument were found to be -0,03 and 0,17 respectively, neither being statistically significant.

The correlation coefficients between class mean attitude scores on the eight-scale ATTM instrument and class mean scores on each of the 12 CAS dimensions were, with the exception of Dimension 6: Insight, found to be low and not significant. The correlation between Insight and pupils' attitudes towards MATHEMATICS was -0,57 , a significant ($p < 0{,}01$) result. (Because of the scoring protocol, the lower the attitude score the more positive the attitude.) Thus, high scores on Insight (defined as 'fostering concern for and understanding of personal problems) were associated with positive attitudes towards MATHEMATICS.

DISCUSSION

The Class Atmosphere Scale, originally designed for use in high school classrooms in America was found to have functioned satisfactorily in high school mathematics classrooms in Zimbabwe. CAS profiles were found to be valuable for comparing the classroom atmospheres perceived by different classes, while teacher-class comparisons served to highlight a teacher's awareness of the classroom atmosphere as perceived by his class.

The attitudes of teachers and pupils alike towards mathematics and teachers' attitudes towards themselves as mathematics teachers, were found to be significantly positive.

There was no relationship between pupils' attitudes towards mathematics and teachers' attitudes towards either mathematics or themselves as mathematics teachers. Similarly, there was no relationship in general, between classroom atmosphere and pupils' attitudes towards mathematics.

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L. TECHNOLOGY

TECHNOLOGICAL CHANGE AND MATHEMATICAL COMPETENCE

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There is an increasing impact of new technologies on mathematics education. What has been the handheld calculator in recent years is the introduction of microcomputers into mathematics lessons at present. The influence of these technologies may be studied in different aspects. This contribution will be restricted to the relation between mathematical competence and the competence of programming a computer: Are these two competencies two completely different domains? May mathematical competence be acquired by using computers? May the competence to program a computer be learnt by using mathematics? What about a new type of knowledge by fostering the combined use of computers and mathematics?

In the studies of DÖRFLER & PESCHEK (1984) generalizing is conceptualized as the central mathematical competence that partly implies and partly regulates the other processes. The Austrian authors distinguish eight stages of generalization:

1) Constructing and displaying the situation that constitutes the problem
2) Analyzing the situation
3) Performing the actions or operations
4) Describing the actions or operations by verbal, iconical, and other means
5) Performing the actions in different situations
6) Schematizing the actions or operations by direct symbolizing of its elements and describing the conditions of action
7) Formalizing the elements of action by explicating the conditions of executability
8) Improving the object-adequate description of the resulting relations and execution of actions or operations by using the acquired elements of action in ever new situations

After reviewing more than 200 investigations concerning programming PEA & KURLAND (1983) summarize the results of the studies in a process model of programming that consists of four stages:

a) Understanding the problem
b) Planning and designing a program
c) Transforming the program design into a programming language

d) Understanding the complete program, debugging, and program maintainance

This is a brief description of both models, but if you compare them in depth you will find a number of similarities despite of the different objects of both activities. While generalizing first a situation is constructed or displayed that constitutes the problem. Then the situation is analyzed. That coincides with the first phase of programming: In this stage a problem representation has to be created in the mental working memory. In both cases domain specific knowledge has to be activated to structure the problem situation and its elements. This analysis of the situation is not performed for its own sake, but is done in order to explore the opportunities of action.

Here we find an additional step of planning in the process of programming: The design of the program has to be sketched briefly. While generalizing the actions or operations are performed experimentally and directly. After that they are described by verbal, iconical, or other means. In the process of programming the problem description is transformed into a programming language (already in symbolical form). This is a respectively simple level of cognitive activity whereas it constitutes a relatively creative act in this stage of generalization because the internal model is tested whether it is adequate in describing the situation and action elements. If the internal representation does not fit and if the reactions from the external world are not as predicted by the model it has to be reviewed and executed again. This is also true for the last phase in the process of programming: the already written program is debugged. It is possible that the program is semantically and syntactically correct for one set of data but not for another one, because one set of data activates certain modules whereas another set of data does not. This stage may be comparable with stage 5 in the process of generalization. The actions are performed in ever new situations (the new situation is a new set of data while programming). The generalizability of a program and hence of the internal representation of the problem is tested.

The following stages of generalization have to be formed very early. While designing a program the actions are already schematized as commands. The relations between commands are described formally by branching. When translating a problem into a programming language direct symbolization and specification of conditions of actions are performed. It is pertinent to programming languages that their elements of actions are formalized as commands and that their conditions of executability are defined by branches.

What distinguishes programming from mathematical generalizing? So far we have only mentioned that there are important similarities in both activities. It is especially the last stage in the process of generalizing where you cannot find a correspondence in the different forms of programming. Usually a program is a product that has to be produced under defined assumptions. Post hoc extensions of the main objective are usually not proposed. Nevertheless there are structural similarities or even equivalences between different programs. This is the reason why there are manifold communication systems between programmers. Why should you start to solve a problem from the very beginning if there is already a solution? But every solution is reached under assumptions which cannot be identical in all details. A competent use of program libraries assumes the ability to generalize these conditions. It further implies a form of programming that facilitates the extension of the program under changed conditions.

This form of generalization, however, has its limits. There are a number of statistical programs which may be described as special forms of the general linear model. It can be doubted that a programmer would be able to find the generalization to the general linear model only by debugging, structuring, and kalligraphic modification of his program for discriminant analysis. It would assume that the mathematical activity of generalizing is embedded in the process of programming and vice versa.

How is the mathematical activity of generalizing modified by the increasing importance of computers in this area? Special forms of visualizations by graphical representations of mathematical problems have been facilitated substantially by the tremendous development of graphical software especially for microcomputers, but also by the development of special graphical central processing units and graphical terminals with a high resolution power. For the domain of statistics BIEHLER (1983) could show that the use of graphical respresentations also changed fundamental views of what is done by using statistics. Whereas the testing of hypotheses was the main topic for NEYMAN & PEARSON (1933) TUKEY (1977) emphasized the explorative view while developing graphical representations of statistical problems.

While learning calculus there is still an emphasis on the learning of routines although recently developed programs like muMATH offer the opportunity to reduce the training of routines to the necessary amount and to apply calculus in extramathematical domains that really are applications and not situations constructed to learn calculus.

It seems to make sense to conceptualize mathematical generalization and programming in one unifying theoretical framework. Both models have their origins in very different research traditions. DÖRFLER & PESCHEK's works are based on Soviet psychology of activity whereas PEA & KURLAND's approach has its roots in US-American theories of problem solving. Nevertheless it is surprising to find that research on programming and on generalizing have so many common features despite of the different research traditions underlying the two models.

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INTEGRATION OF THE CALCULATOR INTO CHILDREN'S EDUCATION

Jan van den Brink

INTRODUCTION

Phases of investigation
In our investigation into the use of the pocket calculator by kindergarteners and primary school children we can distinguish three phases:
1. Children's ideas and activities. This concerns an investigation into children's ideas about the calculator and how they handle the machine. Conversations on a one-to-one basis.
2. Dialogues. Investigation into the use of the calculator by groups of two children.
3. Class. Investigation into the use of the calculator in actual class situations.

Each phase mentioned above can be considered as a preparation for the following one, within the objective we have in mind. What we would like to do, using children's ideas and activities as a starting point, is namely to arrive at interesting primary school class situations centering on the calculator: research for the use of education.

For this we first established a certain disposition which all children have, regardless of age, when using the calculator. With this disposition in mind, errors become comprehensible on the one hand and, on the other hand, we can formulate didactic hints and invent educational situations.

First phase of investigation

A disposition typical of children in regard to the calculator.

On regarding the observations of the first phase of investigation, we are struck by three aspects of this disposition:
1. Children's behaviour with the calculator: motoric, auditive and visual.
2. Children's ideas about the calculator.
3. Certain changes in arithmetical ideas when using the calculator.

In the following section we shall attempt to translate this typical psychological nature into didactic principles.

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Controlling "wild experimentation"
After a while it seems that children get "wild" with the machine. Some punch all sorts of keys at once, try out the strangest things... This goes for all age-groups.

You can help control this "wild experimentation" with three remedies.

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Children become conscious of their activities with the calculator when they talk about what they are doing. The dictating of activities to each other is here a good idea.

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Children's attention to symbols is often directed towards all sorts of details which seem unimportant to us, which is why we don't notice them. Sometimes children notice these details, but sometimes not. Although you may well be mistaken in the second case.

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Children don't usually tend to check the display to see whether they've pressed the right keys.

Exercises should be done whereby the children are compelled to keep an eye on the window. For instance, by having them make jumps until a certain number appears.

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The calculator as arithmetic-partner.

We have already mentioned that the children use the calculator as a verifier and do not easily see it as an arithmetic "partner" or "buddy".

The better a child is in arithmetic, the less likely he/she can be prevailed upon to use the calculator in the second manner. This situation can be resolved by one of the following remedies.

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Kikkie (4;11) has typed in the problem 1+1= which I wrote down for her with the assignment, "you have to press these keys: 1+1= ".

She describes the result, "2" now showing in the display as follows: "a little tiny circle" (she means the decimal point) "and one of these" (the 2), and says, startled, "why is that there? I didn't press this", pointing to the decimal point and to the 2 which she hadn't touched. But a decimal point and a 2 have, indeed, appeared. I'm not quite sure how I should explain it. I just say, "that's because of the + ", because her knowledge of arithmetic is too limited.

It is clear that kindergarteners believe that the only way for something to show in the display is to press the related key. The fact that this is not necessarily so is beyond their comprehension.

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Children's prevailing ideas about arithmetic can be broadened by the calculator.

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Second phase of investigation: dialogues between children

Introduction

The dialogues in the second phase have led to a number of findings which here follow:

a) A meaningful doing of arithmetic cannot be reached by using the calculator exclusively. Arithmetic must always first be learned by means of a variety of models and contexts. The fear that children in the future will only be able to do arithmetic by the grace of a full battery is therefore without justification.

b) The calculator has numerous facilities which are related to traditional ways of doing arithmetic (repeated addition, long division, etc.). Children should be made aware of this.

c) The calculator unexpectedly helps expand the existing knowledge of and insight into arithmetic.

d) Children run the risk of getting stuck in certain very elementary procedures which they have learned on the calculator. By means of games at various levels and on different computers, these

basic procedures can be raised to a higher plane.
e) Basic procedures, which are at the children's centre of attention, often impede the discovery of other, simpler strategies. Handing over the calculator for solving a certain problem must therefore be done with care.
f) If a child is receptive, his/her thought process can be verified numerically by the calculator.
g) The calculator has its own particular limitations with respect to other models and notations of arithmetic (unfamiliarity and lack of capacity). The children experience these limitations in each particular arithmetical subject or problem, and the use of the machine can be attuned to how the other models are used.

We will now further illuminate these standpoints concerning the integration of the calculator into arithmetic instruction by means of:
- the consequences for the knowledge of arithmetic
- the search for strategies in the solving procedure
- the use of models
- the calculator procedures

In passing, we shall inquire into didactic approaches, such as the writing and reading of manuals for the calculator by the children themselves, programming the calculator, working with numbers which are too large for a small machine, etc.

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The estimating and structuring of numbers can be meaningfully done by using the calculator. Connections with existing traditional arithmetical skills are essential and, in some arithmetical subjects, a deeper insight can be reached by using the calculator.

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Strategies

The pocket calculator can obstruct the search for stragegies by a procedure on the machine blinding the children to other possibilities.

In education, this indicates that one should use the machines sparingly, and that the pocket calculator belongs in the first place in the teacher's pocket. He or she can then determine whether the machine should be used or whether other strategies are available for solving the problem.

On the other hand, it occurs repeatedly that the machine will show the child numerically that his/her intended strategy is incorrect.

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Models and notations

The use of the pocket calculator along side other arithmetical models and notations depends in the first place on the familiarity of the other models and notations.

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Calculator procedures

An unexpected didactic problem is caused by the calculator procedures. Children tend to fall back on previously learned procedures on the calculator and attempt to apply these to a given problem. But these ways are often extremely simple, such as + 1 + 1...

The danger of becoming fixed in previously learned games or procedures is no imaginary one. The machine's possibilities are exploited and explored no further. Literally, only a clever manipulation is developed in which the children get stuck.

- This can be broken through by means of a variety of games, such as the altering of digits in a number, dictating to each other, reaching a number by means of a limited number of keys, guess my jump, etc. Here the children introduce their own strategies and rules, whereby they can refer back to previously learned arithmetical structures and discover new possibilities on the machine.
- The use of different computer brands is no disadvantage but, on the contrary, a great didactic advantage in combatting rigidity in calculator procedures.
- Letting the children develop manuals for each other also has this function of finding and notating more and more procedures.

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To sum up, we can say that eight manuals took the following forms:

1. Consisting exclusively of problems (three of them)
2. Consisting exclusively of text
3. Consisting of both problems and text (four of them)

The text can be:

1. The description of one sum, which could have been put in arithmetical symbols (four of the eight manuals)
2. The description of the keys of the calculator - their function and colour (one manual)
3. A generalization of the manual, which is sometimes expressed explicitly in the text for other numbers or operations (one manual)

The problems given contain, for example:

1. One operator per manual (three manuals)
2. More than one operator, sometimes including the results (five manuals)
3. Problems given for the student to check (five manuals)
4. Incorrect results given on purpose (one manual)
5. Problems given without result, where the children had to find it themselves (three manuals)

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Future perspectives

In our following investigation (phase 3: class situations) we will investigate how the calculator can be used for each arithmetical subject. To be considered are the introduction of addition and subtraction, solving of equations, estimating results, memorisation, etc. Models such as counters, number line and manner of notation can be reviewed as well as subjects which are particular to the calculator (decimal numbers, large numbers, negative numbers and all sorts of unfamiliar keys like % and $\sqrt{}$).

M. THEORY

THE ROLE OF THE INDIVIDUAL IN MATHEMATICS TEXTS

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What sense can be made of mathematician René Thom's claim that, "As soon as one uses a textbook, one establishes a didacticism, an academicism, even if the book be so written as to promote individual research" (Thom, 1973, p.196)? By what means, overt or covert, is this achieved and are similar practices visible in the mathematics classroom? Before going on to discuss in detail one clear instance of such a practice, I should like to place the discussion within a wider context, namely a concern with what constitutes the 'discipline' of mathematics education.

MATHEMATICS EDUCATION AS META-MATHEMATICS

There has been considerable informal debate within the mathematics education community concerning whether or not mathematics education can be reasonably said to constitute a discipline in its own right. If so, wherein lies its coherence? Is it derived from the particular phenomena of interest themselves, or the methods of approach to inquiring into them? One alternative way of conceptualising a discipline is in terms of the questions asked rather than the way in which it deals with them. David Wheeler has attempted an informal garnering of questions of Hilbertian status from certain members of the mathematical education community. Some of the individual contributions to his survey can be found in the journal For the Learning of Mathematics, (**4** (1), 1984). A general overall description is in Wheeler, (1983).

One characteristic distinguishing feature of mathematics education is that it can be seen as a meta-discipline; its concern is inquiry about mathematics. I wish to propose that mathematics education might be fruitfully conceptualised as meta-mathematics, where this latter term is afforded a much broader interpretation than is the customary one, namely the formal study of classes of abstract axiomatic systems, and includes for example sociology of mathematics and mathematicians (see eg. Bloor, 1976 or Fisher, 1973) and much else within this scope.

Wartofsky (1980) makes a similar proposal in the context of the natural sciences, where he details instances of what he terms 'the meta-sciences'. He also raises an extremely interesting question about trying to make a distinction between science and ideology. "Are the meta-sciences truly sciences, in that they contribute to the growth of our natural or social scientific knowledge? Or are they part of the ideological superstructure whose task it is to interpret, legitimate, or inculcate certain appropriate beliefs about science for ideological purposes." (Wartofsky, 1980, p.8) If Wartofsky had replaced 'ideological' by 'educational', would this sentence have seemed so threatening?*

The question of ideology raises the notion of resistance. Scepticism towards established dogma is ostensibly a prized asset in science. Where is the tradition of criticism and scepticism in mathematics? I. Lakatos' Proofs and Refutations (1976) provides one notable instance of what might be called mathematical literary criticism. In it he draws attention to precisely the reverse tradition from one of scepticism in mathematics, namely one where mathematical maturity is equated with willingness to suspend disbelief (and questions) until the results are proved, providing a new twist to the doctrine that the end justifies the means. Lakatos stresses the essential role of criticism in mathematical progress, yet it is a sad fact that this is virtually a non-existent area within mathematics education. (See Pimm (1983) for further exploration of this point.) For example, what and how much use is made of examples and counter-examples and to what ends? (Lakatos provides a useful taxonomy of counter-examples, Michener (1978) one of examples.) How are definitions introduced and justified? To what extent is mathematics discussed, rather than merely presented, and in what terms?

In the study of English literature, literary criticism is at the forefront of that study. Where is the comparable educational activity of studying the mathematical literature? An obvious first point is that seldom, if ever, do undergraduates see instances of mathematical literature. Their fare is very much Readers Digest compilations. My intent is to point up the dearth of meta-mathematical discussion in mathematical textbooks (where I am reclaiming the word meta-mathematics for the more general useage implicit in the burgeoning psychological fields of meta-cognition and meta-linguistics), namely discussion concerning knowledge, purposes and beliefs about mathematics itself.

WHO IS 'WE'

In the remainder of this brief paper I should like to present one example of a characteristic of mathematical textbook style which, I believe, contributes to the formation of attitudes about mathematics. In J. van Dormolen's a priori analysis of secondary mathematics texts (1982), he divided characteristics of mathematics into two 'dimensions', that of problem situation and aspects of mathematics and their relations. There is little reference to style or the presence of meta-text, that is writing about the writing in the book. (See Calvino (1983) for an extremely clear instance of this.)

The example I wish to discuss concerns the use of 'we' in mathematics textbooks (and elsewhere). Wills, a linguist, remarks, "'we' seems to have the greatest imprecision of referent of all English pronouns, and therefore is the most frequently exploited for strategic ends" (Wills, 1977, p.279). The question which interests me is to what community is the author appealing when using the word 'we'.

It is commonly employed in formal discourse. "In our attempt to analyse addition and multiplication of numbers, we are thus lead to the idea ..." (Fraleigh, 1967, p.5). The traditions in textbook writing are so strong that mathematician David Fowler, in the preface to his book Introducing Real Analysis, comments, "I, the author, address you, the reader, in a way that may be considered unseemly by my colleagues".

Other instances from Fraleigh are:

"As mathematicians, let us attempt ..." (p.5)

This example follows a paragraph in which "one", "you", "I", "the author" and "mathematicians" have all been used, though I must admit to a certain confusion in places as to precisely whom is being referred to at any particular instant.

"We have seen that any two groups of order three are isomorphic. We express this by saying that there is only one group of order three up to isomorphism". (p.57)

In the first sentence, Fraleigh seems to be expressing a hope about the mathematical state of understanding of his audience, while in the second, he is providing the accepted terminology that "we" use. To end the extracts from this text, here is one which includes many of the preceeding items.

"In the past, some of the author's students have had a hard time understanding and using the concept of isomorphism. We introduced it several sections before we made it precise in the hope that you would really comprehend the importance and meaning of the concept. Regarding its use, we now give an outline showing how the mathematician would proceed from the definition." (p.56).

Part of the effect of using 'we' is to move away from the individual, the personal, in mathematics. Why is there a widespread fear in mathematics of involving, and hence exposing, the self? The public image of mathematics is of something objective and absolute, permanent and impersonal. The inner mental activities of an individual are subjective, partial and relative. In the light of such a belief about the nature of mathematics, it would therefore be reasonable, would it not, to consider it not only appropriate, but the only proper way of behaving, for a teacher to refrain from exposing any personal images or thoughts. This illustrates quite clearly René Thom's claim that "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (Thom, 1973, p.204). Acting on this belief, however, may fail to communicate the possibility that people are working on images when doing mathematics.

This tension, between the author or teacher as individual and as a carrier of the general, arises in other areas of teaching. For example, a colleague who spends much of her time teaching advanced English to foreign students, differentiates between 'I would say ...' and 'we would say ...'. The former phrase she uses to indicate something about the way she uses the English language, about her idiolect. The latter she uses to convey her beliefs about judgements of grammaticality which she feels would be shared by all native speakers of English.

The same can be true of accent in foreign language teaching. Are pupils supposed to model their pronounciation exactly on that of their teacher, despite an awareness that no two people speak in an identical manner. A third instance comes from the teaching of an instrument, such as the piano. How is instruction to proceed, other than merely trying to copy identically what one's teacher does? It is often difficult as a learner to distinguish between the general and the specific, or even to become aware that such a distinction exists.

Thus, one use of 'we' rather than 'I' is perhaps intended as a clue to generality. It can also be employed as a means of spreading responsibility, while at the same time deriving weight and authority from (large) numbers. Such intents can be discerned in politics; 'We, at the ministry, believe ...'; or perhaps in speaking as a true representative on behalf of one's constituents. If a referendum had been taken, if honest, a politician could say 'we' in such circumstances, even if personally is dead against the proposal. Unfortunately the reverse situation is more likely, where the representative gives a personal opinion, yet passes it off as a group concensus. Does the mathematics teacher as politician seem a useful perception? Certainly the same tension between the individual and the representative seems present in both contexts.

In an article dealing with personal and public worlds of experience, Wood (1983) calls attention to how the need for social agreement causes us to seek the least common factor of our personal environments. Mathematics can be taken as the epitome of a search for the least common factor, something we are forced to agree on as common to our experiental environments. This search is for something that is global and possibly co-ordinate free, rather than our idiosyncratic descriptions, which are local and coordinate-full. Possibly, some may even desire to be able to allow the assertion that there is something out there, independent of, because common to, all our experiences. I was reminded of the following extract from Tom Stoppard's play Rosencrantz and Guildernstern are Dead.

Guil: A man breaking his journey between one place and another at a third place of no name, character, population or significance, sees a unicorn cross his path and disappear. That in itself is startling, but there are precendents for mystical encounters of various kinds, or to be less extreme, a choice of peruasions to put it down to fancy; until - 'My God' says a second man, 'I must be dreaming, I thought I saw a unicorn'. At which point a dimension is added that makes the experience as alarming as it will ever be. A third witness, you understand, adds no further dimension but only spreads it thinner, and a fourth thinner still, and the more witnesses there are the thinner it gets and the more reasonable it becomes until it is as thin as reality, the name we give to the common experience ... 'Look, look!' recites the crowd. 'A horse with an arrow in its forehead! It must have been mistaken for a deer'.

Ros: (eagerly): I knew all along it was a band.

Guil: (tiredly): He knew all along it was a band.

Ros: Here they come!

Guil: (at the last moment before they enter - wistfully): I'm sorry it wasn't a unicorn. It would have been nice to have unicorns.

MATHEMATICS AS A SPECTATOR SPORT

There are considerable pressures exerted by the approved style of communication in mathematics for it to be viewed as a depersonalised subject, but also one where there is a Authority. The widespread use of the passive mood (as in Science) allows the active agent effecting the change or making the claim to be omitted altogether. This can be contrasted with the equally common use of the imperative in mathematical discourse, which permits a similar massive deletion of elements.

The effect on me of reading Fraleigh was to emphasize that choices had been made, ostensibly on my behalf, without me being involved. The least that is required is my passive acquiescence in what follows. In accepting the provided goals and methods, I am persuaded to agree to the author's attempts to absorb me into the action. Am I therefore responsible in part, for what happens?

Such linguistic conventions beg the question of the relation of the individual to the mathematics under discussion. Suppose it were widely accepted that mathematics is essentially social and communication about mathematics requires genuine negotiation and sharing of meaning. How might such a conception of mathematics be conveyed through a written medium, the reasons for particular conventional agreements be communicated and explored, while contending with the phenomenon of text authority alluded to be Thom?

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* The same point has been made about the appropriateness or otherwise of having historians and philosophers of mathematics on the one hand and mathematics educators on the other present in a mathematics department. Despite Pascal's claim that "pour inventer, il faut penser à côté", the argument goes, these disciplines are either backward-looking or to one side of the mainstream and hence only a hindrance to the fowards march of new mathematical discovery.

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PHILOSOPHY OF MATHEMATICS AND THE PSYCHOLOGY OF MATHEMATICS EDUCATION

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Philosophical positions and epistemological theories related to mathematics, such as logicism, formalism, constructivism, structuralism, empiricism, have always had a significant influence on the guiding ideas and leading principles in mathematics education. This does not only hold for curriculum development and teaching methodology but also for theoretical work and empirical research related to the mathematical learning process. As Seymour Papert has pointed out in "Mindstorms" Bourbaki's theory of mother-structures is learning theory. In a similar way Plato's and Proclus' philosophy of Euclid's elements with its idealistic ontology and its emphasis on the dialectic between analysis and synthesis being the "capstone of the mathematical sciences" comprises the elements of a mathematical pedagogy and of a theory of learning. This phenomenon can be persued throughout all history of mathematics and its concomitant philosophy. Of course, the observation not only holds for global mathematical philosophies but also for epistemological views of particular parts or concepts of mathematics, such as the set-theoretical foundation of the function concept (functions are - nothing but - particular sets of ordered pairs), the logical interpretation of variables as place holders (variables are - nothing but - meaningless symbols to be replaced by names of objects), the interpretation of geometry in terms of Felix Klein's Erlanger Program (geometry is the theory of invariants of special transformation groups) etc. Every structure of knowledge does carry its specific learning structure.

On the other hand, considering the inverse relation, it is also true what has been said by René Thom in his 1972 Exeter speech: "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics." Such "philosophies" may consist in a teacher's "private" opinion on the nature of mathematics and mathematical knowledge (often indirectly acquired in his own academic studies), and in his thoughts about how this is related to his teaching and to the learning of his students. They are also inherent in didactical principles such as the spiral approach (Bruner), the deep-end principle (Dienes), the operative principle (Aebli), etc. They underlie theories of stages in the learning process (Piaget, van Hiele) or theories of transfer, and they also stand behind hypotheses of empirical studies on how children learn of fail. As was pointed out by Thomas S. Kuhn, each research paradigm has philosophical and ontological assumptions among its different components, and these are parts of the paradigm's bias, its strength and weakness. Often they are a matter of profound discrepancies between competing approaches and of controversies between research groups or individual researchers.

Recent developments in mathematics education show a new dynamics in the field. New philosophies and epistemological theories have entered the scene: the theory of epistemological obstacles (Bachelard, Brousseau), a synthesis of Kuhnian theory dynamics and Piagetian genetic epistemology, a new quasi-empiristic philosophy of mathematics (Lakatos, Sneed, Jahnke), a complementarist philosophy of mathematics (Kuyk, Otte), the epistemology of "microworlds" and of the "society of mind" based on cognitive studies within research on Artificial Intelligence (Papert, Minsky). They claim to provide a better background for an investigation of pupils' real learning behavior both from a cognitive and social point of view. On their bases and supported by new empirical data strong criticisms have been put forward especially against positions elaborated in close connection with the "new-math philosophy" the deficiencies of which are being characterized by terms like "Jourdain effect", "Papy effect", "Dienes effect" (Brousseau), and by exhibiting their one-sided "mentalistic" orientation guided by the assumption of an autonomous, universal, coherent, and homogeneous nature of the human mind (Jahnke), their "puristic" and "static" view in neglecting the intended applications which belong to mathematical theories and concepts in a way similar to physics, in neglecting the representational, the social, the procedural, and the processual dimensions of mathematics, and in overemphasizing the structural and isomorphy concepts when trying to organize and understand mathematical learning.

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N. SUPPLEMENTARY PAPERS

CHILDREN'S MATHEMATICAL FRAMEWORKS

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The data from the CSMS research project (Hart, 1980; Hart, 1981) showed that large numbers of secondary age children gave the same incorrect answer when asked to solve certain mathematics problems. The written testpapers of CSMS provided the evidence of the error but not the reasoning behind it. "Strategies and Errors in Secondary Mathematics" (SESM) sought through interviews to ascertain the reasons for the high incidence of certain errors and then to provide ideas for teachers to use to eradicate these errors (Booth, 1984; Hart, 1984).

Evidence had been obtained from the CSMS interviews (n = 30 on each of ten topics) that children of 15 years tried to solve problems set in the secondary school by methods they had learned as seven year olds. These methods being perfectly adequate for the type of question on which they were expected to work when young but only partially so on the more advanced items presented to them in adolescence. The SESM interviews further confirmed the very restricted view many of our children had of mathematics and the totally inadequate tools with which they were working.

In secondary school mathematics pupils are expected to work within a formal mathematical structure, using a formalised symbol language with which to communicate their ideas. The advent of this expectation can be delayed but at some stage before the age of 15 teachers, textbooks and parents suppose that the child has made the transition and now works within this formal mathematical framework. It is possible that many of the misconceptions and instances of erroneous reasoning come about when the child is expected to move from the concrete, tangible world in which he has been learning to the formalised world of mathematics proper. British Primary and Infant School teachers are encouraged to use concrete materials to introduce the early ideas of mathematics. The use of such aids is seen as "a good thing" since it accords with Piagetian theory and there is research evidence to show that at least in the introduction of some topics, this method of presentation is effective. How the teacher with the children takes the step from the concrete aids to the formal language is a matter for conjecture. How do the teacher and children move from counting the number of squares which cover a rectangle to the formula a x b for the area?

"Children's Mathematical Frameworks", (CMF), a further research project at Chelsea College has been trying to obtain data which could be used to shed light on how some children make the transition as well as why many do not. Teachers enrolled on courses of further study in mathematics education (masters degrees or diploma courses) are asked to fulfil some of the course requirements by being part of the research and also by analysing their own teaching in an essay. They are asked to prepare a scheme of work for use with a group of children (which may be the whole class), the topic being one in which a formalisation is attempted, as in the case of the area of a rectangle. Prior to the teaching of the scheme, six children in the group are interviewed by the researchers to ascertain their knowledge of the topic about to be taught. The teacher then teaches as planned but just prior to the time when the formalisation is broached the researchers again interview the six pupils already identified. The lesson(s) in which the transition from concrete/practical work is to be made, is attended by a researcher and tape recorded. Immediately after the lesson the six children are again interviewed to see whether they have moved (as the teacher hopes) to the new level and then three months later they are interviewed to find in which mode of working they are operating.

The data are in the form of transcripts and the analysis is time consuming and still in progress. The participating teachers are given the set of initial interview transcripts and the transcript of their own lesson(s) to form the basis of their analytical essay. They are always surprised at what they said during the lesson and often identify sections that they did not intend to say.

Many questions are raised on what the children actually take out of the lessons and positively harmful practices are sometimes identified. An example of the latter is in the writing of the common addition algorithm. British children are usually taught a formalisation which requires any "one" carried from a lesser column to be written smaller than the digits in the sum, as in:-

$$\begin{array}{r} 17^{+} \\ \underline{35} \\ \underline{2} \\ {\scriptstyle 1} \end{array}$$

Ailsa (aged almost 8 years) explained what she had done in answer to questions posed by the interviewer (I: Interviewer; A: Ailsa).

I: Um.. shall we go back to that little 1 that sprouted up there. You said, put down the 2 and carry 1, and I said what .. where did that 1 come from, what sort of 1 is it?

A: It's a carrying 1

I: I see, is it the same, you carried it with that big 1 there, is it the same sort of value as that big one?

A: No

I: Oh, it's different, is it?

A: Yes, because it's smaller

I: Because you drew it smaller?

A: Yes

I: I see, what is this big 1 here worth then, 1 what?

A: 1 ten

I: I see, and what's that little 1 worth?

A: Just 1... not 1 ten

I: Not 1 ten, just a 1

A: Yes

I: I see, and is that... alright? How about the 3, what sort of 3 is that?

A: 3 tens

I: I see, so you now start adding up 3 tens and 1 ten and a little 1, is that right?

A: .... 41 ...

I: Oh

A: Because um... because 30 and 1 ten is 40 and if you put the 1 there it's 41

I: Well, you write out, write under here then, what your answer's going to be

$$\begin{array}{r} 17^{+} \\ 35\phantom{^{+}} \\ \hline 412\phantom{^{+}} \\ \hline {}_{1}\phantom{^{+}} \end{array}$$

The fact that the 'ten' is written in a different way would take on significance for many adults let alone a seven year old.

QUESTIONS

In order to promote discussion at PME 8 the following questions are asked:

1) Do we use concrete aids/practical work in order to facilitate learning at that (pre formalisation) level?
2) Do we see the formalisation as a generalisation of a number of concrete experiences and therefore only a step away?
3) Do we use the practical work in order to convince the child of the reasonableness of the formalisation which in fact is a large leap away?
4) Do we expect the child who has trouble dealing with the formalisation to be able to step back and reconstruct the concrete experience (after a period of time)?

Discussion of the last question will be illustrated by data obtained three months after the teaching when the children were asked to explain using concrete materials.

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LEARNED HELPLESSNESS IN ARITHMETIC AN ATTRIBUTIONAL APPROACH TO INCREASED SELF-EFFICACY AND DIVISION SKILLS

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ABSTRACT

The impact of attribution retraining with its relatively short history in psychological and educational research has not yet been adequately explored or understood. This study centres upon investigating the effects of modifying attributional predispositions on achievement outcome. It is hypothesised that such modifications induce concomitant changes, both in attributional perspectives and in achievement outcomes. Altered attributional perspectives influence achievement directly and also indirectly through the mediating variable of self-efficacy.

INTRODUCTION

Recent research indicates that some of the observed behaviour displayed by low-achieving mathematics students may be explained by the antecedent factors and by-product educational outcomes emphasised by Attribution theory and Self-Efficacy theory. The detrimental effects on behaviour and achievement outcome for students displaying learned helplessness characteristics have been documented in a number of studies in which these characteristics were experimentally induced and in naturalistic studies (e.g. Dweck & Reppucci, 1973; Butkowsky & Willows, 1980). In attributional terms, the lack of recognition of the salience of effort expenditure for success or failure and the belief that low personal ability causes failure, promote learned helplessness attitudes. Attribution theory also postules that individuals arrive at causal ascriptions such as ability and effort for success or failure outcomes through the use of informational cues (Weiner, 1977, 1979). Continued failures in particular tend to induce attributions to lack of ability and the belief in effort-outcome covariation is not held by individuals displaying learned helplessness characteristics. Thus marked performance decrements follow negative outcomes and causes for failures, rather than remedies, are dwelt upon (Dweck & Reppucci, 1973; Diener & Dweck, 1978, 1980).

Effort in particular has come under close scrutiny in experimental attribution retraining studies because of its dimensional characteristics, internality and instability, both of which make it more amenable to change than the other causal factors. A series of studies (e.g. Dweck, 1975; Chapin & Dyck, 1976; Schunk, 1981, 1982) proposed to induce behavioural and/or achievement changes through direct retraining of effort attributional stances. Schunk (1981, 1982), although using in part attributional techniques, postulated that such changes were affected through the efficacy variable. His conceptual focus for this hypothesis was Bandura's theory of self-efficacy (Bandura, 1977, 1981, 1982), which deals with judgments about how well courses of action may be organised and executed under ambiguous, unpredictable or stressful situations. Such judgments are based on previous performance histories, socially comparable vicarious experiences, verbally persuasive information from respected sources, and/ or physiological arousal. Schunk's (1981, 1982) results point to the strength of the self-efficacy theory and the relative impotence of effort attribution retraining in inducing achievement changes. These results also failed to demonstrate a connection between attribution and self-efficacy leading to the conclusion that effort attribution retraining had no significant effect on either perceived efficacy or on arithmetic performance.

The purpose of this study was to extend the work in attributional modification particularly that of Schunk (1981, 1982) through an empirical illustration of the association between attribution, self-efficacy and achievement outcome. It is therefore generally hypothesised that the weakness inherent in retraining procedures used in past research lay in the failure to implicate the importance of the attribution variables in bringing about changes in self-efficacy and thereby indirectly inducing positive behavioural and achievement changes in outcome. The treatments proposed for this study involve strategies that enhance the development of skills and self-efficacy (Bandura, 1977, 1981, 1982) as well as changes in attributional patterns and persistence (Dweck, 1975; Chapin & Dyck, 1976; Andrews & Debus, 1978; Fowler & Peterson, 1981; Medway & Venino, 1982).

The proposed treatments were constructed in the light of previous research findings and it is hypothesised that training in attribution to the causal factor of lack of effort for failure coupled with appropriate ability information and conceptual skill training through modelling or self-instructional practice will significantly increase persistence, self-percepts of efficacy and mathematical achievement in division. Groups which receive attribution retraining as well as skill training are also hypothesised to display significant differences in means on the helplessness index both within groups from pretest to posttest and between non-attributional treatment and control groups after treatment. In addition, it is hypothesised that positive changes in achievement performance are causally related to changes in attributional perspectives. These changes alleviate motivational and performance deficits which consequently have effects on achievement that are mediated through the efficacy variable.

METHOD

Subjects

The sample consisted of 562 year 6 children, 301 boys and 261 girls, attending one of eleven private primary Catholic schools in metropolitan Sydney. These schools were systematically chosen to represent different geographical areas of the city and to provide a good cross section of socio-economic status and academic achievement. Of these 562, a total of 84 (42 boys and 42 girls) who exhibited the lowest score profiles on a battery of affective and cognitive tests, were chosen to participate in the study. The 84 subjects were stratified on sex and helplessness score then randomly assigned to each of four treatments or a control group. There were 14 subjects per treatment and 28 control subjects. The final sample consisted of year 6 boys and girls ranging from 11 yrs 0 mths to 12 yrs 10 mths with mean age of 11 yrs 7 mths.

The Research Design

The treatment and training techniques were formulated in consideration of teaching the learning styles (modelling vs self-instructional practice) as well as attribution retraining techniques. Adaptations and improvements on all previously employed effort feedback contingencies were implemented with inclusions of ability feedback and tas difficulty information. Performance changes were analysed through repeated measures techniques. Unlike previous research, this study predicts changes not only in behavioural attributional characteristics, efficacy and persistence but also specific arithmetic outcome changes which are indirectly influenced by changes in attributional stance. Path analysis was implemented to verify the efficacy of proposed causal models.

Instruments and Materials

To determine which subjects within the complete sample, who had low scores on a division test, were most likely to require some form of attribution retraining, a scale was constructed from three Arithmetic Specific Attribution Scale (ASAS) subscales which best reflected learned helplessness attitudes. Its validity was confirmed through factor analysis as it displayed significant loadings on both the effort-failure factor and the internal success factor (Relich, 1983).

In order to gauge persistence then, each subject was required to solve six problems presented in ascending order of difficulty and paralleling as closely as possible the types of problems which make up the Division achievement test and which were used within the retraining procedures (Schunk, 1981). An upper limit of four minutes was allowed on each problem. Time on task was measured in seconds and had a thoeretical range of from zero to 1440 seconds.

The self-efficacy measure consisted of 18 pairs of arithmetic division problems which were successively shown to students who were asked to rate their perceived chances for problem solution. Each member of a pair of problems was of similar form reflecting the content of the Division skills test. This test comprised of 18 division problems graded in ascending levels of difficulty. Difficulty levels were identified according to the number of digits in the divisor, presence of a remainder, and the make up of the quotient with reference to zero values.

For this phase of the study it was also considered essential that some form of skill acquisition treatment be included in the retraining procedures. Several sets of instructional materials were therefore also developed for use during the treatment sequences. A division methodology booklet was prepared which outlined the long division method. Division problems were also devised to be used during the training sessions. Problems were designed at three levels of difficulty.

PROCEDURE

On the first of eleven days of experimental intervention two pretests were conducted; an administration of the self-efficacy measure and a measure of persistence defined as time on task. This was followed by eight training sessions lasting approximately thirty minutes on successive school days. Days 10 and 11, the final two days, were required for post-testing. On the first day each subject was required to see the experimenter (the author) for a half-hour session. All subjects were told that they were being requested to aid, over a two week period for 30 minutes each day, in the production of a book on teaching how to divide using the *long* method. Each subject was also informed that they were chosen because their performance on the division test was poor but that all other measures were indicative of at least average ability and that, with some additional tutoring and the acquisition of proper division skills, they could improve on division performance.

The subjects were then stratified within sex and randomly assigned to one of four treatment conditions and a control group which was double in size in comparison with each individual treatment group. The four treatment groups were Modelling (M), Self-Instructional Practice (SP), Modelling with Attribution (MA), and Self-Instructional Practice with Attribution (SPA). Modelling procedures involved direct observation by the subjects of the experimenter solving and verbalising solutions to division problems followed by practice sessions.

Self-instructional practice included a review of the instructional material without experimenter intervention followed by practice sessions. Attribution retraining when coupled with either of these treatments sought to inculcate all the participants with the impression that success was being achieved on progressively more difficult problems and that this was resulting from at least average intelligence and high effort output.

A success schedule on division tasks was devised which assured an overall performance rate of about 75-80 per cent success and in which performance was a function of increasing success through a manipulation of problem type and time restrictions for inducing failure. Feedback contingencies included: information on task difficulty which was peer normed; attribution feedback which was effort and/or ability related and verbalised by the experimenter or elicited from the subject; and performance on division tasks. A number of individual and group posttests were completed by all the subjects over a period of two days. Control group subjects participated on only the first and last two days of the study.

RESULTS

Repeated Measures Analyses

Repeated measures analysis in conjunction with regression and path analysis were implemented to determine the effectiveness of the treatment programs on division achievement both directly and indirectly through attributional, self-efficacy and persistence variables. Two experimental paradigms were used. The first analytical paradigm, referred to as five group analysis, was carried out using one Control (C) group and the four treatment groups because the question of interest centred on intragroup and intergroup comparisons. The second analytical paradigm was designed to investigate the relative strengths of the attributional and non-attributional treatments by combining the two attributional treatments (MA & SPA) and the two nonattributional treatments (M & SP). Means and standard deviations on pretest and posttest results for the four criterion variables are given in Table 1.

The treatment analyses effectively displayed the general effect of the treatments on a number of criterion variables. For division outcome and self-efficacy, all the treatments were individually powerful enough to elicit significant positive increments from pretest to posttest (Table 2). On the persistence variable this trend was evident for three groups only but the lack of initial equality among group means on persistence pretest scores obviates any firm conclusions being made about significant persistence changes. For the learned helplessness index pretest to posttest results were not as powerful and manifested themselves only when analyses were performed on additively constructed attributional and nonattributional groups (Table 2). In essence, for within treatment comparisons, all treatments were useful in raising significantly the achievement scores on

TABLE 1 Means and standard deviations on pretest and posttest results for four criterion variables by experimental condition and sex

Treatment		Modelling with Attribution				Self-Instructional Practice with Attribution				Modelling				Self-Instructional Practice				Control			
Occasion		Pretest		Posttest		Pretest		Posttest		Pretest		Posttest		Pretest		Posttest		Pretest		Posttest	
Criterion variable		Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls
Division	M[a]	5.57	8.00	11.57	13.17	5.86	7.29	9.29	14.71	7.00	6.71	11.29	12.43	6.86	6.57	10.43	8.43	6.79	8.21	7.57	9.79
	SD[b]	.98	1.26	4.69	2.32	1.86	3.15	2.53	2.50	1.15	2.43	2.93	4.20	1.68	2.57	3.99	3.41	2.19	2.75	3.30	3.17
Self-Efficacy	M	79.43	112.17	137.71	139.17	93.00	113.00	111.71	146.00	108.29	83.29	126.43	124.57	86.29	116.14	104.86	145.43	96.57	114.00	103.50	122.86
	SD	31.20	21.74	32.61	15.72	34.91	25.94	44.77	10.30	13.77	43.72	17.02	47.38	27.26	36.88	33.69	38.34	33.77	32.21	31.31	34.55
Persistence	M	257.57	272.17	479.71	417.83	488.86	455.00	464.71	411.86	272.86	305.00	448.57	458.43	182.71	361.29	403.86	434.29	466.57	363.07	393.86	346.43
	SD	199.56	147.75	154.54	185.06	140.86	162.62	177.89	145.11	210.99	222.58	182.61	138.11	61.26	203.19	296.31	120.74	204.42	189.33	196.00	171.03
Learned Helplessness	M	14.71	18.17	23.43	21.50	16.14	19.57	22.00	25.71	20.29	17.86	19.71	19.71	19.29	16.14	21.71	19.57	20.14	20.00	20.64	22.14
	SD	12.16	3.49	7.93	2.51	6.84	6.60	4.24	7.61	4.82	6.18	4.11	5.28	6.13	10.24	4.61	6.90	4.28	3.46	6.99	4.94

[a] M = Mean

[b] SD = Standard deviation

division and percepts of self-efficacy. The effects of treatment on persistence, in contrast, did not emerge clearly while only the combinational attribution treatments elicited positive changes on the learned helplessness index from pretest to posttest (Table 2).

Differences between treatments were not as substantial as originally hypothesised with three of four treatments, excluding SP, displaying significant superiority on posttest division outcome results when compared to the control group. There were no observed differences across treatments suggesting that both skill training through modelling and attribution retraining are equally effective means of raising standards on achievement in division. A similar superiority of the combinational treatment groups over the control group was observed. With reference to self-efficacy, only the Modelling with Attribution (MA) treatment individually and the two attribution treatments when combined in the analysis displayed significant superiority of self-efficacy gains over the Control group.

This result suggests that percepts of self-efficacy are not as easily changed after a period of attribution and skill training as is achievement on a specific task even though successes are experienced on the task. Between treatment mean differences on persistence are again not clearly delineated due to the nature of the measure and the inequitable pretest means between treatment groups. On the learned helplessness index no specific between treatment effects were evident from pretest to posttest but there was a clear, significant superiority displayed by the attribution treatment subjects over the nonattribution and control group subjects on the posttest learned helplessness scores.

These results suggest that although skill training and attribution retraining are effective in increasing achievement on a specific task such as division, the effects on mediating variables are not as pronounced. Therefore, the indirect influences of treatment on achievement may be less substantial than originally considered. The processes involved in changing percepts of self-efficacy most probably require much more time than that allotted for the retraining portion of this study. It is also interesting to note that the most powerful treatment in producing significant posttest self-efficacy scores compared to control group scores was a combination of modelling and attribution retraining. This positive effect on self-efficacy coupled with this treatment's powerful effects on achievement changes offers a persuasive argument for the inclusion of attribution retraining within teaching programs.

The above analyses have been useful in clarifying the effects of different treatments on a number of interrelated variables but they are not sufficiently sophisticated to reveal the underlying causal connections between variables. The more powerful regression techniques are used in the following section in order to analyse the effects of variables on each of the dependent variables; division outcome, persistence, self-efficacy, and learned helplessness. A hypothetically derived path model is then empirically tested in order to check its consistency with the obtained data.

<u>Path Analyses</u>

The proposed path analysis is a methodological tool for theory testing. Some paths may be assigned values of zero and therefore omitted from the model on the theoretical rationale that one variable does not directly influence another variable. Deleted paths in the model indicate that the researcher has hypothesised that a correlation between variables may be explained in terms of spurious and indirect paths rather than through a direct path. The direct path is therefore arbitrarily set at zero. This more parsimonious model may then be tested through a reconstruction of the original correlation matrix by recalculating the correlation between variables through the addition of direct, indirect and spurious effects. Providing that correlational values within the restructured matrix reflect the values of the original correlations within acceptable statistical levels, it may be concluded that the data are consistent with the proposed model. Although this does not constitute proof of the model, path analysis contributes to theory testing through the rejection of hypothesised models (Kerlinger & Pedhazur, 1973). It is generally recommended that large sample sizes be used in path analysis. The limited number of subjects within this analysis therefore calls for considerable caution in the interpretation of results.

In order to establish the separate influences of attribution retraining and skill training on achievement outcome, two models were hypothesised. The treatment variable in one included a comparison of both skill trained and attributionally retrained groups (MA & SPA) with a control (C) group (Fig. 1). In the other, skill trained only groups (M & SP) were contrasted with the control (C) group (Fig. 2). The direct effects of treatment on division

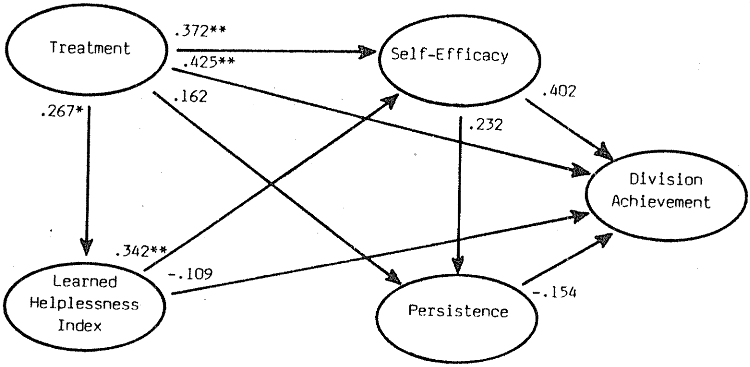

FIGURE 1 The accepted path model contrasting attribution and skill trained (MA & SPA) group subjects with control (C) group subjects
** p < .01 * p < .05

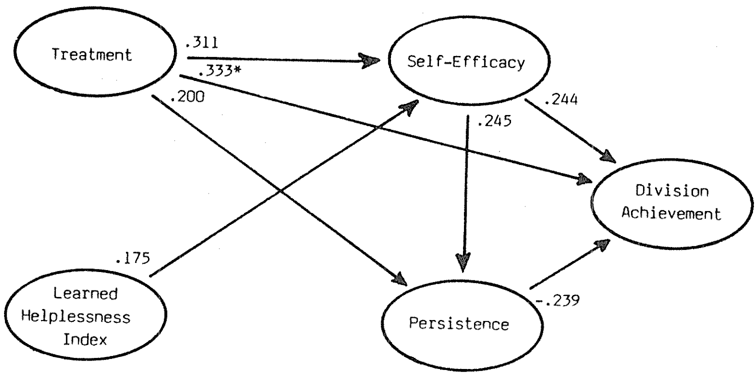

FIGURE 2 The accepted path model contrasting skill trained (M & SP) only group subjects with control (C) group subjects
* p < .05

achievement in both these instances were hypothesised to be significant. The indirect effects when attribution retraining was involved, were hypothesised to be through both the learned helplessness and self-efficacy variables but only through the self-efficacy variable when skill training was contrasted with no training. It is unlikely that treatment conditions which do not include some component of attribution modification would have any direct influence on an attributional variable such as the helplessness index and thereby have an indirect influence on achievement.

To operationalise the models, the categorical attribution variable described the treatment. For the other variables residual scores were used rather than change scores which tend to be unreliable (Cohen & Cohen, 1975). The attribution residual for the model was confined to the helplessness rather than any other attribution index because the largest change in attributional perspective was expected on the helplessness index. For self-efficacy, persistence and division outcome residual scores were calculated from pretest and posttest results.

Reconstruction of the correlation matrix (Table 3) suggests an acceptable fit of the hypothesised path model to the data. Further regression analyses support the hypothesised direct influences such as, treatment to achievement ($p < .01$), treatment to self-efficacy ($p < .01$), treatment to the learned helplessness index ($p < .05$), learned helplessness to self-efficacy ($p < .01$) and self-efficacy to achievement ($p < .01$), which are all significant.

This model (Fig. 1) delineates the role of attribution and skill training from the no treatment condition while affirming the importance of efficacy as a mediating variable for attributional perspectives and treatment effects. It does not, however, address the question of whether these direct and indirect influences on division achievement result from the attributional or skill training portions of the treatments. Therefore, to unravel the influences of treatment type, another model was hypothesised which contrasts skill trained only subjects (M & SP) from control (C) subjects.

Some differential effects from the previous model may be observed (Fig. 2). The only significant direct influences is from treatment to achievement ($p < .05$). The comparison of the two models illustrates clearly the differential effect of including attribution retraining in conjunction with skill training as a teaching strategy in contrast to using only skill training.

TABLE 2 F[1] values for the repeated measures analysis of variance on the set of dependent variables

Source of variation	Division		Persistence		Self-Efficacy		Learned Helplessness	
	I[2]	C[3]	I	C	I	C	I	C
Treatment	1.62	2.15	1.05					
Sex	7.37**	7.37**			5.56*	5.52*		
Treatment x Sex	1.98	3.24*		1.41	1.08			1.19
Occasion	90.93***	86.02***	8.80**	8.15**	25.79***	25.68***	16.79***	17.23***
Treatment x Occasion	6.76***	11.23**	6.38**	7.48***	2.15*	3.46**	2.30	4.29*
Sex x Occasion	1.08	1.02			1.01	1.00		
Treatment x Sex x Occasion	1.52			1.02	1.10			

* $p < .05$ ** $p < .01$ *** $p < .001$

[1]Values < 1 are excluded

[2]F values for the 5 group individual treatment repeated measures analyses

[3]F values for the 3 group combinational repeated measures analyses

TABLE 3 Original and reproduced correlations for the two path models

	Variable	1	2	3	4	5
1	Treatment		.267[1] -.017[2]	.463 .308	.270 .275	.541 .343
2	Learned Helplessness	.267 .000		.441 .170	.077 .020	.170 -.056
3	Self-Efficacy	.463 .311	.441 .175		.308 .306	.503 .274
4	Persistence	.269 .276	.153 .043	.307 .307		.076 -.073
5	Division	.541 .343	.159 .032	.504 .274	.073 -.072	

[1]Contrasts attribution and skill trained group subjects with control group subjects

[2]Contrasts skill trained only group subjects with control group subjects

Note: Original correlations are in the upper half of the table; reproduced correlations are in the lower half.

Those treatment groups which included attribution retraining also demonstrated clear and significant effects on the attribution variable through diminished learned helplessness orientations. Such orientations, in turn, significantly and directly affected self-efficacy and therefore have indirect effects on achievement as hypothesised. When only skill trained group subjects were compared to the control sample, the effects of treatment were only direct on achievement, while the effects on the attribution variable, learned helplessness, were negligible. Consequently, in these analyses, attribution perspectives did not significantly affect either self-efficacy or achievement.

Consideration of the path analyses indicates that the effects of attribution retraining and of attributional perspective on self-efficacy and achievement are in the hypothesised direction. Self-efficacy displays a consistent and significant direct influence on division achievement. Attribution, although displaying some moderate direct influence on achievement, has indirect influences on achievement which are mediated by the self-efficacy variable.

DISCUSSION

All of the above results comply with findings in previous recent research and add further verification to the theoretical expectations for the saliency of attribution retraining (Dweck, 1975; Andrews & Debus, 1978; Medway & Venino, 1982; Schunk, 1981, 1982). None of these previous studies, however, adequately illustrated the interrelationships among these variables and in particular the causal connections between attributional perspectives, changes in self-efficacy and persistence and, consequently, changes in achievement.

As a result the main tenet of this study involves hypothetical propositions about the effects of treatment on achievement directly and indirectly through attributional, self-efficacy and persistence variables. In particular, the question of interest revolves around the mediating function of self-efficacy in inducing changes in achievement. The proposed models alternate the treatment variable by presenting two contrasting scenarios. The two models clearly illustrated the mediating functions of percepts of self-efficacy. Treatment directly influences the attributional variable, the learned helplessness index, and both in turn directly influence the self-efficacy variable, itself demonstrated to have strong and pervasive direct influences on achievement and an indirect influence through persistence. The effects of treatment were clearly outlined by path models which contrast treatment group subjects with control group subjects because skill training alone is an effective educational process for inducing positive achievement changes. If accompanied by attribution retraining, however, treatment effects on achievement are mediated not only by positive changes in percepts of self-efficacy but also by the alleviation of learned helplessness states through attributional perspectives which emphasise the salience of effort in success and failure situations. Therefore, although skill training in itself is sufficient to induce direct positive changes in achievement, the added indirect influence through attributional and efficacy changes presents an arguable case for including such modification techniques in the remedial arithmetic classroom.

The success of the treatments, whether modelled or presented for self-instructional practice, in ameliorating behavioural and achievement variables is due, to a large extent, to the hierarchical mode of presentation of steps necessary for the successful solution of division problems. As well, the treatment program was devised to result in moderate but ascending levels of success from the first to the last day. These are of course educational practices which are prewent in the repertoire of every good teacher and therefore it is not surprising to observe the apparent success of subjects in attribution treatments and/or skill training treatments in contrast to subjects receiving no treatment. The lack of differentiation of effect among treatments presented some problems in extracting the effects of skill training from attribution retraining although this was successfully accomplished through path analysis.

Educational implications for teachers of arithmetic include an awareness of the necessity not only of presenting material in a hierarchical, logical sequence but also of the need to monitor students' progress by presenting material at a level commensurate with their abilities and to reinforce both successes and failures through references to causal attributions of effort and ability. Such references are most effective if related to immediate past experiences rather than future events (Schunk, 1982) and when directly solicited from individuals (Fowler & Peterson, 1981). The success of the treatments used here are indicative of the limitations of educational practices which would seem to produce the less desirable educational consequences. Practices such as lock step sequence of instruction which disadvantage the less able; ability groupings which assure success for the more able students but induce reduced levels of self-efficacy and failure in the less able; and competitive rather than co-operative or individual learning (Bandura, 1981) are all in antithesis to the methodology and consequences of attribution retraining programs.

Future research is required in the area of generalisability although, in one sense, the definition of self-efficacy precludes the expectation that increased percepts of self-efficacy in a specific context are necessarily transferable to another. Nevertheless, it would be interesting to note if in the general area of arithmetic there is a parallel change in helplessness perspectives, persistence and self-efficacy when a complementary basic skill task, for example multiplication, is presented to the student (see Stokes & Baer, 1977). This would seem unlikely with reference to self-efficacy although theoretically changes in attributional perspectives would be expected to be more permanent and may therefore influence percepts of self-efficacy and achievement.

CONCLUSION

Together with Schunk's (1981, 1982) recent work, this study provides further verification that attributional feedback promotes percepts of self-efficacy and arithmetic achievement and is therefore relevant to remedial teachers in arithmetic.

These results further verify that the restructuring of attributional perspectives through reinforcement under normal learning conditions may facilitate ongoing achievement activities by inducing positive changes in percepts of self-efficacy. The training procedures involve a well structured presentation of teaching material in conjunction with verbal reinforcement through effort and ability attributional inputs. For the student, the recognition of the salience of effort coupled with a more positive attitude towards personal ability that is reinforced through moderate success is the first step towards remolding perceptions of past performance histories which have been demonstrated to account for such a large percentage of the association between attributional perspectives and achievement.

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PREFACE

As usual the Proceedings of PME8 will be published in advance of the conference, the book being produced directly from typescripts submitted by the authors. This has a number of advantages.

* The Proceedings are published quickly.
* Speakers can (and are encouraged to) talk around the printed material: clarifying, amplifying and putting into context. Discussion can be more meaningful and form a larger part of the working sessions.
* Participants have a detailed version of a speaker's contribution and can make more informed decisions about which sessions to attend and whom to consult.
* The expensive procedure of packing and posting volumes of proceedings is avoided.

It also has disadvantages.

* Some papers do not get into the Proceedings. (This is not always the author's fault.)
* The book ends up being a book designer's nightmare, with great variations in layout and quality of "setting" from one page to the next.
* The tight production schedule makes it difficult to incorporate features (such as running headings) which make a book easier to use.

While the advantages outweigh the disadvantages, a really satisfactory "Proceedings" will only come about when the disadvantages are reduced: some authors will need to take more care in the preparation of their typescript and in meeting deadlines.

The papers in this volume are arranged under the following headings.

A. Plenary Addresses
B. Theories of Teaching and Learning
C. Cognition/Cognitive Theory
D. Representational Modes
E. Problem Solving
F. Operations
G. Language
H. Proportional Reasoning
I. Rational Number
J. Girls and Mathematics
K. Attitudes
L. Technology
M. Theory

Within these groupings, papers are ordered (alphabetically) by first named author. There is also a section for papers which, for one reason or another, were received too late to be placed under their correct heading, N. Supplementary Papers. These appear under the appropriate heading in the table of contents.

A short description and history of PME and of the local organising committee follows this preface.

The editorial committee thanks all involved in the production of this volume: the authors and their typists, the Mathematical Association of New South Wales (for layout and page makeup), the printers and the binder.

Beth Southwell, Roger Eyland, Martin Cooper,
John Conroy, Kevin Collis

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

At the Second International Congress on Mathematics Education (ICME2) at Exeter in 1982, Professor E Fischbein of Tel Aviv University, Israel, instituted a working group bringing together people working in the area of the psychology of mathematics education. At ICME3 (Karlsruhe, 1976), this group became one of two groups affiliated with the International Commission for Mathematical Instruction (ICMI).

The first conference of the group was held in Utrecht, Netherlands, in 1977. Since then international conferences of the group have been held annually: Osnabruck (West Germnay), Warwick (England), Berkeley (USA), Grenoble (France), Antwerp (Belgium), and Shoresh (Israel).

The major goals of the group are:
* to promote international contacts and exchange of scientific information in the psychology of mathematics education;
* to promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
* to further a deeper and more correct understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Membership is open to persons involved in active research in furtherance of the group's aims, or professionally interested in the results of such research.

Officers and committee members

President:	G Vergnaud (France)
Vice-President:	K Collis (Australia)
Secretary:	L Burton (UK)
Treasurer:	G Becker (West Germany)

Other committee members:

N Balacheff (France), M Behr (USA), J Bergeron (Canada), T Dreyfus (Israel), E Filloy (Mexico), K Hasemann (West Germnay), R Hershkowitz (Israel), C Hoyles (UK), F Lowenthal (Belgium), J Moser (USA), L Streefland (Netherlands), A Vermandel (Belgium), S Vinner (Israel).

THE LOCAL ORGANISING COMMITTEE

The local organising committee for PME8 came into being as the result of an approach by the conference convener, Prof Kevin Collis, to the Mathematical Association of New South Wales (MANSW). The Association was pleased to accept his invitation and the local organising committee was drawn from its officers and members, drew on its financial resources in the initial stages and drew on the technical resources of its publishing division throughout the preparation period. It is expected that (in addition to the usual sources) copies of this book will be available from MANSW for some years after PME8.

MANSW began in 1910 as a branch of the Mathematical Association (of Great Britain) and thus antedates all other Mathematical and Mathematics Education associations and societies in Australia. In the 1960's, MANSW was one of the founders of the Australian Association of Mathematics Teachers (AAMT): it remains one of the largest associations affiliated with AAMT. AAMT provided a major part of the effort which went into the campaign to have Adelaide selected as the site for ICME5; its officers and members have been heavily involved in the organisational work for ICME5.

Local Organising committee members:

Mary Barnes, John Conroy, Martin Cooper, Mary Coupland, Lloyd Dawe, Roger Eyland (Publications), Kevin Ford, Joe Relich (Treasurer), Beth Southwell (Chairperson), Mary Werhoven, Margaret Whight, Joan Wilcox.