

MAKING SURE THAT MATHEMATICS EDUCATION
RESEARCH REACHES THE CLASSROOM

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for the Psychology of Mathematics Education

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EDITORS

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A – J



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**MAKING SURE THAT MATHEMATICS EDUCATION
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RESEARCH REPORTS
A – J

ZPD SUSTAINED BY EMOTIONAL TRAJECTORIES

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In this paper we investigate the emergence of the Zone of Proximal Development from an emotional perspective. We argue to broaden the evidence of learning and thinking by looking beyond the words. Building on Dewey's notion of experience as driven by, featuring emotional intensity, we liken this to learning as internal integration and fulfilment in the ZPD where emotional trajectories generate, sustain and 'fulfil' the learning in the ZPD. We build this argument through analysis of a learning episode between a mother, child and a television remote control following the child's stated discovery of counting in threes. We conclude by suggesting that emotional trajectories are an integral part of the emergence of the ZPD and as such should be included in analysis of learning.

EMOTIONS: NOT SUPPLEMENTARY BUT CORE TO LEARNING

For decades, studies that have used theories of embodiment, encouraged researchers to, while working with the data, look beyond what is said. Instead, we are asked to also attend to gestures, postures and gaze (Sinclair, 2023, 2024 a, b). Sinclair points to Fuller and Weitzman's (2021) conceptualisation of aesthetics and Dewey's understanding of experience to argue that we need to, in the communication of our research, evoke in readers aesthetic experiences and sensory knowing. Her aim by this use is to push both "the envelope of what it means to know mathematics and also in question[ing] categories and sequences of knowing that are taken for granted in Western culture" (Sinclair, 2023). Like Sinclair, we use Fuller and Weitzman's concept of aestheticizing to explore "new conditions of knowing, seeing and doing" (p. 111). In this paper, we delve deeper to Dewey's (1934) understanding of experience to better understand and expand what it means to learn mathematics within the construct of the ZPD, as a tool for analysis and as way to see if and how learning happens. Our key contribution is to suggest an expansion to the elements at play within the notion of the ZPD going beyond the focus on cognitive development drivers (visible in what is being said and actions during problem solving) to emotive drivers (i.e., what is being emotionally expressed and experienced — visible in expressions — facial and bodily and in tone and volume of what is said). Through an analysis of data we argue that emotions are not by-products in learning. On the contrary, emotions are a central driving element, in their own right, to any learning experience and to the establishment and sustainment of the ZPD.

WHAT CONSTITUTES A LEARNING EXPERIENCE?

We explore what constitutes a ‘learning experience’, by first breaking down the phrase into its components of ‘learning’ and ‘experience’. Then with an example we expand on new conditions of knowing, seeing and learning. In making sense of learning, we use Vygotsky’s (1978) construct of the Zone of Proximal Development (ZPD). The ZPD is described by Vygotsky as ‘the distance between the actual developmental level (independent problem solving) and the level of potential development (problem solving under adult guidance or in collaboration with more capable peers)’ (Vygotsky, 1978, p. 69). For Lerman (2014), the ZPD is ‘the mechanism through which learning happens’ (p. 22). Roth and Radford (2010) saw the ZPD as ‘the emergence of a new form of collective consciousness, something that cannot be achieved if we act in solitary fashion’ (p. 306). The more knowledgeable other (MKO) is an integral part of the learning process. The role of the MKO alternates among participants in interactions (Abtahi, Graven and Lerman, 2017). In making sense of an experience, Dewey highlights common patterns in different experiences, no matter how unlike they are to one another. He explains that there are three conditions without which experiences cannot come to be. The first is some sort of interaction between a live creature and some aspects of the world the creature lives in. The second is the emotions involved in the process of experiencing and unfolding of the experience. The third is the closure, the “mutual adaptation of the self and the objects” (p. 44) of the environment. Therefore, experience is not just doing and undergoing in alterations, it is the relation between the two; that is “a perception of relationship between what is done and what is undergone” (p. 47). In an aesthetic experience, for example, this relation is between an art product as a physical thing and the artwork as an experienced thing. Therefore, an experience is not “seeing and hearing plus emotions” (p. 53). Emotions permeate and pervade throughout aesthetic experiences. Here, we argue and show that emotions permeate and pervade throughout *learning* experiences too.

Combining the ideas of Dewey (1934) and Vygotsky (1978), we expand a learning experience within the ZPD as: space to undergo interactions with the environment, in which problem solving under the guidance of a MOK leads to independent problem solving, guided not only by intellect but also, multi-directionally, by emotions (e.g., eagerness, frustration, excitement, disappointment) and culminating with a sense of fulfillment. This is because a [learning] experience comes to its completeness, only in its unity with emotions (Dewey, 1934).

EMOTIONAL INTENSITY OF A [LEARNING] EXPERIENCE

The relation between what is done and what is undergone becomes apparent in highlighting the complexity of emotional intensity, which Dewey (1934) conceptualises through an aesthetic experience while interacting with an artwork. He explains that an aesthetic experience consists of a ‘series of responsive acts that accumulate toward objective fulfilment’ (p. 58). However, there would be nothing called “fulfilment”, “without internal tension” (p. 143). Further in an aesthetic

experience “things bear upon one another, [in] their clashes and unitings, the way they fulfill and frustrate, promote and retard, excite and inhibit one another” (p. 139).

We believe the same conceptualisation of an aesthetic experience applies to a learning experience. A learning experience also consists of a series of responsive acts (interactions between the teachers and the students) that accumulate toward a fulfilment (mathematics being learnt). The learning of mathematics (fulfilment) happens with tension.

In this paper, we illustrate how an understanding of a learning experience cannot (and should not) be separated from the understanding of the clashes and unitings, fulfilment and frustrations, excitements and inhibitions occurring. In order to do so, we follow Dewey (1934)’s set of conditions for an aesthetic experience: “continuity, cumulation, conservation, tension and anticipation” (p. 143), and, like Sinclair (2024b) we apply these to a learning experience. Our aim is to use Dewey’s conceptualisation of an aesthetic experience to further our understanding of a learning experience, through the ZPD, as a holistically and socially intellectual-emotional practice. Here we exemplify these theoretical concepts.

METHOD

Our paper is based on an opportunity sample of a video transcription of a learning event of one child (5 $\frac{3}{4}$ years, Lila), bringing her mathematical discovery enabled by the layout of the numbers 1 to 9 in three rows of three buttons on a television remote to her mother (Mellony) who engaged further with her on it. Mellony hastily video recorded this learning event on her cell phone. This video was analysed by Graven & Lerman (2014) as learning emerging from the perception of the remote by the child, the articulation of perceptions and actions through sharing with another (mother) leading to the bi-directional understanding of the ZPD. Abtahi, Graven and Lerman (2017) re-analysed the data to suggest the ZPD as a multi-directional construct, conceptualising the mother, the child and the remote as alternating more knowledgeable others. Here we re-analyse it using Dewey’s conditions of an experience, namely: continuity, cumulation, conservation, tension, anticipation and fulfilment. We consider facial and bodily expressions and the content, tone and volume of spoken words as indicators of these.

FINDINGS

In this section we show how emotional trajectories are inseparable parts of this multi-directional ZPD, through analyses of the trajectories of continuity, cumulation, conservation, tension and anticipation. The analysis shows their fundamental role in the learning experience.

1	Mellony	Okay, Lila, what did you want to show me, you had something, no but first show me what did you do?
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2	Lila	I am going to count in threes—look (<i>Lila holds up the remote control towards the camera</i>).
3	Mellony	But how did you work that out show me?
4	Lila	Cause everyone is 3. (<i>she lifts the remote up and starts pointing with her finger from left to right across the rows of buttons 1 2 3; 4 5 6; 7 8 9</i>)

As we see, this experience is guided by different participants, at different times — for example, guided by Mellony “Show me”, by Lila “I count in three” and by the remote in its layout (of numbers 1-9 in rows of three). Yet, we clearly notice that the establishment of the emotional trajectories of willingness to continue, of anticipating, of cumulation and of fulfilment with tones of eagerness, excitement and disappointment at times. The learning experience starts with Mellony’s invitation — show me what did you do [Line 1]. Lila is willing and her expressions and voice communicate eagerness to fulfill the invitation. She does not move her body to go back to watching the TV or change the conversation instead she explains: “I am going to count in threes” [Line 2]. Continuity is established by both participants (both are focused on and responsive to each other’s communication acts showing interest and enthusiasm for the communication in expressions and tone of voice). “How did you work that out show me she”, Mellony asks [Line 3]. Mellony invites both **cumulation**, and **anticipation** responses from Lila. Lila still willing to fulfill the anticipation, responds “Cause everyone is 3” [Line 4]. Lila here fulfills Mellony’s invitation. The experience could then have come to an end. Continuity is formed as Mellony **anticipates** that she might further this experience (influenced by gauging Lila’s **emotional trajectory (interest and excitement) evident from her body language and tone**). “So show me how you are counting?” she asks. She **anticipates** a response which is **fulfilled** by Lila “Three, six, nine, twelve. (*Lila holds her three fingers over each of the number rows 1 2 3; 4 5 6; 7 8 9 then over the 3 buttons below the 7 8 9 as she calls out 3; 6; 9; 12.*)” The numbers 3, 6 and 9 are written on the remote. But not number 12. Mellony continues: And how did you know that 12 my angel?” Lila **fulfills** the **anticipation** by responding: “I don’t know, because that’s 10 (*Lila points to the unnumbered button under the 7*) and that’s 11 (*points to the ‘0’ button under the 8*) and that’s 12 (*points to the unnumbered button under the 9*). Lila again fulfills the invitation. Further, continuity is formed as Mellony **anticipates** furthering this experience. Mellony asks: “what do you think comes after 12 if we are counting in threes?” Lila responds *looking to the side, moving her lips mouthing the words one, two, three, four, ..., twelve, in the rhythmic counting ‘in ones’ way then says aloud* 13. Lila signals fulfillment of the experience and declares 13 with confidence indicated by the accompanied smile, declarative confident tone and increased volume of her ‘statement’ of 13 as ‘the answer’.

Mellony continues: “13 is the number after 12. What if you were counting in threes?” Here new levels of emotional intensity are added to the learning experience, including **tension**. We show these emotions with the photos of Lila (Images 1-2).



5	Lila	I'm not sure. <i>(Lila looks at her mom and puts her finger in her mouth seemingly thinking)</i>
6	Mellony	Have a look on the buttons and make <i>(Lila lifts remote up again and looks at the buttons)</i> and see if they can help you <i>(Lila looking at her mom indicating to her with pointing that there are no more buttons)</i>
7	Lila	There aren't any. Only these. <i>(She points to the bottom row of three non-numbered buttons)</i>
8	Mellony	Can we pretend that there are three maybe?
9	Lila	<i>(All the while looking at the remote and placing her three middle fingers over the four rows of buttons 1 2 3; 4 5 6; 7 8 9; *** and as she does this says):</i> Three, Six, Nine, 12 13 14 15. 15! <i>(15 declaratively with a huge smile as shown)</i>

The **tension** starts as Lila is unsure about the answer (what comes after 12, if we count in threes). Lila says: “I’m not sure” [Line 5]. Mellony notices Lila’s **anticipation** and **tension**. We can see from Image 1 that Lila is not sure. Mellony guides Lila, as she invites Lila to “Have a look on the buttons” [Line 6]. Mellony’s guidance is in turn guided by the emotional trajectory of the experience. As Mellony assumes Lila’s willingness to fulfill the invite, and as Lila **fulfills** this assumption, there still is continued **tension**. Mellony’s guidance is limited by the limitations of the remote control. The physical buttons on the remote cannot guide Lila. **Cumulative tension** grows as she struggled to use the configuration of the remote, Lila is unsure as she says: “There aren’t any [more buttons]” [Line 7]. Mellony (guided by the **emotional trajectory** of the experience) prompts Lila: “Can we pretend that there are three maybe?” [Line 8]. Lila **continues**, imagining there are more buttons, she says: Three, Six, Nine, 12 13 14 15. 15! ([Line 9]. Lila is clearly excited and proud (See image 3) as she declaratively and with a huge smile says: 15. She **fulfils** the experience with excited response of 15, with a smile, triumphantly, marking an emotional intensity and fulfillment of this episode in the learning experience.

Within the construct of the ZPD, the above interactions point to a learning experience, happening through collection of emotional and intellectual mediation from Mellony, Lila and the remote. These interactions consist of a series of responsive acts that accumulate toward fulfilment of the learning task of counting in threes, up to 15, using a remote control that only marks 3, 6 and 9. Lila's declarative response comes at the end of these sequence of events involving cumulation (her positive responses to Mellony's invitation to the continuation of the experience, her unsureness of the physical, configurations of the remote), some tension and anticipation (sharing of her idea with Mellony (i.e., 'there aren't any) when asked what comes after 12), and fulfilment (with victorious pride and a visible sense of satisfaction).

REFLECTION ON OUR ORIGINAL ANALYSIS

We thus reflect on why in our earlier writing we had omitted a focus on the emotions within the learning episode that were highly visible to us. Some of this linked to the limited theoretical tools we had available at the time for including a focus on these that was rigorous. Sinclair's (2024) linking of Dewey's (1934) art as experience to learning experiences provided a way forward on this. Another detractor from a focus on emotions linked to indications from journals that the quality of the photos was not high enough for printing which led us to remove all photos of Lila in the first sharing of the learning episode — including only a high quality photo re-taken just before publication of the remote control (Graven & Lerman, 2014) and line drawing conversions of only three images of Lila's expressions in (Abtahi et al., 2017). This prompts us to question why lower quality data images (as have been included in this paper) should be omitted from journal publications. If images are of sufficient quality to generate an aesthetic experience and sensory knowing for the reader why should they be 'barred' from publication. A final aspect is that at the time of earlier writings we had positioned emotions as factors that colour the learning in the ZPD and that while important would not be critical to communicating that a learning episode occurred (the words and actions of the learner would establish this). Re-analysing this learning episode through merging Dewey's notion of experience and Vygotsky's construct of the ZPD reconstitutes the ZPD to include emotional drivers and mediation as central rather than a peripheral by-product of learning experiences.

DISCUSSION

In this report, we used Dewey's (1934) understanding of 'art as experience' to investigate a learning experience with the ZPD. Noting the elements of continuity, accumulation, conservation, tension and anticipation, we articulated how emotional trajectories are central stimuli for the entire episode. We showed how these emotional elements (such as willingness to engage, being unsure and excitement) guided Mellony and Lila throughout their interactions, leading to sustainment of the ZPD. That is, the ZPD emerges with Lila and Mellony's eager willingness to engage around Lila's 'discovery'. Lila communicates through emotional cues her willingness to unpack her learning following the prompts of her mum. Even with the brief disappointment and

‘unsureness’ of her incorrect answer of 13 and how to correct it, she shows willingness (she stays and responds to the prompts) to engage in solving the problem posed. Throughout the analysis of data, we showed evidence of guidance not just in what is said and acted on (exemplifying thinking) but also in intensity of emotions and types of emotion. Dewey’s (1934) work helped us understand learning in its *unity* of the entire experience. Expanding the nature of learning holistically, intellectually and emotionally within the ZPD—we suggest that emotions are driving and enabling the sustenance of a multi-directional ZPD. The point that we emphasise is that a basic tenet of education is that individual learning is dependent on a learner’s emotional state. However, it should be clear from the outset that this is not merely a statement of correlation between individual learning and emotions. This premise should be interpreted in its fullest form, proposing that qualities and the momentum of thinking are inextricably linked to and generated by the emotional trajectories in the interactions. If not interpreted in its fullest, various challenges may arise.

We conclude by raising some challenges and a note of hope in advocating for learning constructs (like the ZPD) to embrace deeper understanding of the learning experience, emphasizing emotional intensities within learning interactions.

As stated by Dewey (1934), emotional trajectories not only communicate interest and excitement — as it was in the case of Lila — but also often reveal hidden potential barriers, such as frustration, fear, or disengagement. If emotions are ignored or dismissed as less important, it becomes increasingly challenging for teachers to recognize the ZPD —the critical space where students require guidance from their teacher to progress to the next stage of mathematical development.

Further, we argue that focusing solely on what is said or done as a marker of rigorous reporting on a learning experience is both incorrect and misleading (even if unintentional). It is incorrect because it overlooks emotions as integral to thoughtful reflection and action within a context. It is misleading because it distorts the reality of the teaching profession, relegating emotions to the margins and undervaluing their significance.

Finally, we believe that when emotions are not analysed with the same rigor as what is said, we risk creating an image of learning as a limited transactional experience—one that focuses on the exchange of information while ignoring the rich, human complexities that make learning meaningful.

We hope that further research into the understanding of the critical importance of the guidance from the emotional trajectory in the emerging and sustaining of the ZPD could contribute to a more comprehensive understanding of students’ (un)productive tensions and uncertainties in learning mathematics. Additionally, we hope this understanding provide a space for the teachers to support the wholeness of students’ states of being, knowing and learning.

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MISTAKES AND MATHEMATICS EDUCATION: NARRATIVES OF CHILEAN MATHEMATICS PEDAGOGY STUDENTS

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In this report, we present a research project in which we analyse the ways mistakes are experienced, handled, and understood in mathematics education. The project focuses on the perspectives of Mathematics Pedagogy students, who had recent experience as both students and pre-service teachers. We pay particular attention to the affective and gendered dimension of their discourses and practices. We share some preliminary findings related to mistake-handling practices, mistake-related affects, and links between mistakes and gender–mathematics stereotypes. We also discuss some of the theoretical and conceptual tools that we are applying when interpreting these data.

INTRODUCTION

Conceptually, in mathematics education mistakes are a resource to enrich mathematics learning (Alvidrez et al. 2022). Despite this, in practice, mathematics teachers and students continue to treat incorrect answers as something to avoid. At school and elsewhere, the advantages of success and disadvantages of failure are constantly reinforced (Zhuang & Conner, 2022). While this is not only the case in mathematics, a negative view of mistakes is especially prevalent in this discipline, and maladaptive patterns in teachers' responses to mistakes are more frequent than in other subjects (Tulis, 2013). Moreover, due to differential socialization and stereotypes, negative responses to mistakes can have a bigger impact on certain groups, particularly girls and women (Kay & Shipman, 2021).

The research presented in this report delves into the ways mistakes are experienced, handled and understood in mathematics education, as observed in the narratives of pre-service mathematics teachers. We pay attention to the affective dimension of these experiences and perceptions, and analyse the role that gender plays in them.

THEORETICAL FRAMEWORK

Why is a negative view of mistakes especially prevalent in mathematics? There is more than one plausible answer to this question. One is related to the nature of mathematics itself. In mathematics, a logical proposition has two possible truth values that are mutually exclusive: true or false. Tulis (2013) argues that this may be linked with the fact that maladaptive responses by teachers are found more often in mathematics than in other domains.

The prevalence of a negative perception of mistakes in mathematics could also be related to the place of the discipline in modern societies. Per modernity, control over the natural and social world is a means to achieve scientific, technological and, thus, economic development. Hence, building technological capacity has become a national priority in many countries, especially in the last century (Gayozzo, 2022). Due to this optimistic view of science and technology, mathematical achievement has acquired an extraordinary value compared to other disciplines (Gates & Vistro-Yu, 2003). Mathematics performance has become a proxy for general intelligence (Gerdes, 1991), which may have made failure in mathematics particularly undesirable.

The negative view of mistakes can have a disproportionate impact on girls' and women's mathematics identity. This is because, for boys, chaotic behaviour is often considered more acceptable. By contrast, through gender socialization, girls learn from an early age that, to be seen and valued, they must project beauty and perfection. The outcome is that girls' socialization encourages safe bets and perfectionism, while boys' prepares them to inhabit imperfection (Kay & Shipman, 2014). On the other hand, gender-mathematics stereotypes attribute different causes to girls' and boys' failures in mathematics. Girls' failures are seen as a sign of inability, while boys' are more often attributed to laziness or lack of maturity (Real-Ortega & Ursini, 2010). Hence, girls tend to associate their failures with a lack of intelligence, and boys with external factors such as having a poor relationship with a teacher (Ursini et al., 2010).

The aim of our research project is to delve into how social perceptions of mistakes materialize in everyday mathematical experiences, in a study involving secondary pre-service mathematics teachers at universities from central Chile. Next, we describe nuances of the methodology, including the participants, the data collection instrument and the analytical process.

METHODOLOGICAL CONSIDERATIONS

Taylor and Bogdan (1986) argue that methodological aspects of research projects should be determined by the way we want to approach the research problem and search for answers. Due to our interest in narratives about how mistakes are experienced, handled, and understood in mathematics education, the design of this research is qualitative, as this methodology best suits the object of study.

Participants and data collection instruments

We conducted in-depth interviews with people who had completed or were enrolled in the Degree in Mathematics Pedagogy at universities in central Chile, who had completed all practical placements included in the degree program. We chose to work with Mathematics Pedagogy students because these interviewees were inhabiting (or had recently inhabited) a role as both teacher and student. We also considered it important to delve into future mathematics teachers' experiences and narratives, given their key role in terms of either perpetuating or transforming mistake-handling in mathematics education.

In total, 14 people responded to an open call for participation, and 10 were finally interviewed in person (6) or online (4). Five participants identified themselves as women, and the other five as men. All interviewees provided written informed consent before their interview. The interviews were carried out from August 2023 to April 2024, had an average duration of 60 minutes, and were recorded and transcribed. Numbers were used to identify participants (e.g. P4 for participant #4).

The interview script was divided into three parts, corresponding to participants' experiences as school students, university students, and (pre-service) mathematics teachers. For each part, they were asked to describe a typical mathematics class, as well as to provide anecdotes related to mistakes. Specifically, they were asked to reflect on the way mistakes were handled by the teacher, their own reactions and those of their peers, and the consequences of the mistake-related event (e.g. class environment, personal implications).

Analysis

Data collection and analysis occurred in two phases. In the first phase, we conducted and analysed six interviews using thematic analysis (Braun & Clarke, 2006). We organized the information around 13 emergent sets of categories (e.g. affective dimension, gender roles and stereotypes, mistake-handling strategies). As the coding did not reach the saturation point, we carried out four more interviews, and analysed them using the category system elaborated in the previous phase, with just a few new codes emerging. The entire analytical process was peer reviewed in order to enhance the validity of the research (Moral, 2006). Finally, we restructured the data around three themes: mistake-handling practices in mathematics classrooms, mistake-related affects and their effects, and mistakes and gender–mathematics stereotypes.

PRELIMINARY FINDINGS

We next present some preliminary findings for each of the aforementioned themes. First, mistake-handling practices are classified into three types. Second, we discuss the affective dimension of mistake-related experiences. Third, we outline the relationship between mistakes and gender–mathematics stereotypes.

Theme 1: Mistake-handling practices in mathematics classrooms

We identified three types of mistake-handling practices in the interviews, which we codified as *punishment*, *avoidance*, and *opportunity to learn*. Some of the practices linked with the first type, punishing the mistake-maker, were humiliation, sanctions, judgement and self-judgement. To give an example, P6 explained that some of his primary school teachers punished mistakes with the aim of using fear to motivate students to avoid future mistakes: “In primary school, there were old-fashioned teachers who would punish mistakes. They wanted us to fear making mistakes. The idea was that fear would prevent us from making new mistakes, or something like that.”

The second type of mistake-handling practice that we identified had to do with covering up mistakes. This was done in several ways: by deleting the mistake (e.g.

literally, from the board), by denying the mistake through humour (e.g. P5 and P8 narrated anecdotes in which, having made a mistake, teachers said that they had made it on purpose, to check if the students were paying attention), by responding to a mistake with the correct answer, by adopting a relativistic position, or by making participation voluntary (which often meant that only those who were sure their answers were correct would participate).

The third type of mistake-handling practice consisted of creating safe spaces for making mistakes. P6 and P9 shared that this could be done by naturalizing mistakes, both in terms of discourse and through concrete actions. More specifically, P9 argued that the board should be transformed into a space in which to practise vulnerability. P2 stated that personalized attention and/or enabling forms of interactions other than the whole-class group could also foster the creation of these spaces, and generate an understanding of mistakes as an opportunity to create an intersubjective space between students and the teacher, particularly when the person who makes the mistake is the teacher. Finally, P1, P2, P6 and P10 underlined the importance of transcending the correct–incorrect dichotomy by focusing on the thought process instead. Specifically, P10 described mistakes as an opportunity for metacognitive practice:

We can analyse a mistake and say, “If a student did this, where did the student go wrong?”
With this type of work, you are practising metacognition, because you are in class thinking about how the student thinks and how you think, and you are comparing them.

The first theme, outlined above, addressed different ways in which mistakes can be handled in the mathematics classroom. However, this leaves another important area unexplored. What emotions arise when dealing with mistakes? And, what reactions do these emotions provoke? Below, we explore the affective domain of mistake management.

Theme 2: Mistake-related affects and their effects

The idea of risk was very present in most participants’ affective mistake-related experiences. The possibility of making a mistake was frequently depicted as a threat, provoking emotions including fear, anxiety, and vulnerability. Sometimes, as in the experience narrated by P6, these emotions were linked to past experiences in which mistakes were handled through humiliation, judgement or punishment:

The teacher made me perform a subtraction on the board, in front of the whole class ...
This was right after the summer recess, so I had forgotten how to do them. I remember that the teacher was really annoying and gave me a hard time for not knowing how to perform the subtraction. I was kind of traumatized because of that, I didn’t like going up to the board from then on.

Other times, emotion was connected to the potential and/or imagined consequences of making a mistake, also related to punishment. For example, P2 said, “I’m afraid to go up to the board, because I feel like I’m going to make a mistake. I think, ‘If I make a mistake, the teacher is going to confront me or tell me off.’” Moreover, in some cases,

this sense of risk seemed to be greater in mathematics, as compared to other subjects. For example, P7 shared the following in his interview: “When we are in a maths class and someone makes a mistake, we think it matters more ... Maybe because of the way maths is seen in society: as very important.”

One possible effect of these emotions is that students may try to diminish their vulnerability, which can be done through self-censorship, for example. Self-censorship consists of reducing one’s participation in the mathematics classroom or other relevant contexts, and can include not asking questions or working alone instead of in a group. For instance, P3 shared an experience in which students who were having difficulties didn’t ask their mathematics teacher questions, out of fear:

In this class, the only ones who asked the teacher questions were those who knew that they had the content under control. Those who had difficulties were afraid to ask. They’d rather work on their own, or ask a classmate.

In the following, we reflect upon this idea of risk from a gender perspective. We argue that the gendered nature of mistake-avoidance practices, such as self-censorship, can be explained by the fact that risk itself is gendered, and that making a mistake can present a potentially higher risk to girls and women.

Theme 3: Mistakes and gender–mathematics stereotypes

Most interviewees believed that the participation of boys and girls in the mathematics classroom tended to be unequal, and suggested that this has to do with the fact that girls were more afraid of making mistakes than boys. P4, for instance, shared, “Boys, at least the ones I was with, didn’t care if they made a mistake, they tried again. Girls, on the other hand, would put their pencil aside, and say, ‘I’ll figure it out later.’ They were more afraid of failing than boys.”

The way a student is perceived, in terms of their talent for mathematics, impacts on how their mistakes are interpreted and handled. When made by a person perceived as mathematically able, a mistake is not seen as a reflection of their basic ability. By contrast, in the case of a person who has not received this recognition, a mistake can serve to ratify their presumed lack of talent. P9 reflected, “Because I was the best maths student in the class, mistakes never impacted me. My teacher was like, ‘well, nobody is perfect, you’ll do it better next time.’ But when some other classmates made a mistake, my teacher was like, ‘well, it seems you haven’t got much talent for this.’”

P4 shared her perception that boys are innately better at mathematics: “In my experience, boys have less difficulties with mathematics than girls. Maybe because mathematics has always been harder for women: men are more focused, they understand better.”

From P4’s and P9’s excerpts, we can conclude that, while not backed by empirical evidence, gender–mathematics stereotypes still have an impact on the social perception of mathematical talent, and that this perception affects how a student’s mistake is interpreted and handled. This differential interpretation and handling of mistakes can

make failure a gendered experience, which leads us to believe that mistakes present a potentially higher risk for girls and women.

DISCUSSION TOOLS AND CONCLUDING REMARKS

We are currently engaged in interpreting our data in greater depth by connecting it with other relevant research. In terms of mistake-handling practices in mathematics classrooms, Alvidrez et al. (2024) argue that the epistemological positioning of teachers regarding mistakes is not always coherent with the way in which these same teachers approach mistakes in the mathematics classrooms. In this sense, contrasting the mistake-handling practices of the interviewees with their epistemological positioning about mistakes may enrich our analysis. With respect to the creation of safe spaces for making mistakes, paying closer attention to work focused on intersubjectivity (Benjamin, 2004) and metacognition (Kramarski & Zoldan, 2016) could help identify productive mistake-handling practices.

In addition, Ahmed's (2014) work on the sociality of emotions is a valuable tool with which to analyse mistake-related experiences by taking into account the performative dimension of emotions. The concept of performativity will allow us to better understand the links between mistake-related emotions and attitudes towards mathematics.

Performativity will also guide our examination of mistake-related experiences from a gender perspective, given that we understand gender and mathematics' identity as constructed performatively in constant, dynamic interaction with one's environment (Butler, 1991; Martin, 2006). Finally, analytical categories such as burden of doubt, burden of representation and super-surveillance (Puwar, 2004), and epistemic injustice (Fricker, 2017), will enable us to engage with the potential effects of gender-mathematics stereotypes in mistake-related experiences.

Overall, this research project aims to shed some light on phenomena surrounding mistakes in mathematics. We hope that it will contribute towards the widespread implementation of more productive and egalitarian mistake-handling practices.

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THE RELATION BETWEEN PROFESSIONAL KNOWLEDGE AND TEACHING PERFORMANCE: HOW PRE-SERVICE TEACHERS TEACH INVERSE FUNCTIONS

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Most models of teachers' professional competence consider professional knowledge a key disposition for teachers' performance. However, cognitive theories suggest that transforming (declarative) knowledge into usable knowledge (that can be applied in instruction) is highly complex. Processes of how subject-specific knowledge is applied in teaching are so far seldom investigated. This study uses role-play-based teaching simulations to examine how pre-service teachers ($N = 14$) apply their professional knowledge in instructional demands when introducing the concept of the inverse function. The results show major inconsistencies between knowledge, lesson planning, and actual classroom implementation. We discuss these and address implications for the operationalization of professional competence.

In research on teacher education, teachers' professional knowledge is one of the most important constructs. Even in recent approaches emphasizing the situation-specific nature of teaching competence, professional knowledge is still considered an important resource (e.g., Blömeke et al., 2015). Consequently, many large-scale studies investigate the relationships between teachers' professional knowledge and other aspects of teaching, such as instructional quality or student learning outcomes (e.g., Hill et al., 2005). However, from a cognitive perspective, the processes underlying the observed relations between professional knowledge and teaching performance are not straightforward: (Pre-service) Teachers first acquire professional knowledge as declarative knowledge, which subsequently must be converted into knowledge for action (know-how) to be usable for mastering (situation-specific) instructional demands. This application of declarative knowledge in teaching demands does not automatically always succeed (Jeschke et al., 2021; Lindmeier, 2011; Scholten & Doll, 2024), and the underlying processes are covert and difficult to observe. Getting a better understanding of the processes can hence be considered a main yet highly relevant challenge of teacher education.

This contribution presents a study in which we examine how pre-service teachers use their knowledge of inverse functions in a variety of professional demands using standardized, role-play-based teaching simulations. Exemplarily, we want to examine the processes underlying the relations between declarative and usable knowledge as well as teaching performance in instruction.

THEORETICAL FRAMEWORK

Teachers Professional Knowledge and Professional Performance

The question of how to model and measure the professional competence of teachers is a central topic in teacher education research. Undoubtedly, teachers' professional knowledge is an essential component of teacher competence and a fundamental prerequisite for professional performance. According to Shulman (1986), it is often conceptualized as comprising content knowledge (CK), pedagogical content knowledge (PCK), and pedagogical knowledge (PK). However, the connection between teachers' knowledge and their ability to apply it in instruction is highly complex (Blömeke et al., 2015; Lindmeier, 2011; Scholten & Doll, 2024). It is assumed that declarative knowledge as it is acquired in teacher education at university must first be converted into *usable knowledge*, meaning knowledge “*that teachers are able to access and use in a classroom situation*” (Kersting et al., 2012). For example, the ACT-R theory is widely used in psychology to explain how declarative knowledge about facts (knowing what) can be transferred into an actionable state as *procedural knowledge* in the sense of knowing how to (re)act (know-how) (*Active Control of Thought*, Anderson, 1983). However, this transformation into usable knowledge is not automatic and may depend on the situational context (Kersting et al., 2012). In addition, the decision-making process for instructional action is not only knowledge-based but also depends on teachers' beliefs and experiences (Ferry et al., 2022).

Dual-process cognitive theories (Evans, 2008) suggest that different instructional demands require different cognitive processes. Lindmeier (2011) differentiates between two types of instructional demands: While classroom interactions require a flexible understanding of the situation and immediate reaction under time pressure (e.g., students' difficulties), preparation and evaluation of teaching (e.g., analyzing textbooks) allow sufficient time to consider several alternative actions. These demands are used to delineate corresponding competences: *Action-related competences* (AC) to master demands under time pressure and *reflective competence* (RC) to master pre- and post-instructional demands that allow for conscious reflection on knowledge. Subject-specific professional knowledge (CK/PCK) is understood to be a central prerequisite for RC and AC (Lindmeier, 2011). Empirical results support the assumption that RC is more closely related to teachers' (declarative) knowledge than AC (e.g., Jeschke et al., 2021). However, these relations have been mostly investigated with broad measures of knowledge and competence and have been observed to be weak for pre-service teachers. Especially evidence that allows us to trace (un)successful processes of knowledge applications on the individual level is missing so far.

Concept Development: Teaching Demands and Practice of Inverse Functions

Introducing new mathematical concepts is a crucial instructional demand for mathematics teachers, and it particularly requires subject-related knowledge reflection processes before and in instruction. For example, during lesson planning, a teacher must consider which terms and sub-concepts are necessary for students to develop an

understanding of the new concept (see also elements of comprehension, Albu & Lindmeier, 2023). Secondary school teachers, in particular, must reduce mathematical concepts learned in initial teacher education at university for instructional use without distorting them. To do so, and also to be able to react spontaneously to students' questions (AC), teachers need a deep understanding of the mathematical concepts and how school and university mathematics are connected (cf. *school-related content knowledge*/SRCK, Dreher et al., 2018). Furthermore, teachers need to consider not only the actual mathematical definition of a concept (*concept definition*) but also the individual cognitive associations (e.g., mental images, processes) that can or should ideally be associated with the concept (*concept image*) (cf. Vinner and Tall, 1981). The teachers can support the individual development of students' concept images through explicit communication or by offering representations as opportunities to acquire mental images. For simplicity, we use the term *representations* as a short-hand for such kinds of learning opportunities.

For example, the inverse function is defined (at university) as follows: Any bijective function $f: X \rightarrow Y$ has an inverse function $f^{-1}: Y \rightarrow X$, for which $f^{-1}(f(x)) = x$ applies (concept definition). In other words, the inverse function f^{-1} *undoes* f . In fact, the dominant concept image in school is a graphical representation, where the graph of the inverse function appears as the *reflection of the graph of the original function across the line $x = y$* (Breen et al., 2017). However, this representation can lead to misconceptions regarding the prerequisite of bijectivity and hide central properties of the concept. In general, such conceptual discrepancies between the definition and the teacher's representations bear the potential for conceptual misunderstandings, as several studies have shown in the context of inverse functions. For example, Breen et al. (2017) showed that students can calculate the inverse function but cannot explain what it is. Another criticism is the algorithmic recipe-like approach of *swapping x and y* in the formula (instead of solving for the dependent variable; Wilson et al., 2011). There is empirical support that developing a conceptual understanding of inverse functions related to "undoing" is more beneficial for student learning than acquiring procedural rules (Bayazit, 2004).

PROJECT AIMS AND RESEARCH QUESTIONS

As illustrated by this example of the inverse function, conceptual aspects that need to be considered and the knowledge reflection processes (for CK and PCK) that teachers need to carry out for concept development are highly complex. Therefore, teachers need profound subject-specific knowledge. At the same time, they must consider the possible consequences of decisions on concept development for students learning in lesson planning. For this reason, and in light of the cognitive theories of knowledge processing (ACT-R Theory/Dual-Process Theories), the questions arise as to which knowledge application processes are underlying and to what extent conclusions can be drawn from (declarative) knowledge about teachers' performance in class.

In the PerformA project, we investigate how secondary-school pre-service teachers apply their SRCK in role-playing-based teaching simulations (planning and implementing instruction) on the topics of inverse functions and limits of sequences (the latter is not part of this contribution; see Albu & Lindmeier, 2023). Therefore, we assess the subject-specific knowledge and its application in various teaching demands; in lesson planning (RC), and in the actual teaching implementation (AC). This contribution will look at these relations at a micro level. To do this, we will focus on how pre-service teachers define the concept of inverse functions. Precisely, we address the following research questions (RQ):

- RQ1: How profound is the pre-service teachers' knowledge of inverse functions, and how do they define it within the teaching simulation in planning and implementation?
- RQ2: To what extent do the chosen definitions from the knowledge test, the intended summary statements (mnemonics) for students (planning), and the presentation in instruction (representations and definition) match?
- RQ3: To what extent is the presentation of the inverse functions in the teaching simulations consistent, especially regarding the concept definition and the used representations?

METHODS

PerformA Teaching Simulations

The PerformA teaching simulations are simplified standardized microteaching scenarios and consist of two parts: First, participants develop a 20-minute teaching sequence to introduce the concept of the inverse function based on a structured planning document. They receive framing information, such as the learning objective, prior knowledge, etc. The planning protocol supports the development of a lesson plan through various tasks (e.g., planning tasks, formulating a summary statement for students/mnemonics). There is no time limit for the lesson planning, and any materials may be used. Afterward, participants teach their lesson to simulated students (simS) played by trained actors. The participants are free to choose the method and structure of the teaching sequence, which should be targeted at an introduction of the concept and need not cover advanced topics like how to determine the term of the inverse function. The simS follow role profiles (without a script) and may ask questions in case of mathematical uncertainties. For example, they will ask whether all functions have an inverse if the teacher addresses no prerequisites. The teaching simulations are videotaped.

In addition to the planning protocol and before teaching, the participants completed a knowledge test with 34 items (complex multiple-choice and open-ended items) covering topics of real analysis to assess the participants' CK, PCK, and SRCK. Eight of these subject-specific knowledge items (K) were related to inverse functions. For example, item K_{def} asks about the evaluation of the three dominant

representations/concept images *reflection*, *undoing*, and *swapping* (cf. Breen et al., 2017): *Decide for each of the following whether they are suitable definitions for the concept of inverse functions in the classroom. The inverse function (1) ... is obtained by swapping the variables x and y . So, $f(x) = y$ and $f^{-1}(y) = x$. (2) ... is the function created when the graph is reflected across the line $y = x$. (3) ... is the function that undoes $f(x)$. So, $f^{-1}(f(x)) = x$.*

Data Collection and Analysis

This contribution considers a subsample for which data from all three assessments are available. Therefore, data from $N=14$ pre-service secondary mathematics teachers (♀ 6 ♂ 8, $M_{\text{age}} = 23.5$ years; $M_{\text{studyprogress}} = 3.8$ years) were analyzed. All students had at least one term of practical teaching experience and were advanced (last quarter) in their teacher study program for the secondary level. The teaching simulations were carried out in the context of regular university courses. However, participation in the study was voluntary.

The video data were coded with MaxQDA by two independent raters using a detailed manual to determine the occurrence and quality of various content-related aspects (e.g., *elements of comprehension*; see Albu & Lindmeier, 2024). In particular, the raters coded which (mental) representation of the inverse function the teacher predominantly used (AC_{rep}) and how the participants defined it in class (AC_{def} ; $\kappa_{AC_{\text{def}}} = .79$; $\kappa_{AC_{\text{rep}}} = .64$). Coding was dichotomous (0 – absent, 1 – present). Similarly, we also analyzed the summary statements in the planning protocol (RC_{def}). Furthermore, the answers to the knowledge test were analyzed (score 0 = wrong, 1 = partially, 2 = fully correct). The following analysis for RQ1–3 relates to the item presented above on the three types of definition (K_{def}).

For RQ2, we looked for (in)consistencies between the pre-service teachers' subject-specific knowledge, as shown in the test, and their performance, as shown in the lesson planning and teaching. Thus, we searched for mismatches between K_{def} and the used definitions ($RC_{\text{def}}/AC_{\text{def}}$). In the analysis presented, we only included mismatches where the teachers had judged a definition as mathematically inappropriate in the knowledge test but still used it, which indicates that the teachers acted against their mathematical knowledge (logical contradiction).

RESULTS

The analysis of the eight items of the knowledge test showed that the participants had little knowledge about inverse functions ($M = 1.12$; $SD = .74$; $\text{min} = .75$; $\text{max} = 1.45$; score 0–2). To answer RQ1, we examined which of the three representation types the participants judged to be adequate for use in instruction (K_{def}) and which they predominantly used for planning and implementation (see Table 1; multiple types can be used). For example, twelve participants considered the definition of the inverse function as a *reflection* to be suitable. However, only four use this as a definition during planning, and only two finally define it this way in instruction (see Table 1). Notably,

the definition most frequently chosen as appropriate (*reflection*) is also the most used type of representation in instruction. Meanwhile, *swapping x and y* was most often used to define the inverse function.

Type	K_{def}	RC_{def}	AC_{rep}	AC_{def}
undo	7	2	4	2
swap	7	10	6	10
reflect	12	4	10	2

Table 1: Frequency definition types judged adequate in the knowledge test (K), used in planning (RC), and implementation (AC); Multiple answers allowed; $N = 14$

For RQ2, we took a deeper look at inconsistencies between professional knowledge and teaching performance at the individual level. For this purpose, we identified differences between the assessment in the knowledge test, in the planning, and in the implementation of the definition (six contrasts) per person. Considering mismatches (between knowledge and performance) per participant reveals a high number of discrepancies (see Table 2). Overall, there was no participant without at least one mismatch. These mismatches occur particularly frequently in the representation of *swapping*. Over a quarter of participants described it as unsuitable but still used it, indicating that the participants could not apply their knowledge when mastering instructional demands. In ten cases, there was also a mismatch between planning the definition and the one they actually implemented ($RC_{\text{def}}/AC_{\text{def}}$; see Table 2), indicating that the pre-service teachers are also challenged in following their lesson plan, even in the complexity-reduced simulated situation.

Type	$K_{\text{def}}/RC_{\text{def}}$	$K_{\text{def}}/AC_{\text{rep}}$	$K_{\text{def}}/AC_{\text{def}}$	$RC_{\text{def}}/AC_{\text{def}}$	$RC_{\text{def}}/AC_{\text{rep}}$	$AC_{\text{def}}/AC_{\text{rep}}$
undo	0	1	1	2	4	6
swap	4	5	5	4	6	4
reflect	1	1	0	4	10	10

Table 2: Frequency of mismatches between knowledge and performance; $N = 14$

For RQ3, we investigated how the selected predominant representations fit the presented definitions ($AC_{\text{def}}/AC_{\text{rep}}$). Likewise, many mismatches can be identified. In ten cases, the representation of *reflection* was primarily used, although it was defined differently (see Table 2). This indicates that the teachers offered opportunities to acquire mental images that were not compatible with the concept definition.

DISCUSSION

The study allows a detailed insight into the knowledge and its application in pre-service teachers' teaching of the inverse function. First and foremost, the knowledge test

results confirm a generally low conceptual understanding, which aligns with other studies (see Breen et al., 2017). Only two participants implement the mathematically correct definition of the inverse function as undoing in instruction (see RQ1). Moreover, a lack of knowledge may also be reflected in the mismatches between knowledge and performance, as well as between the concept definition and representations used. So, our study illustrates how pre-service teachers struggle to apply their knowledge, even in complexity-reduced teaching settings (RQ2 and RQ3, see Jeschke et al., 2021). Our findings also complement prior work by Lindmeier (2011), who showed that teachers draw on their knowledge differently when planning lessons (RC) or teaching under time pressure (AC). The participants may have recognized in the planning process through conscious knowledge reflection that the reflection across $x = y$ is unsuitable as a definition. In instruction, however, under time pressure, they may have fallen back to this supposedly simpler representation, which they are likely to be familiar with from their own schooling (Ferry et al., 2022; Scholten & Doll, 2024). In principle, flexible adaptation to the specific teaching situation may be useful. However, from a mathematical perspective, adaptations that result in offering mathematically distorted learning opportunities are detrimental to teaching quality.

While broad measures may be suited to show relationships between knowledge, RC, and AC on the group level, our study illustrates how weak connections between declarative knowledge and teaching performance can be, at least for pre-service teachers. Using knowledge tests to draw conclusions regarding teaching performance at the individual level seems, hence, not to be advisable. The study gives first insights into the underlying proceduralization (e.g., ACT-R Theory) and decision-making processes (e.g., model by Blömeke et al., 2015; Ferry et al., 2022), yet more work to trace the processes is needed. Our study's scope is generally limited due to the small, non-representative sample. Furthermore, we focused on specific mathematical content (defining the concept of inverse functions), so we can not exclude that the results would differ for other concepts. However, the numerous mismatches identified that all participants show in terms of performance raise further questions and highlight the need for more studies to understand the processes for successful knowledge application. Besides implications for operationalizing professional knowledge when modeling teacher competence (focus on usable instead of declarative knowledge), the results also indicate a need for learning opportunities in teacher training that aim at the application of knowledge instead of mere knowledge acquisition.

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PRIMARY TEACHERS' AND COACHES' BELIEFS AND SELF-EFFICACY FOR TEACHING MATHEMATICS IN INCLUSIVE PRIMARY CLASSROOMS

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With inclusive mathematics classes becoming more prevalent in primary schools, this study investigates teachers' and coaches' beliefs and self-efficacy for teaching mathematics to students with special education needs (SEN). A four-part Likert-scale questionnaire was handed out to both groups, followed by interviews. Three components of self-efficacy were investigated: self-efficacy regarding subject-matter knowledge, for knowledge of content and students with SEN, and for teaching students with SEN. The fourth part investigated beliefs about inclusion. While coaches had a higher self-efficacy than teachers for all three self-efficacy components, beliefs did not differ between the groups and were considered average.

INTRODUCTION

In recent years, increased attention has been given to teaching mathematics in inclusive classrooms (Scherer & Bertram, 2024). Inclusive education aims to offer equal access to quality education for all, especially (but not exclusively) to those with special education needs (SEN) and low attainers (Roos, 2023). In Israel, where this study took place, a new law was enacted in 2018 prioritizing the integration of students with SEN into mainstream schools. In this study, we focus on students with SEN learning in mainstream classrooms.

A main factor in the success of inclusive education is the teacher (Roos, 2023). Previous studies have found that teachers' beliefs and self-efficacy can impact on teacher practice (Beswick, 2007). Regarding inclusive classrooms, Scherer and Bertram (2024) pointed out that while several studies have investigated teachers' beliefs and self-efficacy regarding the general practice of inclusion, few have investigated these variables regarding inclusion in specific subjects, such as mathematics. The aim of the current study is to investigate these variables among two groups of participants: primary school teachers teaching mathematics in inclusive classrooms and mathematics instructional coaches (we shall call them coaches for short). Coaches are knowledgeable colleagues with a deep understanding of mathematics and how it is learned. They are placed in schools to work with in-service teachers and can have a significant impact on motivating instructional shifts, thus enhancing student achievement (e.g., Kho et al., 2019). Thus, investigating coaches' beliefs and self-efficacy for teaching mathematics in inclusive classrooms can be an important step in implementing positive inclusion practices in mathematics classrooms.

TEACHERS' BELIEFS AND SELF-EFFICACY FOR INCLUSION

Beliefs are essentially what we hold to be true, exist at various levels, and impact on our actions (Beswick, 2007). Teachers' beliefs have been found to impact on actions taken and decision-making during teaching (Beswick, 2007). Studies of teachers' beliefs regarding the general, non-context specific practice of inclusion, have shown varied results. In an early study (Avramidis & Norwich, 2002), it was found that general teachers' beliefs were influenced by child-related variables. For example, teachers were more positive about integrating students with physical or sensory disabilities, than they were when discussing the integration of students with learning difficulties and emotional-behavioural difficulties. In a more recent review of general teachers' beliefs about inclusive classrooms, Dignath et al. (2022) found that some studies reported teachers holding positive beliefs, whereas others showed that teachers hold negative beliefs. They also found special education teachers held more positive beliefs regarding inclusive classrooms than general teachers.

Within the context of mathematics education, an early study of middle-school mathematics teachers (Desimone & Parmer, 2006) found that most believed in giving students with SEN every opportunity to learn mathematics in mainstream classes, however, less than half believed that those students would learn mathematics better in inclusive classrooms. Similar to Avramidis and Norwich (2002), Scherer and Bertram (2024) investigated preservice teachers' beliefs about including specific groups of students in mathematics classes. They found that participants agreed that teaching disabled and non-disabled students together is appropriate.

Teacher's self-efficacy for teaching can also impact classroom practice. It has been related to teachers' effort in teaching, persistence, and resilience in the face of difficulties with students, classroom management, cognitive activation, and classroom climate (Burić & Kim, 2020; Gratacós et al., 2023). Teacher self-efficacy also affects students' achievement (Caprara, et al., 2006).

In their survey of research regarding general teacher self-efficacy for teaching in inclusive classrooms, Wray et al. (2022) found that primary school teachers have a higher self-efficacy for teaching in inclusive classrooms than do secondary school teachers. The authors mentioned two possible reasons for this difference, that secondary school teachers are more focused on content than primary school teachers, and that primary school teachers spend more time with their students than secondary school teachers. The study also revealed that teaching experience and teaching context impacted self-efficacy. Dignath et al. (2022) found that self-efficacy beliefs were higher for preservice than for in-service teachers. Within the context of mathematics education, Scherer and Bertram (2024) found that preservice teachers have a medium belief in their ability to teach mathematics in inclusive classrooms. As with beliefs, they addressed the inclusion of specific groups of students, such as inquiring about participants' self-efficacy for including students with severe physical disabilities.

THEORETICAL FRAMEWORK

Although this study investigates affective factors involved in teaching, we begin with Ball et al.'s (2008) framework concerning the mathematical knowledge needed for teaching, and focus on subject-matter knowledge (SMK) (Shulman, 1986) and two elements of pedagogical content knowledge (PCK): knowledge of content and students (KCS) and knowledge of content and teaching (KCT). KCS is "knowledge that combines knowing about students and knowing about mathematics" whereas KCT "combines knowing about teaching and knowing about mathematics" (Ball, et al., 2008, p. 401). In the current study, we refine the PCK components to focus on students with SEN. Thus, instead of KCS, we define KCSSN as knowledge of Content and Students with Special Needs and instead of KCT we define KCTSSN as Knowledge of Content and Teaching Students with Special Needs.

Just as researchers and educators consider it important to investigate and promote different components of teachers' knowledge, we propose (in line with Tsamir et al., 2019) to consider different aspects of self-efficacy related to those knowledge components. Thus, for each knowledge component, our framework includes a corresponding self-efficacy component, differentiating between teachers' mathematics self-efficacy (i.e., their belief in their ability to solve mathematical problems) and their pedagogical-mathematics self-efficacy (their self-efficacy related to the pedagogy of teaching mathematics). Here too, we differentiate between self-efficacy related to knowing mathematics and students with SEN (KCSSN), and self-efficacy for teaching mathematics to students with SEN (KCTSSN). Finally, our framework also considers beliefs regarding the effectiveness of including students with SEN in mainstream mathematics classes.

With this framework in mind, the current study investigates two groups of participants: elementary school mathematics teachers, and elementary school mathematical teachers who also serve as mathematics teachers' coaches. Our research questions are: (1) For each of the knowledge components described in the above framework, what are the self-efficacy levels for each group of participants, and are there differences between the two groups? (2) To what extent do participants in both groups believe that including students with SEN in mainstream mathematics classes will be advantageous to all students, and is there a difference between the groups? (3) Might there be participants who believe in their ability to teach mathematics in inclusive classrooms, but do not believe that inclusive classrooms are effective, and if so, why?

METHOD

Two groups of teachers participated in this study. The first group consisted of 34 primary school teachers from the same district, with 30 having more than five years of teaching experience. The second group consisted of 21 mathematics teachers' coaches, with 20 having over five years of teaching experience. As coaches, they were required to attend specialized professional development programs. Their responsibilities included deepening the mathematical and pedagogical mathematical knowledge of

elementary school mathematics teachers, providing the staff with curriculum materials to meet the needs of their students, and mentoring new mathematics teachers.

Based on the expanded framework presented, a four-part Likert-scale questionnaire (ranging from 1-9) was developed regarding teachers' self-efficacy and beliefs about the inclusion of students with SEN. Because self-efficacy could depend on the mathematical content, we decided to focus on the topics of multiplication and division, central topics taught throughout primary school.

The first part, called Self-efficacy for Subject Matter Knowledge (SMK), consisted of 20 statements related to teachers' self-efficacy for solving multiplication and division problems. The second part included 10 self-efficacy statements, focusing on teachers' self-efficacy related to their knowledge of students with SEN (KCSSN) within the context of multiplication and division. The third part had 12 statements related to self-efficacy and teaching multiplication and division to students with SEN (KCTSSN). The fourth part included 8 statements related to teachers' beliefs about the effectiveness of inclusion in mathematics lessons. Table 1 offers two example statements for each part of the questionnaire.

Self-efficacy for	Example 1	Example 2
SMK	I know how to solve problems and answer exercises in division in different ways.	I can explain the connections between different multiplication problems.
KCSSN	I know typical errors made by students with SEN in multiplication and division.	I can predict the solution strategies students with SEN will use when solving multiplication and division problems and exercises.
KCTSSN	I know how to design tasks based on the difficulty level of students with SEN in multiplication and division.	I know how to build an individual educational program in multiplication and division tailored for students with SEN.
Beliefs	Students with SEN in mainstream classes will have a greater chance of succeeding in society, than students with SEN learning in special education classes.	Students with SEN, who learn mathematics in mainstream classes, will do better in mathematics, than students with SEN learning in special education classes.

Table 1: Examples of self-efficacy statements for each questionnaire part

Each group of statements for each part of the questionnaire was combined to give an average self-efficacy score (see Table 2). Cronbach's alphas were .973, .963, .977, and .662 for parts one through four, respectively. A mixed-model repeated measures

ANOVA was used to test for differences between self-efficacy dimensions (within subject) and between the groups (between-subject). The interaction between self-efficacy dimension and group was probed using Sidak's corrections for multiple comparisons.

In addition to the questionnaires, five teachers (T1-t5) and five coaches (C1-C5) from each group were interviewed. These participants were those that scored lowest on the belief part of the questionnaire. Participants were asked: “What caused you to give low scores on the beliefs part of the questionnaire?”

FINDINGS

As seen in Table 2, both teachers and coaches had a high self-efficacy SMK. Results (see Figure 1) showed a significant SE dimension by group interaction ($F(2, 106)=5.25$, $p=.007$).

Self-efficacy for	Teachers (N=34) M (SD)	Coaches (N=21) M (SD)
SMK	8.37 (1.23)	8.95 (0.11)
KCSSN	5.51 (2.45)	7.22 (1.63)
KCTSSN	4.76 (2.38)	6.86 (1.44)
Beliefs	5.07 (1.21)	5.25 (0.81)

Table 2: Self-efficacy means (SD) for teachers and coaches

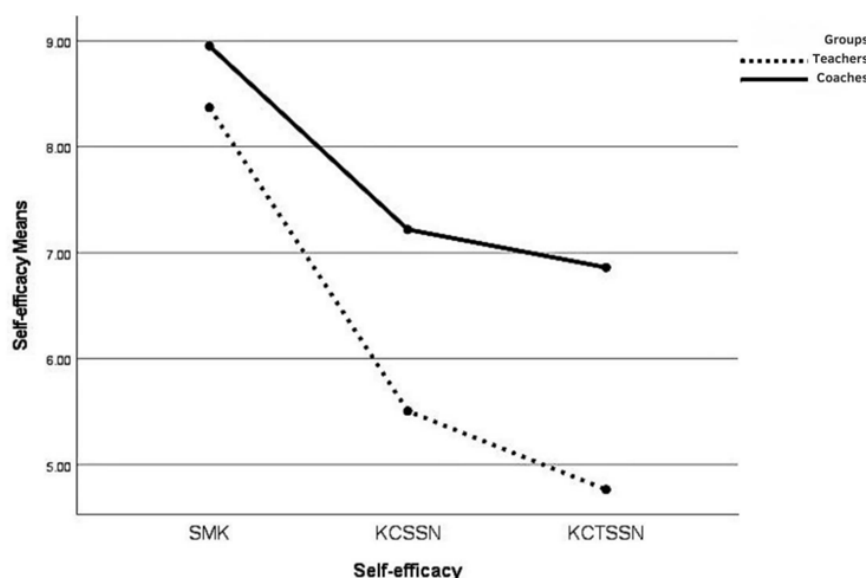


Figure 1: self-efficacy by group interaction

Simple effects analysis showed that the teachers had significantly lower SE in all dimensions compared to the coaches. Among teachers, all dimensions significantly

differ from each other, while among coaches SMK was significantly higher than each of the other dimensions, but no difference was found between KCSSN and KCTSSN. Interestingly, no significant differences were found between teachers' and coaches' beliefs.

Inductive analysis of the interviews led to four major themes for why these participants had a relatively negative view of inclusion in mathematics classes.

Cognitive abilities. Participants expressed their concern that students with learning disorders would not succeed in mainstream classes because they would not be able to keep up. T2 said, "I have no way of reaching everyone and there is no way that I can advance with the material." C1 said, "Special education students find it very very difficult in mathematics lessons... their starting point is the key issue." C2 claimed that "Students with special education needs would receive more [attention] in small classes with an individualized learning program."

Emotional aspects. Participants related both to their students' emotions (those with SEN), as well as their own emotions as they attempt to include students with SEN in their classrooms. Regarding the students, C1 stated, "Even for a regular student, mathematics is a highly emotional subject, even more so for students with SEN." C5 simply said, "They (students with SEN) develop mathematics anxiety." Relating to her own emotions, C2 said, "As one teacher in a class of 30 students, I am frustrated because I cannot attend to each one in time." C3 expressed self-doubt and unease, "I always feel like maybe I don't know enough to help them." T3 simply stated that she felt challenged, while T4 related specifically to children with attention deficit disorder, and said, "There are students whose difficulty is attention, and I've learned to deal with them, but there are some students who are very difficult to include."

Lack of resources. Nearly all those interviews mentioned that the educational system does not give sufficient support to implement successful inclusion. T3 stated, "The school system does not give us enough tools... there is not enough preparation time and not enough support... inclusion requires professional development." C5 said, "It is not easy to actually implement educational ideas. It requires much adaptation from the students, the staff, and the system."

Personal experience: Some participants related to their personal experiences teaching mathematics in inclusive classrooms, as the reason their beliefs were not high. T5 stated that she had 12 students with SEN in her class and relayed, "The children in my class are integrated into classwork and play, but I claim, on a general level, that the education system has failed in the last ten years when it gave parents the decision to integrate into a mainstream class..." C1 said, "From my experience, those students (with SEN) need more mediation and concrete manipulatives to succeed... I say that as one (referring to herself) who has learning disorders."

DISCUSSION

The current study found that while mathematics teachers and coaches exhibit a high sense of competence on subject-matter knowledge (SMK), their self-efficacy significantly diminishes when addressing the specific needs of students with special needs (KCSSN and KCTSSN). Moreover, we found that the teachers' self-efficacy differed for each of the self-efficacy components. Knowing that teacher education and professional development can enhance teachers' self-efficacy (Scherer & Bertram, 2024), this result implies that teacher educators should consider how to enhance the self-efficacy for each of those components separately.

Regarding beliefs, in general, teachers and coaches seemed to be ambivalent about including students with SEN in mathematics classes. Regarding the coaches, this is worrisome. Studies have shown that there is an interplay between coaches' beliefs and teachers' beliefs (Campbell & Malkus, 2014) and that mentoring experiences can potentially mediate belief change in teachers, supporting the implementation of educational reform (Dignath et al., 2022). If we wish to encourage more positive beliefs towards inclusive mathematics education, we should perhaps first attend to the coaches' beliefs.

Some of the reasons for these beliefs, like not having enough support from the school system, and the cognitive demands of students with SEN, were found in previous studies (Desimone & Parmer, 2006). However, in the current study, both teachers and coaches also expressed concerns regarding emotional consequences for their students as well as themselves. This is significant and indicates not only a need for instructional resources, but for emotional support, especially for the coaches who are there to support the teachers. Being that the coaches had a relatively high self-efficacy for teaching mathematics to students with SEN, we suggest professional development for coaches focus more on affective than cognitive issues.

This study has some limitations. First, although we investigated participants' SE for solving mathematical problems and their pedagogical mathematical self-efficacy for knowing about and teaching students with SEN, we did not investigate their self-efficacy for knowing about and teaching students in mainstream non-inclusive classrooms. A future study might investigate if there is a difference between teachers' self-efficacy for teaching in non-inclusive versus inclusive classrooms. Second, as opposed to previous studies (e.g., Scherer & Bertram, 2024) our questionnaire related to students with SEN, without specifying those special needs. From their interviews, we noted that participants mentioned having students with learning disorders, and students with autism spectrum disorder, in their classrooms. However, these are two very different types of SEN.

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LIMIT AS A CHALLENGE: UNDERSTANDING THE FORMAL DEFINITION THROUGH AN INQUIRY-GAME ACTIVITY

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In this paper we present a teaching experiment centred on the concept of limit. It involved a class of grade-12 students, who worked on inquiry-game activities aimed at promoting the understanding of the formal ε - δ definition of limit for real functions. These activities were designed by the authors basing on the Game Theory Logic by Hintikka and involved two players, Mr. Epsilon and Mrs. Delta. One selected episode from the teaching experiment allows to investigate how a pair of students makes sense of the (logical and dynamical) relationships between ε and δ by playing the game: the analysis of their gestures and utterances reveals the complex evolution of their conceptualisation processes.

INTRODUCTION

“It's like a game: 'You tell me how close you want $f(x)$ to be to L ; then I'll tell you how close x has to be to a .' Player Epsilon says how near *he* pleases; then Delta is free to seek his own pleasure. If Delta always has a winning strategy, then $f(x)$ tends to the limit L .” (Stewart, 1992, pp. 105-106)

With this sentence I. Stewart describes in the form of a game between two players the way in which Weierstrass defined the concept of limit. Weierstrass took seriously the expression 'as near as we please'—with which he spoke about the distance between f and L —and treated a variable not as an actively changing quantity, but simply as a static symbol for any member of a set of possible values. In his game-interpretation Stewart adds a dynamic and temporal dimension to the Weierstrass' definition with the personification of the two players. The dynamic game-interpretation makes more operative and possibly more understandable a notion that for its abstractness and logical complexity usually is very difficult for students: in fact, it is the first time that they meet a $\forall\exists\forall$ -sentence in their school itinerary. The literature has widely pointed out this difficulty, and since the eighties, mathematics educators have been investigating the complexities in the teaching and learning of the concept of limit, which is a pivotal concept for Calculus. Early studies have addressed how part of this complexity arise from the fact that the limit is both the process and the result of that process and have highlighted a significant difference between an intuitive, dynamic conception of limit and the formal, more static, definition (e.g., Tall & Vinner, 1981). Sierpiska (1985) identifies five types of obstacles related to limits: what she called 'horror infiniti', obstacles related to the notion of function, geometric obstacles, logical obstacles and obstacles related to the symbolism used. Tall (2013) extensively discusses how a limiting process may be 'intuitive' in a mathematical sense but not in

a cognitive sense and discusses *cognitive roots* of other Calculus concepts, which rely on embodied ways of grasping these concepts and do not depend on the limit definition.

Research studies have explored the source of students' difficulties in the comprehension of the concept of limit and its formal definition. These mostly concern the use of language, graphic representations and the use of quantifiers in the definition (e.g., Bagni, 1999; Cornu, 2002). Lakoff & Núñez (2000) address the concept of limit introducing the Basic Metaphor of Infinity, which accounts for processes that have an imperfective (uncompleted) aspect. They conclude that difficulty with the formal ε - δ definition of limit arises because students are given an intuitive idea of limit in terms of a dynamic metaphor and are then incorrectly taught that the ε - δ definition expresses this same idea, while in fact this is not the case. Their point is coherent with Tall and Katz (2014), who suggest that difficulties with the limit concepts are related to mathematical aspects rather than cognitive ones, and in particular to the fact that sometimes cognitive images conflict with the formal ε - δ definition. Far from being exhaustive, this brief literature analysis points out many slippery aspects that are to be taken into account concerning the concept of limit and potential pitfalls in the teaching and learning of the formal ε - δ definition.

GAME THEORY LOGIC AND DESIGN

The metaphor of the game between the players Epsilon and Delta mentioned in the previous paragraph is also the way in which the logician J. Hintikka interprets mathematical expressions and on which the Game Theory Logic is based. Inspired by Wittgenstein's language games and Herbrand's tableaux, Hintikka (1998) introduced semantical games to study the truth of natural and formal languages. Of particular interest for our investigation is the semantic game used to determine the truth of sentences of the type $\forall x \exists y S[x,y]$. In this case, we must imagine two players, a verifier and a falsifier, playing against each other, with the verifier controlling the variable y , while the falsifier controls the variable x . The verifier wins if he can find a value of y , say y_0 , for each value of x , say x_0 , chosen by the falsifier, so that $S[x_0,y_0]$ is true; otherwise the falsifier wins. The game-theoretic definition of truth is very different from the usual Tarski's recursive definition (1933/1983), which follows a bottom-up model: starting from atomic formulas, it shows the truth of complex formulas by applying definitory rules. In contrast, Hintikka's definition follows a top-down model: starting from complex formulas, it shows their truth by moving to the atomic formulas and applying strategic principles alongside definitory rules. Particularly interesting is the fact that the game-theoretic approach allows a more natural way of using and thinking about the quantifiers, making it possible to capture the functional dependency between x and y in sentences through a sort of 'somersault', which reverses such a dependency: in particular this also happens for the Weierstrass dependence between ε and δ in the case of limit.

Taking advantage of the technological possibilities of the dynamic geometry environment, we designed inquiry-game activities (Soldano & Arzarello, 2017) based

on the concept of limit. For space reason, this paper focuses on the moment in which a pair of students moves from the played-game to the reflected-game in the first inquiry-game activity, which involves a continuous function (a hyperbola, see Fig. 1).

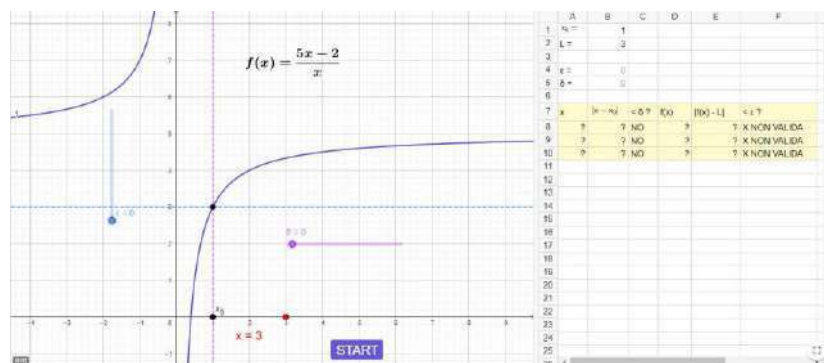


Figure 1. GeoGebra applet of the game (link to the [game](#) and link to the [worksheet](#))

As shown in Fig.1, the GeoGebra applet is divided into two windows: the graphics window on the left and the table window on the right. The game is played in pairs, with one player playing as Mr. ε (the falsifier) and the other as Mrs. δ (the verifier). Each game consists of three moves: in the first move Mr. ε chooses a value of ε by moving the ε slider; in the second turn Mrs. δ chooses a value of δ by moving the δ -slider; finally, in the third move, Mr. ε selects three values of x so that $0 < |x - x_0| < \delta$ and enters them in column A of the table. If the values have been selected correctly, all "YES" must appear in column C of the table. Mrs. δ wins the game if all "YES" appear in column F; Mr. ε , on the other hand, wins the game if at least one "NO" appears in column F.

In this case, the game is played at a point of continuity of the function, i.e., if Mrs. δ plays well, she can always win. To guarantee this outcome, Mrs. δ must choose the value of δ appropriately. If it turns out that the chosen value of δ is too large (Fig. 2), Mr. ε is given the opportunity to win by choosing a value of x that satisfies the condition $0 < |x - x_0| < \delta$ (i.e. is contained within the purple stripe), whose image does not satisfy the condition $|f(x) - L| < \varepsilon$ (i.e., is not contained within the blue stripe).

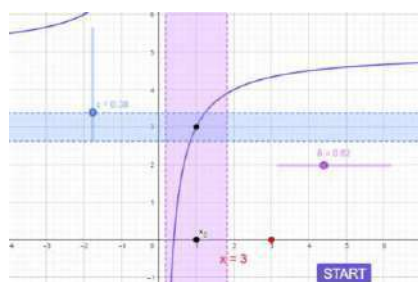


Figure 2. Example of Mr. ε 's winning configuration

The students are asked to play two matches in the role of Mrs. δ and two matches in the role of Mr. ε . By playing, we expect students to graphically discover the strategy for choosing a value of δ that always makes Mrs. δ win. If the result of the game is reinterpreted in the language of mathematics, this means that the function $f(x)$ has a

limit L in x_0 because it is not possible for Mr. ε to find a value of x that lets him win. Of course, the example starts with continuous functions and then goes on with the different cases of discontinuity.

In this study we are interested in how to approach the limit definition in a way that it is accessible for secondary school students. Hence, we try to answer the following research question: “How an inquiry-game activity designed in a digital environment can promote students’ understanding of the ε - δ definition of limit?”.

METHODS AND PARTICIPANTS

This study was conducted in an Italian scientifically-oriented secondary school. The teaching experiment took place in May 2024 and involved one grade-12 class composed by 21 students and their teacher (one of the authors). The game had initially been tested in a pilot study in September 2023 in another grade-13 class. The pilot allowed us to observe the game dynamics of the students and to design a teaching trajectory focused on the introduction to the formal definition of limit through an inquiry-game activity. Parallel to this pilot study, we analysed Italian textbook to have a sense of how limits are usually introduced. An analysis of secondary school textbooks showed that most of them introduces limits with the ε - δ definition: starting from the definition of limit for x that tends to a finite value, the other definitions (for the other cases) are then obtained. On the contrary, in university textbooks we observe a multiplicity of approaches: in particular, some textbooks treat separately the definition of limit for x that tends to infinity from the one for x that tends to a finite value. For the latter, they first introduce the definition of continuity and derive the definition of limit for x that tends to a finite value from that of continuity. Starting from this analysis and addressing the complexities highlighted in the literature review, briefly summarised in the introduction section, the main design choices made for the teaching experiment are the following:

- to start from the limit of sequences, because the iterative processes are the source of all the imperfective processes (even continuous);
- to separate the discussion of the definition of the limit of function for x that tends to infinity from that of the limit for x that tends to a finite value, since the first is derived from the definition of limit for sequences, whereas the second is derived from the definition of continuity;
- to introduce the concept of continuity without the use a dynamic metaphor, to avoid possible misconceptions or interferences;
- to use a geometry software, which offers dynamic representations (e.g., dynamic stripes) and allows for interactive, embodied experiences (e.g, zoom or dragging).

The teaching experiment was held during 9 lessons of one hour each, in the regular mathematics hours. The teacher used GeoGebra to display and discuss examples through the IWB. In the first five lessons the students were introduced to the limits of sequences, and the definition of limit of function for x tending to infinity was

constructed through a whole class discussion. In the last part of lesson 5, the problem of continuity was introduced by presenting examples of non-continuous functions. At the end of this first phase, the definition of limit for x tending to infinity was also formulated using symbolism by teacher. The inquiry-game activities were introduced in lessons 6 and 7, which took place in the computer lab. Students worked in pairs, with each group using one computer. After a brief introduction, they were given time to play following the rules provided in the worksheet. To move students from the played-game to the reflected-game, they were asked the question: “How would you explain to a friend a way to always win?”. At the end of the group work, a class discussion orchestrated by the teacher guided the transition from game language to mathematical language. In lesson 7 the students continued the inquiry-game activities with discontinuous functions. In lesson 8, during collective discussion, the definition of limit for x which tends to a finite value was formulated with the help of the teacher. Since our main interest is in studying how the game we designed can promote the understanding of the definition of limit, collective discussions during lessons 6 to 8 and the work of a group of two students (as well as the screen of the computer they were using) were videorecorded. Teacher notes were also collected during all the lessons. In the next section we present an episode from the classroom that refers to the passage from the played-game to the reflected-game. The analysis focuses on how the game frames students’ understanding of the limit concept, concentrating on the students’ gestures and on how these contribute to their communication.

SELECTED EPISODE

M and G are two students working together on the first game of the first activity. The two girls have been playing different roles, switching between them as requested by the worksheet, and explored (mostly collaboratively) ways in which each player could win (or lose). We now focus on a segment of their activity when they are trying to answer the questions on the worksheet. While doing so, they go back to the game (Fig. 3a). M is inserting values in the first column when G says:

- 1 G: When do [Delta] always win? When this and this are equal, [Epsilon] always win (*moves the δ -slider until it shows the same number of the ε -slider*). No, when this is equal to this... (*moves the δ -slider so that the left border of the violet stripe intersect the function in the same point where the lower border of the blue stripe does and points to that specific point*). [pause] To me, it’s like that. Because if Delta wants to win, she has to put... If Epsilon wants to win [he] has to put a value, to be acceptable, that is included here (*points to the neighbourhood of x_0 on the x -axis*) and if it is included here it has to be less than epsilon. Because it is inside there. To me, it’s like that. Because Delta acts later! (*gesture in Fig. 3b*)
- 2 M: Sorry, but if I put values here, I am Epsilon and Epsilon wins when...
- 3 G: Epsilon sets before what he has to do (*gesture in Fig. 3c*), Delta does it later (*gesture in Fig. 3d*), so Delta can rely on where [it] arrives here (*slides the index finger down vertically on the screen and points to $(x_0 - \delta, L - \varepsilon)$, then repeats the same vertical gesture, following the left margin of the blue stripe*), on what is the first acceptable value included in epsilon and then

[Delta] adjusts for this measurement here (*with index and middle finger points to extremes of the interval of radius δ : gesture in Fig. 3e*).

- 4 M: Yes, but for this one, do values exist with which I can win?
- 5 G: Go on, try!
- 6 M: Here should I get all no? (*points to column F of the table*)
- 7 G: Here you should get all yes (*points to column C of the table*) and here all no (*points to column F of the table*)
- 8 M: (*puts the cursor on column A of the table, looks at the screen*) You're right



Figure 3. The game (a) and G's gesturing on and around it (b, c, d, e)

M seems a little confused, goes back to the worksheets and looks for the choices of values selected in previous games. G, instead, proposes they continue answering the questions in the worksheet and write "Delta always win if, once Epsilon gives a value to Epsilon...". Meanwhile, M asks her to go back to the game, because she has worked out the table so that in column F there are all YES:

- 9 G: In fact, they are all YES, they will all be YES, you win if there is at least one NO and I always win... whatever value you give here they will all be YES and here they will all be YES (*she refocuses on the writing*). Once Epsilon gives a value to Epsilon...
- 10 M: Ah, because all the values...
- 11 G: ... they are necessarily included here (*points to the intersection of the blue and violet stripes*)
- 12 M: they necessarily are included here (*points to the same spot on the screen*) [...]
- 13 G: It must be included... that is, Delta from here down (*move the δ -slider*) can give any value, in this interval [he] can give all these here... Delta has a range of movement that goes from 0, from greater than zero, that is, from x , no, the value of Delta must be greater than 0 but less than 0.39, 0.4, now this point here specifically I don't know how much it is..

The first part of the episode is crucial for observing how the two students, especially G, understand and communicate the meaning of the game and link it to the played-game, represented on the screen. G's gestures show that she has grasped the two aspects, pointed out above, which feature the game-theoretical interpretation of the limit definition: its dynamic and temporal dimension and the 'somersault'. This is shown by gestures b, c, d in Fig. 3, all produced in a short period of time: c is a repetition of b, while d is the inversion of the previous gesture, metaphorically indicating that Delta must act later (as said by G in [3]) and also within a strategy,

which puts constraints both to Epsilon (since he has to avoid cases that would allow Delta to win) and to Delta (who must choose within the bounds given by Epsilon). This strategy, typical of strategic games, forces to think in a ‘reverse’ way and G is expressing it through its gesture d. During the whole episode, a decisive role is represented by the intersection of the coloured stripes in the game: it clearly appears on the screen, to which G refers at the beginning of the episode (gesture in Fig. 3a) and at the end (gesture in Fig. 3e).

DISCUSSION AND CONCLUSION

In the paper we have shown how it is possible to design a didactical situation apt to support students in grasping the ε - δ definition of limit for real functions, so answering to the research question of the paper. We have based on Hintikka’s game-theory logic suitably embedded in a digital environment and have experimented it in a 12-grade class of a scientifically oriented school. Due to space constraints, we could only describe the first part of the teaching experiment. In the presented episode, we have shown how two students grasped some aspects of the game of limits within an embodied approach, where gestures contribute to developing conceptual understanding.

Among the various issues that we have not had the space to discuss is the cognitive framework that allows to properly frame the evolution of students understanding, namely the conceptual blending phenomenon (Fauconnier & Turner, 2002) and the way it evolves through gestures and language (Robutti et al., 2022). This framework is apt to describe how two or more mental spaces, “small conceptual packets constructed as we think and talk, for the purposes of local understanding and action” (Fauconnier & Turner, 2002, p. 40), are integrated into a new, blended mental space. In our example, the two mental spaces are given by the played game and by the mathematical concept of functions. The final blended space is given by the fresh conceptualization of the limit definition after the game. In this evolution gestures play a fundamental role. In our example above, we have shown two types of gestures in the productions of one student, G: iconic (e.g., gestures a, e) and metaphoric (e.g., b, c, d) ones, according to the definition of McNeill (2000). When a gesture refers to the real world or comes from an embodied experience (the played game in our case), then the gesture is primarily iconic; when it is knowledge from an abstract context (here mathematical), then the gesture is primarily metaphoric. What effectively happens is that, while the gesture recurs (as b, c) in what McNeill describes as a catchment (that is a recurring gesture), the mental model evolves, and the two components (played game + mathematics) are blended in a new mental space. Further research is needed to enlarge our understanding on how playing the game contributes to grasp the multiple facets of the limit definition.

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THE ANALYSIS OF TRANSPLANTABILITY BETWEEN DISCOURSES: THE CASE OF SCALES AND EQUATIONS

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This paper aims to report on a theoretical analysis, framed within Commognition. In particular, we elaborate on the definition of discourse isomorphism (Sfard, 2008, p. 122) to study, as a specific example, the relationship between algebraic discourse on equations and discourse on two-pan scales. We analyze the extent to which this inter-discursive relation is defined, preserving intra-discursive relations, and the limits of this correspondence. In our elaboration, developed in a design-based research project on students' participation in algebraic discourse, we present the notion of transplantable discursive elements and the analysis of transplantability, a tool useful to describe correspondences and limits in relationships between two discourses that could guide design and a-priori analysis of didactical activities.

INTRODUCTION

This paper intends to offer a theoretical contribution within the Theory of Commognition (Sfard, 2008). Our theoretical elaboration will refer to specific research objectives which comes from a wider project on mathematical learning with the support of digital environments (e.g., Antonini et al., 2023). In this project, following the results of previous research (Antonini et al., 2020; Baccaglini-Frank, 2021), we studied digital artifacts in the hypothesis that the students' construction of a discourse about these artifacts could promote their participation in algebraic discourse. In particular, we employed the two-pan scale model to promote students' participation into algebraic discourse, specifically on equations.

Activities with two-pan scales are common to introduce equations to students (for a review, see Otten et al., 2019). They are built on the view that equality and the equivalence principles for equations correspond respectively to equilibrium on a scale and actions on its weights that maintain equilibrium. The analysis of this correspondence in commognitive terms was the starting point for this study. More generally, the theoretical problem addressed is to elaborate tools within Commognition for the analysis of relations between two-pan-scales-discourse (hereon 'TPS') and equation-discourse (hereon 'EQ'), for design and a-priori analysis of task situations when students' participation in TPS is used to promote their participation in EQ.

CONCEPTUAL BACKGROUND AND RESEARCH OBJECTIVES

Based on the assumption that mathematics and its learning have a discursive nature, Commognition (see Sfard, 2008) conceptualizes mathematics as particular type of *discourse* characterized by *specific words* (e.g. "equation"), *narratives* (descriptions of

objects, of relations between objects and of activities with them), *routines* (repetitive patterns characteristic of the given discourse) and *visual mediators* (perceptually objects produced for communicative purposes). Within this perspective, learning mathematics consists in becoming able to participate in this discourse and to study students' mathematical learning means to analyze how their mathematical discourse changes. A possible element of analysis is offered by the routines that are employed by students in their discourse. Lavie et al. (2019), focusing on this aspect, propose an operational definition of routine as pair *task - procedure*. In brief, a task is what a person interprets that needs to be obtained when involved in a *task situation* (a setting in which she considers herself bound to do something); a procedure is the prescription of actions that the person recognizes to be done to accomplish the task.

Central in our study is what Sfard calls the *principle of continuity of discourse*, a “pedagogical principle according to which new discourses should be developed by transforming discourses in which the learner is already fluent” (Sfard, 2008, p. 301). Starting from this principle, in our study we are interested in the transformation involving specific couples of discourses (such as TPS and EQ), when the students' participation in one discourse is used to promote their participation in the other one.

Specifically, the objective of the study is to analyze, in commognitive terms, the relationships between TPS and EQ with the aim of framing design and a-priori analysis of task situations involving two-pan scales and equations. The starting point of our elaboration is built on the following definition:

[...] two discourses can be called isomorphic if there is an isomorphism that maps the set A of all the utterances of one of the discourses onto the set B of utterances of the other discourse. Isomorphism i between A and B is a one-to-one mapping that assigns to every utterance u from A an utterance $i(u)$ from B and preserves the truth-value of utterances and the relations between them, that is, fulfills the following conditions: (a) Utterance u of A is endorsed (true) if and only if $i(u)$ of B is endorsed, too; (b) if the utterance w in A is of the form “if u then v ” (or “ u and v ” or “ u or v ,” and so forth), then the utterance $i(w)$ in B is of the form “if $i(u)$ then $i(v)$ ” (or “ $i(u)$ and $i(v)$ ” or “ $i(u)$ or $i(v)$,” and so forth, respectively. Of course, because of the blurriness of discourse boundaries and because of the context-dependence of its utterances, the relation of isomorphism is never as clearcut as required by this formal definition. Still, it is both useful and justified to think about some pairs of discourses as nearly isomorphic (Sfard, 2008, p. 122).

The study in this paper has been developed with the methodological approach of Design Based Research (DBR) (see Bakker & Van Eerde, 2015). This approach consists of different cycles of design-implementation-analysis whose results, which could be either empirical, theoretical or in the form of design products, are used as a starting point for the following cycles. Embedded in this general frame, the study in this paper is built on different experimental cycles that developed in the last 3 years. All these cycles involved the design and implementation of didactical interventions based on digital artifacts explored via tablets (such as the moving scales, see Figure 1), aiming to promote students' participation in algebraic discourse. The interventions, in

the different cycles, involved 10th grade students with difficulties with algebra (and mathematics in general) and 8th grade students with no previous experience with equations. Consistently with DBR, to address our theoretical objectives, we reflected a posteriori on the different conducted cycles focusing both on the design processes and on the analyses of the data empirically collected.



Figure 1: A two-pan scale with colored shapes realizing various weights. Users can drag the shapes across the screen to place them onto or remove them from the pans.

Some shapes are marked with numbers indicating their weight, while others may have an unknown weight. The scale gives feedback to user interactions, dangling to the left/right, or remaining balanced based on the cumulative weight on each side.

A-PRIORI ANALYSIS OF RELATIONSHIPS BETWEEN DISCOURSES

Consider an initial discourse A and the goal of promoting students' participation, that is not limited to repetition of memorized procedures, to a target discourse B. We can distinguish between two cases: B is a discourse whose objects are new for students, or B is a discourse in which students participate only in a *ritualistic* way (in the sense of Sfard, 2008, pp. 222-260). These two cases involve a distinction between two levels of learning as described in Commognition. The first one is the object-level learning which consists of an “expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives” (Sfard, 2008, p. 255). The second one is the meta-level learning, which consists of changes in metarules, those rules that define patterns in the activity of the discursants. This second level of learning is involved in the change described by Lavie et al. (2019) as *de-ritualization*, that is passing from a *ritualistic* to an *explorative* participation in a discourse. In this sense, we can interpret the continuity of the principle quoted above as articulated on the two levels (object and meta): we intervene fostering students' explorative participation in a discourse A with the aim of supporting their participation in a target discourse B (new or whose participation requires *de-ritualization*). In addition to different pedagogical hypotheses about participation to A, to B and about their transitions, interventions of this kind are often proposed on the basis that, as experts, we see a correspondence between A and B. For example, two-pan scales are commonly used in school algebra because, as mathematicians, we see a correspondence between scales and equations. This correspondence should be deeply analyzed, and we intend to frame this analysis within Commognition by elaborating Sfard's notion of discourse isomorphism in two directions. The first one involves a redefinition of isomorphism with respect to the constituent elements of discourse (words, narratives, routines, visual mediators); the second one focuses on the boundaries within which this isomorphism can be defined.

For expository purposes, we use a mathematical notation considering a map f (that is i in Sfard's quotation). Firstly, we describe the isomorphism through some examples from the correspondence between words, narratives, routines and visual mediators in the specific case of TPS and EQ.

Words: For instance, $f(\text{weight}) = \text{number}$, $f(\text{weight of unknown value}) = \text{unknown}$; $f(\text{pan of the scale}) = \text{side of the equation}$; $f(\text{equilibrium}) = \text{equality}$.

Narratives: For instance, $f(\text{'If a scale is in equilibrium and equal weights are put on or taken off the pans, then the scale remains in equilibrium'}) = \text{'if } A=B \text{ then } A \pm C = B \pm C'$.

Routines: For instance, we consider in TPS the task of identifying the value of an unknown weight paired with the procedure based on removing equivalent weights from the two pans, maintaining the scale in equilibrium, until one unknown weight remains on one pan and only known weights on the other. This task and procedure correspond in EQ, respectively, to the task of determining the solutions of a linear equation and to the procedure based on subtracting numbers and unknowns to the two sides, obtaining equivalent equations, until reaching an equation of the kind ' $x = k$ ', with a known number k . In general, considering a routine as a couple task-procedure, the map f acts on it with the relation $f(\text{task, procedure}) = (f(\text{task}), f(\text{procedure}))$.

Visual mediators: For instance, referring to Figure 2, we consider $f(\text{visual mediator for removing weights, Fig. 2a}) = \text{visual mediator for deleting numbers (Fig. 2b)}$.



Figure 2: Visual mediators produced by Aldo and Giulio (10th grade) to indicate weights removed from a scale (a) or numbers deleted in an equation (b).

The map f sets an *inter-discursive* relation between elements of the two discourses (not all of them, as we shall discuss below), preserves truth values (c.f. Sfard's quotation) and preserves also *intra-discursive* relations between elements of each discourse. In other terms, if W is a word of the narrative N , then $f(W)$ is a word of $f(N)$, and the role of W in N corresponds to the role of $f(W)$ in $f(N)$. Also, if R is a routine that is based on N , then $f(R)$ is a routine that is based on $f(N)$. For example, the procedure described above for identifying the value of an unknown weight is based on the narrative about the equilibrium invariance of scales. This procedure corresponds to the procedure for determining the solution of an equation that is based on an equivalence principle, and also the narrative about the equilibrium invariance corresponds to the narrative describing the equivalence principle. Similar relationships between visual mediators and the other elements can be preserved (e.g., see Figure 2).

A crucial aspect characterizing the relation between TPS and EQ is that it is only partial (these echoes with Sfard's remark on discourses being only "nearly isomorphic"). For

instance, there are no elements in TPS that can be mapped to negative numbers or to $x\sqrt{2}$ or to x^2 in EQ. Even if we restricted EQ to linear equations with positive integer coefficients, there would still be elements of TPS that cannot be mapped into EQ. This is particularly interesting for routines. For instance, students from our experimentations, to find the value of an unknown weight, spontaneously used a procedure consisting in emptying the scale, putting on one pan one unknown weight and testing its value by putting known weights on the other pan until the scale resulted balanced. This procedure, based on empirical observation of the scale feedback, does not have a corresponding one in EQ where it is not possible to write ' $x = \dots$ ' and obtain feedback when writing different numbers. In other words, the feedback determines some limits of the isomorphism because it is involved in the metarules of discourse production: in TPS feedback allows to endorse narratives, whereas in EQ we have to construct theoretical relations between them (e.g. about equivalent equations).

Our analysis shows that the relation between TPS and EQ is not an isomorphism but there is a subset of elements of TPS which has a corresponding part in EQ. In this respect, we say that TPS and EQ are *nearly-isomorphic*. In general, if A and B are two nearly-isomorphic discourses, we say that the elements of A that have a corresponding element in B are *transplantable*, and those that do not have it are *non-transplantable*. This term recalls the Sfard' interpretation of metaphor (Sfard 2008, p. 39) as "an action of 'transplanting' words", with the difference that Sfard refers to the act of transplanting a word from A to B, while our focus is on the property - of words, narratives, routines and visual mediators of A - of having a corresponding element in B. Rephrasing the analysis above, we can observe that the isomorphism f determines a correspondence between transplantable elements that preserves truth values and their mutual intra-discursive relations. Using a mathematical metaphor, we could say that the diagram with the inter-discursive map f between A and B and the intra-discursive relations between elements of each single discourse is commutative. This commutativity guarantees that the use of transplantable elements in one discourse corresponds to the use of the corresponding elements in the other discourse.

This whole analysis can be seen as a discourse (a *meta-discourse*) about two discourses and their relationships. We call this meta-discourse *analysis of transplantability*. In general, it consists in identifying relations between two discourses and the metarules that can define patterns in the transition from one discourse to another. In particular, it consists in identifying (i) transplantable words, narratives, routines and visual mediators where the inter-discursive relation is well-defined, and the intra-discursive relations and the truth-values are preserved, (ii) non-transplantable elements and different metarules, specifically those for endorsing narratives.

DESIGN OF TASK SITUATIONS: SOME EXAMPLES

The analysis of transplantability is an a-priori analysis for educational purposes: both the isomorphism and its limits may be used to design task situations for promoting participation to a target discourse. We briefly present some examples in what follows.

Example of task situation for promoting participation in TPS: *Here you see a balanced scale. Could you find out how much a little red ball weighs?* This request is paired with a dynamic interactive scale as the one shown in Figure 1. This kind of task situation aims to expand the students' TPS promoting the emergence of different procedures for determining the value of the unknown weight. It is not important, here, if some of these procedures will be transplantable in EQ and others will not.

Example of task situation for promoting exploration of procedures and of metarules for endorsing narratives: *Could you find out how much a purple square weighs? This is a picture, thus nothing can be moved, but you can write on it if it helps.* The request is the same as in the previous example, but in this case the students have only a visual mediator of a scale, static and without feedback (see Figure 2). One of the aims is to make students question the applicability and the justification of procedures that may become in this way the objects of a meta-level discourse. For instance, since the scale is just a picture, the (non-transplantable in EQ) procedure of weighing an object to find out its unknown value does not work. On the other hand, the (transplantable in EQ) procedure of removing equivalent weights can be imagined, and these could also prompt students' production of visual mediators (note the '*you can write on it if it helps*') to describe the procedure which might correspond to visual mediators in EQ (see Figure 2). Moreover, without feedback, narratives such as 'The scale is balanced because I can see it' cannot be endorsed. Then students could be prompted to make explicit the assumptions that guide their narratives about the scale's behavior and this could prompt the production of transplantable narratives corresponding, in EQ, to equivalence principle for equations (e.g., 'Being balanced at the beginning, if one takes off the same quantity anyway it remains balanced'). In other terms, the shift from dynamic artifact to static visual mediator with the inhibition of feedback can promote students' exploration in at least two directions: a reflection on the applicability of previous procedure and the search for procedures that are not based on the scale feedback and that could be transplantable in EQ; the transition from narratives based on empirical observations to narratives that are endorsed through general properties of the scales (in our project the students moved from saying "I see that the scale..." to "we can understand/know that the scale...").

Example of task situation for promoting a meta-discourse about equations: *Watch the video. Write an equation corresponding to the scale. If possible, translate on the equation each step accomplished within the scale.* The video involves a scale in which all the objects are removed and then one red ball with unknown weight is put on one pan and known weights on the other until, after some attempts, the scale results balanced. The video could be interpreted as showing a routine in TPS for determining the value of the unknown weight with a procedure that is non-transplantable in EQ. The translation on the equation is thus not possible. The objective in this case is to promote the students' participation in a meta-discourse about equations through their production of a meta-discourse about relations between TPS and EQ in which transplantable and non-transplantable elements are discussed.

CONCLUDING REMARKS

In this study, within the framework of Commognition, we deepened the relation of isomorphism between discourses (Sfard, 2008, p. 122), in the specific case of scales and equations. We described the isomorphism in terms of discursive elements, with reference both to object and meta-level learning, and clarifying to what extent it is a nearly-isomorphism. We defined *transplantability* as the property of an element of a discourse A of having a corresponding element in B so that its use in A ‘corresponds’ to the use of the corresponding element in B. Our analysis with the specific example of scales and equations allowed clarifying in what sense the uses of corresponding elements ‘corresponds’: the *inter-discursive* relation (the map f between elements of A and B) preserves the *intra-discursive* relationships. This compatibility between inter- and intra-relations which is established by f corresponds to saying that *the use* of an element in a discourse is transplantable in an isomorphic discourse. And this, coherently with the commognitive adoption of Wittgenstein’s definition of ‘meaning’ (Sfard, 2008), that refers to the way a term is used in language (Wittgenstein, 1953/2003), allows us to say that in this isomorphism, meanings are also potentially preserved. This invariance relationship constitutes the core of the correspondence scales - equations, and we believe that this analysis can be extended to other situations, for which further investigations are needed.

A main point of our elaboration is the distinction between elements that are transplantable and those that are not, a distinction that defines the boundaries of the nearly-isomorphism and that becomes central in the transition from A to B. Indeed, the analysis of transplantability for TPS and EQ has been an effective tool to describe correspondences and limits in relationships between discourses, for design and a-priori analysis of task situations in accordance with the principle of continuity of discourse. This design, of which we have presented three examples, considered both object- and meta-level, and both transplantable and non-transplantable elements. These latter could be exploited in purposely designed task situations to promote meta-level learning, that is a crucial aspect as discussed in Commognition (e.g., Nachlieli & Elbaum-Cohen, 2021; Valenta & Enge, 2022). Specifically, as presented in the examples, some task situations were designed to explicitly address the distinction between transplantable and non-transplantable elements, aiming to promote TPS, EQ, a meta-level discourse on the relationships between TPS and EQ, and, finally, a meta-level discourse on EQ.

We wish to clarify that we are not suggesting that two discourses A and B must be necessarily nearly-isomorphic for promoting students’ participation to a target discourse B through a participation in discourse A. Our point is that in such situations, when A and B are seen as isomorphic in the eyes of an expert, an a-priori analysis of the relationship between A and B is needed to build and develop didactical activities, considering possible actions and discourses of students and teachers. Indeed, even if in this study we have not focused on teachers’ actions, in tune with other studies (Cooper & Lavie, 2021; Nachlieli & Elbaum-Cohen, 2021; Valenta & Enge, 2022), we consider

the teacher's role essential in promoting the meta-level learning related, in our case, to the transition from two discourses and in managing the different use of transplantable and non-transplantable elements of discourses emerging in the classroom.

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MATHEMATICAL MODELLING AND CASE METHOD TO INTEGRATE STEM

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Mathematical modelling has been considered as a bridge to integrate STEM (science, technology, engineering, mathematics) disciplines, because it is aligned interdisciplinarity with STEM and has been recognized as important for students to solve local and global problems in accordance with current challenges. Due to the need for qualified teachers, it was implemented a training program for in-service secondary mathematics teachers using as a learning strategy the case method. The results of a multiple case study on STEM skills of groups of teachers, using deductive content analysis, are presented. The findings show that mathematical modelling helps teachers to integrate STEM skills.

INTRODUCTION

The study has focused on the potential and challenges of mathematical modelling as a bridge to integrate STEM (Hallström & Schönborn, 2019; Stillman et al., 2020) and for authentic STEM literacy (Hallström et al., 2023; Li & Schoenfeld, 2019). Mathematical modelling, as an integral part of mathematics, aligns interdisciplinarily with STEM education because it connects mathematics with the other disciplines, providing a greater understanding in the application of mathematical concepts and processes (Aravena et al., 2024; English, 2017). The search for relations, using mathematical language, allows a balance between STEM areas, because the role of mathematics is deepened and not simply relegated as a calculation tool to make sense of data from phenomena in other STEM areas (English, 2017; Goos et al., 2023).

Today, more than ever, mathematical modelling and the use of models have become a topic of global interest to address major global and local challenges (e.g. water, climate crisis, agricultural production, transportation, health) and for technological advances (e.g. augmented reality, additive manufacturing, data science, artificial intelligence) (Aravena et al., 2022; Aravena et al., 2024). All these implications of mathematics in science, technology and engineering require that students experience training that truly reflects the reality of these challenges (Aravena et al., 2022; Nguyen et al., 2020).

For STEM education to be effective, qualified teachers are required, because teachers are not yet prepared to deal with this scenario (Kurup et al., 2019). Therefore, the study focused on a training program, with in-service teachers, that addresses complex,

unstructured and open-ended problems (English, 2017; Stillman et al., 2020) and uses the case method as a learning strategy (Lattuca, et al., 2006). The case method focuses learning on students, placing emphasis on discussion and construction of knowledge and generating creative solutions (Aravena et al., 2022; Passyn & Billups, 2019). In addition, it encourages STEM problems to be consistent with local needs, which require disciplinary articulations for their resolution (Aravena et al., 2024).

The question that guided the study was: What are the STEM skills that in-service teachers integrate when solving cases close to the sociocultural reality of their students based on mathematical modelling?

THEORETICAL FRAMEWORK

Mathematical modelling to integrate STEM

The term STEM integration is still diffuse and there is no consensus around a definition or terminology to describe STEM integration (Goos et al., 2023; Wang, et al., 2011). However, in the study it has been considered as an opportunity to experience learning in authentic situations (Wang, et al., 2011) where such integration is not mere fusions between disciplines, but rather an interdisciplinary cooperation that helps develop skills that cross disciplines (Hallström and Schönborn, 2019).

Research on the nature of modelling as a bridge between STEM disciplines is still emerging (Hallström & Schönborn, 2019), but it has become a topic of interest for integrating them (English, 2017; Hallström et al., 2023). The use of modelling processes, as a basis for mathematical activity, is of interest to prepare teachers and students for authentic STEM education (Li & Schoenfeld, 2019) because it is in line with the purposes of STEM that emphasize conceptual comprehension and expand its understanding through its use in socially and culturally relevant domains (Baker & Galanti, 2017; Ritz & Fan, 2015).

When referring to mathematical modelling, there is agreement that in order to model a situation, there must be a correspondence that allows generating an interaction between a real problem and mathematics, where the objects and relations of the real problem are in correspondence with the objects and relations of mathematics (Blum & Borromeo Ferri, 2009; Niss et al., 2007). To analytically capture this process, it has been distinguished non-linear modelling cycles, which describe the student's performance in solving mathematical modelling problems (Kaiser et al., 2006).

STEM education and mathematical modelling in teacher training

The need to involve in-service teachers in modelling and STEM is an essential factor for their incorporation into the educational field, so that they can broaden the perspective in this field, making use of technologies, social and cultural aspects of modelling and its teaching (Stillman et al., 2020). It is recommended to recognize the role of mathematical knowledge in solving real problems, mathematics incorporated in other sciences and the real world (Kaiser et al., 2006; Niss et al., 2007), identify the

importance of building models to understand the dynamics of scientific, social and cultural processes (Stillman et al., 2020) and the study of modelling cycles, to understand the correspondence between the real problem and mathematics (Blum & Borromeo Ferri, 2009).


Modelling and using models in STEM scenarios makes teaching more demanding, because it requires teachers to have deep mathematical knowledge, deal with highly complex problems, work on problems in authentic contexts, evaluate student progress, detect obstacles and difficulties, and evaluate skill development (Kaiser et al., 2006).

METHOD

The research method is qualitative with a multiple case study design (Yin, 2014) to comprehend in depth the relations between STEM skills and the phases in the modelling cycle. The participants were 9 mathematics teachers from the Maule Region and were divided into 3 work groups (G1, G2 and G3). Video recordings of each group were analyzed to capture the verbal record of the modelling work, the record of technological tools while solving the case and the engineering proposal. The case called "The manufacture of paper in the Arauco cellulose plant, Constitución", and is framed in the needs of society related to the protection of forests, water and the environment, being close to the sociocultural environment of the teachers.

The design and its relationship with STEM skills correspond to: I. Case context: social problems that require interdisciplinary solutions. II. Questions for discussion: related to scientific skills (using scientific language (S.1), applying investigative processes (S.2); concluding based on evidence (S.3) and communicating (S.4)). III. Mathematical work focuses on mathematical skills (representing (M.1), arguing (M.2) and communicating (M.3)). IV. Engineering proposal: focuses on engineering skills (studying solutions (E.1), designing (E.2), evaluating solutions (E.3) and communicating (E.4)). Technological skills were considered throughout the entire process (using technological applications (T.1), simulating (T.2), using multimedia resources (T.3) and communicating (T.4)) (Aravena et al., 2022; Aravena et al., 2024). The presentation of the questions related to mathematical work (stage III) can be seen in figure 1.

0 Work of Mathematicians, Technical and Engineering



In the final process of the paper making the product is wraps around the tube (see the figure on the left). The roll of paper is called coil. See:

- 1) What is the maximum paper capacity you that Arauco Cellulose produce?
- 2) How much cellulose does it take to create a ton of paper?
- 3) Perform a 3D simulation of the paper roll and establish relationships?
- 4) Model the roll to find the length of the paper and the area so you can know the amount of water needed by ton.








Figure 1: Questions for the mathematical work proposed in the case.

Regarding the modelling cycle, the phases defined by Blum and Borromeo Ferri (2009) are considered as categories: understand/construct (Phase 1), simplify (Phase 2), mathematize (Phase 3), work with mathematics (Phase 4), interpret (Phase 5), validate (Phase 6).

The analysis was carried out by identifying the frequency of co-occurrences between STEM skills and the phases of the modelling cycle that the three groups of teachers (G1, G2, and G3) manifested while solving the case collaboratively. The deductive content analysis (Bingham & Witkowsky, 2022) was used as an analysis strategy based on the theoretical categories and codes previously declared.

RESULTS

The frequency with which STEM skills were observed throughout the 6 phases of the modelling cycle, according to the four dimensions (Mathematics, Engineering, Science and Technology) and the respective subcategories, considering the work of each group, is displayed in Table 1.

Modelling Categories																			
Phase 1			Phase 2			Phase 3			Phase 4			Phase 5			Phase 6				
Groups, Skills	G1	G2	G3	G1	G2	G3	G1	G2	G3	G1	G2	G3	G1	G2	G3	G1	G2	G3	T
S.1	1	0	1	1	0	5	0	2	0	0	0	0	0	0	0	0	0	0	10
S.2	1	0	2	2	0	0	0	0	0	1	0	0	0	0	0	0	0	0	6
S.3	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	3	6
S.4	1	0	1	2	0	2	0	0	0	0	0	0	0	0	2	0	0	0	8
T.1	0	0	0	7	2	3	5	11	2	3	4	3	0	0	0	8	0	3	51
T.2	0	0	0	3	1	3	3	0	1	3	1	1	0	0	0	0	0	3	19
T.3	0	0	0	0	0	0	0	0	0	0	1	3	0	0	6	0	0	6	16
T.4	0	0	0	1	0	1	0	0	0	0	0	1	0	0	1	1	0	4	9

E.1	4	0	0	6	2	4	3	4	0	4	0	1	0	0	2	0	0	2	32
E.2	1	0	1	2	2	7	0	2	0	0	0	1	0	0	0	1	0	0	17
E.3	0	0	0	0	0	0	2	0	0	0	0	1	0	0	2	0	0	3	8
E.4	0	0	0	0	0	0	0	0	2	1	0	0	1	0	0	0	0	0	4
M.1	2	0	3	2	1	6	2	0	2	1	0	2	0	0	2	0	0	4	27
M.2	0	0	0	1	3	3	1	3	7	0	0	2	0	0	9	3	0	4	36
M.3	1	0	2	1	0	5	3	0	4	3	0	5	2	0	7	0	0	6	39
total	12	0	10	29	11	39	19	22	18	16	6	20	3	0	32	13	0	38	288

Table 1: STEM skills according to the phases of the modelling cycle in the three teacher work groups.

Of all the co-occurrences, the most notable are those that show the connection between technological skills T.1 and phase 3 of the modelling cycle (G2), together with those that connect technological skills T.1 and phase 6 of the modelling cycle (G1). In general, the skill of using technological applications (T.1) proves to be the most transversal, reporting the highest frequency throughout the 6 phases of the modelling cycle. Other interesting co-occurrences are observed between mathematical skills M.2 and phase 3 of the modelling cycle (G3), and between these same skills M.2 and phase 5 of the modelling cycle (G3). Although the skill to argue is very important in a collaborative work process, which aims to validate a mathematical solution, its presence varies throughout the phases and is not manifested with the same intensity in the different work groups. Regarding the integration of STEM skills, this can be observed from the frequencies highlighted in each phase of the cycle. In phase 2, the integration between scientific, technological and engineering skills stands out in both G1 and G3. In the latter, integration with mathematical skills is also added. In phase 3, the integration between technological and engineering skills stands out in groups 1 and 2. Finally, in phases 4, 5 and 6, integration between technological and mathematical skills is observed specifically by G3.

Of the three groups, the one that stands out the most in the overall manifestation of STEM skills throughout the modelling cycle is G3. Based on the performance of this group, it can be characterized the actions carried out to solve the case according to the 6 phases of the modelling cycle, as shown in Figure 2.

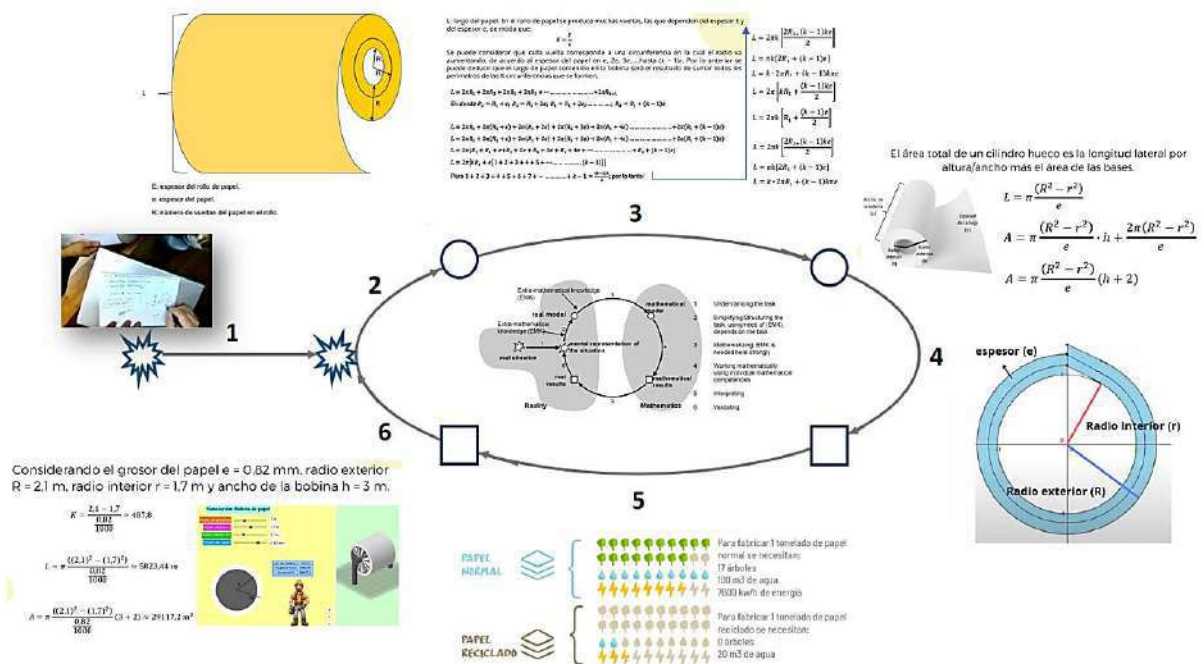


Figure 2: Case resolution according to phases of the modelling cycle in group 3.

In phase 1, the G3 makes a sketch of the paper roll design, considering some variables such as the amount of water and cellulose required to make a ton of paper (understand/construct). In phase 2, they identify variables for the construction of a prototype with the support of a geometric approximation based on a hollowed cylinder, highlighting the thickness of the paper roll, the thickness of the paper and the number of turns in the roll (simplification), thus generating a preliminary model. In phase 3, they propose symbolic representations derived from the preliminary model and point out that the length of the paper roll depends on both the thickness of the roll and the thickness of the paper, thus considering that each turn of the roll corresponds to a circumference in which the radius increases. This allows them to conclude that the length of paper contained in the reel will be the result of adding all the parameters of the successive circumferences that are formed (mathematize). In phase 4, they refine the mathematical idea and focus on three main aspects: number of turns, length of the paper roll and the total area of the hollow cylinder, presenting for the latter an equation that relates the width of the paper roll, the thickness of the sheet, the inner radius and the outer radius of the cylinder. They thus describe that the total area of a hollow cylinder is the side length times height or width plus the area of the bases (working with mathematics). In phase 5, they analyze the relation between the estimated paper area and productivity per ton, linking the total amount of paper with the requirements for water consumption, trees and energy expenditure (interpret). In this way, they inform the great environmental impact involved in the production of a ton of paper. Finally, in phase 6, they test the relevance of the model using a calculation simulation of total area programmed in GeoGebra, which operationalizes variables similar to those

of the model itself. By testing different values in the simulation, they manage to confirm the feasibility of the proposed mathematical model (validation).

CONCLUSIONS

The case method, which was approached in four stages and designed to promote the use of a modelling cycle, allowed us to identify specific types of STEM skills in each of the phases. The analysis was based on a detailed study of the case resolution process used by each of the three groups of teachers. Although the case required an analysis from different STEM skills, integration was predominantly evident in phases 2 and 3 of the modelling cycle, especially between technological and engineering skills. In addition, in one of the groups, it could be seen relevant presences of mathematical skills, together with engineering and technological skills, in the phases that promote mathematical work and the validation of mathematical models.

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WHAT SCHOOL SEGREGATION LOOKS LIKE: MATHEMATICAL DISCUSSIONS IN CONTRASTING SES SCHOOLS

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This study examined mathematical discussions in two third-grade classrooms from Chilean schools with contrasting SES during an early algebra intervention. Through a social semiotic approach, both spoken discourse and students' movements were analyzed. The findings showed that, while both groups made significant progress in generalizing functional rules, their participation dynamics and the teaching strategies required differed. In the high-SES classroom, students exhibited greater autonomy and engaged in peer-directed interactions, whereas in the low-SES classroom, students depended more on direct interventions from the researcher. These findings underscore the importance of considering cultural and contextual factors in research on mathematical discussions, particularly in segregated educational contexts.

Research has consistently documented the gaps in mathematical performance linked to students' socioeconomic status (SES) (OECD, 2023), which are even more pronounced in countries with high levels of school segregation, such as Chile (Otero et al., 2023). Among the disparities in mathematical learning, algebra stands out as a particularly critical area, as it involves a high level of abstraction that often emerges abruptly for students (Kaput, 2008), disproportionately affecting those from disadvantaged groups and limiting their future opportunities (Moses & Cobb, 2001; Morton & Riegle-Crumb, 2019). In response to this challenge, the approach of *early algebra* has gained prominence, advocating for the introduction of foundational algebraic concepts at an early age and in a sustained manner throughout the curriculum (Kieran, 2022).

From this perspective, it has been consistently demonstrated that children aged 5 to 12, through carefully designed instructional interventions, are capable of identifying mathematical structures and relationships, generalizing pattern rules, and representing these generalizations using both formal and informal symbolic systems (Cañadas et al., 2019; Blanton et al., 2019). Furthermore, participation in early algebra lessons during grades 3 to 5 has been shown to provide significant and sustainable benefits to all students, including those living in socially disadvantaged environments (Blanton et al., 2019).

Despite the potential of early algebra to improve mathematical learning outcomes for students from disadvantaged backgrounds, research has paid limited attention to the specific pedagogical practices that can benefit this group. Considering that factors such

as culture and linguistic capital, which are closely tied to students' social class, can significantly influence classroom interactions in mathematics (Lubienski, 2000; Zevenbergen, 2013), it is essential to examine the forms of interaction within early algebra classrooms in relation to SES, as well as the strategies teachers use to manage discussions that foster student progress. This study aimed to understand the characteristics of mathematical discussions observed in two third-grade Chilean classrooms from schools with contrasting SES contexts, while implementing the same intervention based on an early algebra framework.

DISCUSSIONS IN MATHEMATICAL CLASSROOMS

One of the areas of interest in mathematics education research has been the study of classroom dialogues and discussions, emphasizing those interactions that promote student participation, argumentation, and intellectual autonomy as desirable (Smith & Stein, 2011). Teachers play a central role in orchestrating more enriching dialogues by posing open-ended questions and fostering discussions on different solution strategies, encouraging students to actively engage by questioning ideas and proposing alternative solutions (Solar et al., 2022). Furthermore, student-centered approaches advocate that teachers refrain from providing answers or hints, allowing students the opportunity to reach solutions independently (e.g., Brousseau, 2007). These interactional approaches stand in contrast to the traditional Initiation-Response-Evaluation (IRE) pattern, which tends to constrain student participation and limit the exploration of ideas (Drageset, 2015).

Some authors have argued that research on classroom dialogue has often been presented as culturally neutral, showing limited sensitivity to the specific contexts and communication styles of different communities. For instance, Xu and Clarke (2022) highlighted that mathematical discussions deemed effective in classrooms in Seoul and Shanghai—where silence and teacher authority hold intrinsic cultural value—differ significantly from the practices considered desirable in Western contexts.

Similarly, Lubienski (2000) examined the perceptions of students from different SES backgrounds regarding activities based on mathematical discussion in a socioeconomically diverse classroom. While students from high-SES households viewed these activities as opportunities to analyze different ideas, low-SES students perceived them primarily as a means to find the correct answers. Additionally, low-SES students preferred greater teacher guidance, whereas high-SES students valued solving problems independently. In a similar vein, Zevenbergen (2013) demonstrated how social class and linguistic capital influenced student participation in mathematics classes based on dialogue and debate. Students from higher social classes possessed linguistic capital and cultural knowledge more closely aligned with teachers' expectations for desirable mathematical discourse. Both studies were conducted in classrooms integrating high- and low-SES students. However, no studies have been identified that specifically examine mathematical discussions in countries with high levels of school segregation, where schools tend to concentrate students with similar

socioeconomic conditions. This study, therefore, provides an opportunity to delve deeper into this issue.

DESIGN AND RESEACH METHODS

We conducted an exploratory case study (Yin, 2014) to understand the differences in interaction patterns observed during mathematical discussions in two third-grade classrooms from Chilean schools with contrasting SES contexts. The data presented here were collected as part of a broader research project focused on the development of early algebraic thinking in marginalized school settings. As part of this study, we implemented a six-session didactic intervention, with each session lasting 90 minutes, in both schools. The intervention was led by the same researcher-teacher.

The participants in this study were the students from the two third-grade classrooms involved in the research. On one hand, 32 students from a public school located in the western area of Santiago participated; this school primarily served families of low socioeconomic status (SES). According to government data, 87% of the students at this school were in a situation of social vulnerability. On the other hand, 23 students from a private school in a high-income district in the eastern area of the city were included. The monthly tuition at this school was approximately \$700 (in US dollars) per student, a figure higher than the national minimum wage. Regarding prior knowledge, students in both schools were familiar with addition and had begun learning multiplication during the semester preceding the intervention.

We implemented a six-session didactic intervention in both classrooms based on an early algebra approach, aimed at teaching students to generalize functional rules using different representations, including natural language and algebraic expressions. To achieve this, students were presented with various pattern-based contexts and were asked to record data in tables, determine values for distant terms, and express the rules in generalized forms. We facilitated moments of discussion for students to share their findings, encouraged them to explain their reasoning, and express disagreements. Details of the intervention can be found in Araya (in press). All sessions were video-recorded.

The analysis adopted a social semiotic perspective, which posits that mathematical meanings emerge from social interactions that develop not only through spoken discourse but also through gestures, actions, and the use of space, revealing cognitive aspects as well as attitudes and beliefs (Morgan, 2006). Initially, the session recordings were transcribed. Subsequently, segments corresponding to mathematical discussions in both classrooms were identified and selected. A qualitative comparative analysis was then conducted using the constant comparative method, which involves a systematic and iterative comparison of the data to identify patterns, differences, and similarities (Glaser & Strauss, 1967). This approach facilitated the identification of relevant aspects within the observed interactions, providing a deeper understanding of the dynamics present in mathematical discussions across different socioeconomic contexts.

FINDINGS

Although by the end of the intervention several students in both classrooms were able to generalize functional rules and express them using different representations, the way discussions were managed and the support provided by the researcher varied significantly between the two settings. To illustrate these differences, we selected two representative scenes that highlight the distinct aspects identified in the comparative analysis. The scenes correspond to the same moment in the intervention in both classrooms, where students worked on the task shown in Figure 1.A, whose functional rule is $f(x)=3x$. By this stage, the students had completed the value table on the board with the data (Figure 1.B).

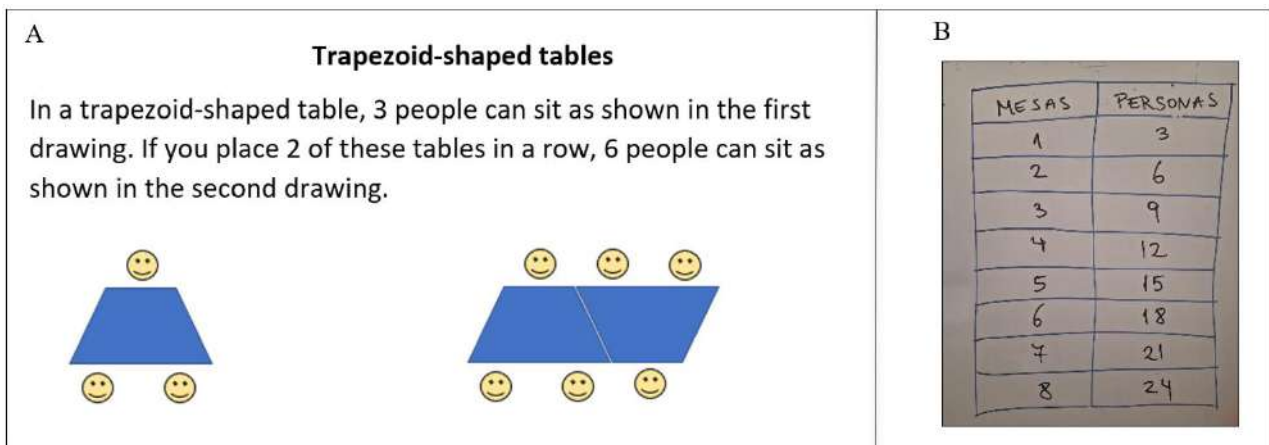


Figure 1: (A) Pattern task. (B) Value table on the board (Translation: Tables/People).

When students were asked to find the number of people that could sit at 21 tables, both classrooms made a recurring mistake that the researcher anticipated. However, the way this error was managed differed significantly, reflecting key elements that remained consistent throughout the intervention. The following dialogue illustrates the interaction in the high-SES classroom:

When Sara stated, “21 tables can hold 24 chairs because the pattern increases by three each time,” the researcher asked the class if they agreed. Several students disagreed, and the researcher gave some of them the opportunity to speak.

Researcher: David, what do you think?

David: I don’t agree because (stands up and walks to the board), for example, Sara (speaking to Sara) you missed that you should be doing 21 *times* three, not 21 *plus* three (emphasizing the words *times* and *plus*).

Researcher: What would be the correct result, David?

David: 63, because... Sara (turning to Sara), how much is 21 plus 21?

Sara: 42.

David: And plus 21?

Sara: 63

Researcher: Why should it be multiplied by three, David?

David: Because the sheet says if three tables are joined together.

- Researcher: Does anyone else want to add something? Matilda? (Matilda walks to the board and picks up the marker from the table).
- Matilda: It couldn't be 24 because, look (speaking to the class and pointing to the value 8 in the table), what's the answer?
- Gabriel: 24. (Marti circles the number 24).
- Matilda: So, if there are more tables (emphasizing more), why would it be 24? (Triumphant expression, some classmates applaud).
- Researcher: So, is anyone still disagreeing that the result is 63? The group that thought it was 24?
- Sara: (laughing) Now we think it's 63.

As seen in this dialogue, it was common for students in the high-SES classroom to address their peers directly rather than the teacher, questioning one another during the discussions. The students did not seem focused on providing the correct answer to the teacher but rather on debating ideas with their classmates. In this interaction, David presented arguments to explain why the answer was correct, addressing Sara directly. Matilda, on the other hand, used Sara's response to highlight a contradiction with the data in the table. In this case, the researcher's interventions were aimed at organizing the conversation and providing different students with opportunities to share their ideas. The students' explanations were clear and convincing enough for the group that initially made the mistake to reconsider their answer.

Another notable aspect was how the students used the classroom space. They would walk to the board, pick up the marker, and use the board without being asked or seeking permission. This behavior suggests that they felt confident and in control of the space as a means of communicating their ideas.

The same incorrect response—"24 people"—was also observed in the low-SES classroom. However, the way the discussion was orchestrated and the teacher's role in managing the interaction varied significantly. The following dialogue illustrates part of that discussion:

- Researcher: Okay, How many people can sit at 21 tables? Carla?
- Carla: 24 people (from his seat, addressing the teacher).
- Researcher: Why is it 24?
- Carla: Because it goes up by three each time, you add three
- Researcher: And does anyone think differently? (students remain silent, looking at the teacher, and no one raises their hand) ... So, I see that most of you are using a rule that is "add three," but does this (points to the 3 in the table) come from adding three to this one? (points to the 1 in the table).
- Various: Noooo (in unison).
- Researcher: It doesn't, right? Is two plus three six?
- Various: Noooo (in unison).
- Researcher: Then why do many of you think that the answer for twenty-one tables is to add three to twenty-one? If that rule doesn't apply here?

After this dialogue, the researcher noticed that several groups were trying to add different values to the independent variable without finding a rule that applied to all the values in the table. Then, the researcher then pointed out, “Last week the rule was to add 1, so you’re thinking that you need to add something. But what if it’s a different operation? Do you know any other operation?” Minutes later, several students began experimenting with multiplication and were able to identify the correct rule. They then proceeded to explain their discovery by verifying it against the table, using the validation process that the researcher had demonstrated earlier.

As in this scene, the dialogues articulated by the students in the low-SES classroom were noticeably shorter than those in the high-SES classroom. Additionally, most of the responses were directed toward the researcher, and spontaneous peer discussions rarely occurred. Furthermore, low-SES students frequently required more direct support and explanations to progress in their tasks. In this scene, the researcher explicitly pointed out the contradiction in the table and provided hints that allowed the students to move forward.

In this context, the researcher’s role shifted significantly. Simply giving students the floor and organizing their ideas, as seen in the high-SES classroom, would have been insufficient. Instead, more guided interventions were necessary. Notably, it was common for students to internalize the reasoning demonstrated by the researcher and later explain those ideas in their own words.

DISCUSSION

The findings of this study reveal significant differences in the interaction and mathematical discussion dynamics among third-grade students from highly contrasting socioeconomic levels when engaging with early algebra tasks. These differences are evident both in how students develop and express their responses and in the teaching strategies required to support their learning. These results align with previous research indicating that students from high-SES households tend to exhibit greater autonomy in mathematical discussions, which is linked to their linguistic capital. In contrast, low-SES students prefer more explicit guidance from the teacher and attribute greater authority to their instructors (Lubienski, 2000; Zevenbergen, 2013).

Furthermore, by incorporating not only the analysis of spoken discourse but also movements and gestures, our study revealed that high-SES students demonstrated a remarkable ability to take ownership of both the classroom space and the discourse. These students directed their contributions primarily toward their peers rather than the teacher and autonomously used available materials, such as the marker and the board, without needing to ask for permission. This behavior reflects a sense of agency that may be linked to their social and cultural experiences. We argue that the integration of bodily and gestural aspects into the analysis of classroom interactions is a key dimension for understanding how participation and communication dynamics are shaped in different educational contexts.

A notable aspect of the study was that, as a result of how discussions unfolded in both classrooms, the researcher's interventions took on different forms. In the high-SES classroom, the researcher adopted a facilitative role, primarily giving students the floor and organizing their contributions. In contrast, in the low-SES classroom, more direct interventions were necessary, including strategies such as pointing out contradictions and providing hints to help students progress with the task. Despite these differences in interaction dynamics, it is important to highlight that both classrooms made significant progress in their ability to generalize functional rules and express them using different representations.

Based on these results, we argue that research on effective mathematical discussions in the classroom should consider the cultural and contextual factors that influence pedagogical interactions. As noted by Xu and Clarke (2019), the promotion of certain types of mathematical dialogues deemed desirable in the classroom has largely been driven by studies conducted in specific contexts, yet they are often presented as universally applicable standards. It is essential for future research on early algebra instruction to delve deeper into how teaching strategies can be adapted to foster meaningful dialogue in highly segregated contexts, recognizing that pedagogical practices must be culturally responsive to meet the specific needs of each student group.

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INVESTIGATING THE EFFECTIVENESS OF A TRAINING MODEL FOR DEVELOPING SEMIOTIC INTERPRETATIVE KNOWLEDGE IN MATHEMATICS EDUCATION

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This study examines the development of Semiotic Interpretative Knowledge (SIK) in a teacher training course designed according to a model focusing on semiotic functions in feedback. The course is structured in three phases, including feedback formulation, exemplar analysis, and reflective discussion on quality criteria. Data analysis revealed an evolution in participants' feedback, from minimal semiotic attention to advanced use of semiotic functions. The results emphasize the model's efficacy in enhancing SIK and its critical role in improving feedback effectiveness.

RATIONALE

Research in mathematics education has shown how interpreting students' reasoning necessitates a robust semiotic competence regarding the patterns of sign use and production (e.g., D'Amore, 2003; Duval, 1993, 2017; Iori, 2018). Iori (2018) particularly emphasizes the need for mathematics teacher education programs to adopt approaches that account for the central role of teachers' knowledge about the semio-cognitive reasons behind students' difficulties in mathematics. Various studies have examined the types of feedback teachers spontaneously provide, emphasising conceptual, strategic, or procedural features (e.g., Galleguillos & Ribeiro, 2019; Stovner & Klette, 2022), while the semiotic aspects remain under-investigated. Building upon Duval's semio-cognitive approach (1993, 2017) and relating it to the notions of Interpretative Knowledge (IK, Ribeiro et al., 2016) and feedback (Hattie & Timperley, 2007), Asenova et al. (2023a) define the notion of Semiotic Interpretative Knowledge (SIK) and analyse prospective primary school teachers' SIK in feedback, categorising it according to semiotic parameters (2023b). Results show that when prospective teachers recur to SIK, they deploy the network of semiotic functions, especially when providing feedback, and clarify that SIK interiorisation requires specific training, especially in relation to feedback. In Asenova et al. (2024), we introduce a model for a teacher training course on SIK operationalisation. In this paper, we discuss the results of a first investigation on the course model effectiveness in developing SIK.

THEORETICAL FRAMEWORK

Ribeiro et al. (2016) introduce the notion of IK as the part of the mathematical knowledge “that allows teachers to give sense to pupils' nonstandard answers (i.e., adequate answers that differ from the ones teachers would give or expect) or to answers

containing errors” (p. 9). Starting from this notion and introducing a semiotic lens in it, Asenova et al. (2023a) define the notion of SIK as “the knowledge needed by teachers in order to interpret students’ answers (be they standard or non-standard), as well as students’ behaviour, and to give an appropriate feedback to them, when conceptual knowledge is hindered, and thus remains hidden behind difficulties related to patterns of sign use and production, including individual creativity in sign use” (p. 11). The definition of SIK is based on Duval’s (1993) theory of semiotic registers. According to Duval (1993, 2017), conceptualisation in mathematics involves a complex coordination of several semiotic systems, rooted in semiotic transformations within the same semiotic system (treatments) and between different semiotic systems (conversions). A semiotic system (or register) is composed by: (1) a set of basic signs that only have meaning when set against or in relation to other basic signs (e.g., the meaning of the digits within the decimal number system); (2) a set of rules for the production of signs, starting from basic signs, and for their transformation. According to Duval, D’Amore (2003) identifies conceptualisation with the following semiotic functions, specific to mathematics: (1) choice of the distinctive features of a mathematical object and its representation in a semiotic system; (2) treatment in the same semiotic system; (3) conversion between semiotic systems. As highlighted above, SIK is particularly important in relation to feedback. According to Hattie and Timperley (2007), feedback is defined as “information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one’s performance or understanding” (p. 81). Galleguillos and Ribeiro (2019) investigate prospective teachers’ ability to use IK in giving feedback, focusing on conceptual and strategic aspects. In Asenova et al. (2023b) the authors develop the kinds of feedback introduced by Galleguillos and Ribeiro, consistently with the notion of SIK. In particular, the authors categorised the collected feedback according to the implementation of the semiotic functions: type I feedback – no mention of semiotic functions, which is framed by Galleguillos and Ribeiro’s categories; type II feedback – use of semiotic representations confined to the recognition of the distinctive features; type III feedback – use of distinctive features and treatments; type IV feedback – use of distinctive features, treatments, and conversions. The semiotic categorisation of feedback does not provide levels of effectiveness per se but represents a tool to tune sign use in producing and evaluating feedback, by identifying levels of complexity of semiotic activity.

TEACHER TRAINING STRUCTURE

Building on the basic structure of the course presented in Asenova et al. (2024), we fine-tuned the model in three main phases (P1, P2, P3). P1 aims to provide participants with the technical tools to develop SIK. It begins with a mathematical problem to be solved and a student’s solution to the same task, to be interpreted by the participants, who are asked then to write a (spontaneous) feedback for the student. A lecture on Duval’s semiotic functions follows. P1 ends with the participants being asked to modify their initial feedback if they see that, after the lecture, this might be necessary. In P2 the focus is on the reflection on quality criteria for feedback. The group activity

is based on *exemplars* (e.g., Carless & Chan, 2017) that are concrete examples of productions, each representing a different level of quality. In this context, exemplars quality is designed by us considering an increasing use of semiotic functions, according to the classification from type I to type IV feedback. Participants, working in groups on the exemplars (given in random order) are asked to evaluate each exemplar's strengths (pros) and weaknesses (cons) with open-ended analysis criteria (i.e., we do not explicitly require participants to focus on semiotic functions). In P3, a whole-class discussion takes place, where each group's analysis is presented and discussed. The discussion concludes with the introduction of an operational diagram for analysing written feedback based on semiotic functions, in which the criteria are instantiated by the five exemplars (Figure 1). The participants are required to comment on their former feedback, referring to the criteria listed in the diagram.

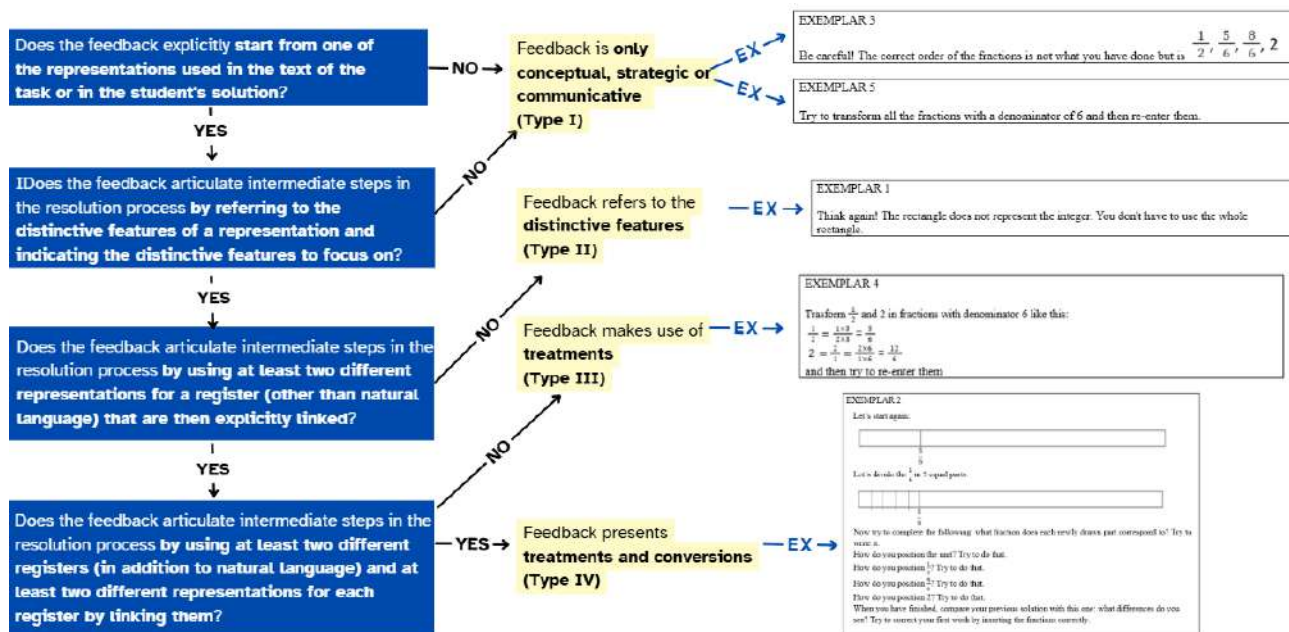


Figure 1: Operational diagram for analysing the feedback.

Thus, P3 focuses on the institutionalisation of the diverse quality criteria for the use of semiotic functions in written feedback. As the focus of the paper is on the course effectiveness, the guiding research question is: To what extent can the development of teachers' SIK be tracked within the context of the training model described?

METHODS

To operationalise the training model, we designed an activity sheet (available at <https://tinyurl.com/SIKPME48>). The activity sheet consists of nine pages. The first three pages articulate P1: page 1 presents the problem (Figure 2, left), page 2 shows the hypothetical student's (Gina) solution requiring feedback (Figure 2, right), and

page 3 asks for revisiting the feedback after the lecture on semiotic functions to allow participants to refine it.

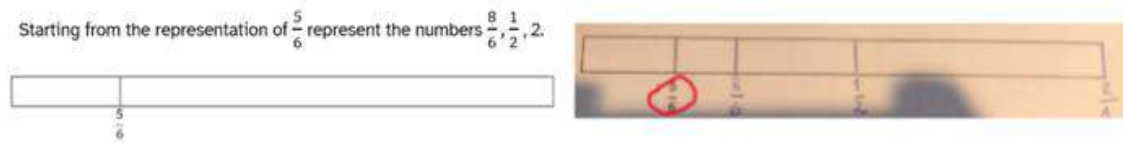


Figure 2: Problem (on the left) and Gina's solution (on the right).

P2 – group work with exemplars – is articulated on pages 4 to 9. Page 4 presents the full list (in random order of quality) of the five exemplars we designed, each of them highlighting different nuances of the semiotic functions, as presented in Figure 1. Participants' written productions in P1 and P2 are then the backbones for the final discussion (P3).

The experiment lasted 3 hours in 1 session and involved 10 pre-service primary school teachers from an Italian university, who volunteered to participate by responding to an email invitation. The activity sheets were printed out and distributed individually, and the photos of all the productions were added to a shared Padlet along with the activity. In P1 and P3, all participants were seated at the same round table. In P2, the participants were divided into three groups. The data collected consist of all participants' productions, the audio-video recording of collective and group work and field notes. For data analysis, we first designed a hypothetical learning trajectory (HTL) for the course and then compared it with the participants' actual learning trajectories (LTs). LTs, to summarise Clements and Sarama's (2004) definition, are descriptions of students' thinking in a mathematical domain along with a pathway of tasks to support their progress towards specific goals. The HLTs we designed assume that participants will solve the problem, provide feedback with little reference to semiotic functions, and then gradually increase their attention to semiotic aspects. The actual LTs are outlined by analysing participants' written productions in P2 and P3, as well as their contributions in the discussions during P3. For the analysis of the LTs, the questions in the left part of the diagram in Figure 1 serve us as a technical tool for identifying the type of written feedback (P1-productions); linking participants' comments in the exemplars to semiotic considerations (P2-productions); identifying the use of semiotic functions in the participants' reflections during the discussion (P3-productions). Thus, the diagram serves us as a lens for understanding the semiotic development observed in participants' LTs. Each of us first analysed all the data individually and independently. We then compared our analyses and reached a consensus by resolving the disagreements and unifying the observations.

RESULTS

We qualitatively analysed the LTs of four participants – Sara, Eva, Gio and Pia – chosen because they explicitly participated in the final discussion. Analysis showed that Sara displays type II feedback before and after the lecture on semiotic functions but focusing more explicitly on representations in the post-feedback. This tendency is

confirmed by her contribution to the final discussion, where she refers to semiotic transformations in her feedback, suggesting a shift to type III feedback, although she seems to consider the transformation from fraction to decimal number as a conversion rather than a treatment. Pia shifts from spontaneous type II feedback to type III feedback after the lecture on semiotic functions. She does not introduce additional semiotic aspects in the final discussion. Gio displays type I feedback both before and after the training on semiotic functions. However, from her group's work on exemplars, explicit quality criteria – which they named coherence, completeness, and clarity – emerge, implicitly referring to the use of semiotic functions. Gio also acknowledges that her written feedback does not reference semiotic functions, although she does not elaborate on this point. Eva displays a switch from type II to type IV feedback from before to after the lecture on semiotic functions and explicitly recognises the characteristics of her feedback according to the diagram in Figure 1 in the final discussion. In the following, we exemplify our analysis by illustrating Eva's LT because her pathway is particularly suitable to highlight how references to semiotic aspects evolve throughout the course. In P1, Eva's spontaneous feedback is the following:

Your answer Gina helps us to reason together more about the fraction. It really convinces me how you placed $\frac{8}{6}$ because from $\frac{5}{6}$ you went to the right. Did you pay attention to the $\frac{6}{6}$ notch? Do you remember what happen in this case? Now you are still convinced about the position of $\frac{1}{2}$ and 2?

Eva refers to Gina's representation of the fraction $\frac{8}{6}$, saying that it convinces her, since she “went to the right”: there is an implicit reference to the correct ordering, which is anchored in the student's semiotic representation (indicator for the quality criterion expressed in step 1 at the top left of the diagram in Figure 1). She also draws attention to the reference to the whole, without naming the term ‘whole’, but speaking of the “notch” corresponding to $\frac{6}{6}$. An attempt is made to highlight the distinctive feature of the fraction as part of a whole by making explicit the reference to the whole and its positioning on the rectangle (indicator for the quality criterion expressed in step 2 in the diagram in Figure 1). There are no explicit references to treatments (explicit rules belonging to the arithmetic and iconic registers) and conversions (no explicit reference is made to a ‘translation’ between the representations of the two registers), as well as relations between representations. Feedback is thus of type II. At the end of the introduction to semiotic registers and functions, the student returns to the feedback and proposes the modification shown in Figure 3. What in the spontaneous feedback remained a connection between representation in natural language (“notch”) and representation in arithmetic register ($\frac{6}{6}$) is now expressed by articulating representations in iconic, arithmetic and natural language registers. The student again explicitly starts from the representation in the assignment (‘the strip’) and the distinctive feature of the fraction as part of the whole (she explicitly represents the unit).

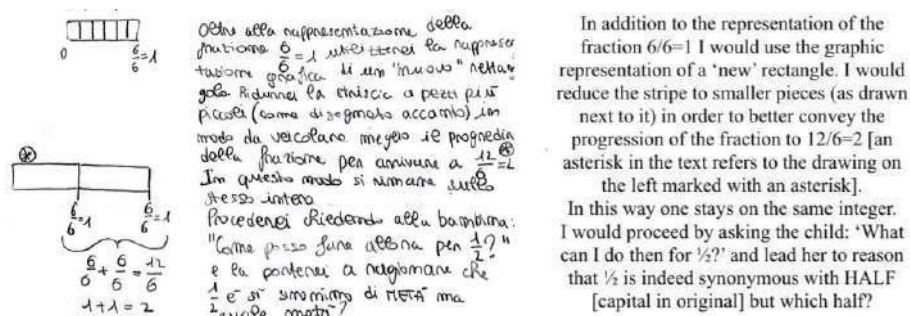


Figure 3: Eva's post-training feedback (left), flanked by the text translation (right).

In doing so, she applies transformation rules in the iconic register, performing treatments; in fact, she constructs the strip step by step, instead of considering it as already given (indicator for the quality criterion expressed in step 3 in the diagram in Figure 1). At each step, the student associates a fraction in the arithmetic register with its representation in the iconic register, performing conversions (indicator for the quality criterion expressed in step 4 in the diagram in Figure 1). The post-training feedback is therefore type IV. Regarding P2, we analyse Eva's group work by referring to the collective sheet. Rather than examining each exemplar in detail, we focus only on exemplar 1 (Figure 1, upper right side) to illustrate the process of analysis and then provide a summary of the group's overall work. Concerning exemplar 1, the students write:

Pros: An attempt is made to clarify the concept of the rectangle unrelated to the concept of the unit, but the concept of the unit must be clear for the clue to be understood.

Cons: It is not acknowledged that Gina has correctly positioned $\frac{8}{6}$ and nothing positive is said, which does not serve the student's motivation. There is no mention of fractions being compared to the unit, no mention of the fact that the unit is in reference to $\frac{6}{6}$.

In commenting on the pros of this exemplar, the students note that it attempts to distinguish the rectangle from the unit. This reflects their recognition of the importance of selecting the distinctive features of the mathematical representation used and of considering the relationship between different representations of the same concept. In this respect, they point out in the cons that there is no explicit mention of how fractions relate to the unit, nor any reference to $\frac{6}{6}$ as the relevant unit. This concerns the lack of a conversion from the iconic register (the unit as shown on the rectangle) to the arithmetic register ($\frac{6}{6}$). Overall, the comments illustrate how for the students recognising distinctive features and facilitating conversions between registers are essential elements of effective feedback. When asked to go back over her own post-semiotic feedback during the final discussion (P3), Eva claims to have reasoned in terms of treatments when he transformed $\frac{6}{6}=1$ and $\frac{12}{6}=2$, and to have "also included the let's say drawing part", thus working "on two registers". She uses terms related to semiotic functions properly, saying that she wanted to "associate a treatment with a conversion as well". She confirms that she recognises the characteristics of type IV

feedback in her own feedback, consistently referring to semiotic functions.

DISCUSSION AND CONCLUSIONS

The analysis shows that the prospective teachers' LT correspond to the designed HLT for three of the examined cases (Sara, Pia and Eva), in the sense that there is an evolution of their SIK in providing feedback, while in one of the examined cases (Gio), the effective LT presents a lack of such evolution. Nevertheless, also in this last case, the student's intervention in the final discussion shows an awareness of a lack of semiotic functions in the provided feedback. This seems to be due to the group work on the exemplars and to the final discussion, culminating with the presentation of the diagram (Figure 1). In this sense, during the different phases of the training course, teacher-students' feedback becomes more and more SIK-informed or at least they become more SIK-aware. The evolution of Eva's feedback is particularly significant in this sense: Eva starts with a type II feedback that does not refer to treatments and conversions, while her feedback after the lecture on the semiotic functions displays a successful coordination of all three semiotic functions becoming type IV feedback. Furthermore, during the discussion phase, Eva reflects on the feedback quality criteria linking them to the characteristics of her own feedback. Eva's LT is a paradigmatic example of how teachers' SIK development can be tracked throughout the training course. In particular, the various phases of the course highlight the key stages of SIK evolution: first through the lecture on semiotic functions, then through reflection on their instantiation in exemplars, and finally through the meta-reflection on the quality criteria presented in the diagram and the subsequent reconsideration of one's own feedback. In this sense, Eva's LT shows not only a growing use of SIK in feedback but also a growing awareness of its role, revealing the course's capacity to operationalise SIK in relation to feedback in teacher education.

Our aim was to examine to what extent the development of teachers' SIK can be tracked within the context of the training model presented in Asenova et al. (2024) and fine-tuned as described in the third section. According to Hattie and Timperley's definition, feedback provides *information* regarding aspects of one's performance or understanding. As the results of our research show, such *information* can be shaped by the SIK: it is the conscious use of the semiotic functions that allows the teacher to purposefully employ SIK in his/her feedback, improving its effectiveness. Further research should be carried out to characterise feedback in reference to SIK and to investigate how the participants transfer the acquired awareness of the role of SIK in feedback to other student-solutions, and to fine-tune more the course structure and implementation (e.g., we realised that more time is needed for the last phase).

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NOT JUST A CHALLENGE: TURNING UNCERTAINTY INTO TEACHER GROWTH IN INQUIRY-BASED INSTRUCTION

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This study examines the uncertainty teachers face while implementing inquiry-based mathematics instruction. We analyzed data collected from four years of a professional development program and its classroom implementation. Two key sources of uncertainty were identified: unpredictable lesson dynamics and unfamiliar student-introduced content. Teachers tackled these challenges using: (1) extensive preparation to foresee potential inquiry paths, (2) leveraging peer support, and (3) adopting a co-learning stance that embraces knowledge gaps and fosters mutual exploration. While uncertainty poses significant challenges, it can also serve as a catalyst for teacher growth in pedagogical knowledge, instructional strategies, and professional identity.

INTRODUCTION AND BACKGROUND

Despite the benefits of student-centered approaches (Darling-Hammond et al., 2017), their adoption in mathematics instruction remains limited (Dorier & Maass, 2020), largely due to the significant professional growth they demand. Teachers must acquire new knowledge, refine existing practices and forge new ones, and rethink their beliefs about teaching and about their role (Horn & Garner, 2022; Avishai et al., 2025). A major challenge is the inherent uncertainty that emerges when the focus shifts from definitive answers to multiple plausible solutions and unpredictable discussions driven by students' autonomous inquiry (Palatnik, 2022). This uncertainty extends beyond mathematical content—where teachers must rely on deep content knowledge to navigate explorations that may surpass their expertise (Cohen, 2011)—to social interactions, classroom dynamics, and facilitation challenges (Horn & Garner, 2022).

However, uncertainty is not merely an obstacle; it can present significant opportunities for growth. In the theory of didactic situations in mathematics, Brousseau (1997) describes uncertainty as “a source of both anguish and pleasure” for students, noting that “the reduction of this uncertainty is the aim of intellectual activity and is its driving force” (p. 45). Similarly, Zaslavsky (2005) highlights the potential of “uncertainty-evoking tasks” to deepen *students'* mathematical understanding, while Staats et al. (2024) emphasize how students' productive struggle can transform uncertainty into discovery. Extending this perspective to *teachers*, we argue that while engaging with the inherent uncertainty stemming from students' inquiry is indeed challenging for teachers, it offers meaningful opportunities for their professional development, fostering growth in knowledge, instructional strategies, and professional identity.

In this paper we (1) explore the sources of uncertainty inherent in student-centered mathematics instruction and (2) examine the strategies teachers used to manage these

challenges. Then we (3) demonstrate how uncertainty, while a significant challenge, can also serve as a powerful catalyst for teacher growth. We conclude with practical guidelines for PD providers and school leadership to help teachers transform these challenges into opportunities for professional development.

CONTEXT AND METHOD

The context of this study is “Practimatics”, a PD program and teacher learning community aimed to expand the use of problem-based and inquiry-based pedagogies in middle school mathematics instruction in Israel. The 30-hour year-long program provided initial training and modeling of relevant pedagogical aspects such as group mentoring and inquiry facilitation, original mathematical contents and materials, and a safe meeting place for peer dialogue and support. Meetings were held in Zoom every three weeks and lasted between two and three hours. In these PD sessions teachers engaged with real-world problems (see example in Figure 1) as learners in an unguided setting. Then, in a guided session that followed, mathematical solutions were analyzed, possible students’ inquiry paths were discussed, and the teachers adapted the problem to their classes. The teachers were then encouraged to take the problem to their 9th grade class, starting with problem-solving activities and gradually moving toward open inquiry, where students generate follow-up questions as basis for autonomous group inquiry. The teachers returned to the PD meetings to share success, challenges, and dilemmas that arose during implementation of the contents and pedagogy in their classes. The PD facilitators – one of whom is the author of this paper – also provided real-time support over Zoom or WhatsApp and responded to individual teacher challenges through personal consultation meetings. Over its four years of operation, 26 teachers engaged in the program (F=22, M=4, teaching experience ranging from 2 to 40 years). Preliminary analysis revealed notable variation in the implementation’s extent and depth. For example, while some teachers introduced problem-solving lessons but minimized or omitted open inquiry, others devoted considerable time and effort to open inquiry despite its challenges (Avishai et. al., in 2025).

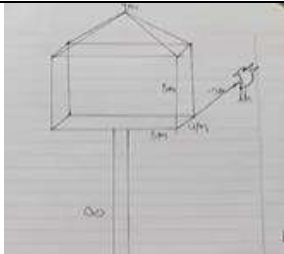
Part 1 – problem-solving: A goat is tied with a 6m rope to a corner of a 4X5m rectangular building in the middle of a very large field. What is the area the goat can feed in?	Part 2 – problem posing and open inquiry: Create and explore a follow-up problem by altering one or more data parameters from the original problem.		Ann’s students follow up question: A pigeon is tied with a 10m long rope to a rectangular cage measuring 4X5X5m. What is the volume of the space in which the pigeon can fly? Note the ∞ simbol, indicating the cage is higher from the rope’s length.
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Figure 1: The goat problem (left, adapted from Mason et al., 2010), the instructions given to the teachers in the PD and later to the students in class (middle), one of the follow-up questions generated by Ann’s students (right)

Data Collection and Analysis

The data for this study was collected during the PD’s four years and includes recordings and transcripts of over 50 PD meetings and one-on-one teacher consultations with PD

facilitators, observations in student project presentations, excerpts from email correspondence and participant WhatsApp group, dozens of post-session reflection forms, over 20 synthesis papers submitted by teachers for PD credits, and eight retrospective teacher interviews. Data analysis included several iterations:

We began by organizing all spoken and written statements made by each teacher throughout the year. These were compiled into detailed, chronological sequences of events for each teacher, noting challenges, dilemmas and decisions taken along the process. We documented the problems selected and the depth of implementation, ranging from brief problem-solving activities (1-2 lessons) to students formulating follow-up questions and engaging in extended open-ended inquiry spanning weeks. Next, we reviewed our dataset to extract utterances related to challenges stemming from uncertainty. Three primary categories emerged: (1) sources of uncertainty, (2) emotional responses to uncertainty, and (3) strategies for managing uncertainty. However, revisiting the teachers' chronological sequences to analyze the differences in their choices revealed a fourth, less immediately apparent category: instances where navigating uncertainty not only enhanced student outcomes but also catalyzed teacher growth. Intrigued by this seemingly paradoxical phenomenon, we analyzed the ways uncertainty triggered change in teachers' pedagogical and mathematical knowledge, their instructional practices and strategies, and their professional identity.

FINDINGS

The following sections present short excerpts from our dataset. We begin by describing the sources of uncertainty teachers encountered, followed by the strategies they used to address it, and finally, cases where coping with uncertainty fostered teacher growth. Teachers' experience (in years) is noted in parentheses next to their pseudonyms.

Lack of clear pathways and shift in classroom dynamics

A major source of uncertainty for many teachers was the lack of a predetermined script, both within individual lessons and across longer-term pedagogical processes. Erin (12), for example, stated in her synthesis paper that:

“It’s difficult to predict and anticipate the course of events, and the teacher must constantly adapt and learn alongside the students to be able to guide and support them in their learning process.” (June 2022).

Leading students' inquiry, she added, challenges familiar teaching methods and creates discomfort, as it “...requires much more effort and energy... the control shifts to the students”. Sara (7) referred to the disruption in the flow compared to traditional lessons, which required her to navigate the slower pace of student discovery:

“It took me twice what I planned... I learned that as a teacher I need to be flexible, be prepared not just to the current lesson, yes? To have a little more.” (PD session 4, 2021).

Summarizing her experience, Sara noted that “it's not like regular math classes where you prepare the unit and teach it. Every class is a surprise”. (last PD session, 2021).

Exposure to unfamiliar mathematical directions

Granting students greater agency through problem-solving activities, let alone through open inquiry, led to uncertainty regarding mathematical content brought up by students. May (5) explained in her synthesis paper that her biggest challenge was "...letting the students explore on their own and ask questions that I may not have an answer to. It was unusual, both for me and for the students." (June 2023)

Such a knowledge gap surfaced when Ann's (5) students, who engaged with the goat problem (Figure 1, right), transformed the original 2D problem to a 3D problem. In a PD session several days later, Ann shared:

"I told them that if I take it to two dimensions, I'll know how to solve it. But to work on it in three dimensions, I kind 'a need to figure out how to solve it, because I'm not really familiar with it—it's not material I usually deal with." (PD session 10, 2023)

In the discussion that followed, the PD facilitator encouraged teachers "...not to be afraid to engage in such a process with the students, even though you don't know everything...inquiry is something we all do together." Nevertheless, the mathematical uncertainty that arose when students generated follow-up questions caused anxiety and hesitation among some teachers, as illustrated by Nora (20+):

"...then I had to continue to the next stage, have them think about a small inquiry project... and that's where I got stuck...I mean, I could have continued, but I didn't want to go somewhere I don't know...and then get stuck." (PD session 6, 2021).

Despite the uneasiness some teachers felt in face of uncertainty, many coped with it successfully. In the following sections we present three strategies identified through our analysis that helped teachers address and navigate these challenges.

Extensive preparation

Teachers invested substantial time and effort to learn new mathematical content and attempt to foresee as many potential student inquiry paths as possible. Pam (20):

"I also sat down and researched and checked... I mean, at nights or in the evenings, or while they [students] were working, I would also sit and check to find [mathematical solutions, because] I needed to be their mentor and give them some guidance." (Interview)

Similarly, Erin (12) described her process of watching YouTube videos to anticipate and research directions students might explore:

"At home, I sat and watched [the videos] ... and tried to think of all the possibilities they might raise." (Interview)

Amy (34) also dedicated significant time outside of class to learning mathematical and technical content to support her students' projects:

"I had to learn new things in various technological areas—different resolutions and their differences, pixels, tiling methods, and more, so that I could support the different groups in their projects." (Interview)

But she highlighted the need for mental preparation as well:

“It also required a lot of...mental preparation...for when you enter the classroom and you're not really sure what you're going to do, how you're going to deliver it.”

Relying on peer support and mentoring

Teachers often sought advice and support from fellow participants in the program. Nora (20+), for example, obtained ad-hoc support by observing Sara's classes that were held in Zoom due to Covid imposed restrictions. She described this experience as invaluable for refining her own teaching practices and admitted she learned a lot from Sara (7), who, despite being newer to the profession, demonstrated great confidence in her teaching. In one of the PD sessions Nora reflected that

“It really, really helps me that I attend Sara's Zoom and see her dilemmas in front of the students... and then I try to correct, improve, add... in my group” (PD session 6, 2021).

In May's (5) case, support was formally initiated by the school's principal who assigned Jade (8), a more experienced teacher, to mentor her. Jade not only assisted May with lesson planning but also supported her during actual implementation in class, providing practical guidance and bolstering May's confidence in managing the challenging classroom dynamics. Similarly, Erin's (12) support came from Ira (40), the school's math coordinator and a physics teacher, which enabled Erin to effectively support her students' inquiry despite limitations in subject matter, as Ira described:

“At every stage, Erin updated me on the students' progress, her own dilemmas, the challenges that arose, as well as their achievements, and I was there to listen and provide support when needed... [When] one of the groups chose a topic related to physics – a subject Erin lacks knowledge about, she asked me to mentor this group, and I agreed.”

Openly admitting knowledge gaps and embracing a co-inquiry approach

Another common strategy was for teachers to openly admit their knowledge gaps to students and embrace a co-learning approach, as Ann (5) shared during a PD session:

“I told [the students] ‘No, I have no idea and that's fine, you investigate, and if there are things you need my help with, ask. But the goal is that *you* will ultimately investigate because *I* don't have all the answers.’ That's what I told them.” (PD session 6, 2023).

Admitting knowledge gaps required courage and marked a significant shift in some teachers' beliefs about their role, traditionally viewed as authoritative figures expected to provide definitive answers. For Amy (34), this shift proved rewarding, as she described in her interview (July 2022):

“... At first, I was afraid it might undermine my authority a bit, but it turned out to be a wonderful experience. Truly, the kids are open-minded, and they understand their teacher isn't God. She has her shortcomings; she has a lot to learn... I have to say, it's a very interesting experience and I enjoy it, learning from the students as well, as young and curious as they are.”

She openly admitted to her students that she, herself, is in constant learning:

“I remember a situation where I said, 'I don't have an answer right now. I'm going to check it, and I just want to remind you that the [inquiry] process that you're going through, I went through it the day before [in the PD session]”.

Uncertainty as a catalyst for teacher growth

When Eve's (13) students explored shade balls used to reduce evaporation in a water reservoir near LA, they asked her why the designers chose a certain ball size, to which she replied, “I don't know, let's investigate.” Eve's embrace of uncertainty empowered her students to lead the inquiry, fostering a dynamic classroom environment where curiosity and discovery thrived. By acknowledging her knowledge gap, Eve created space for students to take ownership of their learning, exploring alternative explanations such as cost efficiency and transport logistics, thereby transforming uncertainty into an opportunity for student authentic learning and exploration.

For Eve, who usually provides close guidance to her students, leading inquiry without knowing the solution became a positive learning experience: “I liked not knowing because otherwise, I lead them too much... I give them too much direction.” While relinquishing control was uncomfortable at times, she embraced it, reflecting “I wasn't in control, but that's okay. I need to know how to let go.” Her reflections illustrate professional growth in both her practice and beliefs.

May's case (described in detail in Avishi et. al., 2025) provides another example of how coping with uncertainty fostered teacher growth. May (5) entered the PD program with no prior experience in inquiry-based instruction. She adhered to a traditional, teacher-centered pedagogy, believing that good teaching meant always having the right answers. Cautiously engaging with the PD activities, she implemented several structured problem-solving tasks in her class. Allowing her students to independently grapple with the complexities of the problems despite knowing only partial solutions herself marked her first steps toward embracing uncertainty. Yet moving beyond structured problems into open inquiry proved daunting, causing sincere helplessness at times: “This week I canceled the meeting because I didn't quite know how to continue” she told the PD facilitator during a personal session. She candidly expressed her concerns about appearing uninformed: “How do you get the teacher to be able to honestly deal with a situation where they just don't know?”

The turning point occurred after May heard Ann (5), a fellow teacher at the PD, sharing her own experiences of embracing uncertainty in the classroom. Ann (quoted above) described openly admitting her lack of knowledge to students, framing it as an opportunity for collaborative investigation. Ann's sharing inspired May to contain her concern of her knowledge gaps being exposed. From that point May led her students to select research directions, conduct autonomous inquiry for several weeks, and refine their findings through iterative presentations. Reflecting on her journey, May recognized how pivotal was Ann's description for her own professional growth:

"What I heard from Ann, like, what she experienced. She told the students 'I don't have an answer to your question, I don't know'. I heard her experience and wanted to experience it

myself...[moving] from a traditional instruction to a more creative way of teaching required me to change my role as a teacher” (interview, August 2023).

May's journey illustrates how embracing uncertainty can catalyze not just a change in teaching practice, but a fundamental reimagining of one's role as a teacher.

DISCUSSION

This study explored the inherent uncertainty teachers faced when engaging in inquiry-based mathematics instruction. The uncertainty stemmed from two primary sources: (1) anticipating the dynamics of individual lessons and the broader trajectory of the inquiry process, and (2) unpredictable mathematical content introduced by students during open inquiry. These findings align with Horn and Garner's (2022) assertion that shifting to ambitious mathematics instruction creates a more complex terrain for teachers, disrupting lesson smoothness and introducing uncertainty as mathematical discussions may evolve in directions that can sometimes surpass their expertise.

Our analysis identified three key strategies that helped teachers manage these uncertainties: (1) Extensive preparation, which involved learning new content and anticipating potential student inquiry paths to mitigate uncertainty's impact and equip teachers with tools to address the unexpected. (2) Relying on peer support, ranging from informal advice exchanges to structured mentoring. Peers offered practical guidance, emotional encouragement and assistance in preparation for class and students' mentoring during lessons, and enabled teachers to better navigate challenges stemming from the shift toward problem-based and inquiry-based instruction, and (3) Adopting a co-learning approach, where teachers openly acknowledged their knowledge gaps and positioned themselves as learners alongside their students. This approach, while demanding courage and vulnerability, often led to rewarding outcomes, illustrating how uncertainty can drive growth by shared exploration.

An especially intriguing finding was that uncertainty, far from being solely a challenge, can drive meaningful professional development for teachers, fostering growth in their practice, knowledge and beliefs. While Brousseau (1997) referred to uncertainty as a driving force for student learning, our findings extend this perspective to teachers. Similarly, while Zaslavsky (2005) highlights the role of uncertainty in deepening students' mathematical understanding, we showed that navigating uncertainty also holds transformative potential for teachers. Many of the teachers in our study turned the uncertainty they encountered when engaging with students' inquiry into a catalyst for professional growth. As demonstrated in the cases of May, Eve, and Amy, this growth manifested in changes to their teaching practices, instructional strategies, and their professional identities.

To facilitate the transformation of uncertainty into professional growth, PD providers and school leadership should: (1) help teachers recognize and embrace uncertainty as an inherent and valuable aspect of inquiry-based mathematics instruction, (2) actively encourage teachers to model authentic exploration by transparently acknowledging

knowledge gaps with students, fostering a genuine co-learning environment, (3) create supportive networks and spaces for teachers to collaboratively reflect, share experiences of supporting student inquiries and adapt to challenges, backed by clear administrative support to reduce isolation.

At the same time, it is evident that different teachers will respond differently to uncertainty depending on various individual and contextual factors. Future research should explore how factors like teachers' beliefs and attitudes, prior knowledge and professional phase, institutional conditions, cultural background, and social networks influence their capacity to transform uncertainty into professional growth. Such insights could inform PD programs to better support diverse teacher needs effectively.

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BRIDGE-BUILDING AS A DIMENSION OF MATHEMATICS TEACHER EDUCATORS' EXPERTISE

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Mathematics Teacher Educators' (MTEs) expertise involves designing formative tasks that address mathematics, teaching mathematics, and/or academic research for teachers. Connecting two or more of these scopes, termed bridge-building, is central to their role. This study explores and classifies bridge-building forms demonstrated by experienced MTEs in 20 booklets written for novice MTEs and prospective teachers. The corpus was categorized based on the number of scopes (binodal/trinodal), the arrangement (sequential/integrated), and the purpose (informative/engaging). The proposed categorization contributes to understanding MTE expertise and advancing research on teacher educators' professional practices.

INTRODUCTION

Research on mathematics teacher educators (MTEs) has gained increasing attention in the mathematics education agenda (Dona & Ribeiro, 2024; Helliwell & Chorney, 2022; Helliwell et al., 2024; Goos & Beswick, 2021). Mathematics Teacher Educators (MTEs), responsible for mathematics teachers' education and professional development, play a vital role in pre-service and in-service teacher education (Goos & Beswick, 2021). The notion of expertise involves how MTEs develop their professional practices in relation to teachers. (Helliwell & Chorney, 2022; Helliwell et al., 2024; Barbosa & Chapman, 2024). Barbosa and Chapman (2024) identified three scopes of MTE expertise – mathematics, teaching mathematics, and academic research – which refer to "what" MTEs address with teachers in their role. A key aspect of MTEs' expertise is bridge-building, i.e., connecting two or more of these scopes, which is fundamental to improving the quality of teacher education programs. Revisiting prior studies on MTEs reveals the presence of this aspect in empirical descriptions of their professional practices (Wasserman et al., 2023). Barbosa and Chapman (2024) argue that the theoretical notion of bridge-building sheds light on our understanding of the complexity of MTE expertise and merits further investigation. This study aimed to address the question: How do MTEs demonstrate bridge-building as part of their professional expertise? After clarifying the theoretical perspective of the study in the next section, I will restate the research question. The research report outlines the related literature, theoretical perspective, context, methodology, findings, discussion, and final remarks, contributing to the ongoing research agenda on MTEs' expertise.

RELATED LITERATURE AND THEORETICAL PERSPECTIVES

There is broad agreement that Mathematics Teacher Educators (MTEs) play a pivotal role in the initial education of prospective mathematics teachers and the professional development of in-service teachers (Goos & Beswick, 2021). Broadly, MTEs are professionals who guide mathematics teachers' education and professional growth (Barbosa & Chapman, 2024). These professionals work in universities or teacher education institutions and, in schools, often take on mentoring roles to support novice teachers (Ebbelind & Helliwell, 2024).

Much of the research on MTEs focuses on characterizing their specialized knowledge (Chapman, 2021; Dona & Ribeiro, 2024; Goos & Beswick, 2021), leading to theoretical models defining MTEs' knowledge domains as extensions of models for mathematics teachers' knowledge (Chapman, 2021). For example, Superfine et al. (2020) detail the domains of the Mathematical Knowledge for Teaching Teachers (MKTT) model, extending the Mathematical Knowledge for Teaching framework. Similarly, Martignone et al. (2022) introduced the Mathematics Teacher Educators' Specialized Knowledge model, building on the Mathematics Teachers' Specialized Knowledge model. While these approaches provide important insights, Chapman (2021) argues that relying solely on models for teachers' knowledge may limit understanding of MTEs.

To expand this understanding, some researchers use the concept of expertise (Helliwell & Chorney, 2022; Helliwell et al., 2024; Barbosa & Chapman, 2024). Inspired by distributed cognition, Helliwell and Chorney (2022) conceptualize MTE expertise as extending across individual, social, and material dimensions. Building on this, Barbosa and Chapman (2024) define MTE expertise as "an amalgam of knowledge, social participation, and communication, reflecting the specific know-how of MTEs in their professional role" (p. 40).

In this study, expertise is understood, as outlined by Ericsson (2018), as encompassing professionals' performance within their domain. In the case of MTEs, expertise entails engaging with teachers through pedagogical communication. This process involves purposefully exchanging perspectives, skills, and values to support teacher education and professional development shaped by social and institutional contexts (Barbosa & Chapman, 2024; Helliwell & Chorney, 2022). MTEs enact their expertise through pedagogical communication (Barbosa & Chapman, 2024). Following Basil Bernstein's sociology, Barbosa and Chapman (2024) see pedagogical communication as a socially situated interaction, where, in this case, MTEs are educators and teachers are learners. This dynamic is fluid, acknowledging that teaching teachers involves learning and learning to teach involves teaching.

Barbosa and Chapman (2024) identified that the expertise of MTEs manifests itself in three scopes of pedagogical communication: mathematics, teaching mathematics, and academic research. These scopes reflect "what" MTEs might communicate in their interactions with teachers. Mathematics involves tasks to deepen and expand

mathematical understanding; teaching mathematics focuses on aspects of mathematics teaching practices; and academic research explicitly brings findings from mathematics education research into teacher education.

The authors use the bridge-building metaphor to describe instances where MTEs explicitly connect two or more scopes of pedagogical communication. For example, solving a mathematical problem to deepen understanding and then analyzing school students' responses bridges the scopes of mathematics and teaching mathematics. In turn, tasks that encourage teachers to explore research findings and identify their implications for teaching help connect the scopes of teaching mathematics and academic research. The literature on teacher education emphasizes the need to connect advanced mathematics with school mathematics (Wasserman et al., 2023), raising questions about how MTEs educate teachers to achieve this. Following Barbosa and Chapman's (2024) framework, this effort constitutes bridge-building mathematics and teaching mathematics scopes.

Hence, it is essential to examine MTE's expertise in designing and implementing formative tasks that promote bridge-building between different scopes. Recognizing its potential impact, this study identifies and characterizes how MTEs demonstrate bridge-building across the scopes of pedagogical communication in their role.

CONTEXT AND METHODOLOGY

The data comprised 20 booklets authored by ten experienced Brazilian mathematics teacher educators (MTEs), most of whom hold PhDs in mathematics education. Here, I assume that these booklets encapsulate the meanings that MTEs assign to their professional practices, which aligns with qualitative research (Creswell, 2015). These booklets, ranging from 30 to 50 pages, were written to support newly hired MTEs at the University of the Federal District, located in Brasília (Brazil), which began operating in 2023. The university funded the production, and because of time constraints, the authors autonomously decided the structure and perspective of their booklets. The series consists of four booklets for each theme, covering the topics of Numbers, Algebra, Geometry, Probability and Statistics, and Measurement.

The booklets exhibit significant variation, ranging from those that focus solely on one scope of pedagogical communication to others that address two or three scopes, with examples of bridge-building appearing in the latter. They embody the collective experience of this group and provide insights into their expertise as MTEs. This dual purpose – supporting new MTEs and prospective teachers – offers a unique opportunity to examine how MTEs design bridge-building across different scopes. Instances of bridge-building were found in 15 out of the 20 booklets.

Inspired by Creswell's (2015) approach to qualitative data analysis, I examined the booklets through a line-by-line reading. This process involved identifying patterns and grouping passages related to bridge-building based on specific characteristics. I argue that this study is trustworthy due to the heterogeneity of the data, derived from booklets

authored by MTEs from different universities, adherence to Creswell's (2015) analytical steps, and the consistent use of a theoretical background to interpret the findings.

FINDINGS

The analysis of how Mathematics Teacher Educators (MTEs) demonstrate their expertise in bridge-building across different scopes of pedagogical communication in the booklets could be categorized using three criteria: number of scopes, arrangement, and purpose. I will illustrate them with a booklet example (Figure 1).

(a)	(b)	(c)
<p>Access the GeoGebra link: https://www.geogebra.org/m/d3srety.</p> <p>Vary a and b, and in groups, analyse and answer the following: Regarding the graph of the linear function $f: R \rightarrow R; f(x) = ax + b; a, b \in R$.</p> <p>a) What does b represent in the graph? b) What happens when b changes? c) What is the intersection of the graph of f with the Ox-axis? What is its relationship with solving first-degree polynomial equations? d) What characteristics do you observe in the graph when $a > 0$? And when $a < 0$? e) If a_1, a_2, \dots, a_n positive real numbers are the slopes of linear functions, what is the relationship between the steepness of their graphs? And if the numbers are negative? f) How would you justify the answers given in the previous items using more "formal" arguments?</p>	<p>The previous problem illustrates an inquiry-based task, aligning with what Skovsmose (2000) calls "landscapes of investigation". According to the author, this type of task requires that "[...] students are invited to engage in processes of exploration and justified argumentation" (SKOVSMOSE, 2000). The proposed investigation activity leverages software to enable simulation, hypothesis construction, and inference. Specifically, the task would encourage students to explore, comprehend, and interpret images (graphical representations of functions), make inferences, and, after testing, translate these inferences into verbal and symbolic language. This process fosters justification through multiple forms of communication. Ponte, Brocado, and Oliveira (2019) propose that an inquiry-based task should be structured in three phases: (1) introduction of the task; (2) conducting the investigation (individually, in small groups, or with the whole class); and (3) discussion of the results. Engaging in and implementing inquiry-based tasks in mathematics teaching can significantly enhance students' learning (PONTE, BROCADO, & OLIVEIRA, 2019).</p>	<p>1. Read the following paper: Milani, R. (2020). Transformar exercícios em cenários de investigação: Uma possibilidade de inserção na educação matemática crítica. <i>Perspectivas da Educação Matemática</i>, 13(31), 1–16.</p> <p>2. Discuss in groups what an inquiry-based task is and how exercises can be redesigned into inquiry-based tasks.</p> <p>3. Choose an exercise from a textbook on functions and transform it into an inquiry-based task.</p>

Figure 1: Excerpts translated to English from an Algebra booklet.

Following the classification by Barbosa and Chapman (2024), the pedagogical communication of MTEs with teachers might involve three scopes: mathematics, teaching mathematics, and academic research. Figure 1 illustrates a progression of texts designed by a booklet author to engage teachers across the three scopes. In Figure 1a, the presented task emphasizes mathematics understanding without explicitly linking it to classroom context. Figure 1b shifts the focus to teaching mathematics and introduces comments on how teachers can use such tasks in classroom. Finally, Figure 1c integrates academic research by asking teachers to analyze an academic paper on transforming exercises into inquiry-based tasks and apply this by redesigning a textbook exercise, bridging teaching mathematics and academic research scopes. The excerpts are displayed as parallel figures due to space constraints, even though they were originally presented sequentially in the booklet.

To present the bridge-building classification based on the data analysis, I will only refer to the example presented in Figure 1 because of space limitations. As previously

mentioned, the identified bridge-building passages were categorized into three groups: the number of scopes, the mode of arrangement, and the underlying purpose.

Number of Scopes. The bridge-building demonstrated by the MTEs in Figure 1 crosses all three pedagogical scopes – mathematics, teaching mathematics, and academic research – making it "trinodal". In contrast, if the sequence involved only the texts from Figures 1a and 1b, the bridge-building would involve just two scopes, mathematics and teaching mathematics, and be classified as "binodal". Notably, the analyzed corpus shows that most instances of bridge-building were binodal, highlighting more difficulty connecting all three scopes.

Arrangement. It was possible to identify “how” MTEs arrange their bridge-building across different communication scopes. Returning to Figures 1a and 1b, they correspond to the mathematics and teaching mathematics scopes. Their sequential organization allows this approach to be classified as "sequential" bridge-building.. In contrast, Figure 1c features a single task that involves reading and discussing a scientific paper integrated by its application to redesign a textbook exercise. This organization merges the scopes of academic research and teaching mathematics, classified as "integrated" bridge-building.

Purpose. The sequencing in Figures 1a and 1b connects a mathematical task with commentary on inquiry-based tasks and the activities students might do. However, no formative tasks are provided to support teachers in developing this perspective; they are simply informed about inquiry-based tasks. This type of bridge-building is classified as "informative". Conversely, Figure 1c invites teachers to explore the idea presented in the paper by redesigning an exercise into an inquiry-based task through a hands-on activity. This type of bridge-building is "engaging", as the MTE involves teachers directly articulating the connection between the scopes.

Summing up, through the analysis of the booklets, it was possible to identify passages where MTEs demonstrated the ability to bridge-building across different scopes of pedagogical communication. I developed a classification for MTEs’ bridge-building, as illustrated in Figure 2.

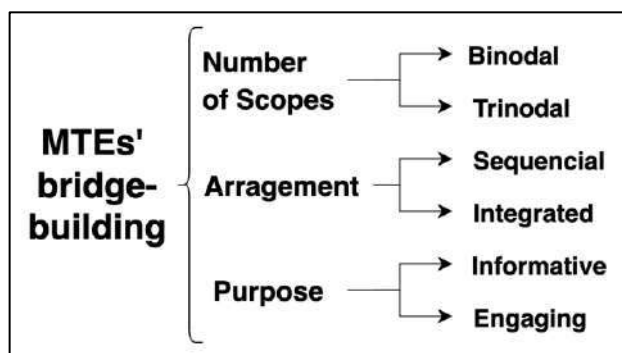


Figure 2: Classification of MTEs' bridge-building

While this classification is derived from booklets authored by MTEs to support new MTEs and prospective teachers, it is not limited to this specific context. Instead, it can

be theoretically extrapolated to other contexts, serving as analytical tools for researchers investigating MTEs in different settings.

DISCUSSION AND FINAL REMARKS

This study aimed to investigate how a group of experienced MTEs demonstrate bridge-building as part of their professional expertise, contributing to the growing research agenda on MTEs (Barbosa & Chapman, 2024; Helliwell & Chorney, 2022; Helliwell et al., 2024). The findings contribute to this field by proposing a classification of *bridge-building* based on the analysis of booklets authored by experienced MTEs. Although this classification was derived from written materials, it provides a preliminary analytical tool to help us examine MTE expertise in any context.

The proposed classification identifies various forms of *bridge-building*, characterized by the number of scopes involved (e.g., binodal or trinodal), the mode of arrangement (sequential or integrated), and the purpose (informative or engaging). This framework sheds light on how MTEs might articulate the scopes of mathematics, teaching mathematics, and academic research in their pedagogical communication with teachers. Such articulation is central to the quality of teacher education programs, fostering a more interconnected and comprehensive approach to mathematics teacher education (Barbosa & Chapman, 2024; Georgiou et al., 2023; Wasserman et al., 2023).

The findings emphasize that the role of MTEs extends beyond supporting teachers in their current or future teaching practices. MTEs also facilitate the deepening and expansion of teachers' mathematical understanding and the connection of research findings to mathematics teaching practices. The articulation between advanced mathematics and school mathematics emerges as a critical area of focus in the literature (Wasserman et al., 2023), aligning with the concept of *bridge-building* across the communicative scopes of mathematics and teaching mathematics. Similarly, the view that research should inform teaching (Georgiou et al., 2023) aligns with MTEs' expertise to bridge-build teaching mathematics and academic research scopes.

This dimension of *bridge-building* as part of MTEs' expertise is particularly significant for its capacity to provide comprehensive and interconnected formative experiences. Therefore, *bridge-building* occupies a central role in the MTE expertise perspective, comparable to the concept of *unpacking* within the Mathematical Knowledge for Teaching model (Ball et al., 2008). The development of this dimension should be emphasized in the professional growth of MTEs. Teacher education institutions are encouraged to provide opportunities for MTEs to reflect on and enhance their *bridge-building* expertise. For instance, collaborative discussions among MTEs could focus on strategies to integrate mathematics, teaching mathematics, and academic research scopes in teacher education settings. Mentorship programs for novice MTEs should emphasize the development of this dimension, equipping them to design formative tasks that connect the different pedagogical communication scopes.

While this study offers insights into the concept of *bridge-building*, further research is needed to deepen our understanding of this dimension and its implications for teacher education. Research involving MTEs in different contexts could expand, refine, and better characterize the classification of bridge-building presented in this study. Future studies could explore how MTEs enact *bridge-building* in diverse contexts, including in-person teacher education settings. Additionally, longitudinal research could also investigate how MTEs develop this ability over time and assess how it unfolds into teachers' professional growth. By advancing this line of inquiry, the field can continue to expand its understanding of MTE expertise (Barbosa & Chapman, 2024; Helliwell & Chorney, 2022; Helliwell et al., 2024) and its role in enhancing mathematics teacher education programs.

Acknowledgement

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SCHOOLS' RATIONALES FOR DECISION-MAKING IN SELECTING PRIMARY MATHEMATICS CURRICULUM RESOURCES IN ENGLAND

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In England, in contrast with jurisdictions where textbook use may be mandated, schools choose their mathematics curriculum resources. Following a government initiative promoting 'mastery-style' mathematics textbook schemes, in this paper we analyse interviews with 12 mathematics subject leaders to examine reasons for schools' resourcing decisions. Drawing on an existing model of teachers' interactions with curriculum materials, we identify two distinct pathways of mathematics curriculum resource decision-making, contingent on schools' institutional context and perceived priorities and underpinned by high levels of professional reflection.

INTRODUCTION

This paper examines the rationales underpinning mathematics curriculum resourcing decisions of subject and school leaders in English primary (age 5-11) schools.

Whilst internationally, a textbook may be considered “a fundamental tool” in the learning and teaching of mathematics (Parra-Fica et al., 2024, p. 1), the English context stands apart from many jurisdictions. In England, schools have autonomy in how to resource their mathematics curriculum; historically, this has resulted in a relatively limited use of mathematics textbooks. Indeed, Mullis et al. (2008) found that whilst internationally 65% of 4th grade (age 9-10) teachers reported using textbooks as primary sources, in England this figure was just 15%.

The project from which this paper is drawn (Marks et al., 2023), mapped the landscape of the use of textbooks, schemes and other mathematics curriculum resources in England within the context of a government funding initiative (Department for Education [DfE], 2016) and the drive to give primary schools access to “the south Asian ‘mastery’ approach to teaching maths [...] supported by the use of high-quality textbooks” (DfE, 2016, n.p.).

Marks et al.'s (2023) project generated data via a nationwide mixed-methods survey of primary teachers who lead mathematics in their schools, (mathematics subject leaders, MSLs), a subsidiary nationwide quantitative teacher survey and semi-structured interviews with MSLs. In this paper we report on data arising from the semi-structured interviews with reference to key outcomes of the mixed-methods survey.

Drawing on Marks et al. (2023), we use the following terms. A *mathematics scheme* is a published resource, with or without a physical textbook, written to support the

teaching of the full primary curriculum without the need for supplementation. *Curriculum resources* thus include mathematics schemes and other resources linked to curriculum delivery, e.g. workbooks, worksheets, and teaching materials.

LITERATURE

In English primary schools, the historic low rate of textbook use (Mullis et al., 2008) reflects, in part, a longstanding antipathy towards textbooks with concern expressed that their use may render the teacher’s role to that of “technician” (Boyd & Ash, 2018, p. 221) whose function is only to deliver prepared lessons. This is accompanied by fears of loss of autonomy (Gear, 2022). Furthermore, evaluations of previously available textbooks identify a “restricted and impoverished diet” (Haggarty & Pepin, 2002, p. 584) focusing on development of fluency via drill and practice tasks, with little focus on mathematical language and the development of investigative thinking.

Following a national funding initiative (DfE, 2016) to introduce mastery-style textbook schemes into English primary schools, Marks et al. (2023) updated schools’ reported resourcing of their mathematics curriculum (Table 1):

Schools’ Resourcing Approach	% of Schools
Curriculum resources sourced from variety of places	46
One scheme (with/without physical textbook) used exclusively	3
One scheme (with/without physical textbook) mainly used with supplementation	51

Table 1: Schools’ resourcing of mathematics curriculum in England

Whilst variability in schools’ resourcing decisions remains evident, the data presented (Table 1) indicates an increase in the use of textbook schemes in England from the figure reported by Mullis et al. (2008). Marks et al. (2023) found that hybridity in textbook use is high; of the 54% of schools using a textbook scheme, 94% (51/54) supplement their use with materials drawn from elsewhere, a figure not dissimilar to that found in Silver’s (2022) U.S. study. Despite the recent textbook initiative in England, 46% of schools choose not to use any textbook as a main resource. Instead, teachers in these schools curate their own mathematics curriculum resources, drawing materials from a range of sources, including occasional use of schemes, but also drawing on online resource banks and materials of their own creation.

Despite Remillard et al. (2024, p. 64) finding that teachers make “selective and purposeful use” of curriculum resource components, the sourcing of mathematics curriculum resources has, nevertheless, triggered concern internationally. With teachers frequently using online resource banks (Silver, 2022), the quality of materials from these sources is a legitimate consideration. Polikoff & Dean (2019) examined 300 of the most downloaded materials on three sites popular with US elementary teachers, rating the majority (64%) as either “should not be used” or “mediocre, probably not

worth using” (p. 39). In England, with Marks et al. (2023) finding that primary teachers had access to at least 107 mathematics curriculum resources, similar concerns may be valid. Beyond the quality of individual resources, Foster et al. also (2021) note that “a collection of great tasks does not necessarily make a great collection of tasks” (p. 624); despite individual merits the design principles of different tasks will vary and may lead to poor coherence of the curriculum overall. Teachers’ time spent on curating a curriculum is an additional concern and the plethora of resources available potentially overloads teachers; Foster et al. (2021) suggest that this time may be better spent analysing and improving existing resources to better match pupil needs.

These concerns, in conjunction with the historic and current picture of the range of approaches to resourcing the primary mathematics curriculum in England, led us to formulate the following research question for this paper:

When primary (age 5-11) schools have autonomy to choose their mathematics curriculum resource(s), what factors underpin their resource selection?

THEORETICAL FRAMEWORK

Our research draws on Remillard’s (2012) model of teacher-curriculum interactions and relationships. In this model the features of the curriculum resource used, including its pedagogical emphasis, embedded support for the teacher, and its structuring of topics and tasks, forms one key influencing factor. The second key influence is the teacher’s own personal resources, for example their sense of agency, the extent to which they are viewed as a professional, their own capacity for pedagogic design and their social capital. Both are underpinned by the teacher’s perception of the demands of their particular institutional context. Together these factors combine to produce instructional outcomes in the classroom. Of necessity, we have adapted this framework to render it appropriate to focus on the initial decision-making regarding which curriculum resource to use which is the focus of this paper.

METHODOLOGY AND METHODS

To understand the reasons underpinning primary schools’ selection of mathematics curriculum resources, twelve 40-minute, semi-structured interviews were conducted with primary MSLs during 2022. Interview questions were developed to elicit the reasons underpinning schools’ curriculum resource decision-making. Following ethical approval, MSLs were selected from English primary schools to reflect a range of situations with regards to their curriculum resources (Table 2).

Interviews were audio recorded and transcribed. Data were coded in Nvivo, drawing on coding developed by Marks et al. (2023) with additional inductively developed codes. Codes were arranged into thematic categories drawing on Remillard’s (2012) conceptual model for teacher curriculum interactions, adapted for school-level decision-making. This resulted in three overarching themes. Theme 1, institutional context, focused on the school context at the point of decision-making e.g. inspection outcomes, attainment. Theme 2, key criteria for choosing a resource, focused on the

embedded teacher support, pedagogic emphasis and perceived quality of the curriculum resource. Theme 3, key criteria for staff, comprised two key areas, teacher autonomy and teacher workload.

School Group	Schools' Resourcing approach	Number of Schools
A	Uses one scheme exclusively	2
B	Uses one or more schemes with hybridity	7
C	Does not use a scheme	3

Table 2: Profiles of schools' mathematics curriculum resourcing

FINDINGS AND DISCUSSION

We present findings and discussion in relation to the three themes identified above, beginning with the institutional context (Remillard, 2012), the starting point common to all schools' decision-making.

Institutional context

Eight MSLs from schools in categories A and B described an institutional context in which evaluations of pupil progress and attainment, arising from external inspections or school-based analysis, revealed subject knowledge weakness in staff or insecure teaching and/or learning:

One of the main targets [from inspection] was to improve maths. [MSL, School Group B]

I think [teacher] subject knowledge was a big thing [MSL, School Group B]

For three MSLs, all in schools in group C, a key factor in their current context was the presence of non-standard and unusually diverse attainment profiles arising from having mixed-age classes and/or identifying gaps in pupils' mathematical understanding, which they perceived to arise in part from prior use of a scheme. For example:

We saw that there were huge gaps; we were finding that children, although they may have had an understanding, it wasn't a deep understanding of maths. [MSL, School Group C]

Here differences emerge between schools in different categories in terms of the aspects of the institutional context that underpinned their resourcing decisions.

Key criteria for choosing a resource

Several features emerged as important for MSLs in terms of what their mathematics instructional materials needed to provide. Here again, differences emerged between schools in different groups. The majority of MSLs (8), all from school groups A and B, identified a need for greater consistency; this resulted in them buying a scheme. For some, the consistency required related to aspects of pedagogy, for example in the use of mathematical representations, explanations, vocabulary:

Bringing in a scheme has enabled me to gain that consistency of representation through the entire school. [MSL, School Group A]

For others, the identified need for consistency related to the support of a consistent structure for teachers in building subject knowledge and planning for progression:

You could look back and see exactly how things have been taught so you know how to build on it. [MSL, School Group B]

For schools choosing a scheme, some key features relating to the quality of the scheme, as perceived by the MSL, were instrumental. For some the quality of application of variation theory evident in the children's independent tasks was noted:

You can see if you're looking for it, really careful variation in the way that they've done the questions. [MSL, School Group B]

For some schools, professional development (PD) embedded within the scheme through its lesson structure and accompanying teacher guides was an important factor in curriculum resource selection because of its value in supporting generalist primary teachers' subject and pedagogic knowledge:

We know that the better subject knowledge the teacher has, the better the learning experience for the children. [MSL, School Group A]

Another MSL, in Group A, derived confidence in the materials from their evident research informed design, the high quality of published materials constituted a clear rationale not to create their own curriculum resources:

You knew it was grounded in years and years of educational research that you just couldn't argue with. It gives you a really good base and vehicle for teaching mastery. This is why we don't try and write things ourselves because actually everything's deliberate.

This view of the quality of current textbook schemes presents the antithesis of the "restricted and impoverished diet" of those previously available (Haggarty & Pepin 2002, p. 584).

Research was also valued by Group C MSLs that opted to curate their own curriculum; research articles accompanying materials supported teachers to understand the activity value and intention and were instrumental in curating their own curriculum:

It comes with quite an easy bite size article which is steeped in research, you can give people a short burst to read that is helping them to understand what the point of certain elements are and what activities there are. [MSL, School Group C]

Such articles and resource analysis supported teachers in this school to better match resources to pupil needs (Foster et al., 2021).

For schools who chose not to use a published scheme, PD was also central to their approach in two ways. First, if they had previously invested in mathematics PD, they recognized that staff had the expertise to construct and collate their own curriculum resources. Second, PD remained an ongoing priority for continued success in collating

their curriculum resources. There were examples in these schools of carefully curated approaches to PD.

Key criteria for staff

Two criteria relating to teachers' needs emerged as central to schools' resourcing decisions. For some MSLs this related to affordances for workload reduction through the use of a scheme. For others, maintenance of teacher autonomy and professionalism, arising from either not using a scheme, or using one with hybridity, was key.

For some MSLs, all of which were in Group A or B, the use of a scheme was seen as having a positive impact on teacher workload in terms of the time spent preparing lessons and sourcing or creating activities. Importantly, this was seen as a means to enable teachers to spend more time focused on tailoring teaching sequences:

We were looking for teacher wellbeing and workload as well. Because we wanted our teachers to be spending time thinking about the lesson, the delivery of the lesson as opposed to scrabbling around, looking for [resources]. [MSL, School Group A]

For other MSLs, in both Group B and C, the maintenance of teacher autonomy was a central factor to enable teachers to make apposite resourcing decisions to meet their pupils' needs. Whilst curriculum resource quality remained important, this quality arose from teachers' professionalism to curate and interpret curriculum resources, rather than from a reliance on individual resource quality (Foster et al., 2021).

For some MSLs in group B, teacher autonomy took the form of freedom to either use the scheme or not, or freedom to adjust the length of time on a unit, or to inject hybridity through the use of other teaching materials. Thus, a key aspect of teacher autonomy was found in the freedom to exercise professional judgment in the form of decisions to adjust scheme materials to suit the needs of pupils (Gear, 2022):

I think it's really good for teachers to have the flexibility to do their own thing. We've got loads of really very good teachers...But it doesn't mean we need to stick [to the scheme]. It doesn't mean it's always the best for those children. [MSL, School Group B]

Phrases such as 'making it your own' were used to describe the decision-making process needed to adjust a scheme appropriately for pupils' needs; however, one MSL indicated the complexity underpinning this so as not to lose the consistency and quality provided by the scheme (Foster et al., 2021).

For Group C schools, preserving teacher autonomy and the exercising of professional judgement was paramount in the decision to curate their own curriculum:

A lot of the teachers were confident in themselves, were teaching maths to [a] deeper level, making sure all children have that really deep understanding, conceptually understanding maths. So [moving away from the published scheme] helped teachers fine-tune their teaching. [MSL, School Group C]

This decision also removed the perceived risk of teachers being deskilled (Boyd & Ash, 2018) or engaging less thinking in lesson planning through the use of a scheme.

MSLs in these schools viewed their colleagues as having high levels of subject confidence that would be further enhanced through the decision not to adopt a scheme.

CONCLUSION

In the context of primary schools in England, in which schools, and sometimes individual teachers, have autonomy to select which curriculum resources to use, schools' institutional contexts were the starting point in making decisions about the adoption of mathematics curriculum resources. Two distinct hierarchies of influence in decision-making (Figure 1), arose from the schools' institutional context.

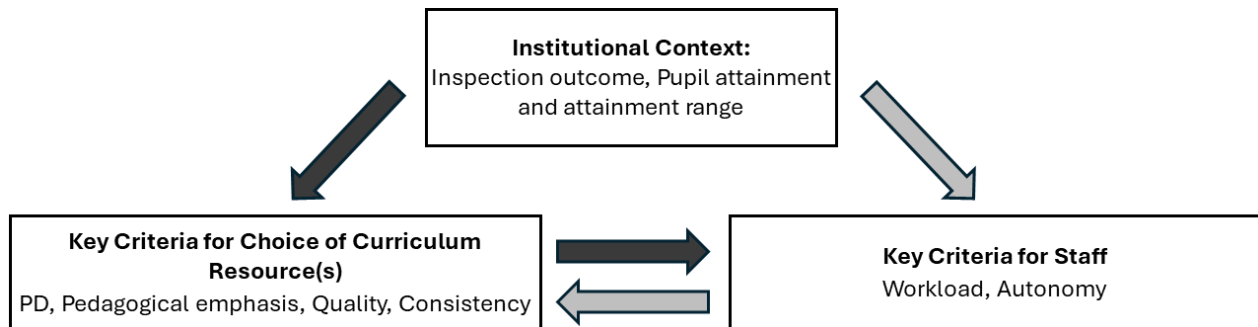


Figure 1: Hierarchies of influence in schools' decision-making about mathematics curriculum resources

Where schools articulated an institutional context in which they were responding to external or internal judgments based on pupil progress and attainment measures, the strongest influence was typically the drive to improve consistency. This led to decisions to align curriculum resource choices with a recent government initiative (DfE, 2016) and to adopt the use of mastery-style textbook schemes. The perceived high quality of these schemes and associated professional development, in conjunction with anticipated reduced workload, were cited as beneficial attributes. This met the need for consistency; some schools also valued maintaining teacher autonomy through the freedom to inject hybridity into scheme use.

A second hierarchy of influence in decision-making manifested in schools whose institutional context was characterised by a diverse range of attainment and/or gaps in pupils' mathematical understanding. The strongest influences here were school and teacher autonomy, with recognition of teacher professionalism. Whilst curriculum resource quality remained important, the overall curriculum quality was also seen to arise from the professionalism of staff to curate and interpret curriculum resources.

Notably, there was a high level of focus on the effectiveness and limitations of curriculum resources and, implicit within this, teacher confidence in resource evaluation. Whilst this does not address Foster et al.'s (2021) concern about teachers' skills in curriculum and resource design, it does demonstrate a high level of teacher reflection in the selection of curriculum resources to meet the needs of pupils, staff and schools; this may mitigate concerns about resource quality (e.g. Polikoff & Dean, 2019).

Finally, there is synergy between Remillard's (2012) conceptual model for teacher-curriculum interaction and the hierarchies of schools' mathematics curriculum resource decision-making (Figure 1) - both have the institutional context as an underpinning factor, but our findings indicate this to be central to subsequent decision-making.

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INVESTIGATING STRATIFICATION IN MATHEMATICS CLASSROOM DISCOURSE: A SECOND LANGUAGE MATHEMATICS CLASS IN CANADA

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While it is not controversial that mathematics classroom interaction is shaped by broader social forces that leads to marginalisation of some learners, it is not obvious how to analyse interaction in a way that connects it to these broader forces. In this research report, I present findings from analysis of interaction in an elementary school second language mathematics class, drawing on concepts and methods from contemporary sociolinguistics. Specifically, I analysed a lesson on fractions by looking for how meaning-making worked on different scales, in relation to an indexical order, and multiple centres of authority (polycentricity). Among the findings, the analysis revealed centres of authority relating to the English language, standard mathematical discourse, the Cree language, and students' locally produced mathematical discourse.

There has been ample research examining the fine microlevel detail of mathematics classroom interaction. In the context of language diversity, this kind of work has clearly established, among other things, that second language learners or bilingual learners are able successfully to participate in mathematical meaning making, and that they use multiple forms of language to do so; that learners benefit from being able to make use of their full linguistic repertoire in learning mathematics; and that teachers have an important role to play in facilitating and mediating the use of these repertoires (see, for example, El Mouhayar, 2021; Moschkovich, 2015; Uribe & Prediger, 2021; Zahner et al., 2021). Meanwhile, macrolevel research has sought to understand how the politics of language affects mathematics learning and teaching. For example, research has shown how national language policies affect what happens in mathematics classrooms and in mathematics teacher education programmes, and how these policies create differential outcomes for learners (e.g., Planas, Chronaki & Svensson Källberg, 2022; Setati, 2005).

Much of this work (including my own) is attempting to move away from assumptions that may limit what analysis of mathematics classroom interaction can accomplish. These assumptions include (1) languages are stable, relatively standardised systems (2) mathematical discourse is a stable, relatively standardised subset of any given language of instruction, and (3) an important role for teachers is to guide learners from whatever languages or idiosyncratic forms of the language of instruction they bring into the classroom towards 'the' standard form of mathematical discourse which occupies a hegemonic position. Even research in language diverse mathematics classrooms tends to adopt some version of these assumptions; while the use of multiple languages is

increasingly recommended, for example, it is often in the service of supporting learners to become proficient in using mathematical discourse in the language of instruction. Learners may use multiple languages in mathematics class but will be assessed in a single language.

Alternative perspectives focus on the multiplicity of language, such as recent contributions that draw on Bakhtin's work. These studies emphasise the heteroglossia of language, discourse and voices in mathematics classrooms (e.g., Barwell, 2018) and consider ways in which working with this multiplicity can support learners of mathematics, such as through the use of translanguaging pedagogies (Planas & Chronaki, 2021). Bakhtin's work also highlights the stratification of language in society, with some languages, discourses, genres or other features considered more valuable, desirable or powerful than others (Barwell, 2017). In this research report, I present the results of some preliminary analysis of mathematics classroom interaction that draws on sociolinguistic concepts consistent with a Bakhtinian perspective and designed to examine in more specificity the connection between mathematics classroom interaction and social stratification.

THEORETICAL FRAMEWORK: SCALE, INDEXICALITY AND POLYCENTRICITY

Contemporary perspectives in sociolinguistics emphasise the fluidity and diversity of human interaction across time and space. Some of the most interesting work has been conducted in superdiverse spaces in which multiple forms of language – e.g., named languages, discourses, registers, genres, styles, accents – come into contact and interact with each other, such as in markets or on social media. This work has challenged established ideas about how language and human meaning-making work. For this research, I particularly drew on the work of Blommaert (2010), who developed some valuable concepts based on extensive ethnographic observations in Belgium, Tanzania and Japan among other places. Blommaert explicitly challenges the assumptions mentioned above, based on an underlying conception of human meaning-making as fluid, situated and multiple.

Seeking ways to analyse discourse data from this kind of perspective, Blommaert proposes three related notions. First, meaning operates on different *scales*. Some forms of language are meaningful at a very local level (e.g., family in-jokes, local slang), while others may be so globally (e.g., mathematics). Crucially, because language is meaningful in multiple ways, some utterances may be differently meaningful on different scales simultaneously. In a mathematics classroom, some terms may come to have specific local meanings within the class, while also having more normative meanings in the wider mathematics community. Such differences can lead to ambiguity. Explicit shifts between scales, known as *scale jumping*, are particularly revealing.

Second, language use is *indexical*. Any utterance 'points to' various broader social affiliations. Accents, word choices, or turns of phrase all point to different groups,

activities, times and so on. Teachers have ways of talking that index them as teachers, for example. Furthermore, indexicality is stratified: while any utterance can index multiple associations, some associations or interpretations are privileged over others. This stratification links language to broader forms of social organisation: “systematic patterns of indexicality are also systematic patterns of authority of control and evaluation, and hence of inclusion and exclusion by real or perceived others” (Blommaert, 20120, p. 38).

Finally, interaction is *polycentric*. Although, as Bakhtin suggested, discourse is shaped by centripetal forces, it is a mistake to think of these forces as all relating to a single centre. Multiple centres are in play, in relation to different scales, and different situated indexical meanings. For example, in a school setting, various centripetal forces may be in play relating to norms of the language of instruction, norms within friendship groups, subject-related norms, and so on. Participants orient to these different centres which, through centripetal forces, depend on implicit or explicit social authority. These authorities can be institutional, such as a mathematics curriculum, or individual such as a teacher or “the coolest guy in class” (p. 39).

These three aspects of human meaning-making are interrelated and connect interaction to broader forms of social organisation and centres of authority. Indexicality is ordered differently on different scales and in relation to different centres of authority. Put more simply, multiple norms available to participants in meaning-making relate to different scales, centres and interpretive clusters. Based on these ideas, I conducted an analysis of interaction in a lesson in a Grade 3 French immersion mathematics class in Canada. The research question for this preliminary analysis was: How do broad social forces shape mathematical meaning-making in the lesson?

RESEARCH CONTEXT: A SECOND LANGUAGE MATHEMATICS CLASS

The data come from an ethnographic study of four second language mathematics classrooms in Canada (see Barwell, 2020). Data include classroom observations, audio recordings of whole class interaction and of some small group interaction, students’ written work, and interviews with teachers and students. Class A2 was a Grade 5/6 class in an anglophone schoolboard. The students in class A2 at the end of the year were all from Cree First Nations in the north of Quebec but were attending school in a major urban area because their parents had moved to work or study. The students all spoke Iiyiyuu Ayimuun (the Cree language) as well as English and were considered to be learners of English as a second language. The analysis presented in this report represents a first step towards comparing scales, indexicality and polycentricity in the classroom interaction in all four classes in the dataset. For each class, a lesson from the latter stages of the school year has been selected. In class A2, the lesson was part of a unit on fractions. Five students were in attendance. The transcribed audio-recording of the lesson was the primary focus for analysis, with other data used to inform interpretations of the interaction.

FINDINGS

Two broad overlapping orderings emerged from the analysis relating to the use of languages and mathematical discourse. These are dimensions of classroom interaction in which scale, indexicality and polycentricity are clearly involved in shaping processes of collective mathematical thinking.

Use of languages

I have reported before that students in this class regularly used Cree during mathematics classes. I observed such usage during the lesson analysed for this study too. I summarised part of the lesson in my notes:

Almost an hour was devoted to the idea of mixed fractions and improper fractions. For each, Teacher M explained at the blackboard, did an example, did a shared example and then gave some questions to do. With mixed fractions, the class worked together to come up with a representation of $4\frac{1}{4}$ – Teacher M did not tell them, she asked various students to come to the blackboard and try. As they did so, other students commented or made suggestions, sometimes in Cree. It was quite an engaged discussion. Teacher M then gave each student another one to do on the blackboard (in parallel).

There is evidence of a clear centre of authority relating to the use of English. Twice during the lesson, the teacher asked students to speak in English, saying, for example “say what you’re saying in English”, so the teacher acts as the voice of this authority. Institutional authority is also captured by a notice in the classroom that said “remember to speak in English”, which indexes English as the language of mathematics, and as the language of instruction and assessment for the school and the schoolboard. As one of its official language, it also indexes Canada as a nation. The students were clearly aware of this notice – on one occasion, a student satirised it by writing on the blackboard “remember to speak in Cree”. Moreover, the class has a designation of English as a second language and learning English was a key objective. I have argued before that the privileging of English contributes to the marginalisation of these students. This privileging is related to the global-scale reach of English as a national and international language.

In their use of Cree, however, the students are orienting to another centre of authority. Students were clearly sometimes talking among themselves in Cree about the mathematics they were working on. They also spoke among themselves in Cree during time allocated for eating a snack before the lesson started. Cree, therefore, served as an important aspect of social cohesion among the students as well as being valuable in their learning of mathematics. The teacher acknowledged the positive aspects of the students’ use of Cree but also noted that she couldn’t understand them:

the Cree thing is a good thing (.) they can explain something if they don’t understand and they can explain to each other better (.) and often (.) a lot of the time (.) they would get it and (2.0) it can also be a bad thing too because (.) they’re I know they’re thinking (.) differently in their like their own mind maybe maybe in Cree maybe not (.) explaining it to themselves in Cree in a different way than I am explaining it in English and to me I

would love to hear what they (.) what they like be able to understand what they are saying and and share that as a as a group (.) um rather than you know (.) within one or two people because (.) it gives me an insight on the way that they are thinking too

The teacher's comments about wanting to understand what the students are saying highlights how Cree operates on a different, more local scale. Cree is widely used in the students' communities, and in interviews they confirmed that they had attended schools where Cree was used as a language of instruction for at least part of the time. Within the context of School A, the Cree language indexes students' communities and distinct heritage, the Cree nation more broadly (as distinct from Canada) and their geographical home. The inscription "remember to speak in Cree" only works as satire for the Cree-speaking students. The teacher's requests to "say what you're saying in English" are instances of scale-jumping, revealing the indexical order that gives English higher status than Cree within the context of the school and mathematics class. Among the students, however, in relation to this alternative centre of authority, it is possible that this ordering is inverted.

Mathematical discourse

The use of mathematical discourse was also ordered in relation to two centres of authority: standardised mathematical discourse occupied a higher position than students' non-normative interpretations. Two examples illustrate this ordering. The first example arose at the start of the lesson, when the teacher reviewed two components of a mixed fraction – the whole number and the fractional part – referring to the example $1\frac{2}{3}$ written on the blackboard:

- Teacher M okay what do you think the other side is called (.) this part here (.) what's that called (.) if that is a whole number (.) what's this called
- Student 2 two thirds
- Teacher M no not thirds (.) what do you think
- Student 2 one two thirds (.) one two thirds
- Teacher M just this part here (.) right here (.) what do you think this is called
- Students 1, 2 two thirds
- Teacher M it is called two thirds you are right (.) but what is it what is it known as (.) what's it called
- Student 1 fraction
- Student 2 fraction
- Student 3 fraction thirds haha
- Teacher M its actually called (.) yeah it's called a fraction (.) but it's a fractional part (.) part

There is a clear but brief struggle in this exchange about how to refer to the fractional part of the expression. The students correctly name the fraction (two thirds) but the

teacher is looking for a higher-order term for that part of any mixed fraction. To call it two thirds works in situ – all participants know what is being referred to. The teacher is looking for what it is “known as”, so that the desired term indexes people, perhaps mathematically educated people (as well as the curriculum), who know what term to use or what term should be used. The teacher’s implicitly evaluative response to the students’ situated attempts highlights a centre of authority for mathematical discourse to which she is oriented and which presumably relates to the provincial mathematics curriculum.

The second example comes from later in the lesson, when the students are attempting to draw a representation of $4\frac{1}{4}$. One student draws a whole divided into four and shaded one part, and then draws another whole divided into four and shaded four parts:

- Teacher M ‘kay (.) are you going to add anything to that? (2.0) what does [student name] right now (.) so far (.) what does [student name] have
- Student 1 one fourth
- Teacher M one fourth (.) you got it
- Student 1 draw another (colour)? (all of it)
- Teacher M say what you’re saying in English
- Student 1 colour it all
- Teacher M ahh (.) why do we colour (.) why do we colour all this?
- Student 1 cause it’s fo- four wholes
- Teacher M ‘cause its four wholes right (.) we need four wholes (.) we need four whole parts (2.0) okay (.) what does [student name] have now?
- Student 1 one (.) one fourth
- Teacher M oh nope (.) not yet
- Student 1 one fourth four
- Teacher M Dylan has now what (.) excuse me (.) stop talking (.) what does [student name] have right now
- Student 4 ^one two three^
- Student four wholes
- Teacher M no he doesn’t have four wholes (.) he has
- Student 3 one whole
- Teacher M one whole (.) so right now [student name] has (.) one and one fourth (.) how can we make it four fourths=four wholes

The word ‘whole’ appears to be used in different ways in this exchange. Some participants interpret four wholes as referring to four quarters of the same whole, while for others, this representation shows one whole. The former usage indexes students’ locally meaningful, situated and emergent mathematical discourse in English. The

teacher scale-jumps when she negatively evaluates the students' non-normative discourse, such as when she says "no he doesn't have four wholes". She unpacks the expected normative interpretation of the diagram, "so right now Dylan has one and one fourth", indexing a standardised mathematical discourse, as well as, presumably, curriculum expectations. Once again, the teacher orients to a centre of authority relating to standardised mathematical discourse in English. The students' local interpretations of the diagram are subordinate to this authority. On the other hand, the students orient to a local centre of authority in which their locally produced interpretations of the diagram make sense.

DISCUSSION

Analysis of this elementary school lesson on fractions reveals some of the detail of the stratification of the interaction. The interaction includes layers of meaning at different scales, with English in general, and an implicit sense of standardised mathematical discourse both relating to provincial, national and international scales, while Cree and students situated forms of meaning-making relating to more local scales. The identification of scale jumping is a valuable method for identifying stratification, such as when the teacher asks student to speak in English. These various scales are related to an indexical order through which English indexes Canada, Canadian national identity, and global culture, while Cree indexes students' specific First Nations and cultural heritage. Mathematical discourse indexes provincial curriculum requirements and, arguably, a particular Eurocentric notion of what mathematics is and how to do it. A particularly valuable outcome of the analysis is the identification of multiple centres of authority in the class. While the teacher generally orients to standardised mathematical discourse and English, the students sometimes orient to non-standard mathematical discourse and a mixture of Cree and English. In the latter case, these forms of language use are not just about mathematics; they are about social relations and identities. Previous research has perhaps tended to assume that mathematics classroom interaction is monocentric, oriented around the teacher and to standardised, large-scale languages and discourses, the above analysis shows that other centres of authority can be (and probably are) present. In this class, for example, the students are able to converse about mathematics in their own language, with its own norms, in which, unlike the teacher, they have expertise.

In conclusion, this analysis shows in detail how mathematics classroom interaction operates at different scales, related to an indexical order and potentially multiple centres of authority. Comparison of different classrooms will provide insight into how these features vary across contexts.

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ACQUIRING A CONCEPT AND ITS SUBCONCEPTS: INSIGHTS INTO THE RELEVANCE OF PREREQUISITES FOR LEARNING

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Learning prerequisites play a significant role for achievement in mathematics. So far, the effect of such prerequisites throughout a whole teaching unit for a mathematical concept and its subconcepts has received little attention. For this reason, the present study examines the effects of learning prerequisites on achievement during a teaching unit on the concept of derivative implemented in 15 mathematics classrooms. For this, we administered a pretest, midterm tests, and posttest. The results highlight that prior knowledge, cognitive abilities, and subject-specific interest are key predictors for the overall concept. However, their effects on the subconcepts are more nuanced, with some predictors showing no influence for some subconcepts. This supports the knowledge-is-power hypothesis while revealing differential impacts at the subconcept.

Learning prerequisites influences achievement. Hambrick and Engle (2002) and Stern (2015) emphasize prior knowledge as a key learning prerequisite, with Stern also emphasizing cognitive abilities. With respect to prior knowledge, domain-specific prior knowledge is particularly important for understanding the concept of derivative in mathematics (Litteck et al., 2023). Furthermore, mathematical concepts not only build on each other across topics, but can also be structured into subconcepts. Yet is still unclear to what extent the effects of learning prerequisites differ when the effects on the overall concept or on the subconcepts are considered. For instance, the content of the subconcepts can vary significantly, one subconcept may focus on real-world context, while another may emphasize algebraic manipulation in mathematical context. These differences likely influence the impact of various learning prerequisites for the subconcepts. In addition, the subconcepts are often hierarchically related to each other and are thus prerequisites of each other (Sfard & Linchevski, 1994). What has been missing so far is the investigation of the effect of learning prerequisites on achievement for a concept and the different subconcepts. For this reason, the present study examines the influence of learning prerequisites throughout a teaching unit on the derivative aligned with its subconcepts, using a standardized (digital) learning environment to generate evidence for identifying the key learning prerequisites for successfully learning the concept of derivative and its subconcepts.

PREQUISITES FOR LEARNING

Several prerequisites can influence the achievement of students during the acquisition of knowledge of mathematical concepts. Different types of prerequisites can be distinguished, such as metacognitive, cognitive, and other general individual characteristics that are important factors in predicting students' achievement. The

importance of prior knowledge as a prerequisite for learning is reflected in the knowledge-is-power hypothesis, suggesting that it may be one of the strongest, if not the strongest, learning prerequisites (Hambrick & Engle, 2002). However, this hypothesis is also faced criticism, and initial evidence suggests that the hypothesis strength might not be fully tenable (Simonsmeier et al., 2022). Nonetheless, learning mathematics is characterized by building knowledge of new concepts on that of previously acquired and related concepts (Sfard & Linchevski, 1994). In this respect, the significance of the knowledge-is-power hypothesis for mathematics is particularly powerful, and it is therefore not surprising why prior knowledge as a domain-specific factor (*domain-specific prior knowledge*) is so important in learning mathematics (e.g., Aubrey et al., 2006). In addition, many concepts in mathematics can be structured in very closely related subconcepts and it can be assumed that knowledge is acquired hierarchically (Sfard, 1991). This means that the influence of domain-specific prior knowledge may vary depending on whether it is considered for the overall concept or for the hierarchically structured subconcepts. The difference arises because the content and focus of the subconcepts can differ. For example, one subconcept may emphasize real-world examples in functional contexts, while another may focus more on mathematical abstractions and operations. Additionally, prior knowledge might play a stronger role during the introduction of a concept, where it serves as a learning context to integrate new ideas, compared to later subconcepts where this integration may be less emphasized. In addition to prior knowledge, there are several other factors that can have an influence on achievement in mathematics. Cognitive abilities, interest, and metacognitive awareness are particularly frequently named predictors of achievement in research (see, for a detailed discussion, e.g., Kim et al., 2017; Stern, 2015; Ufer et al., 2017).

THE ROLE OF PREREQUISITES FOR THE CONCEPT AND SUBCONCEPTS OF THE DERIVATIVE

The concept of derivative can be considered to contain three underlying subconcepts (Zandieh, 1997). These subconcepts can be interpreted as a hierarchical structure of knowledge about the concept of derivative, starting with the difference quotient, continuing with the differential quotient, and ending with the derivative function (Sfard & Linchevski, 1994; Zandieh, 1997). This division of the concept of derivative into three interconnected subconcepts has important consequences for the relevance of learning prerequisites. The introduction to the derivative begins with the difference quotient, which is based on the slope triangle used to calculate the slope of linear functions. As prior knowledge, the idea of the slope triangle is then employed and extended to compute the average slope for non-linear functions in a pre-defined interval (Weber et al., 2012). That means that students are encouraged to think of the change of a function in intervals rather than as an overall characteristic of a (linear) function. Generally, this approach introduces the idea of the average rate of change or the average slope of a function, respectively (Weber et al., 2012). This suggests that prior knowledge of the slope triangle could play a significant role in acquiring the

difference quotient, especially if the slope triangle is not addressed explicitly during the introduction. To simplify the computation, the slope triangle is generalized to a point $(x_0, f(x_0))$ and a second point $(x_1, f(x_1)) = (x_0 + h, f(x_0 + h))$ reflecting a change of h in x -values. This perspective on the average rate of change provides the knowledge to consider the idea of the instantaneous rate of change or the local slope of a function at a specific point and the definition of the differential quotient. This requires the introduction of a limit process and the limit (Zandieh, 1997). A possible approach is to first illustrate the limit process by assuming that the change in x -values (h) for the average rate of change can (always) be chosen to be smaller. For dealing with these quotients and the infinitesimal change a solid understanding of rational numbers and fractions is crucial for grasping the limit process, algebraic manipulations are essential for calculating the differential quotient, and a thorough knowledge of functions is fundamental to comprehending the local slope of a function at a specific point. Based on the understanding of the limit process, the limit can then be introduced, and the differential quotient enabling the determination of a function's rate of change at a specific point. In the following, instead of considering only one point of a function, as for the differential quotient, the idea of the instantaneous rate of change of each point or the local slope of each point of a function is introduced, which results in the derivative function. Therefore, abstraction and complexity are increased for this subconcept, especially in the dual consideration of association and covariation of the function and the derivative function.

Cognitive abilities and metacognitive awareness seem to be important in understanding the concept of derivatives, both for empirical reasons—as they are key predictors of learning outcomes (Kim et al., 2017; Stern, 2015)—and because of the varying complexity of its subconcepts, which likely requires a solid cognitive foundation to grasp more demanding content in some subconcepts. Furthermore, interest is an important predictor for learning (Ufer et al., 2017). According to the Person-Object Theory of Interest, interest always relates to a specific object or context. This framework highlights that students' interest in mathematics, or the concept of derivatives can influence their learning success. While real-world examples, such as speed and slope in contexts like sports, transportation, or economics, are often used to illustrate the derivative, these applications may not directly reflect students' interest in mathematics itself. Instead, it is the interplay between general mathematical interest and these contexts that could predict achievement in understanding derivatives (e.g., Ufer et al., 2017).

THE PRESENT STUDY

The importance of learning prerequisites can be found consistently in the literature. However, the strength of the effect of important prerequisites is unclear and may vary according to content aspects (subconcepts). We address the following research questions: RQ1. What influence do learning prerequisites have on the achievement of

the concept of derivative? RQ2. What influence do learning prerequisites have on the achievement along the subconcepts of the concept of derivative?

Based on the previously elaborated theoretical considerations, we expect that prior knowledge, and cognitive abilities will have the strongest effect on students' achievement.

METHOD

The sample consists of 365 students from 15 classes from Germany (52.3% female, 46.2% male, 1.5% other). On average, students have had a good grade in mathematics ($M = 2.49$ with 1 = very good; 6 = unsatisfactory). The study has a high mortality rate, which is explained by the complexity and length of the study. Of the 15 classes that started, only 5 completed the unit. A critical factor in assessing the influence of prerequisites on achievement is ensuring that learning conditions are consistently controlled across all classes. Without such control, the effects of the learning prerequisites may vary with different learning opportunities—for instance, if the concept is introduced without revisiting the slope triangle or if specific real-world examples are chosen that either enhance or diminish students' interest. To address this, we developed and implemented a digital learning environment that ensures comparability across all classes. We structured the three subconcepts of the derivative into three lesson sets, which together form the unit for the concept of derivative. The first lesson set, focused on the difference quotient, consists of approximately 5 lessons. The second lesson set, covering the differential quotient, spans about 7 lessons. The third lesson set, focusing on the derivative function, includes 6 lessons.

Instruments

To capture the learning prerequisites, we administered a pretest for potentially relevant prerequisites. Firstly, we assessed domain-specific prior knowledge with items on knowledge of functions, the slope of functions, and algebraic manipulations, which Litteck et al. (2023) have shown to be important prior knowledge of the derivative concept. For these we used a test with 19 items, which was scaled using a unidimensional IRT-Model and showed a WLE-Reliability of .68 (test adapted from Litteck et al., 2023). In addition to prior knowledge of the derivative concept, other important predictors of achievement mentioned in the theoretical part were selected. For example, cognitive abilities (20 items, $\alpha = .91$), subject-specific interest (4 item Likert scale, $\alpha = .87$), and metacognitive awareness (18 item Likert scale, $\alpha = .77$) were assessed (e.g., Kim et al., 2017). As control variable, reading comprehension was assessed using a standardized, country-specific reading comprehension test. The commercial instrument demonstrates retest reliability ranging from .72 to .89.

To investigate the influence of the learning prerequisites on the achievement of the concept of derivative, an item pool covering all three subconcepts was utilized. From this pool, suitable items were selected to assess achievement in the first midterm test after learning the difference quotient (5 items, $\alpha = .67$), in the second midterm test after

learning the differential quotient (5 items, $\alpha = .65$), and the posttest at the end of the entire unit (18 items, WLE reliability .70).

Data analysis

For the analysis, the prior knowledge test and the posttest were independently of each other first scaled using a one-dimensional IRT model. To ensure comparability between regression models, the posttest scores and midterm test scores were z-standardized. The reading comprehension test was analyzed according to the test instructions. For the other variables, scale means were calculated and z-standardized. Due to the longitudinal design, missing values occurred. We used the mice package in R to impute missing values using the random forest method, which is suitable for handling complex relationships between variables and is a robust method capturing nonlinear relationships and interactions in the data. We calculated regression models modelling the relationship between the achievement scores as the dependent variable and the independent variables prior knowledge, cognitive ability, subject-specific interest, metacognitive awareness, and reading comprehension.

RESULTS

The first research question focused on the influence of the prerequisites on achievement for the concept of derivative. For this purpose, multiple linear regression analyses were conducted with students' achievement for different regression models as dependent variable and the independent variables prior knowledge, cognitive ability, subject-specific interest, metacognitive awareness, and reading comprehension. The corresponding regression models are presented in Table 1. The conducted regression model was significantly superior to a null-model $F(5,359) = 22.18$ ($p < .001$). The model explained 23.6% of the variance in the posttest scores. Table 1 presents the effects of the predictors across the different regression models for the various tests. As shown in Table 1, the regression model with the achievement in the posttest as the dependent variable indicates that prior knowledge has the strongest significant effect, followed by cognitive ability. Subject-specific interest also has a significant effect, though it is the weakest among the significant predictors.

Predictor	Midterm test 1		Midterm test 2		Posttest	
	β	SE	β	SE	β	SE
Prior knowledge	-0.06	0.06	0.16**	0.06	0.28***	0.05
Cognitive ability	0.14*	0.06	0.05	0.06	0.22***	0.05
Subject-specific Interest	0.09	0.06	0.09	0.06	0.16**	0.05
Metacognitive awareness	-0.00	0.05	-0.05	0.05	-0.09	0.05

Reading comprehension	0.05	0.05	0.08	0.05	0.04	0.05
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Table 1: Results of the regression analysis for midterm test 1 (model 1), midterm test 2 (model 2) and posttest (model 3). Note. * $p < .05$; ** $p < .01$; *** $p < .001$.

The aim of the second research question was to investigate the influence of the prerequisites, especially prior knowledge, on the acquisition of knowledge for the subconcepts of the concept of derivative. Therefore, we performed two regression models, one for the score at the first midterm test after the first lesson set for the difference quotient and a regression model for the second midterm test after the lesson for the differential quotient (Tab. 1). The model with the score for the first midterm test after the second lesson set was significant, $F(5, 359) = 2.44$ ($p < .05$), but explaining only 3.3% of the variance in the first midterm test. The analysis showed a significant main effect for cognitive abilities. The other predictors did not show a significant effect on the achievement for the first midterm test. For regression model with the score of the second midterm test as dependent variable was also significant $F(5, 359) = 4.14$ ($p < .005$). The model explained a little more variance with 5.5%. The analysis revealed that prior knowledge was the only significant predictor of the score for the second midterm test.

DISCUSSION

The results regarding the learning prerequisites for achievement in a unit on the concept of derivative show that prior knowledge, cognitive abilities, and subject-specific interest are significant predictors. Supporting the knowledge-is-power hypothesis, prior knowledge emerges as a stronger predictor than cognitive abilities, including for specific concepts such as the derivative (Hambrick & Engle, 2002; Litteck et al., 2023). However, cognitive abilities also play an important role, consistent with studies highlighting their influence on learning success (Stern, 2015). The significant effect of subject-specific interest on achievement may be attributed to their interest in mathematics and the relatability of the use of real-world examples in the learning environment. These examples likely resonate with students' everyday experiences, making the contexts of the tasks more relevant and engaging. Conversely, when real-world examples do not align with students' interests or experiences, they may fail to capture attention and engagement. This contextual sensitivity appears to amplify the influence of interest, as students who perceive the material as relatable and meaningful are more inclined to invest cognitive effort and actively engage with the content (Leyva et al., 2022). Therefore, the thoughtful integration of real-world applications within the learning environment plays a crucial role for the impact of interest on students' achievements.

The results of the second question about the influence of learning prerequisites when considering the subconcepts of the derivative concept showed that only the cognitive abilities significantly influence the achievement for the difference quotient in the first

midterm test. The significant effect of cognitive abilities on the first midterm test after the difference quotient may be explained by the importance of cognitive abilities in grasping and processing new content, particularly when students encounter unfamiliar concepts, like for a new topic, and an unfamiliar digital learning environment. Surprisingly, prior knowledge had no effect on achievement for the first midterm test, which may support other research that takes a more critical view of the influence and effect size of prior knowledge (Simonsmeier et al., 2022). It is important to note that our designed teaching unit repeated content from lower secondary school, such as calculating the slope of linear functions using the slope triangle, before progressing to curved graphs of functions. The non-significant effect of prior knowledge may be explained based on these learning opportunities to acquire essential knowledge, potentially compensating for gaps in prior knowledge.

A different effect emerges for the second lesson set. Here, prior knowledge of the differential quotient as the second subconcept of the derivative concept is the only predictor that significantly affects achievement in the second midterm test. This may also indicate the potentially complex interplay of the effect of cognitive abilities and prior knowledge for acquiring concepts, which is nuanced for various subconcepts (Stern, 2015). The effect of prior knowledge is evident here, even though it did not play a role as a predictor for the first midterm test. One potential explanation for these results could be that specific prior knowledge for the difference quotient, such as the slope triangle, was explicitly addressed in the learning environment, because of the close conceptual connection between prior knowledge and unit content. This could not be designed as directly for the activities for the differential quotient. This may explain why the importance of prior knowledge is particularly evident in the second midterm test. Furthermore, this aligns with findings from previous studies. For example, understanding real numbers is crucial for grasping the limit process, while algebraic transformations also play a significant role in calculating the differential quotient (Litteck et al., 2023).

This study has some limitations. The midterm tests included only four items for the test and five items for the second test for the subconcept, restricting the ability to capture students' achievements in detail. Expanding the number of items for each subconcept could provide a more accurate assessment of learning achievement. Additionally, the posttest was administered at the end of the unit, making it impossible to isolate the differential effects of prerequisites specifically for the derivative function subconcept. The midterm tests also explained relatively little variance and were conducted in classroom settings for practical feasibility. This may have limited their ability to effectively capture the intended constructs. Refining the tests in terms of both content and format could improve their explanatory power in future studies.

The analysis of prerequisites of a concept and its subconcepts requires a nuanced analysis. This study quantified the impact of prior knowledge, cognitive abilities, and interest on learning achievements for the concept of derivative, revealing that these

factors influence overall achievement and vary across subconcepts. The findings provide insights into development of students' learning trajectories and underscore the potential for targeted support, such as fostering prerequisite knowledge through adaptive learning environments and promoting interest with real-world examples.

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EXPLAIN WHY “TIMES 1000” FROM KILOGRAM TO GRAM: CONTROLLED TRIAL ON LEARNING TO EXPLAIN IN TWO DIGITAL ENVIRONMENTS WITH DIFFERENT TOPIC FOCUS

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Learning to explain procedures (e.g., convert kilogram into gram) has topic-independent discursive components (general readiness to articulate ideas) and topic-specific conceptual components (understanding this procedure). To study if topic-specific components need to be explicitly treated, we conduct a controlled trial with $n = 282$ fifth graders and compare the effects of two digital teaching-learning environments on topics of mass measurement: The topic-specific environment on the mass unit conversion procedure yielded a medium intra-group effect size ($d = 0.73$) on explaining why the conversion works. Meanwhile, the environment on applying and explaining estimation strategies yields a small transfer effect ($d = 0.14$) on explaining non-treated conversion procedures. Thus, explaining can only slightly be fostered topic-independently.

INTRODUCTION

Whereas nearly all students learn to engage in discourses practices of *reporting how to conduct a procedure*, only few students get productive learning opportunities for *explaining why a certain procedure works*, by drawing upon the conceptual foundations needed for this justification, e.g., in a visual model and its underlying structures (Moschkovich, 2015; Fuson et al., 1997). Design research studies and qualitative classroom observation studies contributed to identify design principles and teachers' moderation practices for enhancing students' competence of explaining (Erath et al., 2021; Moschkovich, 2015), often independent from the content in view (Erath et al., 2018; Mercer & Sams, 2006; Walshaw & Anthony, 2008). However, there is still a lack of interventions with *quantitative evidence* for effects on students' learning gains in explaining (see survey Erath et al., 2021), and no research on how topic-specific the learning opportunities for explaining need to be. Thus, the current study aims at providing quantitative evidence that students *can* indeed learn to explain why a procedure works, and that topic-independent interventions can contribute much less than topic-specific interventions that focus students attention to the conceptual components underlying the procedure in view. For this, we have chosen the procedure of mass unit conversion (e.g., from kilograms to grams and back) as the topic in view.

We first present the theoretical background of our controlled trial before we articulate the research question and hypothesis to be tested, present the methods and the findings.

THEORETICAL BACKGROUND

Mass measurement and its different topics, among them explain unit conversion

Students' competences for dealing with measurement units comprise several topics, among them the big idea of measuring through iterating standard units, benchmarks for typical measures, estimation strategies, and the *procedure of converting between units* which is focused in this paper (Smith & Barrett, 2017). For the strategies and procedures in view, students do not only need the procedural skill to conduct them, but also to *explain them and justify why they work* (Freudenthal, 1981, for unit conversion: Smith & Barrett, 2017). For the *procedure of converting mass units*, students are offered the visual model of fine-grained and coarsely grained base-ten blocks on the balance scale (Figure 1). To explain why 4.3kg is converted into 4300g by “times 1000”, students can visualize the 4.3kg by four 1kg-blocks and three 0.1kg-flats (which is grounded in place-value structures and bundle structures, Fuson et al., 1997). They have the same weight on the balance scale as the finer-grained four 1000g-blocks and the three 100g-flats. When refining a kg-block into a 1000-g-block, each block is split into 1000 finer 1g-cubes, so four blocks into four thousands, thus we calculate 4×1000 , and 0.3×100 . This explanation is grounded in the refinement structures (Bielinski & Prediger, submitted).

Explaining why procedures work as discourse practice with topic-independent discursive and topic-specific conceptual components

Explaining is one of the most important discourse practices that is relevant for many mathematical topics (Moschkovich, 2015). Students' competence to explain procedures and their conceptual foundation has *topic-independent discursive components* (general readiness for articulating mathematics ideas and their conceptual foundations) and *topic-specific conceptual components* (understanding the procedure in view by connecting it to the underlying structures). The general readiness refers to the overall willingness to articulate mathematical ideas (students often start with only naming facts rather than elaborating and skip writing explanation tasks) and topic-independent sociomathematical norms for good discourse practices (e.g., when asked to explain, my teacher wants me to refer to visual models, not only symbols, Erath et al., 2018).

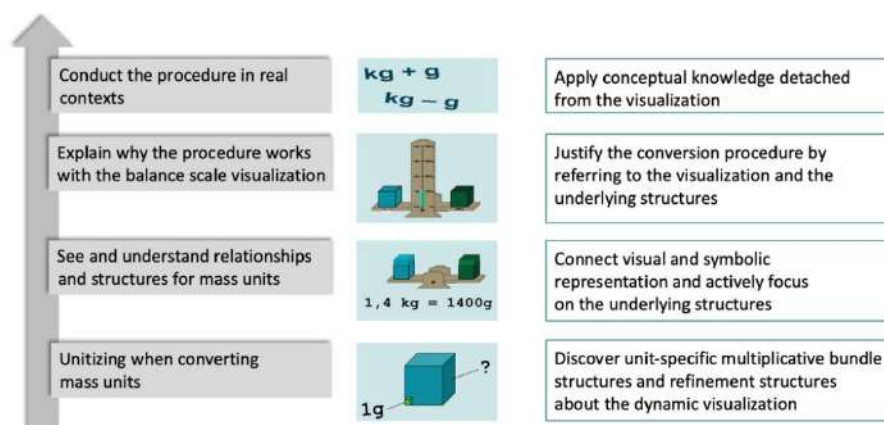


Figure 1: Learning trajectory of the digital teaching-learning environment for IG-CP

Furthermore, topic-specific conceptual components are relevant as students need to learn about the meanings of the visual and symbolic representations and the underlying structures of the topic in view (Fuson et al., 1997). For converting mass units, the place-value structures, bundle structures and refinement structures were identified as relevant to focus for students (Bielinski & Prediger, submitted).

Design principles for learning to explain in digital teaching-learning environments

Students can learn to explain strategies or procedures in teaching-learning environments with two design principles: engaging students in rich discourse practices and connecting multiple representations, as to be explained in the following. The principle of *connecting multiple representations* (Lesh, 1979; Fuson et al., 1997) suggests to flexibly work with visual, material, symbolic, verbal and other representations, in our case the visual representation of finely and coarsely grained blocks on the balance scale and their symbolic translation. Digital tools and *digital learning environments* have been shown to bear potential for connecting representations by multi-representation tools with dynamic links between representations (Kaput, 1986; Drijvers et al., 2016). A condition is that not uncommented (or only automated) translations between representations are conducted because enhancing students' meaning making requires the explicit articulation of how these representations are connected (Renkl et al., 2013). The essential role of teachers' moderation for these explicit articulations is emphasized by the term *teaching-learning environment*. Effects on students understanding have been shown to be larger when the tool is embedded in more structured teaching-learning environments, which sequence the sub learning goals and tasks towards conceptual understanding for procedures in carefully designed learning trajectories (Sacristán et al., 2010), as for our environment (Figure 1). These trajectories follow approaches of progressive schematization, starting from informal experiences in material and visual representations over concept-based strategies and progressively develop towards justifiable symbolic procedures (Freudenthal, 1981; Bielinski & Prediger, submitted).

The principle of *engaging students in rich discourse practices* is realized by tasks and teacher prompts that elicit students' contributions for collective or monological discourse practices, but also by (written and oral) scaffolds that support students' engagement in the collective practices, and by language models that demonstrate what is expected (Erath et al., 2021; Walshaw & Anthony, 2008). Digital teaching-learning environments have been shown to bear potential for fostering students' discursive readiness by digitally assisted communication (Geiger et al., 2023), e.g., with appropriate visual representations as scaffolds, and with strategy conferences on different approaches into which students are introduced and then have to explain to each other.

Empirical evidences for the efficacy of digital environments with these two principles were provided for various mathematical topics (Drijvers, 2018; Sacristán et al., 2010), including explaining measurement formulas such as for volume (Huang & Wu, 2019). But so far, no empirical evidence exists for explaining unit conversion procedures, so

Smith and Barrett (2017) suggest to “harness technological capacity to support learning” (p. 379) and to explore how multi-representation tools “can be used as productive components of ... measurement teaching” (p. 379).

RESEARCH QUESTION AND HYPOTHESIS

In studies on enhancing students’ discursive readiness, learning to explain has sometimes been implicitly treated as if requiring mainly topic-independent learning opportunities and being transferable from one topic to the next (Erath et al., 2018; Geiger et al., 2023; Mercer & Sams, 2006, Walshaw & Anthony, 2008). However, research about explaining arithmetical procedures substantiates the counter hypothesis that learning to explain is topic-specific in that explaining the conceptual foundation of a particular procedure includes specific conceptual components that need to be understood (Fuson et al., 1997). So, we ask the following research question:

RQ. *To what extent do students’ explanations of the mass unit conversion procedure improve in two digital teaching-learning environments that both engage students in rich discourse practices about mass measurements, but with different topic focus?*

In a classwise randomized controlled trial, we test the following counter hypothesis:

H. *Students working in the topic-specific digital environment with topic focus on conceptual foundations of the conversion procedure improve their explanations more than students working in a digital environment with topic focus on estimation strategies.*

METHODOLOGICAL FRAMEWORK

Research design of the controlled trial. To test the hypothesis, we conducted a classwise randomized controlled trial. As the *independent variable*, we compared the effects of two digital teaching-learning environments about mass measurement which lasted 90 to 135 minutes, each. As summarized in Figure 2, both environments shared the same design principles and the visual representation of the balance scale, but had different topic foci: the topic-specific intervention IG-CP focused on conducting and explaining conversions of mass units as introduced in the theory section, whereas the topic-deviating intervention IG-ES focused on conducting and explaining estimation strategies. With explanatory videos and strategy conferences, explaining to peers was introduced and trained more intensively in IG-ES than in IG-CP, as documented in the example tasks in Figure 2. The conceptual components underlying unit conversions were not addressed, but benchmark knowledge and strategies.

As the *dependent variable*, students’ explanations on the topic of unit conversion were assessed before and after the interventions, all within the digital teaching-learning environments. No significant difference occurred in the pretests. As *control variables*, students’ self-reported gender, immigrant background (student or one parent born outside the country) and multilingual background (other languages at home).

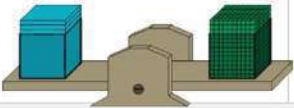
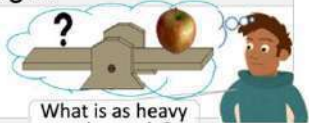
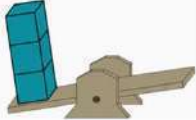

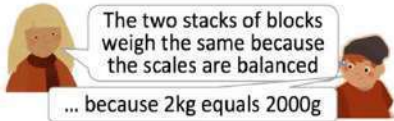

Topic-specific intervention IG-CP on conducting and explaining the conversion procedure	Topic-deviating intervention IG-ES on conducting and explaining estimation strategies for mass units
Connecting multiple representations: visual model balance scale for visualizing equal weights	
<p>Explain with the blocks on the balance scale: Why is 1.3kg the same as 1300g?</p> 	<p>Jimmy shows how he uses benchmark knowledge for estimation. Try similarly.</p>  <p>What is as heavy as the apple?</p>
Engaging students in rich discourse practice of explaining	
<p>Convert 3kg into grams and explain your transformation using the blocks on the scale.</p> 	<p>Estimate how heavy the smartphone is and explain your way of estimating.</p> 
Discursive scaffolds for explaining by language models	
<p>Here are two students' explanations. Reformulate in your own words.</p> 	<p>Strategy conference: Here are four brief silent videos. Watch one of them and explain the strategy to your three neighbors.</p> 

Figure 2: Same principles, different topic foci: Example tasks on two interventions

For the *sampling*, 11 classes were randomly assigned to the two intervention groups. In total, $n = 282$ fifth graders (aged 10-12 years) completed pretest and posttest, they form the intervention whole sample, ($n = 124$ in IG-CP and $n = 158$ in IG-ES). Both intervention groups were comparable in gender, multilingual background and immigrant background (Chi² tests with $p > .05$).

Methods of data analysis. Students' explanations were scored according to explicitly addressed structures, the precise/imprecise articulation of representations, and the richness of the discourse practices, with a maximum score of 11.5 and satisfactory interreliabilities of Cohen's κ between 0.71 and 0.77. Statistical analysis determined descriptive data and intra group effect sizes d (counting as small < 0.50 , medium between 0.50 and 0.80, large > 0.80). An ANOVA with repeated measures was conducted to test the hypothesis on a 5%-level.

EMPIRICAL FINDINGS

Figure 3 shows the learning gains from pretest to posttest in explaining the conversion procedure for both intervention groups. In the *topic-deviating intervention group IG-ES* (working on estimation strategies for mass measures) students started with an average explanation score of $m_{\text{pre}} = 1.06$ (and standard deviation $SD_{\text{pre}} = 1.62$) and ended with $m_{\text{post}} = 1.34$ ($SD_{\text{post}} = 1.90$), so indeed, learning to explain on another topic can still reveal a small intra-group effect (with Cohen's $d = 0.14$) on explaining the non-treated conversion procedures, even if it is not significant in the ANOVA. Meanwhile, the topic-specific intervention group IG-CP (working on mass unit conversion) started with a higher average explanation score of $m_{\text{pre}} = 1.61$ ($SD_{\text{pre}} = 2.23$) and ended with

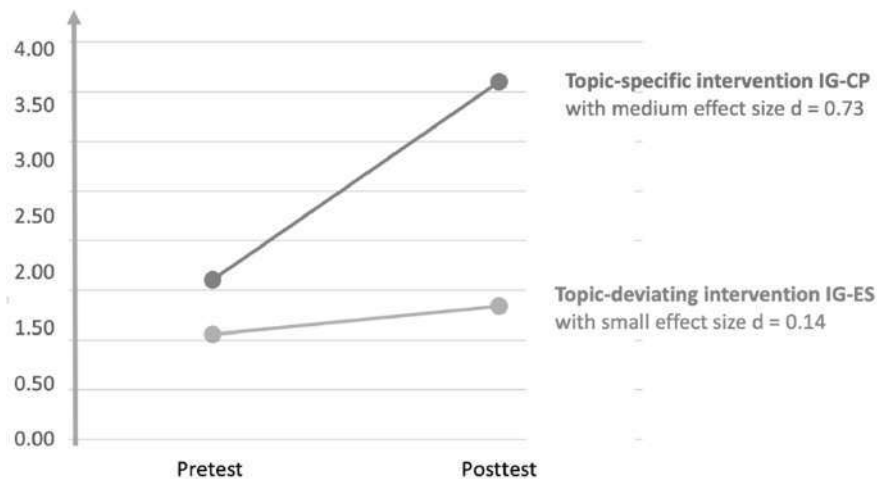


Figure 3: Higher learning gains in explanation scores in topic-specific intervention

significant gains to $m_{\text{post}} = 3.60$ ($SD_{\text{post}} = 2.25$), with a medium intra-group effect size ($d = 0.73$). According to the ANOVA with repeated measures, the intervention group IG-CP had significantly higher learning gains than the IG-ES ($F_{\text{time}}(1, 547) = 13.27$, $p < 0.001$; $F_{\text{time} \times \text{group}}(1, 547) = 5.67$, $p < 0.02$). Thus, hypothesis H can be validated: Students working in the topic-specific IG-CG improved their explanations more than those working in IG-ES.

DISCUSSION AND OUTLOOK

Learning to explain is not at all easy, as many qualitative studies have shown (Moschkovich, 2015; Erath et al., 2018), nor is understanding why a procedure works (Freudenthal, 1981; Huang & Wu, 2019). Our digital teaching-learning environments for mass measurement have improved students' competence to explain, so we can provide further empirical evidence for the efficacy of digitally assisted interventions for measurement, as called for by Smith & Barrett (2017). By this, we have replicated findings from other mathematical areas that the design principles of connecting multiple representations and engaging students in rich discourse practices can be effectively realized in digital teaching-learning environments (Drijvers, 2018; Sacristán et al., 2010) by transferring them to the under-researched area of explaining why the procedure for mass unit conversion works. When comparing existing review studies, Drijvers (2018) identified various aspects that might contribute to the efficacy of digital tools for learning: younger students (primary and lower grades of secondary school) seem to benefit more, just as interventions that seem to focus more on higher-order learning goals and short interventions (p. 173). All of these aspects apply to our interventions.

Beyond these replications, the novel contribution of our study is the validation of the topic-specificity hypothesis: Students working in the topic-specific digital environment with a topic focus on conceptual foundations of the conversion procedure improved

their explanations more than students with a topic focus on estimation strategies. Although it may sound obvious, this finding is not trivial, neither in the literature, where explanation learning processes are often studied independent of the topic under consideration (e.g., by Erath et al., 2018; Mercer & Sams, 2006), nor obvious in our data: The topic-deviating intervention also yielded a small (yet not significant) effect (of $d = 0.14$), presumably achieved by familiarizing with mass measures, using the same visual representation of the balance scale and establishing the sociomathematical norm that explanations should refer to the visual representation and any attempt is better than no answer, these are topic-independent learnings. However, the topic-specificity hypothesis was validated as students in the topic-specific intervention group had significantly higher gains in their explanation scores (with a medium effect size of $d = 0.73$). This indicates that treating topic-specific conceptual components leads to much higher gains, which in our study were the visual representation of place-value structures and refinement structures underlying a concept-based justification of the conversion procedure (Bielinski & Prediger, submitted).

Due to *methodological limitations*, the results must be interpreted with caution. The intervention was taught by relatively unexperienced teachers who were not yet familiar with the digitally-assisted approach nor with the classes, thus they might have not fully exploited the potentials of the digital teaching-learning environments. In the future, we should analyze the teachers' enactment in terms of teacher moves and provided oral scaffolds. The class-wise random assignment did not result in fully comparable prior explanation scores. Although this was controlled for in the ANOVA with repeated measures, future studies should examine whether more comparable intervention groups lead to similar results. In future analysis, we will investigate also procedural knowledge and language knowledge and the impact of background variables.

But already now, the findings are so promising that a transfer to digital teaching-learning environments for other topics with the same design principles should be sought in order to further investigate the transferability. Future studies should also examine longer interventions and verify sustainability through retention tests.

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EMPIRICAL INSIGHTS INTO TEACHING PRACTICES: A MULTI-METHOD STUDY

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This study investigates the empirical reconstruction of teaching practices in mathematics problem-solving using a sequential multi-method design. Integrating video-based observations, interviews, and video-stimulated recalls, the study captures self-reported, observable, and articulable elements of practices. Findings validate a theoretical model of teaching practices and highlight the complementarity of methods in addressing the complexity of professional teaching actions.

INTRODUCTION

Teaching practices are central to effective mathematics instruction. In problem-solving, experienced teachers rely on recurrent patterns of action that create supportive learning environments, present challenges suitable to students' knowledge, and foster heuristic skills (Rott et al., 2021). However, despite the significant role of teaching practices in fostering mathematical problem-solving, their complexity and conceptual vagueness (Charalambous & Delaney, 2020) have made them an elusive research object in teacher professional research. The study we report on aims to address this gap by empirically reconstructing selected teaching practices, providing insights into their components and dynamics. In mathematics education research, the term "practice" is often conceptualized differently. Charalambous and Delaney (2020) outline four perspectives on practices: as distinct from theory, as rehearsal, as an exercise of a profession, and as habitual action. This study adopts the latter perspective, defining practices as recurrent patterns of teacher actions in response to specific classroom demands and requirement situations. These practices, shaped by a teacher's cognitive and affective dispositions, are refined over time and serve as markers of professional expertise (Blömeke et al., 2015; Bromme, 2001).

The multifaceted nature of teaching practices calls for a robust methodological framework. This study uses a sequential multi-method design, integrating video-based classroom observations, semi-structured interviews, and video-stimulated recall to capture the self-reported, observable, and articulable dimensions of teaching practices in problem-solving lessons. Focusing on two experienced secondary-level mathematics teachers in Germany, we examine how they plan, enact, and reflect on their lessons, addressing requirement situations, adaptive actions, and the tools they use. By reconstructing these practices, we aim to validate a theoretical model of teaching practices, that was developed by an integrated literature review (Hirsch & Buchholtz, 2023) and contribute to a deeper understanding of teacher expertise.

THEORETICAL BACKGROUND

Teaching Practices and Their Empirical Reconstructability

According to Hirsch and Buchholtz (2023, p. 3779), teaching practices are defined as routinized recurrent patterns of utterances and actions of teachers for coping with specific requirement situations in a specific teaching context. They are learned through deliberate practice and internalization and relate to the adaptive planning, enacting, and the reflection of pedagogical decisions. Constitutive elements of practices are teachers' affective and cognitive dispositions and situation-specific skills. For the performance of practices, a teacher draws on his or her own body and appropriate pedagogical tools.

Two primary research traditions have influenced the study of teaching practices: the sociocultural perspective and the cognitive science perspective. The sociocultural approach views practices as socially embedded and context-specific activities that emerge through interaction (Lave & Wenger, 1991). From this perspective, practices are best analyzed through ethnographic methods or (video-based) classroom observations (e.g. Häsel-Weide & Nührenbörger, 2022). Conversely, the cognitive science perspective, as highlighted a.o. by Bromme (2001), Stigler and Miller (2018) and Blömeke et al. (2015), focuses on how teachers develop expertise and competence to handle professional demands. This perspective emphasizes the interplay of knowledge, beliefs, and situation-specific skills like perception, interpretation, and decision-making that a teacher draws back on in adaptively responding to a specific requirement. These processes often occur unconsciously, yet through reflection, teachers can bring both implicit and explicit knowledge to awareness, enabling them to adapt and transfer these insights to similar situations (Cianciolo & Sternberg, 2018). Reflection also helps uncover affective dispositions, such as emotions and beliefs, that influence teaching practices. Consequently, practices can be systematically investigated through self-reports.

The complexity of teaching practices necessitates a multi-method approach. A sequential design that integrates video-based observations, interviews, and video-stimulated recall provides complementary insights. For example, observations capture observable actions, while interviews reveal self-reported elements of practice. Video-stimulated recalls allow teachers to reflect on their actions and articulate implicit knowledge.

Professional Demands for Teaching Problem Solving

Teaching problem-solving requires addressing unique professional demands, such as selecting and presenting suitable mathematical problems, scaffolding student learning, and fostering heuristic strategies. Teachers must navigate the dynamic interplay of students' prior knowledge, task complexity, and classroom interactions. Effective problem-solving instruction therefore emphasizes structuring lessons around key phases: understanding the problem, planning a solution, executing it, and reflecting on the process (Rott et al., 2021). When teaching, teachers must employ adaptive

strategies to support students' cognitive and metacognitive processes, such as self-regulation and heuristic reasoning (Schoenfeld, 1985). Recent research underscores the importance of encouraging collaborative exploration and leveraging pedagogical tools, including digital and analogue artefacts, to enhance student engagement (Jacinto & Carreira, 2017).

RESEARCH QUESTIONS

This study investigates the empirical reconstruction of teaching practices in problem-solving lessons, focusing on the elements of practice. The research questions are:

RQ1) What elements of teaching practices do teachers report in semi-structured interviews directly following their problem-solving lessons?

RQ2) What elements of teaching practices for teaching problem solving can be captured through video-based classroom observations?

RQ3) Which elements of selected teaching practices do teachers articulate in the context of video-stimulated recalls?

These questions aim to provide a comprehensive understanding of teaching practices by triangulating data from multiple sources, addressing both theoretical and empirical dimensions of teacher expertise.

METHODOLOGY

Study Design

This study employed a sequential multi-method design to empirically reconstruct teaching practices in problem-solving lessons. The approach integrated video-based classroom observations, semi-structured interviews, and video-stimulated recall, enabling triangulation across data sources (Creswell & Plano Clark, 2011). Two experienced secondary teachers participated. Each conducted two 45-minute problem-solving lessons (L1 & L2), which were video-recorded. The data collection comprised initial observations and interviews in December 2023, followed by video-stimulated recall sessions in January 2024, after a preliminary analysis of the first data set.

Data Collection

Three cameras (one portable, two stationary) recorded each lesson to capture comprehensive classroom interactions. Teachers selected tasks from a predefined problem-solving task catalogue to ensure alignment with the research focus. Tasks were tailored to students' grade levels and designed to encourage exploratory and collaborative problem-solving. Interviews were conducted immediately after each lesson, guided by a protocol derived from teaching practice literature (Hirsch & Buchholtz, 2023). Initial questions encouraged open narratives about the lesson, while subsequent prompts targeted recurrent requirement situations, teachers' responses, and their rationale for specific actions (e.g. *In today's problem-solving lesson, was there a situation that you dealt with recurrently or that required an action that you do*

frequently or regularly?). Two video clips per teacher, showing similar patterns of actions across lessons, were used for recall interviews. Teachers reflected on their actions (e.g. *What did you intend with the action?*), cognitive and affective dispositions, and situation-specific skills (e.g. *What experience or knowledge do you draw on in this situation? What did you perceive in this situation?*). The interview guide furthermore incorporated themes from the theoretical framework, such as the development of practices and expertise.

Data Analysis

Data were analyzed using qualitative content analysis (Mayring, 2022). Self-reported elements of teaching practices were identified and categorized inductively, focusing on recurrent requirement situations and corresponding action patterns. One example of an inductive category in this system is *guiding the problem-solving process*, when the teacher interrupts the problem-solving process of all students with an intervention in order to achieve a specific goal (e.g. securing, clarifying tasks, focusing). For the video data recurrent observable actions were again inductively coded, followed by an analysis of pedagogical tools employed (e.g., *analogue/digital artefacts*, *verbal impulses*, *body language*). The video-stimulated recall interviews aimed to validate and extend the theoretical framework, thus, teachers' reflections on their practices were categorized deductively based on the theoretical model of Hirsch and Buchholtz (2023), addressing cognitive, affective, and situation-specific skills. The data and results were then triangulated, leading to a representation of teaching practices that integrates self-reported, observable and articulable elements of practices.

Participants and Tasks

Two male teachers, aged 39 (T1) and 55 (T2), with 13 and 30 years of professional experience respectively, participated in the study. Tasks included open-ended problem-solving scenarios designed to foster reasoning and exploration, such as the “Cube melting task”, which was used by T1 in L1 (adapted from Bruder, 2003):

Two metal cubes with a given integer edge length, e.g. 2 cm and 4 cm are melted together to a cuboid. Which integer dimensions could have such a cuboid? Find all the possibilities.

These tasks encouraged heuristic strategies and collaborative problem-solving while allowing for teacher adaptation.

RESULTS

RQ1: Self-Reported Elements of Teaching Practices

As elements of teaching practices, the participating teachers mentioned six recurring requirement situations in their lessons, each addressed through routinized patterns of action: *encouraging student engagement with mathematical problems* (1), *guiding the problem-solving process* (2), *instructing retrospective reviews* (3), *providing individual* (4) and *group support* (5), and *promoting mathematical language* (6).

One example of a self-reported element is the recurring instruction of a retrospective review of students' problem-solving processes on which we concentrate the presentation of results. T1 described this with regards to L1 as a routine action involving the systematic collection of student results on the whiteboard, followed by a discussion of these results to highlight key mathematical ideas.

The collection of the students' results on the board [...]; I didn't do such a good job [...] because it was ad hoc, but then it was also routine to just collect it and then also guide it further, yes. [0:02:16]

This action pattern evolved naturally during the lesson, with students taking turns to share their solutions. T1 reflected: “after that it was a no-brainer, because then they also took turns with each other, which I had also originally initiated.” [0:02:30]. This process fostered an interactive and collaborative environment, allowing T1 to steer the discussion toward recognizing patterns and discussing key mathematical ideas. T1 also emphasized their intention to help students make progress with the review without leaving them to struggle aimlessly:

It's part of my routine to regularly motivate the students to engage with the problem, to go around, see the students' solution and still give another hint. [...] So, to let them fumble around for 15 minutes without getting anywhere is no fun for me, nor for the students. [0:01:26]

In total, with the retrospective review, T1 provided students with an opportunity to reflect on their approaches, compare solutions, and consolidate their understanding.

RQ2: Observable Elements of Teaching Practices

In the video data, 126 sequences were identified in which both teachers addressed requirement situations through recurring patterns of action (66 sequences for T1, 60 sequences for T2). Five of the six self-reported elements of teaching practices were observable in the video data, confirming alignment between self-reported and observed elements. Additionally, two new elements of practices emerged: *activation of students' prior knowledge* (7) and *giving an outlook* (8) on future lessons, either through announcements or homework assignments that bridged current and upcoming content.

The videos captured detailed examples of observable elements of practices, particularly T1's collection of students' results when instructing a retrospective review. In L1, when the students had worked in groups for about 20 minutes on the cube melting task, T1 initiated the practice by ringing a small bell to get the attention of the class:

So, roughly speaking, you are all equally on track now (*he stands in the middle of the class*). [...] as you have already found various solutions. I would like to briefly summarise what we have had so far. [0:30:21] (*analogue artefact, body language & verbal impulse*).

T1 proceeded to record results in a table on the whiteboard (*analogue artefact*), transforming the process into a structured collaborative activity, where students shared valid combinations of integers. T1 emphasized clarity and accuracy and highlighted heuristics. For instance, when one student provided a result, T1 prompted: “What did

you consider, or what was important for your table. Um, what the combination definitely has to fulfil?” [0:31:28] (*verbal impulse*; see Figure 1). This led to reflections on unit consistency and mathematical reasoning, such as confirming that their solutions matched the task requirements. When duplicate solutions arose, T1 guided a discussion about equivalent solutions, prompting students to justify why certain configurations could be considered identical due to rotations of the cuboid (*verbal impulse*).

Throughout the activity, T1 combined *verbal impulses*, *body language*, and the use of *artefacts* to scaffold learning. By structuring results into a visible table on the whiteboard (see Figure 1), the teacher not only organized the students’ contributions but also highlighted problem-solving heuristics and critical mathematical concepts.

RQ3: Articulate Elements of Teaching Practices

The video-stimulated recall lasted 27:40 minutes for T1, while T2 took 33:09 minutes. Both recalls provided insights into the articulable elements of practices, highlighting teachers’ reflections about their actions, underlying decisions, and cognitive and affective dispositions. The deductive analysis showed how teachers perceived, interpreted, and adapted their practices in response to requirement situations.

T1’s reflections on the videos of L1 and L2 revealed both planned and intuitive aspects of their teaching practice during this process. In the case of the retrospective review, T1 emphasized the intuitive questioning that shaped his practice. This recursive questioning encouraged critical reflection and comparisons between group solutions:

What happened intuitively then was that I questioned the values that the students gave me. That I took one value, and passed it on to the other group, asking: Okay, is that possible, what did you find out? [...] So, it’s about the questioning of a solution. [13:11]

The teacher reflected that this habit of transferring results for group validation has become an implicit routine: “I think it’s intuitive by now that I pass this on to other students and ask, ‘What do you think about this? Do you have something else?’” [13:30]. T1’s questions furthermore encouraged students to critically reflect on their problem-solving approaches and explain their reasoning. During the retrospective review, T1 guided students to articulate mathematical connections between the quantities they presented:

We asked, what is the mathematical relationship between these three quantities? Some students even expressed the approach verbally, like how to solve it using an equation. [31:11]

This reflective questioning allowed students to conceptualize the relationships underpinning their solutions.

Additionally, T1 noted a successful example of backward reasoning heuristics, where students worked backward from the desired outcome to determine which combinations would satisfy the requirements of the task:

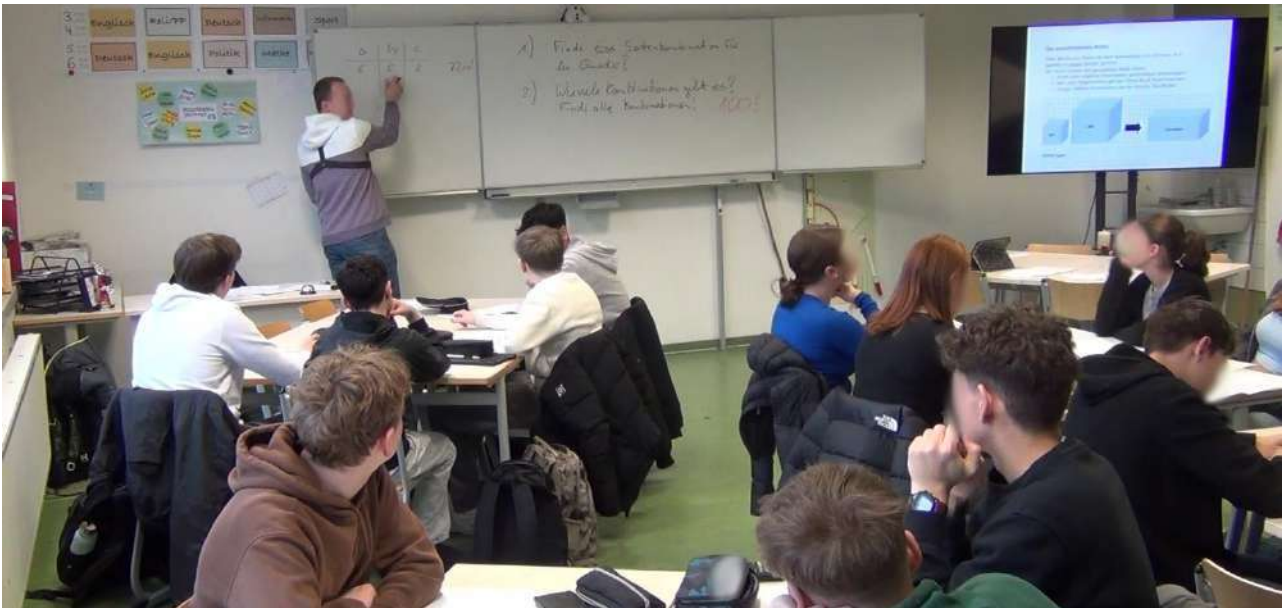


Figure 1: T1's retrospective review of the students' problem-solving process

Only some of them mentioned the prime factorization, but in the end we also had a very nice backwards approach, where they knew what the result should be and which combinations they would find. They succeeded very well in this approach of solving backwards. [31:46]

DISCUSSION

The sequential multi-method design in this study highlights the complementarity of video-based classroom observations, semi-structured interviews, and video-stimulated recalls for reconstructing teaching practices. Each method offered a distinct lens to examine self-reported, observable, and articulable elements of practices, providing a comprehensive view. Triangulation revealed how these methods complement each other in capturing the complexity of teaching practices (Charalambous & Delaney, 2020). For example, self-reports provided insight into teachers' perceptions of recurring patterns of action, such as T1's collection of students' results on the whiteboard as part of a retrospective review. Observations, however, revealed additional elements, such as the emphasis on heuristics and structuring problem-solving through verbal prompts and artefacts. This underscores the value of combining methods to validate and refine theoretical constructs. Similarly, video-stimulated recalls added depth by capturing discrepancies and similarities between in-situ judgments and post-lesson reflections. For instance, T1 critiqued their strong leadership during discussion noting that an open format might have fostered more student autonomy, though in the moment, they prioritized focus as student concentration waned. The integration of findings demonstrated the alignment of empirical evidence with the theoretical model of teaching practices (Hirsch & Buchholtz, 2023). Each method contributed unique elements, validating the model and illustrating its practical value for analyzing and refining teaching practices.

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CROSS-AGE COLLABORATION IN MATHEMATICS

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This report explores a cross-age mentorship program in a Norwegian multicultural school through the lens of positioning theory and storylines, a novel approach in this context. In this case, seventh-grade mentors and third-grade mentees collaborate during weekly mathematics classes. The analysis reveals storylines indicating that mentoring could positively impact the mentors' positioning as mathematics learners.

INTRODUCTION

One of the United Nations Sustainable Development Goals is to ensure inclusive, equitable, and quality education while promoting lifelong learning opportunities for all. D'Ambrosio (2001) contended that much of what schools taught felt disconnected from students' lives. This work is framed within the context of the research project *Mathematics Education in Indigenous and Migrational Contexts: Storylines, Cultures, and Strength-based Pedagogies* (MIM) with the objective to investigate the dynamics of learning in mathematics education when students come from varied linguistic and cultural backgrounds. As we seek to influence the existing (often deficit) discourses surrounding mathematics education and aim to change or pivot (often deficit) existing storylines (Gerbrandt & Wagner, 2023), we investigate storylines (Herbel-Eisenmann et al., 2015) and societal grand narratives like media storylines (Andersson et al., 2022). We do this because storylines play a significant role in shaping and influencing the positions individuals are assigned or have access to (Herbel-Eisenmann et al., 2016). In this report, we focus on the Norwegian tradition of cross-age mentorship programs. This tradition, aligning with Slavin and Cooper's (1999) aims to foster cross-ethnic friendships and reduce racial stereotyping, discrimination, and prejudice through positive social interactions in heterogeneous student groups, ultimately creating a sense of belonging and a positive school environment. While "cross-age peer mentoring" involves younger and older students learning together, "cross-age peer tutoring" focuses on academic learning. To align with the school's goal of fostering developmental relationships, we use the term 'mentor/mentoring'.

Cross-age peer cooperation in mathematics education

Most existing research on cross-age mentoring focuses on its impact on mentees (Burton et al., 2021; Herrera et al., 2007). While mentees are integral to this research, our focus is primarily on mentor outcomes. Academic results were found in a five-week study with students aged 7 and 11 (Topping et al., 2003). Using mathematical games, an increase in mathematical terminology, strategic dialogue, and praise, along with a decrease in procedural talk, was documented. Also, students raised self-esteem and improved the quality of interactive mathematical discussions. Additionally, it

enhanced the tutors' general social and communication behaviors. More recently, Rougeau (2016) found that tutoring pairs, where tutors received training, were more structured, employed more problem-solving strategies, and stayed on task better than other groups. A third study involving middle school tutors and first-grade tutees with low mathematical skills did not report improvements in the tutors' mathematical abilities. However, all the tutors' teachers noted increased leadership skills and confidence and reported that the tutors found the tutoring experience rewarding (Haynes & Brendle, 2019). The latest study on mentors' outcomes found that the program's strength was tied to positive emotional experiences. Relationships between tutees and tutors were characterized by warmth and support, and tutors derived satisfaction from teaching (Barahona et al., 2023). This brief review indicates that tutors notably benefit from increased leadership skills, enhanced confidence, and a rewarding experience, which aligns with Riessman's (1965) helper's theory.

OBJECTIVES AND RESEARCH QUESTION

This report explores how cross-age collaboration impacts mentors' opportunities to assume positions beneficial for learning mathematics by investigating the following research question: How can a cross-age peer mentorship program in primary school help mentors position themselves as mathematics learners?

THEORETICAL LENS: POSITIONING THEORY

Herbel-Eisenmann et al. (2015) define positioning as a discursive process involving action and communication to create social structures. In contrast to the static nature of the concept of "role", Davies and Harré (1990) propose that "positioning" directs attention to dynamic aspects, emphasizing how we understand ourselves in dialogues through the terms "positioning" and "subject position" based on existing narratives.

As most mathematics teachers probably recognize, it is usual to have students who desire to learn mathematics as well as those who do not (Andersson et al., 2015). Positioning theory provides a framework for discussing how students accept or reject roles as mathematics learners. With this understanding, educators can create learning contexts that offer opportunities for students to pivot (Gerbrandt & Wagner, 2023) or change their positions to enhance their engagement in learning mathematics.

Storylines: What are they, and why are they useful in mathematics education?

We have observed that students' positions often align with one or more narratives that support their views on their position as learners or non-learners (Andersson et al., 2023). Positioning theory (Davies & Harré, 1990) refers to these types of narratives as storylines. Herbel-Eisenmann et al. (2015) define a storyline as a culturally shared narrative derived from an individual's lived experiences. Interactions among participants contribute to the creation of storylines, which serve as prerequisites for different positions in people's lives. Storylines play a significant role in shaping and influencing the positions individuals have access to (Herbel-Eisenmann et al., 2016).

Describing a storyline concisely is challenging, as they can be explicitly spoken or implicitly recognized—in this case, spoken or recognized in student interviews. Within the micro-sociological unit (Delgado-Gaitan & Trueba, 2023) of the cross-age mentorship groups, we searched for storylines explaining how the program helps mentors assume beneficial positions for mathematics learning. We define a storyline as ongoing repertoires in students' stories that are (1) recognized by others, (2) cultural, as they are connected to the specific micro-sociological unit, and (3) impactful on individuals' actions.

Methodology

The data collection site is a Norwegian primary school. Approximately 20-30% of the students are multilingual, some arriving in Norway as refugees and others as second—or third-generation immigrants. In recent years, a deliberate effort has been made to place greater emphasis on diversity within the school community. One key strategy for achieving this is the cross-age mentorship program.

Observation and context description: The cross-age mentorship program

In 2023, when Kaja began observing the mentorship collaboration, the two classes were already acquainted. From previous observations, Kaja was familiar with the mentor class and was aware that they were struggling with strongly challenging behavioural issues and adherence to classroom norms. To provide a deeper contextual understanding, we include the mentor teacher Grete's description of the class:

A collection of remarkable individuals. [...] In peacetime, they are all lovely children [...]. It is demanding because, what can I say, the combination of students in this class is very unfortunate, quite simply. That's it. The parents were very determined that they should not be together, but then there has been some mismatch here in terms of communication [from kindergarten]. [...] so it has to do with that, quite simply, that those kids should not have been together.

While this depiction may seem harsh, it aligns with Kaja's empirical observations. The students undeniably exhibit remarkable capabilities and skills. However, the persistent presence of negative tensions significantly undermined the classroom environment, resulting in a less, or rather very bad, conducive learning atmosphere. The form of observation of the cross-age collaboration varied. As the students engaged with mathematics, Kaja sometimes circulated among the groups, observing the interactions between the elder and younger students. At other times, Kaja assisted the groups or the teachers with diverse tasks. Throughout this period, detailed field notes were kept, subsequently informing the creation of structured interview guides employed in later interviews with the teachers, mentors, and mentees.

Interviews

Twenty students and two teachers were interviewed concerning the cross-age collaboration in mathematics. All interviews were semi-structured and audio-recorded. All names used are pseudonyms. The joint interview with the teachers, Sigrid (the mentees' teacher) and Grete (the mentors' teacher), aimed to gain insights into their

motivation for choosing this specific form of collaboration in mathematics and to understand the practical aspects associated with its implementation. The student interviews concentrated on the students' experiences of the mentorship program. Ten students from each class were interviewed: the mentees in pairs and the mentors individually (because they wished to), in pairs, or in groups of three.

Analysis

Inspired by Herbel-Eisenmann and Wagner (2010), we utilized positionings as an analytical tool to understand the experiences, relationships, and emotions derived from the program. To achieve this, we identified narratives that are (1) recognized by others, found in multiple interviews, (2) cultural, related to this specific micro-sociological unit, and (3) impactful on mentors' actions. We then formulated collective storylines. Furthermore, we discussed how these storylines can influence the mentors' opportunities to ascribe and claim positions for themselves and others by defining roles, rights, and obligations (Moghaddam & Harré, 2010) within the context of the mentorship program. In this report, we present a cluster of three storylines in which the mentors describe the process of guiding and teaching mathematics to the mentees.

INITIAL RESULTS: LEARNING MATHEMATICS TOGETHER

Storyline A, “Mathematical explanations are challenging,” highlights the mentors' experiences adapting or relearning methods and algorithms and their efforts to elucidate their functionality to the mentees.

Table 1: Example of excerpts that support storyline A

Bertine: We have to learn other methods to calculate or explain in a different way.
Kaja: Why is that?
Bertine: Because they don't understand adult language like we do.
[...]
Sofie: Explain it in a different way. It's harder for us but easier for them.
Kaja: What's more challenging for you and easier for them?
Bertine: It's harder for us to explain it.
Sofie: Because they have other methods.
Bertine: But it's easier for them to understand it.
Thea: So it's a bit difficult because they have a completely different method that we probably did in 3rd grade, but we have now become accustomed to a completely different method, and it's a bit difficult to help them when they don't understand the way we're trying to help them.

Storyline B, "Challenges bring mathematical thinking forward," refers to the complex explanations identified in Storyline A. In this storyline, the term 'challenges' suggests that mentors' efforts to explain mathematical concepts to mentees enhance their mathematical understanding. As mentors work on explanations, they simultaneously must process and expand their mathematical understanding. These cognitive processes can potentially enhance the mentors' learning and comprehension of mathematics. For example, Madelen's strategy of writing down her preferred method before helping the mentees suggests her recognition of the need to understand a problem before she can effectively assist the mentees.

Table 2: Example of excerpts that support storyline B

Bertine: [...] I learn how to do it in other ways, and I learn more about how to simplify questions.

Madelen: Sometimes I just have to stop and then I have to write it down in my method, and they ask, "What is that, what is that?" And I say, "I just have to calculate it this way, I can help you after that, I just need to get the answer for myself first".

Storyline C, "Mastering a task on your own signifies learning", was explicitly articulated in all the interviews and was concurrently reflected in dialogues with the mentees. For instance, Jesper underscores the importance of explaining the problem-solving process to the mentees rather than simply providing the mentees with solutions. Eirik expands on this discussion, suggesting that without mastering the method, mentees might struggle with similar problems when their mentors are not around. This observation underscores the long-term goal of mentorship: to equip mentees with the necessary skills and knowledge for independent problem-solving. The example from Eirik and Jesper's dialogue reflects the mentors' recognition of the importance of process over product in the learning journey and the ultimate goal of their mentoring: to assist the mentees in mastering independent thinking and problem-solving skills.

Table 3: Example of excerpts that support storyline C

Jesper: [...] we have to show them how to arrive at the answer!

Kaja: Why is it not okay to just say that the answer is 12?

Jesper: Because they won't learn anything.

Eirik: They won't learn anything.

Kaja: No. So, that would be poor mentoring?

Eirik: Yes. They wouldn't learn the method. In case we're not there.

Kaja: Yes, because what you teach them, they should be able to do...

Eirik: ...themselves.

DISCUSSION OF THE STORYLINES

In the following sections, we discuss the implications of each storyline separately to provide a more comprehensive understanding of how cross-age cooperation can support mentors to position themselves as mathematics learners.

In the first storyline, "Mathematical explanations are challenging," the mentors position themselves as tutors by taking on the challenging role of explanation providers. To overcome this challenge, they must actively explore various methods or approaches to explaining mathematical concepts at a level the younger mentees can understand. From a mathematics teacher's perspective, the challenge of explanation serves as a gateway to the goldmine of mathematics learning. The mentors further state the goldmine in the second storyline, "The challenge brings mathematical thinking forward". This storyline is particularly significant for two reasons: first, it states that when working with mentees in mathematics, the mentors are compelled to think about and use mathematical terminology in ways they likely have not done before; second, doing that can help the mentors recognize the practical value of what they have learned, which can serve as a substantial source of motivation.

The mentors' readiness to overcome these challenges is likely linked to the third storyline, "Mastering a task on your own signifies learning." When mentors support mentees in acquiring the necessary mathematical understanding and skills to work on mathematics independently, they feel a sense of accomplishment and pride. In this process, mentors are positioned as enablers who empower their mentees in mathematics. This aligns with Rougeau's (2016) findings that mentors working with mentees of the same age group attempted more problem-solving strategies and demonstrated better task adherence than other groups. Collectively, these three storylines support the increased use of mathematical terminology and strategic dialogue observed by (Topping et al., 2003).

CONCLUSION

The storylines discussed in this article show that the cross-age mentorship program can aid mentors in positioning themselves as mathematicians during collaboration sessions. Still, as this example is in an early process and single study, we cannot make extensive claims about the transferability of mentors' positioning as mathematics learners. However, the program provides an opportunity for the mentors to feel more closely connected to the subject because, for at least a few hours each week, they have the chance to assume role-modeling positions, as the program provides opportunities for the mentors to discuss and explain mathematics to others and to help younger students. This process empowers mentors to exercise agency and assume responsibility in

mathematics learning processes. For future studies, we see potential in examining cross-age collaboration in mathematics education from a positioning perspective on a larger scale, ideally by analyzing both mentor's and mentees' perspectives.

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PROFESSIONAL DEVELOPMENT LESSONS FROM AN INFORMAL MATH LEARNING LONGITUDINAL STUDY

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Collaboration can be a highly effective pedagogy for supporting the development of students' mathematical reasoning skills. Pedagogies centering mathematical discourse can be challenging for teachers to implement in regular classrooms. Teachers, presented with an opportunity to interact with students in an informal mathematics learning environment, reflect on their professional development engaging with students working collaboratively in problem-solving sessions and report on the impact of their earlier experiences on their beliefs and current practices as mathematics educators and professionals.

INTRODUCTION

Meaningful mathematical discourse among students is critical to building mathematical reasoning. In this discussion-centric setting, students have an opportunity to build a collective understanding of the mathematics involved in solving problems. Mathematical ideas are challenged and shared through discussion of approaches and providing justification (Francisco & Maher, 2005). The environment of the Informal Mathematics Learning (IML) study was designed to challenge students with a sequence of carefully selected open-ended tasks engaging students in meaningful discussions that emphasize sense-making and justification of proposed solutions (Mueller & Maher, 2009). This paper focuses on the lessons that researchers and teachers learned regarding pedagogical practices that promote discourse and reasoning.

This investigation follows up on a previous project completed by the Robert B. Davis Institute for Learning called Rutgers Research on Informal Mathematics Learning (IML). The original study occurred over a 3- year period with middle school students in an economically depressed community serving grades 6, 7, and 8. The IML project consisted of several after-school enrichment classes taught throughout the school year and two weeks during the following summers. There were two cohorts of students. The first group had sessions led by researchers while teachers from the district observed. Sessions for the second cohort were led by the teachers. These sessions were all video recorded and stored in a repository, for study. The goal of the current study is to investigate the long-term impacts of IML on researchers' and teachers' pedagogical beliefs and practices.

THEORETICAL FRAMEWORK

A socio-cultural constructivist perspective guides this analysis on two important levels. Firstly, socio-cultural constructivism informed the design of IML (Maher et al., 2023). Secondly, the premise of the learner as an active participant in the construction of knowledge of mathematics (Loyens & Gijbels, 2008) guides the analysis of supporting learning environments highlighting students' engagement with the tasks and each other. Constructivism calls for deep understanding as a focus of learning rather than behaviors or skills/procedural fluency (Fosnot, 2005). This conceptual understanding is achieved through self-regulation and the building of conceptual structures through reflection and abstraction as knowledge is adaptive (Glaserfeld, 1995). Striving to support classroom learning focusing on breakthroughs in reasoning is central to this analysis.

We use the term “informal mathematics learning” to describe the environment of the original study. As described by Shay (2008), the informal mathematics learning environment consists of well-defined, open-ended tasks completed collaboratively and with minimal interference from the teacher(s) or researcher(s) administering the tasks. The collaborative learning environment for the completion of these tasks lends to the development of student reasoning through natural argumentation and justification among peers (Francisco & Maher, 2005). The design of the informal mathematics learning environment is specifically to foster rich, mathematical discourse to highlight understanding and reasoning. Careful consideration was employed in the selection and implementation of the mathematical problem solving tasks. After identifying the mathematical goal of a task, Smith (2018) argues that tasks should be structured to highlight student reasoning. Using the model proposed by Davis & Maher, (1990), this means creating tasks that focus on building representations from given data, calling on relevant knowledge, applying relevant knowledge to the representation, verifying that the model makes sense, and then, as appropriate, using technical devices to help create a solution. Student understanding is developed in the journey to the solution, not in simply providing the correct answer. As such, tasks should be designed to highlight reasoning and solution strategies. Good tasks should also encourage discourse by having multiple avenues to arriving at a solution, be sufficiently approachable but challenging, have multiple possibilities for representation, and should elicit justification (Ball & Bass, 2003; Francisco & Maher, 2005). Carefully selected tasks engage students and set the stage for meaningful mathematical discourse.

Despite the general acknowledgment of the importance of teaching students to problem-solve, pedagogies that incorporate inquiry-based learning are not always implemented in the classroom (Cheeseman, 2018). In some cases it is assumed that direct instruction works better for some students (Dean Jr. & Kuhn, 2007). According to Schoenfeld (2022), most students experience mathematics through mathematics instruction making opportunities for students to express their understanding through engagement in classroom discussions imperative. Incorporating all student

contributions in meaningful ways can be incredibly overwhelming to classroom teachers. Karp (2010) and Bastian et al. (2024) suggest that more research is needed to address teachers' pedagogical knowledge to teach via problem solving. Video footage of students working collaboratively and expressing their justifications for problem solutions has been used to challenge pre-service teachers' beliefs about what students can do (Maher et al., 2023). A follow-up study with teachers and researchers involved in the IML project provided a unique opportunity to understand long-term effects from their experience of engaging students in collaborative learning environments on participants' current beliefs and practices. This study adds to the knowledge base for teacher preparation and professional development regarding the implementation of informal student collaborative problem-solving environments.

METHODS

Four researchers and one teacher were interviewed in the current study. The teacher, IMLT1, has over 20 years of experience in the middle school mathematics classroom at the site of the original study. His career began before IML and continued afterward for 18 years in the middle-school mathematics classroom until he transitioned to teaching STEM at the k-8 level. Two of the researchers interviewed, IMLR2 and IMLR3, who were principal investigators in the original study, are professors of mathematics education with IMLR2 focussing on urban studies. Researcher IMLR5 works in workforce development and continues to be involved in publications focusing on mathematics education. The final researcher, IMLR4, spent much of her career working in teacher professional development. Interviews with IMLT1 and IMLR3 occurred in person and were audio-recorded for transcription. Interviews with IMLR2, IMLR4, and IMLR5 were conducted on Zoom and recorded for transcription.

Due to the nature of observing a specific community of individuals who were involved in the Informal Mathematics Learning project, a purposeful selection method was used. Access to contact information for participants in various roles provided a convenience sample. Multiple participants in various roles were interviewed to ensure multiple perspectives for analysis. The following criteria were used for participant selection: (1) The individual was a researcher or teacher involved in the original IML study; and (2) After the completion of the study, the individual published research in the discipline of mathematics education or continued to work in a role as mathematics educator or teacher educator. Each interview lasted between 40 and 90 minutes. Prior to engaging in the formal interview, participants were reminded of the purpose of the study, reminded that all information would remain confidential, and consent was obtained to record the interview. A broad interview protocol was employed ensuring key questions were discussed while allowing for flexibility in the natural progression of the discussion and for clarification questions. Questions and probes explored participants' role in the original study, their continuing role in mathematics education, and reflected on how participation in IML impacted their pedagogical beliefs and practices about conditions for student learning. After the data were collected and organized, grounded

theory coding was used for initial coding. Second and third coding cycles followed pattern coding and focused coding to group specific labels into broader codes and group those codes based on thematic or conceptual similarity.

RESULTS

In the interview with IMLT1 about his participation in IML he commented, “I feel like it definitely impacted how I teach.” The findings from the current investigation predominantly address the first two research questions that framed the study: (1) How are researcher and teacher beliefs about learning through collaboration impacted by interacting with informal mathematics learning? and (2) To what extent and how does seeing effective implementation of informal mathematics learning impact teachers’ continuing pedagogy? One important finding addressing these questions is how, from the beginning of the study teachers began transferring elements of IML to their own classroom through the IML tasks. Addressing the long-term impact of IML, IMLT1 affirmed, “I still, I mean, even last year, still, every year, would do at least some sort of intervention with the rods, the unifix cubes, and the counting problems.”

The theme of teachers transferring the tasks from IML to their classrooms appeared in multiple interviews. IMLR2 stated that this was an unintentional outcome, as the goal of the study was primarily focused on studying student reasoning:

IMLR2: One of the interesting things that was a byproduct of the work that we did with the teachers and students is that teachers witnessing the ideas that students came up with when they worked in this collaborative environment, began to take some of the tasks and try them in their own classroom because they wanted to, they wanted to see their students behave mathematically in the ways in which they were witnessing the students in the after school informal math program.

Seeing the mathematical reasoning students exhibited when presented with carefully designed tasks to elicit justification motivated teachers to bring those tasks into their classrooms. IMLR4 adds that teachers were intrigued by the tasks of IML because they were a shift from how mathematics was taught in their classrooms, “They were tasks that we gave them, that were quite different from what they were doing in their classrooms...They tried to use what they were learning in their classrooms as well.” Teachers were noticeably choosing to bring elements of IML to their classrooms and, as IMLT1 confirms, continued to do so in a lasting manner after the conclusion of the study. The transfer of the tasks highlights the use of carefully selected tasks to encourage student collaboration and rich mathematical reasoning in the classroom.

The tasks in IML were implemented in connected strands. For example, the first five sessions involved students developing fraction reasoning using Cuisenaire rods to model problem solutions. In these interviews, conducted almost two decades after the conclusion of IML, interviewees referred to specific tasks and strands of tasks that were durable over time. For example, tasks about fraction reasoning were mentioned the most frequently (11 times by 2 interviewees), followed by tasks from the

combinatorics/counting strand (6 times by 3 interviewees), and then probability tasks (4 times by 3 interviewees). While covering different mathematical concepts, each strand of tasks had certain structural similarities, discussed below.

The first structural similarity that was mentioned by participants in each interview, was that the tasks provided opportunities for various representations, such as making a diagram or building a model using manipulatives, or otherwise required students to show their reasoning. IMLR3 explained this was intentional for the purpose of investigating student reasoning, “And I guess one of the things here is all the tasks lend themselves to external representations that link to the representation that we believe linked to the internal representations that the students held.” In addition to being helpful for researchers to observe student reasoning, IMLT1 commented that having opportunities for using representations to show reasoning was valuable for students, “they [students] benefit from having to explain and show.”

The second structural similarity among tasks mentioned by four of the participants was that each task had multiple points of entry and exit. IMLR5 describes this as each task not having one exact way of needing to be solved, “something common across all of the tasks is that, none of them had like a single way that you had to approach it in order to make some progress. So every task had multiple points of entry.” In addition to multiple entry points, success is not defined as arriving at one specific conclusion. IMLT1 describes multiple exits as providing safe opportunities for students to be engaged rather than emphasizing correctness, “They're set up in a way where, if you don't map the numbers to it, it's not a failure, because you're still engaged, and there's still some level of informal mathematics occurring.”

The last similarity as mentioned by four participants, is that the strands of tasks were designed to guide students to see patterns. Rather than addressing misconceptions by telling students they were wrong, teachers and researchers created tasks to address students' learning needs by helping them discover obstacles and try again. In the words of IMLR3, “But in other cases, there would be kind of new tasks that we might try to design. They're having trouble with this. Let's give let's give them this task, and they'll come to see it.” Carefully designed tasks elicit student reasoning, are approachable for students and can be specifically designed to help address misconceptions by inviting students to investigate topics further.

When sharing about his experience using IML tasks in the classroom after the conclusion of the study, IMLT1 reported that he often encountered pushback:

IMLT1: I had to get explicit permission to implement the rod tasks in my classroom because we were using the connected math books, and it was called bits and pieces, and I was, I had to write a formal proposal to replace bits and pieces with the rods. She granted it, but because the math coach was saying that I was not using the curriculum.

The importance of supportive administrators, the restrictions of standardized testing, and the constraint of district curriculum were all mentioned as challenges faced by teachers bringing elements of IML to the formal learning space. When asked what elements of IML were transferable to their formal learning space, IMLR2 said, “All that we did with students in this setting is transferable to what goes on during the day, if what goes on during the day, isn't highly constrained by high stake testing.” IMLR5 adds on that the formal learning space is restricted by many other elements reducing teachers’ autonomy to teach how they want, “Because the classroom setting is governed by all this other stuff and teachers have increasingly less choice over the decades about what they can and can't do in their classroom.” While the mathematics and reasoning that occurred in IML was impressive, the freedom of time and exploration is not always available to teachers in formal learning spaces. As we can see from IMLT1’s experience, IMLR4 points out, “And it so depends on the leadership in your school.” If teachers want to implement teaching methodologies like those in IML, they require permission, flexibility, and support from administrators.

DISCUSSION AND CONCLUSION

When teachers witnessed student reasoning prompted by agency in problem-solving, they began to bring the tasks from the study into their own classrooms to encourage rich, mathematical discussion. By bringing these tasks to their classroom they transferred collaborative problem-solving to their formal learning spaces. Students' reasoning was then elicited by providing and discussing their justifications (Francisco & Maher, 2005). This model of instruction aligns with NCTM directive for mathematics teachers to facilitate meaningful mathematical discussion (NCTM, 2014). This suggests that teachers saw the value in bringing elements of IML such as rich, engaging tasks and student collaboration to their classrooms for the benefit of their students’ learning.

Tasks involving opportunities for representation and multiple points of entry and exit are critical for developing student reasoning (Ball & Bass, 2003). Sources like the Video Mosaic Collaborative, an online repository that stores videos, tasks and video narratives with worldwide open access, provide a starting point for examples of tasks that can be used in classrooms, what those tasks look like being implemented, and examples of student reasoning. IMLT1 reminds us that, "the way we teach is how we were taught, unless you participate in these types of activities." Participating in IML was important to developing a teaching style that worked well for him.

Another significant takeaway from the findings is that teachers need support from administrators to bring rich tasks into the classroom. Administrators need to be supportive of opportunities for professional development where teachers see examples of student reasoning and provide resources that support educators teaching subjects they haven’t seen since they were in school themselves (Karp, 2010). Teachers require support to continue refining their craft and exploring pedagogies that are effective for

them and their students. Administrators serve a critical role in supporting or stifling efforts for professional development and learning.

With all of the important insights gained from the study, it is important to acknowledge that IML occurred outside of the confines of the formal learning environment. However, it is clear that certain elements such as well- designed strands of tasks that encourage student collaboration are certainly transferable to the formal learning space. The discussions surrounding IML also invites reflection on how researchers can help tackle some of the constraints teachers face such as curriculum restrictions, standardized testing pressure and administrative support. It is imperative to combat challenges that teachers face implementing the pedagogical styles they choose. Without support, good teachers are leaving the mathematics classroom. IMLT1, who remained in the mathematics classroom for eighteen years after the conclusion of the study transitioned to a new position with less pedagogical restrictions. Students deserve dedicated teachers, and dedicated teachers deserve support.

IMLT1: I've been teaching for 25 years. I'm not sure, you know, I'm not sure if I can stand another iteration of these kind of bureaucratic structures. And it's kind of nice to be able to teach the way I'm teaching, because I think I will be doing a lot more informal math.

It is recommended that teachers and teacher leaders receive support to develop, implement, and study student growth in mathematical understanding similar to the Informal Mathematics Learning Project environment and make use of the resources available on the Video Mosaic Collaborative Repository (Agnew, et al.,2010).

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INFLUENCES ON TEACHER-TEXTBOOK RELATIONSHIP

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The study examines the teacher-textbook relationship, emphasizing the factors that influence it based on teachers' general perceptions of the mathematics textbook. By integrating the concepts of agency and affordances, as well as considering the significance of the pedagogical context, this qualitative research was conducted using the theorEM System method. The data produced was derived from questionnaire responses provided by 10 public school teachers. The analysis revealed four key influences on this relationship: pedagogical context, curriculum, teacher education, and personal conceptions. These findings offer considerations for improving textbook design and teacher education in the present and in the long term.

WHY STUDY THE RELATIONSHIP BETWEEN TEACHERS AND TEXTBOOKS?

The textbook is one of the most used resources by teachers, which means it can influence an educator directly in a classroom, consequently, teaching and learning process. Because of it, in this PhD research, the authors decided to focus on the relationship between teachers and textbooks once Fan (2013) pointed out the gap in research related to this focus, which is currently maintained.

Choppin (2004) identified an increase in textbook research around the time of his investigation and noted that in the 20 years before this study, there had been intense development of research discussing the use and reception of textbooks with questions such as What kind of use is made of them? Do teachers follow them faithfully, step by step, or do they take certain liberties with their proposed organization? And, in that case, which liberties and for what reasons?

Despite this, years later, Lloyd, Remillard, and Herbel-Eisenmann (2009) point out that although the research on teachers' use of curricular materials is growing, it still needs to be developed. There has been a recent increase in interest in this field regarding how teachers use curricular materials and whether and how these materials may influence classroom practices and teaching more broadly.

In a survey of research involving textbooks in various countries, Fan (2013) noted that research in this area is still very new compared to other areas of Mathematics Education. According to him, it would be important to consider textbooks as a variable and, more precisely, as an intermediate variable in education. From this perspective, the fundamental question about textbooks as a research object is not just about what the textbooks are or their characteristics but also about how they affect other factors, such as the teaching process by the teacher, for example.

Focusing on research about curricular materials within the Brazilian context, Januario (2017) mapped 59 dissertations and theses defended between 1989 and 2012 in the fields of Science and Mathematics Education and Teaching that examined the relationship of the teacher-curricular materials as the subject of study. It was notable that only one of the 59 was a doctoral dissertation. Additionally, of these 59 studies, 80% (47) addressed descriptive research questions, focusing on Mathematics curricular materials and describing their characteristics.

In this direction, it was searched for the keywords “teacher”, “use of textbook” and “mathematics”, resulting in 25 productions. From this, only one research was about the teacher’s use of a textbook in mathematics in their practice but focused on their beliefs and its influence on the use of a textbook.

Since it is possible to perceive a fruitful field for research, it was decided to focus on the teacher-textbook relationship. This dyad term is the dynamic interaction between the teacher and the textbook, considering one influences the other. Therefore, the research question (RQ) is: *What are the influences on the teacher-textbook relationship based on the general view that teachers have of the mathematics textbook?*

THEORETICAL FRAMEWORK

Focusing on a review of research on the use of mathematics curriculum to examine how key constructs in this body of work are conceptualized, Remillard (2005) identified four different types of curricular material use: i) following or subverting the text, meaning assuming that close fidelity between the written curriculum and the enacted curriculum can be achieved under ideal conditions; ii) drawing on, with an emphasis on teacher agency, where textbooks are seen as one of many resources that teachers use in constructing the enacted curriculum; iii) interpreting, with the teacher acting as the interpreter of the enacted curriculum; and iv) participating with, assuming that teachers and curricular materials are engaged in a dynamic interrelationship that involves both the teacher and the text. The author emphasizes that these contrasting conceptions of curriculum use, based on varying theoretical assumptions, influence what can be learned about how teachers use Mathematics curricular materials. In this way, she embraced her discussion about teacher-curriculum relationships.

Exploring beyond these four conceptions, Aguiar and Oliveira (2017) suggest a fifth conception, called 'recontextualizing the text.' This can be understood as a process through which the teacher repositions the texts of educational curriculum materials into pedagogical practice. In other words, teachers selectively act on the texts of these materials, considering the principles present in the pedagogical contexts. Once this process occurs, it becomes clear how the context is crucial and can directly influence how the teacher uses curriculum materials in the classroom.

Considering the dynamic interaction between teachers and curricular materials, Brown (2009) argues that a teacher might interact with curriculum materials in three ways: offloading, adapting, and improvising. Offloading occurs when teachers use literal

parts of the material, such as activities, instructions for their execution, or guidance and interventions, to resolve doubts or help students build knowledge. However, during this process, she/he may intervene in what the didactic material proposes, for example, altering the sequence of activities, suggesting a different classroom organization, or omitting instructions on performing an activity. This is known as adaptation.

Improvisation occurs when a teacher mediates/promotes learning by creating new situations and actions based on the student's learning needs (Brown, 2009). In a single class period, all three may occur dynamically, with none being superior to the others. He proposed a framework Design Capacity for Enactment (DCE) when discussing this interaction, with the various components of the teacher-tool dynamic being captured by this framework. It represents the different interactions between teachers and curricular resources as teachers offload, adapt, or improvise with the textbooks.

The author adds that the DCE encompasses the design resources and the embedded knowledge that comprise the curricular materials – procedures, content representations, and representations of physical objects. These aspects reflect the intentions of curriculum developers. The framework encompasses teachers' knowledge, skills, goals, and beliefs and how they influence the ways teachers perceive and appropriate different aspects of curricular designs. Therefore, it is emphasized the teachers' personal conceptions and the content within these materials. This content is important once the textbooks, for example, are an expression of the prescribed curriculum (Amaral et al., 2022) and it can serve as a basis for teacher's education.

This conception from Brown (2009) highlights how teachers interact with curriculum materials, balancing design intentions with teacher agency as Januario (2017) expended. Beyond observing different uses and applications, to him, it is essential to understand how teachers interact with the textbooks and the reasons for making interventions in the materials. In this case, with such interventions, it can be said that the teacher has agency, understood as the factor that has authority over mathematics and its teaching. According to this author, the agent has the power of decision and authority over the mathematics to be taught; therefore, the one who has agency determines the different types of use.

Thus, the teacher's agency is characterized by recognizing weaknesses in the material, identifying the relationship between expectations, objectives, and activities, redesigning activities, and understanding mathematics and its teaching. On the other hand, the agency in the materials is characterized by contextualization, mobilizing prior knowledge, connecting learning situations, expectations, and objectives, didactic-methodological guidelines, and clarity regarding underlying didactic-methodological issues. Januario, Lima and Manrique (2017) based on Januario (2017) present a scheme which show this idea associated to Brown (2002; 2009) – Figure 1.

In addition to the idea of agency, Brown (2002) discusses affordances, which take on the meaning of certain aspects of the materials that enhance their use by teachers. This means that in curricular materials, teachers can identify the content in these resources

and using their knowledges, beliefs, and values, assume an identity and understand the affordances of these materials, as Figure 1 shows. Depending on their resources, each teacher will understand the affordances differently, which can lead to different uses.

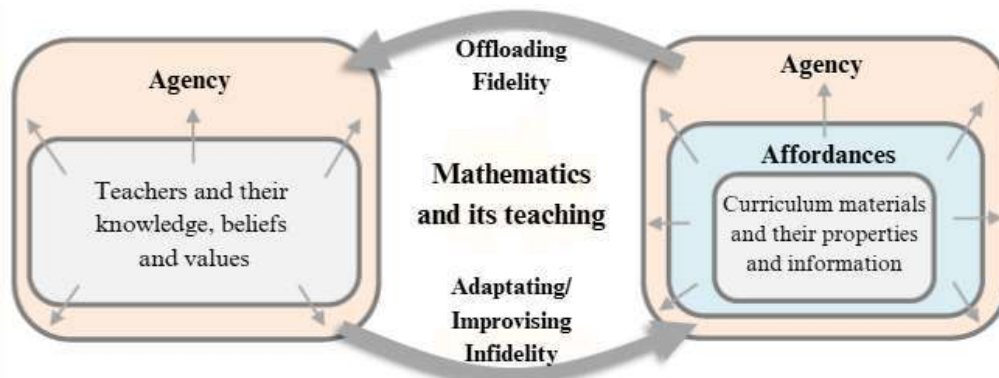


Figure 1: Teacher-curriculum materials relationship.

When developing the curriculum, the more knowledge teachers have and mobilize when planning and conducting lessons, the more they assume agency, as illustrated in Figure 1, with arrows. On the other hand, when they have little or no knowledge about a particular content, the curricular materials assume agency. Integrating this relationship, the affordances are perceived by teachers, through the identification of aspects that enhance the use of materials (Januario; Lima; Manrique, 2017).

These ideas align with Brown (2009), who suggests that the more agency a teacher has, the more likely they are to adapt or improve curriculum material. However, if textbooks themselves possess agency, the tendency is for teachers to remain faithful to the material, essentially “offloading” the content. Since textbooks are a form of curriculum material, these discussions will provide the basis for analysing the data produced.

METHODOLOGY

This research is characterized as qualitative, an approach that allows for exploring the complexity of educational phenomena, standing out for its ability to investigate interactions, contexts, and meanings in depth (Denzin & Lincoln, 2018). By addressing the dynamic relationship between teachers and textbooks, the qualitative methodology was chosen as the most appropriate to understand teachers' perceptions, actions, and decisions in their pedagogical practices.

The study is based on the responses to questionnaires administered to 10 public school teachers in Brazil, enabling the investigation of how they perceive and use textbooks in their practices. This methodological choice reflects an interest in understanding the meanings attributed by teachers to their pedagogical experiences, considering the influences of the school context and the specificities of the curriculum.

For data analysis, the theorem system was employed, consisting of three stages. The first stage is dedicated to planning the research, represented in Figure 2. It illustrates a synthesis of the Theorem System and the various questions guiding the definition of research objectives and the development of the questionnaire.

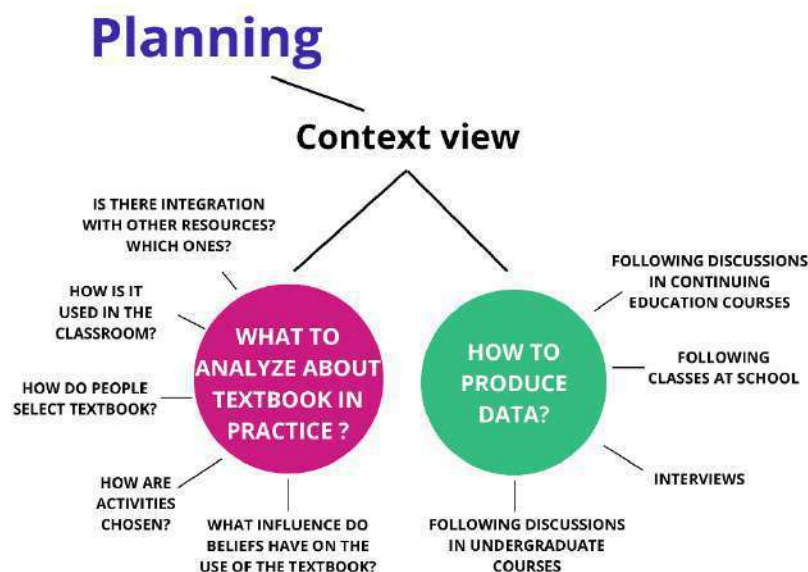


Figure 2: Planning research that focuses on the context view (Theorem System).

The second stage on this System focuses on exploring the material, in this case, considering how teachers engage with it. Finally, the third stage is devoted to data analysis, which, in this case, draws on the analysis of daily life (Amaral et al., 2022). This approach was particularly pertinent due to its capacity to integrate teachers' perceptions with theoretical and contextual principles (Smith et al., 2017).

TEACHERS' VIEW OF THE TEXTBOOK: WHAT INFLUENCES THIS RELATIONSHIP?

The 10 answers receive from the teachers at public schools in Brazil (named as T1, T2, ..., T10), and based in the theoretical framework, it was seen four influences on teacher-textbook relationship as presented in Figure 3.

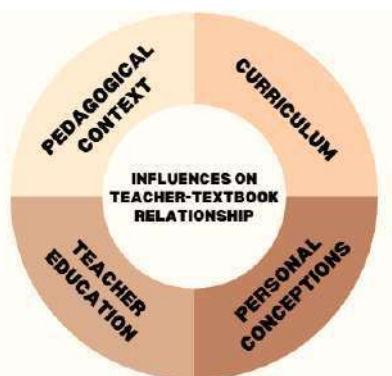


Figure 3: What can influence the teacher-textbook relationship.

It is important to highlight that in Brazil, the National Textbook and Teaching Material Program (PNLD) is managed by the federal government. This program is responsible for distributing textbooks to all students in Brazilian public schools, representing approximately 75% of the total student population, following a thorough evaluation process. Teachers collectively (sometimes the education network of a city or a state) select the textbook that best aligns with their school's political-pedagogical project.

However, due to the large-scale procurement of textbooks, negotiations with publishers often result in the acquisition of books that differ from the ones originally chosen. The books are currently purchased in four-year cycles: kindergarten, elementary school, middle school, and high school. Each year, one level is evaluated and acquired by the government, while the other levels receive only a few replacement copies, as the textbooks are meant to be reused.

Considering this Brazilian background, in the teacher's responses, it was evident how the **pedagogical context** influences the teacher-textbook relationship, considering both internal and external factors. External factors include public policies, as some teachers do not receive the textbooks they wanted to. Additionally, the non-consumable nature of textbooks in Brazil (for middle and high school) was highlighted, as T3 mentioned her difficulty in using them. Regarding internal factors, each teacher can consider their specific classes and students, adapting their use of the textbooks accordingly, if they think this is important for students, as T7 wrote.

T3: "[...] unfortunately, I can't use it [...] because the textbook is non-consumable."

T9: "[...] the choice of textbooks for schools is not an individual decision of each school, but a collective decision of the schools in the network I work in."

T7: "I consider the textbooks as teaching materials that assist in the [...] students' learning. [...] In the second case, it is sometimes the only source of consultation for the students."

In according to Aguiar and Oliveira (2017), the way teachers use textbooks is regulated by principles that already operate selectively in the pedagogical context, related to rules present in specific contexts that act in a regulatory manner on pedagogical practice. In this way, the relationship between teachers and textbooks can be seen in how they appropriate, select, transform, and position them in relation to the principles present in the school context, whether in their classroom, school, city, state, or country.

Besides that, teachers can see the textbooks as a **curriculum** that they have to follow. It was clear how these materials can help them to prepare their classes and organize themselves. So, it can influence the way their lessons and the sequence they adopt to giving classes. As Januario (2017) said, if textbooks have agency, they can act as curriculum inducers. And more teacher sees the affordance in these textbooks, as T1, more agency they have.

T1: "Fundamental material for preparing lessons, mainly for the structured way of presenting concepts."

T6: "It is very good for preparing the lesson."

T10: "They help a lot with the sequence of content topics."

Although, teachers have their own agency by incorporating their **personal conceptions** into how they use these textbooks, as their experience, knowledge, beliefs and values (Remillard, 2005; Januario, 2017). As observed from T10, he acknowledges the

affordances of the textbooks but also identifies certain issues that may impact the effectiveness of teaching the topics mentioned.

T10: “I think they are very good. [...] But I see some [...] points that I think are not given as much attention as they should, considering the importance of the topic.”

Lastly, a teacher considered how he learns from the textbook and its importance to **teacher education**. They can be used to recall certain contents or even to learn some content that the teacher may not be familiar with. Aguiar and Oliveira (2017) highlighted how these materials can serve as learning objective for teachers, not only for students. T7 is a good example of this.

T7: “I consider the textbook as teaching material that helps in teacher education [...] I learn with certain topics that are forgotten.”

Therefore, we considered these four conceptions, which can influence the teacher-textbook relationship from their perception of textbooks, as revealed by the data produced. This establishes essential context before exploring the teachers’ practical pedagogical approaches, while the research progresses.

FINAL CONSIDERATIONS

In conclusion, the analysis of the teachers' responses reveals that the teacher-textbook relationship is influenced by pedagogical context, curriculum, teacher education and personal conceptions. The factors exemplified included public policies, the non-consumable nature of textbooks, and the teachers’ individual context, such as their class dynamics and teaching perceptions. The findings underscore how textbooks can be perceived not only as teaching tools but also as curriculum guides that structure the lessons and content delivery.

However, teachers also have agency by adapting the textbooks to better meet the needs of their students and their own pedagogical approaches, since the textbook cannot and will not be 100% the way each teacher wants. This dynamic relationship highlights the complex role that textbooks play in the teaching process, both as instructional materials and as resources for teachers' own professional development. Since pedagogical practice cannot be separated from the real conditions in which it occurs, this encourages an approach that promotes meaningful learning for both teachers and students. There can be no dissociation between material resources and teacher resources, as these interact mutually in the relationship between materials and teachers (Remillard, 2005) and are directly influenced by the pedagogical context. These insights deepen our understanding of the teacher-textbook relationship, an area that still requires further exploration, and offer valuable considerations for improving textbook design and teacher education in the present and in the long term.

ACKNOWLEDGMENTS

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INTEGRATING INTERACTIVE DIGITAL ENVIRONMENTS INTO A HYPOTHETICAL LEARNING TRAJECTORY FOR UNIVERSITY STUDENTS' UNDERSTANDING OF TRIGONOMETRIC EQUATIONS

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Our research aims to determine how interactive digital environments, integrated into a hypothetical learning trajectory (HLT), facilitate the construction of trigonometric equations among university students. We conducted a design-based investigation with 90 first-year engineering students by implementing an HLT consisting of four learning tasks incorporating interactive digital environments. Data collection included written protocols, interaction logs in the digital environment, and interviews. The results show that the integrated digital environments in the HLT facilitated student learning by enabling (1) the dynamic visualization of trigonometric relationships, (2) the development of mathematical intuitions through the manipulation of representations, and (3) the immediate verification of conjectures.

INTRODUCTION

Trigonometric equations represent a complex area of university mathematics. The study by Castro and Cárcamo (2023) on first-year engineering students found that the main errors in solving trigonometric equations are related to the use of false trigonometric identities, incorrect images of trigonometric functions for known angles, poor handling of algebraic properties, and the erroneous representation of the solution set.

Interactive digital environments have emerged as tools that could help minimize these errors because they enable the dynamic visualization and manipulation of abstract mathematical concepts. Uygun (2020) notes that using interactive digital environments such as dynamic geometry software allows students to create and manipulate geometric objects through simple clicks and drags, facilitating the exploration of mathematical relationships and the formulation of conjectures. This capacity for direct interaction with mathematical representations can help students build connections between various forms of mathematical knowledge representation.

A hypothetical learning trajectory (HLT) provides a theoretical framework for designing learning sequences that effectively integrate these technological tools. Cuevas-Vallejo et al. (2023) demonstrated that systematically incorporating interactive digital environments into an HLT can transform abstract mathematical concepts into meaningful concrete experiences for students.

Within this context, our research aims to determine how interactive digital environments integrated into an HLT facilitate the construction of trigonometric

equations among university students. For designing these environments, we chose p5.js as a tool, given that it meets the characteristics described by Tran et al. (2017) by providing technologies that allow for the creation of dynamic visualizations and promote learning through direct interaction with mathematical representations. These features can enhance each learning task in the HLT we designed for this study.

THEORETICAL FRAMEWORK

The theoretical framework is based on the hypothetical learning trajectory (HLT) as a university teaching tool, and the emergent models design heuristics for creating the HLT and understanding the progression of learning trigonometric equations.

According to Simon (2020), an HLT is an instructional model comprising three components: a learning objective, learning tasks, and a hypothetical learning process. Uygun (2020) highlights that interactive digital environments in HLTs enable students to create and manipulate mathematical representations, thereby fostering more profound conceptual intuitions and connections. In the university context, the HLT becomes a powerful tool for designing effective teaching strategies. For example, the study by Cuevas-Vallejo et al. (2023) developed an HLT with interactive digital environments for teaching abstract concepts in Linear Algebra. These authors found that these environments helped achieve learning objectives by allowing students to transform abstract concepts into concrete experiences. Moreover, the simulation of real problems motivated the students.

The emergent models design heuristic proposes a learning process that allows students to start from known representations, gradually transform informal strategies into more formal mathematical reasoning, and construct a mathematical reality from their own experiences (Gravemeijer, 2020). To move from informal reasoning to more formal mathematical reasoning, Gravemeijer (1999) establishes four levels of activity (situational, referential, general, and formal), which guide the design of learning tasks and interactive digital environments to facilitate students' transition through these different levels of mathematical reasoning. In the context of trigonometric equations, the emergent models design heuristic makes it possible to design environments that activate prior knowledge, facilitate the manipulation of representations, and promote the autonomous construction of knowledge.

METHODOLOGY

Our research was conducted using a design-based research (DBR) methodology, which comprises three phases: preparation and design, teaching experiment, and retrospective analysis (Gravemeijer & van Eerde, 2009). During the preparation and design phase, we developed an HLT (Simon, 2020) on trigonometric equations for first-year engineering students, following the preliminary HLT model proposed by Cárcamo and Fuentealba (2023). The learning objective of this HLT is for students to solve trigonometric equations. For each of the four learning tasks in the HLT, we designed interactive digital environments (developed in p5.js) to facilitate the resolution of

trigonometric equations. Table 1 describes the learning tasks, their corresponding interactive digital environment, and the key levels in the hypothetical learning process (HLP) of the HLT focused on trigonometric equations.

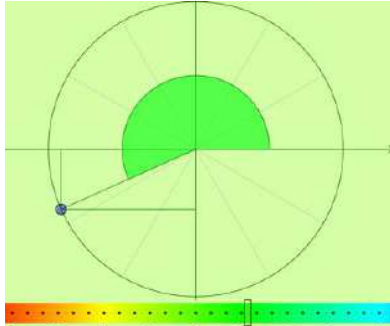
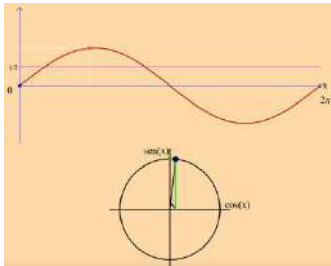
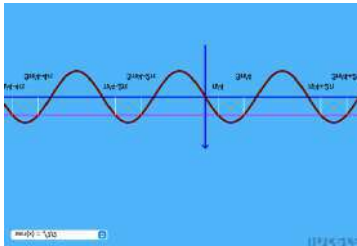
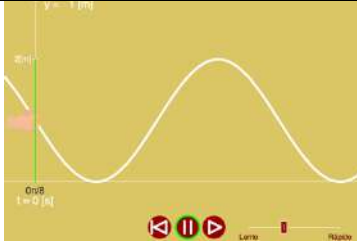
Description of the Learning Task	Interactive Digital Environment	Key Level in the HLP
Learning Task 1. A table with four columns is presented. The first column contains angle measures, while in the remaining columns, students are asked to complete the unit circle graph (column 2) and calculate the sine and cosine trigonometric ratios (columns 3 and 4).		The student identifies angles in all quadrants, compares results, and determines which angles are co-terminal.
Learning Task 2. Students are asked to solve four trigonometric equations and indicate their solution set in a bounded interval.		The student solves trigonometric equations in bounded intervals.
Learning Task 3. (3.1) Students are asked to solve a direct trigonometric equation by starting from its graph. (3.2) Students are then asked to solve a second-degree trigonometric equation.		The student solves trigonometric equations in unbounded intervals.
Learning Task 4. Students are asked to solve an applied problem that involves a trigonometric equation with infinitely many solutions. Precisely, they must determine when the rope of a CrossFit exercise—modeled by a trigonometric function—reaches a height of one meter.		The student solves a trigonometric equation with infinitely many solutions.

Table 1: Description of the learning tasks, their corresponding interactive digital environment, and the key levels of the hypothetical learning process in the HLT for trigonometric equations.

We conducted three iterative cycles during the teaching experiment phase. In this study, we focus on analyzing the third cycle. Ninety first-year engineering students

from a geometry course participated in this phase. Data collection included students' written protocols, screenshots of their interactions with the digital environment, and semi-structured individual interviews.

During the analysis phase, we sought to determine how interactive digital environments facilitated the construction of trigonometric equations through an inductive qualitative analysis. We identified emerging categories related to dynamic visualization, manipulation of representations, and conjecture verification, triangulating data from written protocols, interactions, and interviews. In this study, we present the data analysis results from the first three learning tasks in the HLT.

RESULTS

The results show that the interactive digital environments enhanced student learning by enabling them to (1) visualize trigonometric relationships dynamically, (2) develop mathematical intuitions through manipulation of representations, and (3) immediately verify their conjectures. To illustrate these findings, we present the case of student Peter, who exemplifies how interactive digital environments contributed to building trigonometric knowledge.

Dynamically Visualizing Trigonometric Relationships

The goal of Learning Task 1 in the HLT aligns with the situational level (Gravemeijer, 1999) because it aims for students to activate their prior conceptions about co-terminal angles and reference angles on the unit circle through a meaningful experience. In this case, it involves placing angles on the unit circle, determining the value of trigonometric functions at those angles, and indicating which quadrant they belong to. Peter demonstrated a dynamic visualization of trigonometric relationships when interacting with the digital environments designed for this task. For example, by moving the cursor along the unit circle (Figure 1), Peter explored angles intuitively, activating his prior conceptions of trigonometric concepts. For instance, by placing the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (Figure 1b) and identifying the angle $\frac{\pi}{6}$ (Figure 1c), Peter was able to visually experience the relationships between coordinates, position on the circle, and angular measure. Subsequently, when asked which quadrant each angle was located in, Peter repeated the process to determine the quadrant for each angle.

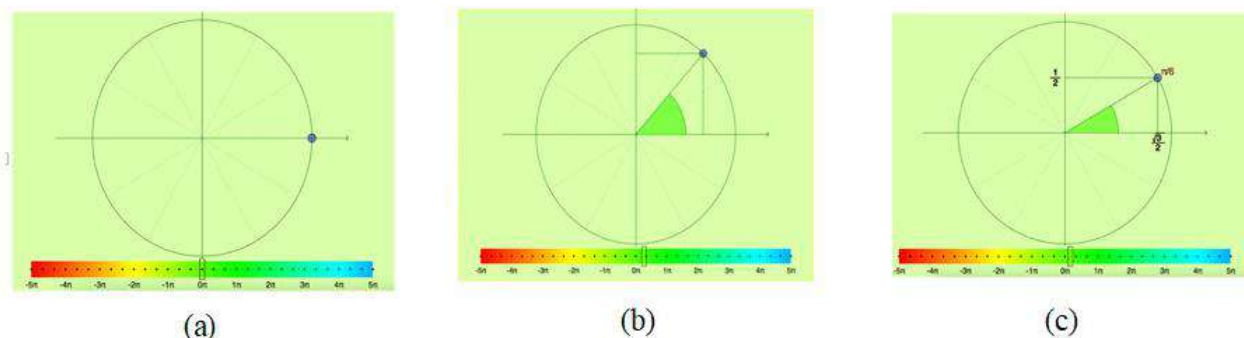


Figure 1: Screenshots of Peter's interactions with the interactive digital environment linked to Learning Task 1.

In addition, Peter established a relationship between trigonometric functions with the same result and the values of their angles. For example, when asked about co-terminal angles, he related the angle $-\frac{\pi}{3}$ and $\frac{5\pi}{3}$ by moving the cursor around the unit circle, indicating how dynamic visualization facilitated the understanding of abstract concepts. This interaction allowed him to begin constructing trigonometric equations from his own experience (a concrete and exploratory experience), which is a fundamental characteristic of the situational level (Gravemeijer, 1999).

Developing Mathematical Intuitions Through Manipulation of Graphical Representations

The goal of Learning Task 2 relates to the referential level (Gravemeijer, 1999) because students were expected to transition from their concrete experiences, visualizing and identifying angles on the unit circle (Figure 2a), toward procedures for solving trigonometric equations in bounded intervals. Peter developed mathematical intuitions when interacting with the digital environments designed for this task (representations of trigonometric functions). For example, to solve the equation $\sin x = \frac{1}{2}$ in the interval $[0, 2\pi[$, he used cursor movement on the unit circle (Figure 2a) and the graph of the sine function (Figure 2a–b), suggesting that manipulating multiple representations facilitates the development of mathematical intuitions.

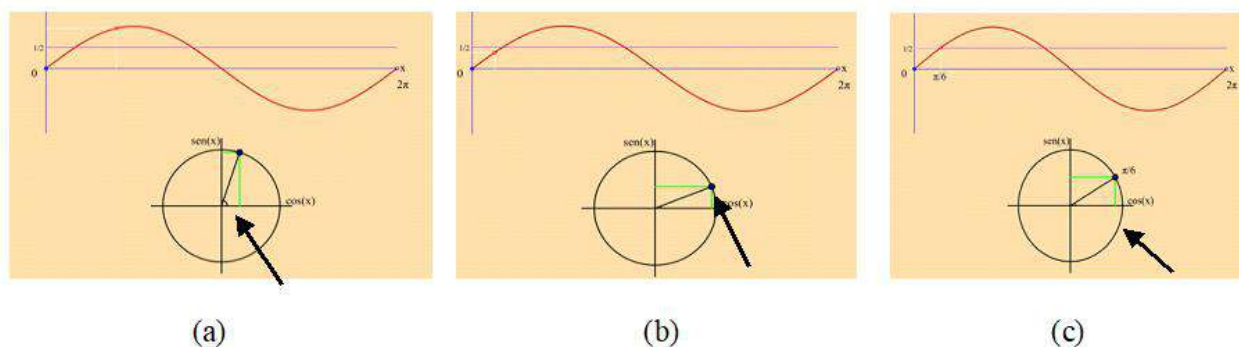


Figure 2: Screenshots of Peter's interactions with the interactive digital environment linked to Learning Task 2.

In particular, Peter provided evidence of mathematical intuition development by visually establishing connections between points on the unit circle and the corresponding values of the sine function. This connection allowed him to graphically verify the solutions in different quadrants and identify interactively the angles that satisfied the equation (Figure 2c). Peter correctly indicated that these angles were $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. He then solved the equation algebraically by substituting x with $\frac{\pi}{6}$. Consequently, substituting $\frac{\pi}{6}$ by $(\pi - \frac{\pi}{6})$ and writing the solution set in the requested

interval $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$. The manipulation of the interactive digital environments facilitated a visual and exploratory understanding that led to a mathematical resolution (writing the solution set of a trigonometric equation in a bounded interval), which characterizes the referential level in the emergent models heuristic.

Immediately Verifying Conjectures

The goal of Learning Task 3 aligns with the general level (Gravemeijer, 1999) because it aims for students to transition from solving trigonometric equations in bounded intervals to the generalization of infinite solutions using the period of trigonometric functions and multiple representations. Peter demonstrated immediate conjecture verification when interacting with the digital environments designed to solve trigonometric equations with infinitely many solutions. For example, Peter systematically explored different intervals of the equation $\sin(x) = \frac{\sqrt{2}}{2}$ by moving the cursor (Figure 3a–c), providing evidence of how dynamic interaction facilitated the visual verification of the solutions to this trigonometric equation.

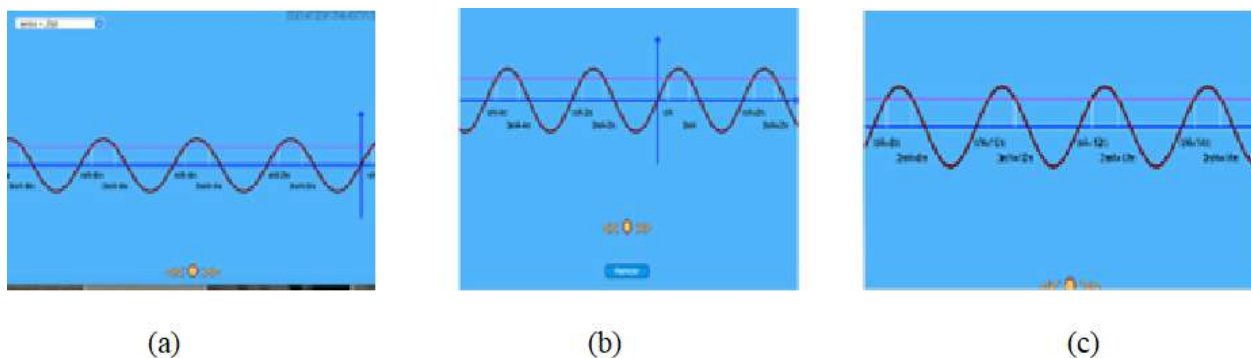


Figure 3: Screenshots of Peter’s interactions with the interactive digital environment linked to Learning Task 3.

For example, Peter systematically explored different intervals of the equation $\sin(x) = \frac{\sqrt{2}}{2}$ by moving the cursor (Figure 3a–c), providing evidence of how dynamic interaction facilitated the visual verification of the solutions to this trigonometric equation. When asked about his strategy for finding the infinite solutions to a trigonometric equation, Peter pointed out where the sine function graph intersects the line $y = \frac{\sqrt{2}}{2}$ and noted, “I was moving along the graph, as if it were between $[0, 2\pi]$. Then, when asked about negative angles, he said that by “moving the cursor to the left” (indicating the digital environment on his computer) (Figure 3a), he could see solutions in that direction. This immediate verification capability allowed him to validate his conjectures about the periodic behavior of the solutions and generalize the pattern to obtain the infinite solutions of a trigonometric equation. Thus, Peter wrote that the solution set of the equation was: $\left\{\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}\right\}$ indicating that the equation “has infinitely many solutions.”

The interactive digital environments facilitated Peter's transition from identifying solutions in bounded intervals (referential level) to understanding the solution set in unbounded intervals (general level). The ability to immediately verify his conjectures through dynamic visualization enabled Peter to develop a deeper understanding of the periodic nature of solutions to trigonometric equations.

CONCLUSIONS AND DISCUSSION

Our research objective was to determine how interactive digital environments integrated into an HLT facilitate the construction of trigonometric equations among university students. The findings provide evidence that these environments fostered learning through three aspects: dynamic visualization of trigonometric relationships, the development of mathematical intuitions via the manipulation of representations, and the immediate verification of conjectures.

The dynamic visualization of trigonometric relationships enabled students to understand the connections among different mathematical representations, confirming Uygun's (2020) argument that these environments facilitate mathematical exploration and conjecture development. This approach helped overcome previously identified difficulties by Castro and Cárcamo (2023) regarding the use of incorrect images of trigonometric functions. Moreover, developing mathematical intuitions through manipulating graphical representations led to more active and meaningful learning, aligned with Gravemeijer's (1999) emergent models heuristic. Students moved from concrete mathematical reasoning to more formal reasoning through situational, referential, and general levels. Additionally, the immediate verification of conjectures provided instant feedback, allowing them to validate their hypotheses and adjust their strategies.

The main contribution of our research is the integration of interactive digital environments into an HLT for teaching trigonometric equations at the university level. The results suggest that these environments can serve as inclusive teaching tools, enabling students to explore and understand complex mathematical content through multiple representations and forms of interaction. As a practical implication, we recommend incorporating activities based on graphical visualization and manipulation into the planning of trigonometry courses, considering level progression to facilitate the transition from informal to formal reasoning.

Additional information

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MECHANISMS FOR EARLY-CAREER TEACHER LEARNING IN AN ONLINE PROFESSIONAL LEARNING COMMUNITY

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Building on a communities of practice framework, this study explored the opportunities for early-career teacher (ECT) learning within a virtual professional learning community (vPLC). We examined the experiences of eight early-career teachers (PK-8) in a video-based, networked inquiry group focused on mathematics teaching through one school year. Data included annotated videos of teaching, individual and focus group interviews. Analysis explored the structures and practices that supported teachers to engage in pedagogically productive talk through asynchronous interactions. Findings revealed that shared language and common instructional vision facilitated these interactions. We also found that ECTs enacted agency related to their learning and leveraged their own expertise to support the learning of peers.

INTRODUCTION

Teaching practices that are responsive to student thinking are central to efforts to create mathematics classrooms that engage all students in deep mathematics learning (Horn & Garner, 2022). Because the development of responsive teaching takes time, support, and consistent practice, early-career teachers need opportunities for ongoing professional growth, beyond initial teacher preparation (Ebby et al., 2023). This paper explores one approach to supporting early-career teachers (ECTs) to develop and refine responsive teaching practices they learned in teacher education. We examine the potential of an asynchronous video-feedback inquiry community to support ongoing learning and collaborative dialogue across classroom and school boundaries through an analysis of eight ECTs participating in a virtual professional learning community (vPLC). Leaning on a “communities of practice” framework (Lave, 1991), we seek to address the following question: How does participation in a vPLC using cycles of video-based feedback support ECTs’ learning and growth towards their responsive teaching practices? Our findings provide insight into the process of teacher learning as well as mechanisms that support it.

THEORETICAL FRAMEWORK

We adopt a view of teacher learning as socially situated within a community of practice (Lave, 1991; Wenger, 2000). Communities of practice (CoPs) are comprised of members engaged in joint enterprises, involving shared goals and common repertoires of practice that are continually developed through collective activity. Learning is “a process of becoming a member of a sustained community of practice” (Lave, 1991, p. 85) through increased participation in this work over time. Wenger (2000) also emphasizes that learning in a CoP is “an interplay between social competence and personal experience,” and involves “a dynamic relationship between people and the

social learning systems in which they participate,” combining both “personal transformation with the evolution of social structures” (p. 227).

This perspective on learning is especially valuable for understanding teacher development within professional learning communities, as it highlights how teachers simultaneously refine their own practices while engaging collaboratively with their peers. Scholars who have applied a CoP lens in studies of teacher communities (e.g., Ebby et al., 2023; Horn et al., 2017; Lefstein et al., 2021) have illustrated how deliberating with others about instructional episodes and problems of practice, individual teachers contribute to the development of the group’s shared understandings. This negotiation of ideas within the community helps individual teachers internalize and transform these understandings for use both within the community and in their own teaching. Adopting a CoP lens when studying ECT learning through participation in an asynchronous video-based feedback community guides us to examine participants’ individual and collective learning.

LITERATURE REVIEW

Professional learning communities (PLCs) have the potential to play a large role in teacher learning, especially for ECTs (Munter & Wilhelm, 2021; Ebby et al., 2023). PLCs are typically built on principles of CoP, with groups of educators working collaboratively to engage in regular, systematic cycles of inquiry (Brodie, 2021). The potential of PLCs to effectively improve individual and collective capacities to teach in ways that deepen students’ mathematical understanding is dependent on multiple factors, including the depth of collective engagement and the degree to which participants are able to analyze instruction in connection with future practice (Horn et al., 2017). Lefstein and colleagues (2021) refer to this type of discourse within PLCs as *pedagogically productive talk*, which they define as “discourse that is rich in pedagogical concepts and reasoning that has the power to develop participating teachers’ professional judgment (p. 361).

One approach to prompting pedagogically productive conversations around instructional practice is the use of video (Borko et al., 2008). By slowing down the teaching process, videos allow for the decomposition of practice in ways that support the development of pedagogical reasoning. When used in a learning community as an artifact for discussion, “video can support collaborative learning focused on reflection, analysis, and consideration of alternative pedagogical strategies in the context of a shared common experience” (Borko et al., 2008, p. 419). Thus, PLCs designed around video artifacts of practice can provide important space for pedagogically productive talk.

ECTs have differing opportunities to engage in PLCs that foster continued learning of responsive teaching practices, depending on their new school environments (Condon, 2024). Networked or vPLCs can cross school boundaries, linking teachers across schools, allowing groups to form that share common learning experiences (Carpenter & Munshower, 2020). vPLCs can also leverage the availability of online

platforms that enable discourse around video artifacts, allowing participants to share their instructional practice and provide one another feedback (Ebby et al., 2023).

Taken together, these studies point to key opportunities for learning in PLCs and as well as potential of vPLCs as a way to cross boundaries and build communities around shared practice. However, very little research exists on understanding the dynamics of the individual and collective learning opportunities afforded to teachers within vPLCs.

METHODS

Inquiry Group Design

Building on prior work exploring novice teachers' take-up of responsive teaching practices introduced in their methods courses in their classroom settings (Ebby et al., 2023), we designed an inquiry group consisting primarily of asynchronous learning cycles using video records of practice and peer feedback. We recruited participants from two different graduate level teacher education programs at one university. All participants had experienced similar learning activities about responsive teaching practices and learned to facilitate number sense routines in their mathematics methods courses.

ECTs were grouped by grade level, and engaged in up to five inquiry group cycles, two in the Fall of 2022 and three in the Spring of 2023. For each cycle, participants were asked to: (1) post a video excerpt of a number sense routine (2) pose a focus question for peers to consider, (3) view and respond to two of their peer's videos, and (4) reflect on the feedback peers posted to their video.

Data Collection & Analysis

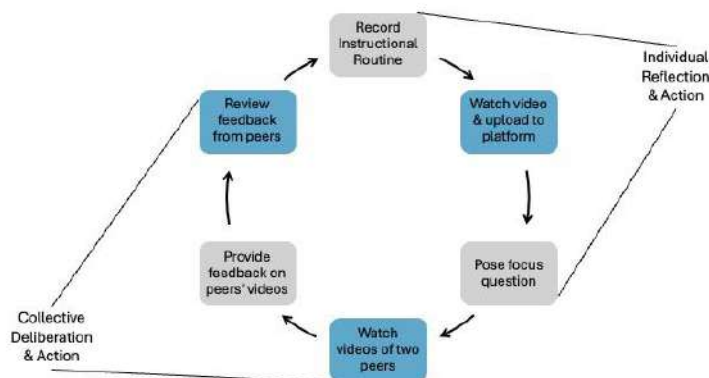
For this study, we analyze data from eight participants who took part in at least four inquiry cycles. Data include annotated video recordings of number talks and transcripts from participant interviews and focus groups. The videos were analyzed for the presence and nature of responsive teaching practices using a coding scheme developed for a previous project (Ebby et al., 2023). We focused primarily on three codes based on their structure of increasing depth (1-4): eliciting student thinking, orienting students to each other's thinking, and orienting students to mathematics concepts. Each video was coded independently by two researchers who then met to discuss and resolve any disagreements.

We analyzed the relative frequency of the codes, creating color-coded diagrams for each participant to examine shifts in teachers' practice. We then created analytical memos for each participant, tracking trends in their responsive teaching practices visible in their teaching, their focus questions, and the feedback that they provided on other teachers' videos. Analysis of interviews and focus group transcripts enabled us to examine the specific goals participants set for their own learning as well as how participants interpreted their experiences. We used this data to note connections between the video and interactional data recorded in the memos and to shed light on the role of inquiry group support in novice learning and growth towards their goals.

FINDINGS

Our analysis revealed how multiple inquiry cycles of reflection and action centered on videos of practice in the vPLC helped ECTs to internalize and enact responsive teaching practices on both individual and collective levels (see Figure 1). The gray boxes indicate elements of the inquiry cycle where participants had opportunities to take action, making visible their knowledge and vision of responsive teaching practices. The blue boxes indicate the elements of the inquiry cycle where participants were prompted to reflect and deliberate on problems of practice. Taken together, these cycles of reflection and action built around a shared repertoire of practice created consistent opportunities for pedagogically productive talk in the vPLC that promoted learning on both an individual and collective level.

Figure 1. Cycle of Reflection and Action in the v-PLC



Individual Learning

In our analysis, we found a correspondence between the goals that participants set for themselves and the responsive teaching practices visible in their actions. Specifically, the goals that they set for their enactment of number talks at the beginning of the inquiry group cycles used a shared language centered on responsive teaching practices. These goals guided their individual process of reflection and action throughout each inquiry cycle. For each participant, we found evidence of specific responsive teaching practices and/or a unique combination of practices that aligned with achieving that goal. In this section we use one case study of one participant to illustrate these patterns of growth.

Jana: Helping students to learn from one another

As a middle school math teacher, Jana sought to expand her instructional practice by helping students to access one another's strategies and build their own conceptual understanding, positioning them as mathematical experts and shifting their reliance away from her as the mathematical authority in the classroom. Jana was able to pursue this goal in distinct and interrelated ways through her participation in each inquiry cycle.

In her initial video, Jana focused primarily on eliciting questions that surface students' understanding of key mathematical ideas and highlighting those ideas to make them clear for students. For example, Jana often engaged in questioning with an individual student then extending questions to the class about the mathematics involved in the strategy: "How did you get from 4×8 to 8×16 ?" (E2) [*The teacher draws a visual representation to represent the student's thinking.*] "Ok, so they are equivalent ratios" (OM2).

Her focus question for this first number talk asked for insights into how to help students in making connections to each other's strategies and to the relationships between the problems they were solving. In subsequent number talks, she continued to seek advice about helping students to use a variety of strategies and see the mathematical connections across them, but with more focus on different elements of her own practice that could support that goal.

As the inquiry cycles progressed, Jana began to incorporate more facilitation that supported students in making connections across one another's strategies, aligning with her goal of students learning from one another rather than relying on her as a teacher. She also uses questioning techniques to orient students to the math rather than making statements telling the students the connection:

"Okay, so can someone connect what [Student 1] did to the drawings that I made (*gesturing to area models of 7×50 and 7×8*) (OI3) Where do we see the 350 in the standard algorithm (OM3)?"

Jana's progress toward these learning goals can also be seen in the opportunities that she had to view others' practice and provide feedback. Her goal of helping students to connect their strategies can be seen in the feedback that she gave to her peers. For example, when providing feedback during Cycle 1, she suggests: "I would repeatedly ask students to restate each other's thinking + connect what everyone says back to your goal (multiplication, in this case) if you want to really make those connections super clear."

Her feedback continues to progress throughout the cycles, as she also looks to support students in making more broad connections to each other's strategies and to the mathematical connections that underscore the number talk. For example, in Cycle 4, she suggests: "This could have been a good place to bring in one or a few other students to connect what this student is doing to the previous problem and/or restate the strategy so far." These examples of feedback from the initial round to a later round demonstrate the consistency of her focus on orienting students to each other's strategies and to mathematical goals. In her reflection following the final inquiry cycle, Jana described her experience of how participating in the IG helped her move towards this goal:

It helped me to find ways to get students to listen to their peers and learn from them rather than relying on me for knowledge, which I think is helpful, because

I got what I said at the very beginning like it reminds me of the value of having an accessible entry point for the math.

Collective Learning

According to interview and focus group data, the vPLC provided participants with multiple opportunities to leverage a shared knowledge base of responsive teaching practices. Across the individual and collective elements of the inquiry cycle, participants engaged in pedagogically productive asynchronous conversations (Lefstein et al., 2021) that supported their learning.

Watching peers' videos. According to almost all of the participants, their current school environments provided minimal space to observe other teachers' practice. As such, the vPLC gave them unique opportunities to engage in professional learning tied to instruction. The process of watching one another's videos helped to support collective deliberation around instructional dilemmas, giving ECTs space for reflection on their own practice as well as an opportunity to support one another. Becca, a second-year teacher, described her experiences of watching other teachers in the vPLC: "I enjoyed it in the sense that I was like. Oh, what would I have done in that situation? Well, how would I have handled it?" For all participants, having a window into others' practices, especially those who were also in their first years of teaching and had learned the same approaches to mathematics instruction was a key element of *why* they joined the vPLC and also a key mechanism in their learning.

Giving peers feedback. Beyond viewing and reflecting on videos of other teachers' practice, the process of giving feedback in response to a specific focusing question prompted collective deliberation and helped participants to leverage their own pedagogical reasoning. This process, in turn, also supported ECTs in making connections with their own practice while they decomposed others' practice. David, a math teacher in his third year explained his own learning that took place through the giving feedback:

I always struggle with reflecting on my own process and my own number talks, so kind of like that exercise of looking at someone else's and reflecting on it, and picking out where they're meeting their focus question, or what they can improve on, it always helps me like, think back to when I watch my videos to like go in like that bird-eye view of someone else looking in.

Within these asynchronous interactions, participants drew on their shared language of responsive teaching practices and talk moves that they learned during their methods courses in teacher education.

Receiving feedback from peers. Six of the eight participants in the study noted that they did not have opportunities to receive subject-specific feedback on their mathematics instruction. For Nick, a first-year teacher, receiving feedback from peers in the vPLC provided him with specific insights he could apply to his mathematics instruction, helping him to see where he could "be adding in different things, or

phrasing things a different way, or approaching a concept in a different way that might like help students stay engaged and also build those [mathematical] connections.”

Community Dynamics. Across all elements of the inquiry cycle, participants found community in their shared experiences both in teacher education and in their work as ECTs. Because of their shared experiences, they were able to be vulnerable with their practice while also being positioned as experts who can learn from one another. Jordan, a second-year teacher described the value he found in the vPLC:

It can bring in a little bit of spark that, just have me thinking about things differently, not getting complacent, and, like you know, being in in fellowship with a lot of other like-minded colleagues who are trying to figure out the same things, and are in the same position that I’m in really and there's that that sort of there's affinity in that.

DISCUSSION & CONCLUSION

Finding opportunities for professional learning of responsive mathematics teaching practices beyond preservice teacher education is a continuing problem for ECTs (Condon, 2024). Findings from this study demonstrate the importance of structuring a vPLC on principles of communities of practice. The experiences of the ECTs in this study highlight key aspects of the structure of the vPLC that afforded space for pedagogically productive talk.

First, participants’ shared background knowledge from their teacher education program served as an important factor in their ability to engage in collective deliberation. Though many of the participants were from different cohorts and had very little in-person interaction, they were able to learn from and with one another on the basis of their shared knowledge and vision of high-quality mathematics (Munter & Wilhelm, 2021). The process of engaging in pedagogically productive talk, in turn, facilitated a bridge between participants’ goals for their instruction and their current practice.

Further, the structure of the inquiry group was a key mechanism in facilitating effective learning opportunities. Within each cycle, the use of video, the requirement of providing a focusing question that centered on a problem of practice, and the ability to provide time-stamped feedback gave space for collective deliberation even though participants engaged asynchronously. Furthermore, the structure of the inquiry group positioned ECTs as having expertise that enabled them to provide feedback to others. This approach empowers ECTs to guide their own individual learning while simultaneously supporting others’ learning.

It is important to note that our findings are based on a relatively small sample size of participants in a vPLC that took place across a single academic year. Further studies are needed to determine if these inquiry cycles would continue to be as successful over longer periods of time, as well as how the size of the group or other

external factors such as school-based requirements and place-based professional learning shapes teachers' learning and practice.

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THE HYPOTHETICAL LEARNING TRAJECTORY AS A THEORETICAL LENS FOR PROFESSIONAL NOTICING DEVELOPMENT: FROM EMERGING TO MIXED PROFILE

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In this study, we characterize the development of the professional noticing of a prospective mathematics teacher using a hypothetical learning trajectory for teaching geometric translation. In the theoretical framework, we use professional noticing and the hypothetical learning trajectory. Through two cycles of a teaching experiment, we analyze attention to mathematical thinking and characterize the skills of attending, interpreting, and deciding. The results show an evolution from an emerging profile to a mixed profile and indicate that the hypothetical learning trajectory influences professional noticing development by providing a structured framework for interpreting mathematical thinking and justifying instructional decisions.

INTRODUCTION

Professional development is crucial for improving the teaching and learning of mathematics, especially in contexts that require attention to students' mathematical thinking. The teaching competency known as Professional Noticing (PN) defines how teachers attend, interpret, and make decisions based on students' understanding (Jacobs et al., 2022). This competency requires linking theoretical knowledge with classroom practices, necessitating specific tools to support its development.

Llinares (2019) points out that training programs can use representations of practice as tools for teacher learning. In this regard, the Hypothetical Learning Trajectory (HLT) stands out as a theoretical lens that guides attention to students' mathematical thinking and supports the skills of attending, interpreting, and deciding (Fernández & Choy, 2020). Research has shown that the HLT enhances teachers' ability to address mathematical thinking within a specific theoretical framework (Dindyal et al., 2021). In this study, we characterize the development of PN in a prospective mathematics teacher using an HLT for teaching translation (HLTt). We analyze classroom interventions and demonstrate how theoretical artifacts facilitate transitioning from an emerging profile to a mixed profile in the three PN skills (Jacobs et al., 2022). Additionally, the methodological approach documents these changes and provides empirical evidence supporting this transition.

THEORETICAL APPROACHES

Teacher Competence in Professional Noticing of Children's Mathematical Thinking

Jacobs et al. (2010) define professional noticing of children's mathematical thinking as a set of interrelated skills: attending to students' strategies, interpreting their understanding, and deciding how to respond based on these interpretations. Attending involves recognizing the mathematical knowledge in students' strategies within

specific contexts. Interpreting relies on reasoning about these strategies and the development of mathematical thinking. Deciding refers to formulating responses based on interpreting students' mathematical understanding.

Jacobs et al. (2022) expand the theoretical framework on professional noticing by introducing profiles that characterize different levels of professional competence development. These authors identify three profiles: emerging, mixed, and accomplished. The emerging profile describes teachers who demonstrate limited skills across the three areas. These teachers struggle to identify relevant details in students' strategies and to interpret their reasoning coherently. The mixed profile refers to teachers who exhibit progress in one or two skills, such as attending to students' strategies or interpreting their mathematical thinking but find it challenging to articulate pedagogical responses. Finally, the accomplished profile corresponds to teachers who have developed a balance across the three skills. These teachers can identify details in students' strategies, interpret their reasoning, and decide on responses that support learning.

Hypothetical Learning Trajectory for Teaching Translation

According to Simon (2020), the Hypothetical Learning Trajectory (HLT) is a theoretical model for planning mathematics instruction. The HLT consists of three components: the learning goal, the learning tasks, and the hypothetical learning process. In this study, we focus on how a HLTt supports the development of professional noticing in prospective mathematics teachers. The hypothetical learning process of the HLTt is based on the heuristic of emergent models (Gravemeijer, 2020), which provides a structured framework to guide students from intuitive models to formal models of translation through four levels of mathematical activity: situational, referential, general, and formal.

The learning goal of the HLTt is that students to recognize the translation of two-dimensional figures in design contexts. The learning tasks and the hypothetical learning process are integrated into a dynamic geometry environment that incorporates guided dragging to support the recognition of invariant geometric properties of translation (Cumbal et al, 2024). Table 1 presents the levels of the hypothetical learning process of the HLTt and their associated learning tasks.

Task 1A-1B Level 1	Task 2A-2D Level 2	Task 3A-3F Level 3	Task 4A-4B Level 4
The student identifies the visual characteristics of translated figures. The figures related by translation maintain their shape, size, inclination, and	The student verifies the conservation of size and shape in translated figures, the preservation of inclination and orientation of the final figure	The student recognizes that the coincidence of vertices and corresponding segments through a straight-line slide determines the translated figure's conservation of orientation, shape, and	The student recognizes the fundamental property of translation: the vertices A, A', B, and B' of the initial and final

orientation. Additionally, the student identifies visual characteristics of non-translated figures, which exhibit orientations, shape, inclination, or orientation changes.	relative to the initial figure. Furthermore, the student recognizes the length, representing the sliding distance from the initial figure to the final figure, depicted by a segment.	inclination. Subsequently, the student configures the three elements of a translation—inclination, orientation, and sliding distance—within an oriented segment. Finally, the student identifies the construction of the final figure.	figures form a parallelogram. Additionally, the student defines translation as a geometric configuration within a design.
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Table 1: Levels of the hypothetical learning process of the HLTt.

Fernández and Choy (2017) explain that HLTs act as a theoretical lens to develop the skills of professional noticing of students' mathematical thinking. This allows connecting students' mathematical expression with specific learning goals and provides a framework for teachers to make informed decisions on responding to classroom situations.

METHOD

Participants and Data Collection

In this study, the participant was a prospective mathematics teacher (hereinafter Sofi) completing her professional practice in a public educational institution. For data collection, Sofi participated in two cycles of a teaching experiment where she used the HLTt to plan lessons, implement class sessions, and carry out retrospective analyses. The experiment was conducted with sixth-grade students aged between 11 and 12 years (designated as S1, S2...). Sofi's classes were video recorded. Additionally, a semi-structured interview was conducted at the end of each cycle. Sofi and her students participated in five class sessions during the first cycle and six sessions in the second cycle, each lasting approximately one hour.

Analysis

In this study, we analyzed 34 teaching segments from cycle 1 and 91 from cycle 2, which were identified through transcripts of classroom interactions and a semi-structured interview. Teaching segments were defined as units of teacher activity focused on students' mathematical expressions, with conceptual breaks in the dialogue serving as the basis for segment delineation (Jacobs & Morita, 2002). These conceptual breaks were interpreted as moments where students' mathematical ideas were modified.

We coded the teaching segments based on Sofi's consistent actions and grouped them into 15 analytical categories, which allowed for an examination of the mathematical elements Sofi attended to, her interpretations of students' mathematical ideas, and the

decisions she made based on those interpretations. The categories were linked to her knowledge of the HLTt, evidencing how she applied the theoretical framework.

The analysis evidenced the evolution of six categories across cycles: *Ask for an explanation (C1)*, aimed at having students express their explanations about a geometric phenomenon; *Ask for precision (C3)*, focused on eliciting detailed responses from students; *Ask for verification (C4)*, aimed at prompting students to confirm the accuracy of information; *Ask for a description (C6)*, intended for students to describe what they observe in a given situation; *Ask for confirmation (C7)*, which seeks validation of previously presented information; and *instruct to guide (C10)*, providing step-by-step instructions for following a process. For example, category C1 transitioned from questions centered on verifying mathematical ideas (cycle 1) to questions directed at executing procedures (cycle 2).

The three PN skills were defined about the HLTt: attending focused on identifying the mathematical elements expressed by students; interpreting involved relating these elements to students' needs or misconceptions, and deciding entailed making decisions based on these interpretations to facilitate understanding of translation. To analyze the skills in each cycle, we included the description of the elements that Sofi attended to, her interpretations of misconceptions, and her decisions. We complemented this analysis with interview excerpts where Sofi justified her interpretations and decisions, focusing on those that allowed a contrast between action and reflection when using HLTt knowledge.

RESULTS

Sofi's professional noticing evolved from an emerging profile to a mixed one. This evolution was characterized by gradually developing her skills to *attend*, *interpret*, and *decide* on the student's mathematical thinking.

Sofi's Emerging Profile

For the component skill of attending, Sofi paid limited attention to mathematical elements. Specifically, she focused on size, shape, and inclination. However, more complex mathematical elements, such as direction, length of the oriented segment, and the construction of the final figure, were considered superficially. For example, Sofi worked on *Task 1B*, where students were required to observe non-rectilinear slides to identify changes in *size*, *direction*, and *inclination*. In teaching segment 57, Sofi initiated the interaction by asking, “*What do you remember from the tasks we worked on?*” Some students responded with “*distance*” and “*inclination*”. Then, Sofi asked, “*And the figures that preserved the size, were they the same?*” S2 responded, “*Yes.*” and Sofi continued, “*And the shape?*” This interaction showed that Sofi superficially attended to the conservation of size and shape without considering the student's expression regarding the length of the oriented segment and inclination. The mathematical elements identified by Sofi corresponded to levels 1 and 2 of the HLTt. For the component skill of interpreting, Sofi made imprecise interpretations when students expressed confusion or a lack of clarity. For instance, in teaching segment 58,

Sofi worked on Task 2A, where the student had to determine which drawings represented translations by sliding the initial figure and verifying the coincidence of the vertices. S2 asked, *“Which is the figure that is the same? These two, right?”* referring to the initial and final figures in Drawing 1, representing an example of translation. Sofi interpreted this confusion and responded with an exploratory question: *“Do you think they are the same?”* Similarly, during Task 2D, Sofi interpreted that the student needed to clarify the conservation of inclination. In segment 72, Sofi asked, *“Does the figure stay on the straight line or not?”* After S2 stated, *“It goes up.”*

Although Sofi’s interventions confirmed the student’s confusion, she interpreted it as a need for confirmation without addressing the meaning of *“being the same”* in terms of conservation of size and shape. Her attempt to extend the student’s expression with the question *“It does not stay, so the vertices will not coincide because...”* did not show a clear interpretation. Sofi’s intervention reflected partial use of the HLTt at levels 1 and 2, where visual identification and verification through guided dragging were used to check for conservation of size, shape, and inclination. However, her questions, such as *“Do you think they are the same?”* and *“Does it stay on the same line or not?”* were limited to visual recognition.

For the component skill of deciding, Sofi chose to ask questions that led the student to confirm the conservation of size and shape. For example, in teaching segment 59, upon interpreting the student’s confusion—when the student stated that the figures were not the same—Sofi decided to revisit the task instructions: *“In each drawing, slide the initial figure to superimpose it,”* and clarified the term *“superimpose.”* Although Sofi’s decision to ask questions and confirm allowed the student to complete the task, her intervention did not fully recognize the conservation of *shape, size, or inclination*, limiting the student’s expression to affirmations because she used closed-ended questions. Faced with confusion about whether the figures were the same, Sofi revisited the task instructions and explained *“superimpose,”* clarifying, *“Superimpose means placing one on top of the other.”* which led the student to conclude, *“Yes, teacher, they are identical.”*

Similarly, in teaching segment 72, Sofi decided to ask the student to clarify the conservation of *inclination*. Upon interpreting S2’s response, who stated that the figure *“goes up,”* Sofi confirmed the observation and asked whether the figure remained on the same line, to which the student S2 responded negatively. Sofi decided to confirm the response and introduced the coincidence of vertices as a verification criterion. However, her intervention focused on the observation and visual confirmation of the inclination. Later, after the student’s statement that the figure *“goes up,”* Sofi asked, *“Does the figure remain on the same line?”* and reinforced the verification with *“It does not stay, so the vertices will not coincide because...”* without leading the student to recognize the conservation of *inclination*. These decisions reflected partial use of the HLTt framework. Although Sofi attempted to focus on what the student observed, she did not use dragging, as proposed in levels 1 and 2 of the progression.

Perfil mixto de Sofi

For the component skill of attending, Sofi identified mathematical elements such as the conservation of size and shape (already present in the emerging profile), the length of the oriented segment, certain aspects of inclination, and the configuration of the oriented segment. However, other elements, such as the construction of the final figure and the direction on of the translation, were either nonexistent or poorly developed. This progress was evidenced, for example, during Task 2B (teaching segment 214) when Sofi guided a student to identify the segment corresponding to each translation, emphasizing, *“I have to check that the size here is the same... how should these three sliders be?”* S1 responded, *“equal”*, and Sofi indicated, *“Adjust them so that they are equal.”* This intervention evidenced that Sofi attended to the length of the oriented segment, which represented progress in identifying mathematical elements, as she incorporated elements corresponding to levels 2 and 3 of the HLTt.

For the component skill of interpreting, Sofi identified imprecisions and recognized when students needed guidance in their procedures. In teaching segment 216, during Task 3B, she supported S1 in calculating the translation length and confirmed the value: *“Five point sixty-six, right?”* In teaching segment 222, during Task 3C, she guided S2 in configuring the oriented segment by suggesting, *“Set one horizontally and three vertically, do it again.”* Finally, in teaching segment 226, during Task 3E, she helped S3 recognize the components of the oriented segment by asking, *“What three concepts does the oriented segment have?”* and S3 responded, *“Inclination, length, and direction.”*

Sofi’s interpretations were sometimes unclear, as she occasionally limited herself to confirming answers, such as in the case of the length or adjusting the values of inclination. However, compared to her emerging profile, these interventions evidenced progress by using procedures to identify the oriented segment and inclination elements. This progress corresponded to level 3, which involves configuring inclination, measuring, and constructing the oriented segment—processes suggesting progress using the HLTt.

For the component skill of deciding, Sofi decided to focus on guiding through procedures and on-screen observation to verify those procedures. For example, in teaching segment 216, Sofi guided the student in measuring the length of the translation when S1 asked, *“How long is the segment?”* Sofi responded, *“Five points sixty-six, right?”* and added, *“If I move it over here, it should measure the same.”* The student S1 commented, *“And if I put it down here?”* and Sofi reaffirmed, *“It should also measure the same.”* These decisions showed how Sofi interpreted that the student needed to verify the measurement and avoid errors in the process, reinforcing the proper use of the ruler. In teaching segment 222, Sofi decided to guide the adjustment of the oriented segment when S2 said, *“A little more here, right?”* while moving the sliders. Sofi instructed, *“Set one horizontally and three vertically; do it again.”* S2 asked, *“Is it okay like this?”*. Sofi responded, *“Click translate and check if it matches.”*

The student S2 added, “*Now it is correct,*”. Sofi concluded, “*When you do it again, remember to count the units.*” This decision demonstrated how Sofi interpreted S2’s difficulty and directed the procedure until the correct adjustment was completed, ensuring that S2 carried out the process.

These decisions allowed the students to slide the figures and execute procedures such as measuring and configuring the oriented segment. However, they did not foster explanations that contribute to understanding the translation. In this sense, the use of the HLTt framework was limited to level 3, focusing on applying procedures and verifying results without advancing toward a conceptual understanding that relates length, inclination, and direction. Sofi partially achieved the recognition of these elements according to level 3 of the HLTt. However, her interventions did not promote articulating these characteristics into an integrated translation model.

DISCUSSION AND CONCLUSIONS

In this study, we characterized the development of PN of students' mathematical thinking in a prospective mathematics teacher using the HLTt. The results show that Sofi evolved from an emerging profile to a mixed profile in the three skills of PN related to students' mathematical thinking, illustrating how the HLTt supported this transition. The HLTt acted as a tool that supported the development of professional noticing by providing a structured framework to make sophisticated mathematical elements visible, a professional language to interpret students' mathematical thinking, and a structure for making pedagogical decisions.

Regarding the skill of identifying, Sofi, using the HLT, progressed from attending to essential elements (size and shape) to identifying more sophisticated mathematical elements (length of the slide and inclination). This aligns with Jacobs et al. (2010) and Fernández and Choy (2017), who highlight that the HLT acts as a framework to guide attention toward relevant mathematical elements. Meanwhile, in the skill of interpreting, Sofi progressed toward establishing connections between more complex mathematical elements, relating properties such as the conservation of inclination to the oriented segment. This finding is consistent with what Jacobs and Empson (2016) propose, stating that interpreting involves relating properties to understand students' mathematical thinking. In the skill of deciding, Sofi transitioned from immediate instructional decisions to decisions that trigger procedures for recognizing geometric characteristics. This is consistent with Amador (2020) and Jacobs and Empson (2016), who emphasize that decisions must dynamically adjust in real time to guide conceptual development and connect broader mathematical principles.

This study contributes to understanding how theoretical tools like the HLTt can support the development of professional noticing in prospective mathematics teachers. While Sofi's evolution from emerging to mixed profile demonstrates the potential of using structured frameworks, her not reaching an accomplished profile suggests that HLTs, while effective for identifying mathematical elements and interpreting student thinking, should be complemented with additional pedagogical tools and reflection opportunities to fully develop sophisticated instructional decision-making. These

findings have important implications for mathematics teacher educators in designing comprehensive approaches to support teachers' professional noticing development.

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PREDICTIVE RELATIONSHIPS BETWEEN CONCEPTUAL AND PROCEDURAL KNOWLEDGE ACROSS THE DOMAINS OF FRACTIONS AND ALGEBRA

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Relationships between conceptual and procedural knowledge within a domain have long been studied. However, less is known about how conceptual and procedural knowledge in one domain relates to conceptual and procedural knowledge in another. Because research shows that fraction knowledge predicts algebra knowledge, we investigated this question in these domains in a sample of 357 middle-school students, using a longitudinal design. Conceptual fraction knowledge at time point 1 predicted conceptual but not procedural algebra knowledge at time point 2. Procedural fraction knowledge predicted procedural but not conceptual algebra knowledge. The findings underscore the importance of conceptual and procedural fraction knowledge for algebra learning.

INTRODUCTION

Conceptual knowledge refers to the comprehension of mathematical facts, basic concepts or principles, and their interrelationship within a particular domain (Baroody et al., 2007; Hiebert & Lefevre, 1986). A *procedure* is a “step-by-step instruction that prescribes how to complete a task” (Hiebert & Lefevre, 1986, p. 6). *Procedural knowledge* traditionally refers to “the ability to execute action sequences to solve problems” (Rittle-Johnson & Schneider, 2015, p. 6). It includes knowledge about rules, strategies, and algorithms to complete a task. Both educational psychologists and mathematics educators agree that developing proficiency within a domain requires conceptual and procedural knowledge (e.g., Byrnes & Wasik, 1991; Hiebert & Lefevre, 1986).

Research on conceptual and procedural knowledge has been a longstanding topic of interest and debate (e.g., Castro et al., 2016; Rittle-Johnson & Schneider, 2015, for reviews). In the past, research and debate on this topic have focused on how prior conceptual and procedural knowledge in a domain (e.g., equation-solving) predicts subsequent conceptual and procedural knowledge in the same domain (e.g., Castro et al., 2016; Rittle-Johnson & Schneider, 2015, for reviews). The prevailing view is that prior conceptual and procedural knowledge iteratively predict both subsequent conceptual and procedural knowledge (Rittle-Johnson et al., 2001). However, the extent to which conceptual and procedural knowledge in one domain are relevant prior knowledge for learning concepts and procedures in another domain is unclear. Answers to this question can inform discussion on knowledge transfer across domains and how

educational practice can leverage prior knowledge effectively. Fractions and algebra are two mathematical domains well-suited to addressing this question.

Relationships between fraction and algebra knowledge

Fractions and algebra are fundamental cornerstones of students' mathematical education (e.g., Common Core Standards Initiative, 2023). However, both are notoriously challenging for students (e.g., Booth et al., 2017; Lortie-Forgues et al., 2015). At the same time, algebra is often considered a « gatekeeper » to success in high school, postsecondary education, and career paths in STEM fields (e.g., Chen, 2013). Knowledge of fractions, on the other hand, is considered important prior knowledge for learning algebra (e.g., Bush & Karp, 2013; Empson et al., 2011).

Supporting this view, numerous studies found strong predictive relationships between students' fraction and algebra knowledge, even after controlling for several potential confounds (e.g., Barbieri et al., 2021; Booth et al., 2014; Liang et al., 2018; Siegler et al., 2012). Despite this strong evidence, the exact nature of the relationship between fraction and algebra knowledge remains somewhat unclear (DeWolf et al., 2015). One reason is that previous studies often treated knowledge of each domain as unitary constructs. That is, they used single-measure outcome variables (Viegut et al., 2024) or focused on specific knowledge facets like fraction magnitude understanding (e.g., Booth et al., 2014). However, both knowledge of fractions and algebra are heterogeneous constructs involving multiple concepts and procedures.

Conceptual and procedural knowledge of fractions and algebra

Conceptual and procedural knowledge of fractions has been conceptualized in the past (for an overview, see Lenz et al. (2019)). Central aspects of conceptual fraction knowledge involve understanding various interpretations of fractions, such as understanding fractions as representations of part-whole relationships or as numbers with magnitudes that can be compared and ordered on a number line. It also includes recognizing how fractions differ from integers, for example, regarding their density property: Unlike integers, there is always another fraction between any two fractions. Procedural fraction knowledge refers to the ability to perform fraction arithmetic (addition, subtraction, multiplication, and division) and operations like simplifying or expanding fractions (e.g., $18/24 = 6/8 = 3/4$ and vice versa).

In contrast to fractions, algebra is a much broader field of mathematics. The present study focuses on (linear) equation-solving as students typically continue with linear equation-solving after being instructed in fractions. Here, conceptual algebra knowledge includes understanding fundamental concepts such as variables, equivalence, and equations. Procedural algebra knowledge includes skills like simplifying algebraic expressions, calculating values of algebraic expressions for given variables, and solving multi-step linear equations.

Separately conducted studies in fractions and algebra show that conceptual and procedural knowledge in both domains are partly distinct latent constructs rather than

components of a single unified latent factor (e.g., Lenz et al., 2019; Schneider et al., 2011). Together with findings that students' fraction knowledge predicts their algebra knowledge, this raises the question of which type of fraction knowledge predicts which type of algebra knowledge (Hurst & Cordes, 2018). Consequently, the present longitudinal study investigates how conceptual and procedural knowledge of fractions predicts conceptual and procedural knowledge of algebra.

METHODS

Sample and procedure

Participants were 357 students from academic-track secondary schools in Germany (178 girls, 179 boys). Their average age was 12.40 years ($SD = 0.56$). Before data collection began, the study received approval from an institutional review board (reference: Ethics Committee of the University of Trier, application no. 16/2020) and the local school authorities (reference: IV.7-BO5106/246/11). Informed parental consent and student assent were obtained from all participants. Data were collected in whole-class settings over two 45-minute sessions using paper-and-pencil tests. Conceptual and procedural fraction knowledge were assessed after fraction but before algebra instruction. After administering the fractions test, students received standard algebra instruction on variables, algebraic expressions, equivalence, and linear equations over three to four months. Conceptual and procedural algebra knowledge were assessed after algebra instruction had ended.

Measures

In fractions and algebra, we measured three central facets of conceptual and procedural knowledge, respectively. For conceptual fraction knowledge, the facets included understanding *fraction magnitude*, *fraction density*, and *fractions as representations of part-whole relationships*. For procedural fraction knowledge, the facets were *fraction addition and subtraction*, *fraction multiplication and division*, and the ability to *expand and simplify fractions*. For conceptual algebra knowledge, the facets included understanding *variables*, *equivalence*, and *equations*. Procedural algebra knowledge facets were *calculating values of algebraic expressions*, *simplifying algebraic expressions*, and *solving multi-step linear equations*. Each facet was measured with four to seven items.

We asked two external and independent experts in the field of conceptual and procedural knowledge to judge for each of the items we used whether the item was suitable for assessing conceptual or procedural knowledge. We only used items for which both experts agreed that they were appropriate for measuring conceptual or procedural knowledge. Sample items for conceptual and procedural knowledge in each domain are given in Table 1.

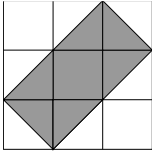
Knowledge facet	Sample item
<i>Conceptual fraction knowledge</i> ($\alpha = .77$; 15 items)	Which portion of the white square is colored gray? Write your answer as a fraction. 
<i>Procedural fraction knowledge</i> ($\alpha = .77$; 16 items)	Calculate: $1/3 + 1/4 + 1/5 =$
<i>Conceptual algebra knowledge</i> ($\alpha = .62$; 16 items)	Circle all <i>linear</i> equations: a) $8x - 4$ b) $(x - 3) - x \cdot (x + 1) = 9$ c) $(x + 1)^2 = 0$ d) $x - 3 = 2$
<i>Procedural algebra knowledge</i> ($\alpha = .74$; 12 items)	Solve for x : $2x - (x - 4) - 9 = -4$

Table 1. Sample items.

Data analysis

We modeled conceptual and procedural knowledge in both domains as latent factors. The three facets per knowledge type were used as indicator variables for the respective latent factors. Indicator variables were operationalized as sum scores over the four to seven items. For example, a sum score was computed for the five items used to measure students' ability to add and subtract fractions. This sum score was used as an indicator variable for the latent factor *procedural fraction knowledge*. To investigate predictive relations, we defined a cross-lagged model with *conceptual* and *procedural fraction knowledge* as exogenous latent factors and *conceptual* and *procedural algebra knowledge* as endogenous latent factors (see Figure 1). Robust maximum-likelihood estimation was used to account for non-normal distributions. Missing values were handled using full-information maximum likelihood estimation.

RESULTS

Preliminary analysis

To ensure that conceptual and procedural knowledge in both domains are partly distinguishable latent factors, we defined a two-factor model with latent factors for conceptual and procedural knowledge and a one-factor model with a single latent factor and compared the two models' fit (Table 2).

Global fit indices suggest that the two-factor models fit the data better than the one-factor models. Significant χ^2 -difference tests confirmed this, showing that the two-factor models provided a significantly better model fit than the one-factor models (fractions: $\chi^2(1) = 28.17, p < .001$; algebra: $\chi^2(1) = 26.96, p < .001$).

χ^2	df	CFI	TLI	RMSEA	SRMR	AIC	BIC
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<i>Fractions</i>								
2-factor model	23.6**	8	.960	.926	.080	.037	12	84
1-factor model	51.6***	9	.891	.818	.126	.056	39	108
<i>Algebra</i>								
2-factor model	16.4**	8	.964	.933	.061	.044	-229	-158
1-factor model	38.6***	9	.882	.803	.105	.062	-209	-142

Table 2. Model fit indices comparing two- and one-factor models in fractions and algebra to assess empirical separability for conceptual and procedural knowledge.

Predictive relationships between conceptual and procedural knowledge of fractions and algebra

The estimated cross-lagged model (see Figure 1) provided a reasonable model fit ($\chi^2(48) = 93.84, p < .001$; CFI = .936; TLI = .912; RMSEA = .061, 90% CI: [.042, .079]; SRMR = .058). Standardized factor loadings ranged between .32 and .84 and were statistically significant (all $ps < .001$). The explained variance for conceptual and procedural algebra knowledge were $R^2 = .71$ and $R^2 = .38$, respectively.

Conceptual fraction knowledge predicted conceptual algebra knowledge ($\beta = .55^{**}$) but not procedural algebra knowledge ($\beta = .03, p = .882$). Procedural fraction knowledge predicted procedural algebra knowledge ($\beta = .60^{**}$) but not conceptual algebra knowledge ($\beta = .35, p = .066$).

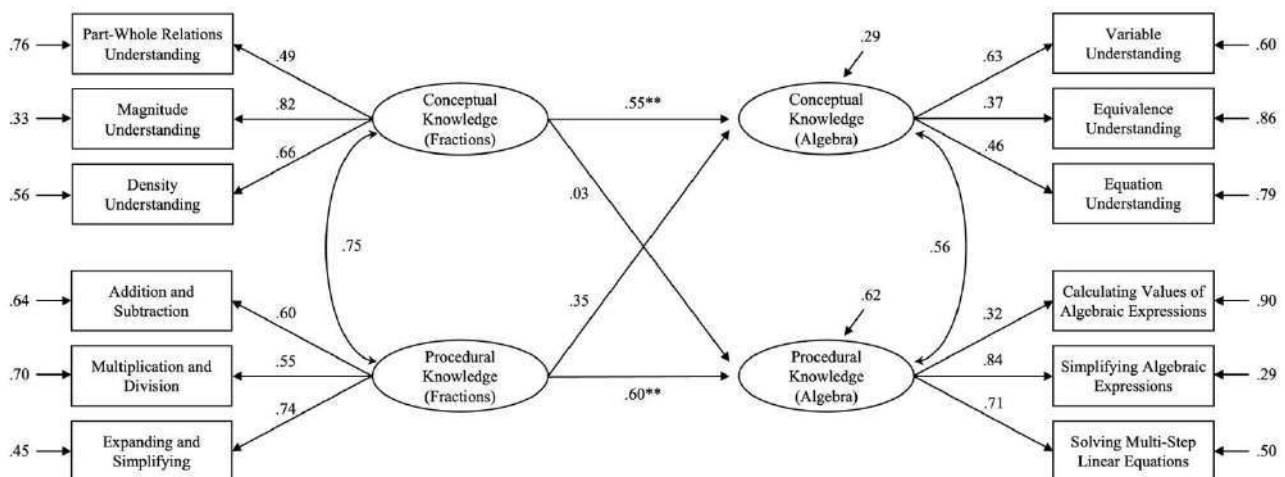


Figure 1. Predictive relationships between conceptual and procedural knowledge across the domains of fractions and algebra.

DISCUSSION

The present study underscores that fraction knowledge is important prior knowledge for algebra learning. Going beyond previous results, which often used aggregated

scores for fraction or algebra knowledge, we investigated to what extent conceptual and procedural fraction knowledge are relevant prior knowledge for conceptual and procedural algebra knowledge. Conceptual fraction knowledge predicted conceptual but not procedural algebra knowledge. Procedural fraction knowledge predicted procedural but not conceptual algebra knowledge. The results highlight that not only procedural but also conceptual knowledge of fractions is important prior knowledge for algebra.

Implications

The findings contribute to discussions on the measurability of conceptual and procedural knowledge in mathematics. While measuring conceptual and procedural knowledge separately is often considered challenging, our results provide evidence that conceptual and procedural knowledge can be empirically distinguished, as these two knowledge types are better understood as two distinct latent factors rather than a single latent factor reflecting fraction or algebra knowledge, respectively.

The results are also relevant for educational practice. Previous research has shown that students often struggle with conceptual understanding of fractions (e.g., Hecht & Vagi, 2012; Lenz et al., 2024), likely because instruction tends to prioritize procedural skills over conceptual understanding. Consequently, researchers and educators advocate for teaching that puts equal emphasis on both conceptual and procedural knowledge (e.g., Kilpatrick et al., 2001). Our findings support this call, as both types of fraction knowledge were found to be important prior knowledge for learning algebra. Additionally, the findings emphasize the close relationship between fraction and algebra knowledge. To promote knowledge transfer across content domains, classroom instruction in algebra could highlight similarities between reasoning with fractions and algebraic thinking, such as how preserving equivalence in fractions (manipulating numerator and denominator identically) parallels maintaining equality in algebraic equations (manipulating both sides of an equation identically).

Limitations and future directions

The present study has limitations, which simultaneously highlight directions for future research. First, our findings can only quantify relationships between conceptual and procedural knowledge across the domains of fractions and algebra but cannot explain *why* these relations exist. Given that both fractions and algebra require, for example, fluency in whole-number arithmetic or relational reasoning skills, it may be that these skills mediate the relationships between fraction and algebra knowledge. We are currently addressing this question in a follow-up study.

Second, the relationships found are not causal. Future intervention studies could explore whether improving students' conceptual or procedural fraction knowledge leads to improvements in algebra knowledge or whether algebra instruction that emphasizes similarities between reasoning with fractions and algebraic thinking is more effective than traditional algebra instruction.

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AN EXPLORATION OF EQUITY IN INTERNATIONAL PME PROCEEDINGS FROM 1978 THROUGH 2024

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*This study examines how the nature of equity has changed in proceedings of the International Group of the Psychology of Mathematics Education from 1978 through 2024. Digitized versions of the proceedings were searched for words that contained the root *equit* (e.g., *equitable*), referred to as instances. These equity instances were categorized using Gutiérrez's (2012) framework, consisting of achievement, access, identity, and power. The number of pages on which equity appeared has experienced an upward trend since 1992-1993, with some variability. Several countries with high levels of inequality and few authors, such as Chile, suggest a need for future research on equity. Achievement was the most frequent category. Identity appeared more frequently than access, with power coming to the fore in more recent proceedings.*

INTRODUCTION

Researchers have known for some time that students with different backgrounds do not achieve at similar rates in the U.S. (Pearman, 2019) and other countries (e.g., Education Assessment Research Unit and New Zealand Council for Educational Research, 2015). Studies addressing or attempting to understand these inequalities fall into the broad category of equity-based research. Not surprisingly, research associated with equity has appeared in the proceedings of the International Group of the Psychology of Mathematics Education (IGPME). Indeed, inclusion and diversity was the theme for the 2004 annual meeting of IGPME held in Bergen, Norway.

The digitization of text and the word search capabilities of software programs now enable researchers to investigate many documents of considerable size for the presence of particular words and how these words appear with other words as in the process of topic modeling (Inglis & Foster, 2018). This study investigates how equity has evolved in proceedings of the IGPME from 1978 through 2024 using resource digitization and searches for words involving the root *equit*. As such, this study fits within a broader array of studies (e.g., Gökçe & Güner, 2021) that seek to understand historical trends in mathematics education research.

BACKGROUND

Inglis and Foster (2018) used topic modeling to examine five decades of research from the founding of the *Journal for Research in Mathematics Education* in 1970 and the founding of *Educational Studies in Mathematics* in 1968 until 2015. They found that research in Euclidean geometry has declined significantly since the 1970s. There was a social turn in the 1990s involving sociocultural theories that have continued to the

present. Since 2000, the field of mathematics education has experienced a diversity of theories involving semiotics and embodied cognition. Experimental methods were prominent in the 1970s but have declined sharply thereafter. Overall, there has been increased research involving teacher knowledge, curriculum reform, and novel assessments.

Gökçe and Güner (2021) examined over 1000 articles published in English in the Web of Science database from 1980 to 2019. They found that early research focused on problem-solving, while more recent studies have investigated technology, motivation, teacher training, and equality/diversity. The focus of research initially involved student learning, then moved to curriculum and teachers during the 1990s through 2010, and later moved to various foci consisting of students, teachers, policy, and assessment. Equity started appearing as a top keyword in studies beginning in 2000-2004, focusing on equality and accessibility. Issues of gender, ethnic groups, and social class about achievement gaps or unfair opportunities were common study topics in the 1980s and 1990s. In the 2000s, the concept of *teaching for equity* started gaining a foothold in reform-focused curriculum studies. More recently, from 2010 to 2019, equity has reemerged as a research issue, with more studies involving motivation, attitude, and self-efficacy across gender and demographic groups. Additionally, more recent studies have focused on explicitly improving mathematics education accessibility for disadvantaged groups like minority students, students with disabilities, and refugees.

Parks and Schmeichel (2012) conducted a two-stage literature review that identified four main obstacles in mathematics education research in discussing issues of race and ethnicity related to identity and power. First, there is a tendency to marginalize discussions of race and ethnicity, relegating them to special equity-focused journal issues rather than integrating them throughout the field. Second, many studies treat race as a static categorical variable rather than recognizing its intricate, historically contingent nature, leading to oversimplified conceptualizations of race. Third, the absence of race and ethnicity as relevant analysis sites is a common omission in research, normalizing whiteness and imposing undue justification burdens on scholars who do address these factors. Lastly, even within equity-focused work, there is often a hesitance to directly engage with issues of race, racism, and racialization.

More recently, Vithal, Brodie, & Subbaye (2024) reviewed equity research in mathematics from 2017-2022. They found that conceptualizations of equity have expanded to focus more on identity, power, recognition, and representation. The largest volume of studies also focuses on mathematical practices and teacher actions within specific classes and schools to address inequality. They report that some studies have examined how mathematics pathways can promote or address inequalities. Furthermore, international studies have revealed how societal inequalities play out to support or constrain student achievement in school systems.

Davis (2024) examined the presence of a collection of 18 equity-based terms (EBT) (e.g., ethnomathematics) in the North American chapter of the *Psychology of*

Mathematics Education (PME-NA) proceedings from 1981 to 2000. Davis found that EBT peaked in frequency in 1995 and decreased thereafter. Qualitatively, he found four themes in the EBT. First, he noted a shifting sample. That is, diverse samples were included in studies to demonstrate the effectiveness of an intervention or to understand a phenomenon. In these studies, it was not common to differentiate results by different groups, nor were attempts made to understand why differential achievement occurred. Later, studies used samples that focused on one specific group of students, such as African American students to understand their capabilities better or to ameliorate differential achievement. Second, in 1991, equity was expanded in the proceedings. There was a growing awareness that several factors affected the achievement of different groups of students, such as teacher expectations and classroom processes or societal inequalities. Similarly, there was an awareness that students bring cultural resources with them into the classroom that can be used to construct mathematics curricula. Third, research initially focused on students and later moved to teachers and understanding their role in promoting or reducing classroom inequalities. Examinations of pre-service and in-service teachers illustrated that while equity was gaining importance, many teachers were not prepared to teach in ways that promoted classroom equity. Fourth, many projects attempted to promote equity in the mathematics classroom were initiated by national (e.g., Equity 2000) as well as more local projects (e.g., San Diego Mathematics Project Leadership Institute).

The following research questions guided this study. 1) How has the frequency of equity pages in proceedings from 1978 through 2024 changed? 2) Which countries produce equity papers in proceedings from 1978 through 2024? 3) How do equity categories proposed by Gutiérrez (2012) fluctuate over time in proceedings from 1978 through 2024?

FRAMEWORK

The nature of equity was analyzed using the framework proposed by Gutiérrez (2012). Gutiérrez divides equity into four dimensions: access, achievement, identity, and power. Access consists of five types of resources students have available in the classroom. These include (1) teachers, (2) supplies, (3) curriculum, (4) technology, and (5) support for learning outside of school. Achievement involves mathematics success at different levels and, for analysis purposes, was broken down into three categories. These include (1) scores on various assessments (standards-based or other), (2) course-taking patterns from elementary through college, and (3) employment in mathematics-based careers. The identity dimension was broken down into (1) consideration of students' pasts, (2) balance between self and others, (3) how forces outside the classroom shape students' identities, (4) whether students may be able to use their cultural and linguistic resources in the math classroom, (5) understanding the larger context that the mathematics classroom is situated within, (6) noticing whose perspectives and practices are valued, and (7) balancing an understanding of oneself with others. The power dimension addresses social transformation across different

levels. Specifically, this dimension was broken down into four categories for this analysis. (1) Voice in the affairs in the mathematics classroom. (2) Using mathematics to critique societal issues. (3) Different forms of knowledge such as ethnomathematics. (4) Reconsidering mathematics from a more humanistic perspective.

For Gutiérrez (2012) the access and achievement components represent the dominant axis of mathematics education, emphasizing preparation for economic participation and maintaining the status quo. The identity and power components form the critical axis, recognizing students' perspectives and resources to foster their development as critical citizens.

METHODS

The proceedings for each international Psychology of Mathematics Education meeting beginning in 1978 in Osnabrück in Germany and continuing until the 2024 meeting held in Auckland, New Zealand, were analyzed for the presence of equity by searching proceedings with the keyword *equit** where the asterisk represents a wildcard enabling the words equity, equitable, inequities, etc. to be located. I could not analyze the first proceedings held in the Netherlands as these were not available. In some cases, PDFs of the proceedings had to be converted into searchable text using a software program, PDFify. Proceedings that were published in separate volumes were searched as separate volumes. In some cases, different types of papers were separated from one another. For instance, some proceedings had plenary papers separate from research reports, such as the 28th meeting of IGPME in Bergen, Norway 2004. In these cases, all PDFs were assembled within one large document, and then the wildcard *equit** was searched for. Each occurrence of words associated with the *equit** wildcard in a paper's title, abstract, and body (e.g., research report, etc.) was considered an instance. Equity terminology with the keyword *equit** was not analyzed further or counted if it occurred within the table of contents or the reference sections of individual papers. In addition to the Gutiérrez (2012) framework above, an open coding process was used to account for instances that did not fit the above framework. The results of the open coding process are not reported in this paper.

The year, authors, title, countries where authors resided, type of paper (e.g., research report), location in the paper where equity terminology was located (e.g., introduction), sentence or sentences containing equity terminology, and code using the Gutiérrez (2012) framework or an open coding process were incorporated into a document. A codebook containing the four categories of the Gutiérrez framework was created to facilitate coding. If an understanding of how equity was used was unclear from the sentence containing the word, sentences near the equity-containing sentence were examined. If this meaning was still not precise, other components of the paper, such as the abstract, were examined to discern how equity terminology was being used in the paper. When coding these sentences, phrases were extracted and added to the codebook to facilitate coding. Equity connected to students and classroom practice was counted, while other instances of equity were not. If multiple instances of *equit** were present

on a single page, they were counted as one page. Equity instances were analysed using the Gutiérrez framework, and it was possible for one instance to be coded with multiple categories (e.g., achievement and access). To facilitate the presentation of the equity page frequencies, two-year spans were used beginning in 1978 and continuing through 2024, with 2020 omitted as there were no conference that year.

RESULTS

There were 265 papers in which at least one word associated with equity appeared in the proceedings from 1978 through 2024. The number of pages on which equity appeared in the proceedings documents by two-year time spans appears in Figure 1. As can be seen in the graph there has been a general upward trend in the number of pages starting with 1992-1993. Despite this upward trend, there has been variability in the number of pages devoted to equity. For example, we see a see-saw movement in equity page frequencies. For instance, in 2002-2003, 21 pages with equity appeared in those proceedings, while there were 13 pages in 2004-2005. There was a steep drop in equity pages in the 2010 and 2011 proceedings held in Belo Horizonte, Brazil, and Ankara, Turkey. In 2008-2009, there were 21 pages where equity appeared, and only four pages involving equity in 2010-2011.

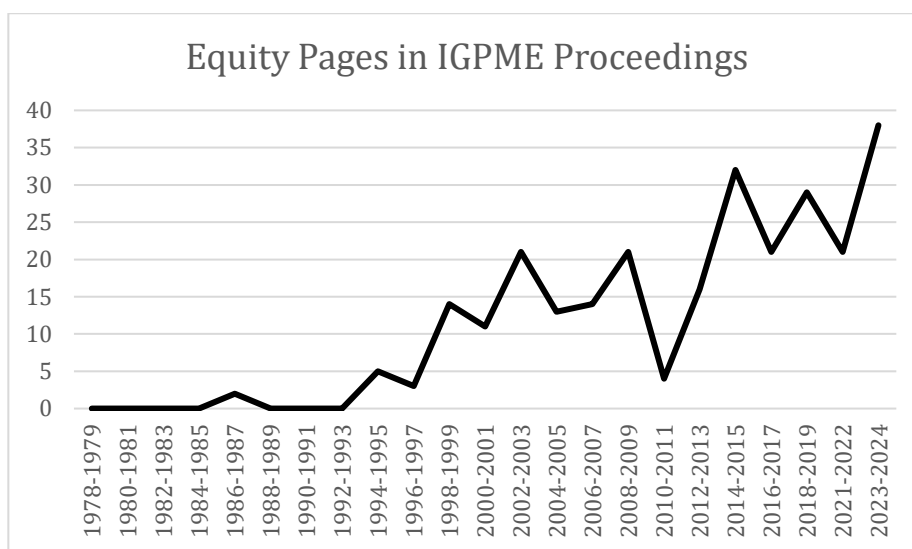


Figure 1: Equity Pages by Two-Year Time Periods

The countries from which the authors of papers involving equity are seen as a tag cloud in Figure 2. Authors from 39 different countries wrote articles with at least one equity page. The country with the highest number of authors writing about equity was the U.S. with 91; Australia had the second most frequent number of authors at 47; Canada had 27 authors; the UK at 22 authors; New Zealand at 21 authors; South Africa had 15 authors; and Israel with 14 authors. There were six countries with only two pages each and 15 with just one page each.



Figure 2: Tag Cloud of Countries Where Equity Authors' Universities Reside

Table 1 shows the breakdown of equity instances using the Gutiérrez (2012) framework. Only two equity instances were in the proceedings from 1978 to 1993, so these years were omitted from this table. Within the dominant axis, achievement typically had a higher frequency than access. While within the critical axis, identity typically had a higher frequency than power. In nearly every period, achievement frequencies were the highest category except for 1996-1997, 1998-1999, and 2012-2013. In the first two time periods, there was an equal number of achievement instances as identity instances. In 2012-2013, there were more instances of identity than achievement. In this period, the critical axis had a higher frequency than the dominant axis. Notably, in 2012, Gutiérrez published her framework involving these categories. However, this emphasis on the critical axis over the dominant axis was short-lived, as proceedings from years thereafter had a higher frequency of instances in the dominant axis than the critical axis. Overall, we see the emphasis on identity as this had a higher total frequency than access within the dominant axis.

Table 1: Frequencies of Achievement, Access, Identity, and Power

Years	Achievement	Access	Identity	Power	Totals
1994-1995	3	3	1	1	8
1996-1997	2	1	2	1	6
1998-1999	8	5	8	5	26
2000-2001	8	5	1	3	17
2002-2003	13	4	7	5	29
2004-2005	10	3	6	2	21

2006-2007	12	3	4	1	20
2008-2009	14	5	6	4	29
2010-2011	2	1	1	1	5
2012-2013	9	6	11	7	33
2014-2015	16	7	4	5	32
2016-2017	11	7	5	8	31
2018-2019	16	13	10	8	47
2021-2022	8	6	7	6	27
2023-2024	22	17	17	12	68
Totals	154	86	90	69	399

DISCUSSION

This analysis showed a gradual upward trend in equity pages starting in 1993-1994, illustrating the importance of equity in the eyes of international mathematics education researchers. A notable exception to this trend was the proceedings in 2010-2011. In 2011, the conference was held in Turkey. The lack of attention to equity in this conference may have been due to the perception by potential submitters that organizers may not have had an interest in mathematics education research in this area. Indeed, the theme for this conference was Developing Mathematical Thinking, which does not suggest an equity focus. The lack of equity pages in 2010 is more difficult to explain. Ubiratan D'Ambrosio and Paulo Freire are from Brazil and dedicated their lives to equity issues. The theme, Mathematics in Different Settings, also suggests that learning in different settings may result in equity issues, providing an opening for research in this area.

Authors of equity papers came from 39 different countries. In this respect, IGPME's diversity and reach are seen. However, as of 2025, there are 195 countries around the world. Thus, there are many countries worldwide where little research involving equity in mathematics education is conducted. A Gini (1912) coefficient of 0 represents complete equality, while a value of 1 represents complete inequality. Given this measure, the high number of U.S. authorships makes sense with a Gini coefficient of 0.4. South Africa has a Gini coefficient of 0.63, suggesting a high level of inequality and the need for more research on addressing schooling inequality in mathematics in this country. Seven authors came from Chile, which has a high level of inequality at 0.45. Only two authors came from Brazil, with a Gini coefficient of 0.48. In both countries, there exists a need for more research on equity.

Despite a call to move beyond gap gazing (Gutiérrez, 2008), achievement was the most frequent category overall and for most years. Thus, there was a preoccupation with what Gutiérrez (2012) refers to as the dominant axis. Nonetheless, identity along the

critical axis had a higher frequency across all proceedings, suggesting this construct's importance to researchers. While power was less frequently addressed when equity was mentioned in proceedings over the years, researchers since 2012 have shown a greater sensitivity to this construct, suggesting the influence of Gutiérrez's work on the field. This analysis suggests that more research on power and access in mathematics education is needed. Despite these findings, this study may have unintentionally underreported the presence of papers associated with equity, as only the reports that contained variations of equity were examined in this study. Authors may have chosen to examine equity or issues associated with equity with words such as diversity, inclusion, etc.

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MOBILIZING MATHEMATICAL KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE THROUGH INTERDISCIPLINARY EDUCATIONAL ACTIVITIES

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This paper explores mathematics teachers' knowledge focused on teaching proportionality while reflecting on interdisciplinary educational activities for primary and middle school. Through the Mathematics Teacher Specialised Knowledge model, we analyze teachers' responses in interviews to identify aspects of their Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK). Findings emphasize the value of real-world contexts in teaching and learning of proportionality and the interconnection between the sub-domains of MK and PCK.

INTRODUCTION

Research within mathematics education has shown the efficacy of collaboration between teachers and didacticians, namely researchers in mathematics education who work with teachers to improve teaching and learning practices in the classroom (Jaworski, 2012). Crossing with this theme, research on the use of resources in teacher professional development has explored various aspects of how different types of materials, tools, and strategies contribute to enhancing teachers' knowledge, practice, and understanding of mathematics teaching. Interdisciplinary resources, like those bridging mathematics with other subjects such as science, can be particularly valuable in teacher education. An example of research on the use of interdisciplinary resources for mathematics education teacher professional development is the work of Swan and Marshall (2005). In their study, Swan and Marshall examined how teachers could use resources related to real-world phenomena (such as the physics of motion or engineering structures) to explore mathematical concepts like rates of change, proportions, and geometry. As this example illustrates, resources related to rich and complex contexts may constitute suitable tools that didacticians may exploit for teacher education.

Grounding from these hypotheses, our research exploits the field of sunlight and shadows in a mathematics teacher education program. Specifically, we will refer to a research project carried out in the 90s by Paolo Boero and his colleagues (Boero et al., 1995). The project involved experiences and observations of the shadows of the sun, it is well-known to Italian researchers but remained largely unknown to teachers (except those involved in the project). The investigation of phenomena related to the sun shadows offers rich ground for exploring proportionality, ratios, similarity, angles, and measures, while also fostering interdisciplinary connections, particularly with science.

In this paper we will focus on the first part of the teacher education program and investigate what teachers' specialized knowledge (related to proportionality) is mobilized when teachers discuss educational activities centered on the observation of the sun and the produced shadows.

THEORETICAL FRAMEWORK

Since the 1980s, research on mathematics teacher knowledge has been influenced by the foundational studies of Shulman (1986), that introduced the concept of *Pedagogical Content Knowledge (PCK)*, emphasizing that effective teaching requires more than just subject matter expertise. Within this line of research, in recent years, Carrillo-Yañez and colleagues (Carrillo et al., 2018) introduced the *Mathematics Teacher Specialised Knowledge (MTSK)* model. This model focuses on understanding the teacher's specialized knowledge used or developed at any particular time, rather than focusing on the assessment of this knowledge. The starting assumption of the MTSK model is that teachers need specialised knowledge to fulfill their role. Therefore, knowledge developed and used for teaching is considered to be specialised. Another feature of this model is that it emphasizes that teachers' actions are strongly related not only to what they know about mathematics, but also to their ideas and beliefs about mathematics, how it is learned and how it should be taught (Thompson, 1992). The MTSK model distinguishes between *mathematical knowledge (MK)* and *pedagogical content knowledge (PCK)*, which are categorized into different subdomains. The subdomains of MK are: *Knowledge of Topics - KoT* (e.g. knowledge of definitions, properties, procedures, representations and applications of mathematics); *Knowledge of the Structure of Mathematics - KSM* (e.g. knowledge of how to connect activities in different domains of mathematics); and *Knowledge of Practices in Mathematics - KPM* (e.g. knowledge of how to prove, justify, define, make inferences and inductions, give examples and counter-examples). PCK is divided into three subareas: *Knowledge of Mathematics Teaching - KMT* (e.g. knowledge of theories of mathematics teaching or knowledge of teaching resources, materials and technologies, but also knowledge of strategies for introducing and presenting content and concepts, etc.); *Knowledge of the Characteristics of Mathematics Learning - KFLM* (e.g. knowledge of theories of mathematics learning or knowledge of how students interact with mathematics); and *Knowledge of Mathematics Learning Standards - KMLS* (e.g. knowledge of expected learning outcomes and teaching goals in different school segments).

In our study, we will exploit the interpretative tools provided by the MTSK model to identify some aspects of teachers' specialized knowledge related to the concept of proportionality.

Vergnaud (1990) defines *concepts* through three components: 1) *Reference Situations*: situations that give meaning to a concept. In education, a situation is a reference if a student can recall and use it during an activity to reason, explain, understand, or solve problems. 2) *Operational Invariants*: the stable elements that support the functioning

of mental schemes. They connect reality with practical and theoretical knowledge, showing how a person's thinking reflects reality. 3) *Representations*: the signifiers like verbal language, symbols, diagrams, and gestures. Tools and materials can also serve as external representations, helping express and support concepts.

Referring to such a view, we can identify aspects of the concept of proportionality when analyzing teachers' specialized knowledge. More precisely, we will answer the following research question: "What aspects of teachers' mathematical knowledge and pedagogical content knowledge can emerge when teachers discuss (reflect on) educational activities set on the field of experience of sun and shadows and focused on proportionality?"

METHOD

This study was conducted within a teacher professional development (PD) course involving 16 primary and middle school teachers in the Piedmont region of Italy. The second author serves as the course coordinator, while the first author collaborates in the program as part of his PhD research. Aligned with the Meta-Didactical Transposition model (Arzarello et al., 2014), teachers actively collaborate with didacticians and fellow teachers, forming a community of inquiry (Jaworski, 2008) and sharing meta-level reflections on the content to be taught and the corresponding didactical activities.

During the course, the didacticians - specifically, the first two authors of this paper - provided teachers with study materials based on educational teaching experiments developed in primary and middle schools by Boero and his research team in the 1990s (Boero et al., 1995). This study material consists of tasks and activities designed to engage primary and middle school students in real-life situations, observations, and interactions related to how sunlight creates and alters shadows. The term 'field of experience' was coined to emphasize the importance of students' engagement with observed phenomena (ibid.). This experiential foundation supports the learning of scientific concepts, particularly in mathematics, physics, and astronomy.

An example relevant to the theme of this article involves students being invited to hypothesize about the lengths of shadows cast by gnomons (also called poles) of varying heights at the same time of day (see Figure 1).

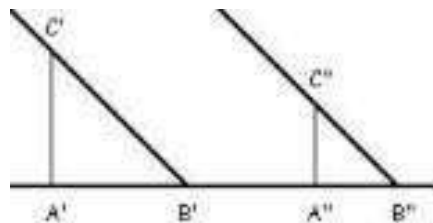


Figure 1: Two gnomons of different heights at the same time of the day

The PD course teachers were not familiar with these activities, whereas the didacticians were. Teachers were organized in 4 small groups and asked to analyse the materials

and to reflect on their teaching practices. The discussions among the teachers in the working groups were audio-recorded. During these discussions, the teachers reflected on identifying the mathematical concepts involved in the proposed activities and tasks for students. This is an example of task for students (T1): “Based on what we have learned about the sun and shadows, can we determine how long the shadow of a smaller nail would be compared to the one we used, if measured at the same time?”.

The analysis of the transcripts of the teachers’ dialogues during group work revealed that they identified proportionality as a central mathematical concept in the sun-and-shadows tasks. To explore their specialized knowledge of proportionality in greater depth, we conducted a follow-up interview. The interview questions are presented in Table 1.

Question	Content
Q1	How do you usually approach the concept of proportionality with your classes? Can you give an example of an activity you use to introduce or explore this topic?
Q2	What difficulties do you expect in task T1, and why?
Q3	Do you expect to encounter similar or different errors and difficulties compared to the activities you usually do in the classroom (or those you described in Question 1)? Why?
Q4	Has analyzing the activities from the 'Sun and Shadow' project and reflecting with your group changed your view on the concept of proportionality? Will it affect how you introduce this concept to your students?

Table 1: Questions of the interview

We decided to interview two teachers simultaneously. As described by Arksey and Knight (1999), involving multiple interviewees can offer two perspectives on the same events, allowing for cross-verification. Each participant may add unique details, resulting in a more thorough and dependable account.

The interview was conducted online by the first authors of this paper. It was recorded and subsequently transcribed. The analysis employed Qualitative Content Analysis (Mayring, 2015), utilizing a deductive coding approach. This involved systematically assigning predefined categories to specific passages within the transcribed interviews. The goal was to identify evidence of phrases that might indicate aspects of teachers' knowledge corresponding to the MTSK model categories (Carrillo et al., 2018), with a focus on the concept of proportionality (framed with Vergnaud’s definition of concept). The content analysis was initially carried out by the authors working independently, followed by a group discussion until a final consensus was reached. In the following section, we present key examples for each MTSK sub-domain, illustrating the application of these categories.

ANALYSIS AND DISCUSSION

Two middle school teachers (grades 6–8), pseudonyms Elena and Chiara, proposed themselves as volunteers for the interview. They work within the same school complex, and they have more than 10 years of teaching experience. We present examples of their responses to questions Q1-Q4. These sentences reveal teachers' Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK), described using the categories proposed in the MTSK model (Carrillo et al., 2018).

Knowledge of Topics (KoT)

Answering question Q1, Elena talks about possible representations of the concept of proportionality.

Elena: However, the concept (of proportionality) is also related to a formula setting that we can give, or directly to the equation of direct and inverse proportionality.

Answering Q4, the teachers highlight the potential of activities with the “Sun and shadows” field of experience to foster meaningful dialogue between mathematics and science.

Chiara: And then another strength. The fact that it is related to science and therefore the fact that you can integrate a mathematical concept with a natural phenomenon

Elena: It makes them touch on how the phenomenon of sunlight is a phenomenon that can be related to a mathematical law.

They note that this interdisciplinary approach can offer deeper mathematical insights, and they recognize this field of experience as a good reference situation for the mathematical concept they are talking about (i.e. proportionality).

Knowledge of the Structures of Mathematics (KSM)

In discussing her answer to Q2, Chiara connects different mathematical items and contexts. Her response reveals also her knowledge of mathematics teaching.

Chiara: The concept of variation of the length of the shadow it is true that we can introduce it as ratio and proportion, but even in the geometric context it comes up with the Pythagorean theorem, with similarities [...] it is really a topic, let's say, across the teaching of topics

Knowledge of Practices in Mathematics (KPM)

In terms of means of production and mathematical functioning, we have also identified references to the teacher's practices in relation to mathematics. The idea of exploring the same problem from different perspectives reflects the teacher's conceptualization of mathematical practices.

Chiara: (answering Q2) The concept of proportionality is really taken up again and again [...] and so it's really a concentric way of working, in concentric

circles, because gradually we go deeper and deeper looking at the same problem but from different points of view.

Knowledge of Mathematics Teaching (KMT)

In response to Q1 and Q2, the teachers show their knowledge of the potential of various activities and approaches to teach proportionality. In particular, Chiara talks about reference situations, and Elena describes possible schemes to be implemented to promote learning.

Chiara: (answering Q1) To introduce the concept of relationship, I choose an example related to reality [...] a set of problems that can be solved, related to real life. The typical example I use is recipes. [...] finding real examples facilitates.

Elena: (answering Q2) If he had to deal with this kind of question, surely the antecedent would have been: if I doubled the nail, if I halved the nail, that is, I would have made them think about these cases first.

Knowledge of Features of Learning Mathematics (KFLM)

Answering Q1, the teachers talk about possible difficulties students may have with the concept of proportionality, emphasizing how these difficulties can be overcome by referring to real situations.

Chiara: It is not easy (the concept of proportionality) for our students. And so finding real examples makes it easier [...] the moment they know the example they are able to find the mathematical aspect of it as well.

Elena: $y = kx$ is a concept that is difficult for kids in our school order to understand on its own, without precisely, referring to a real case.

In Chiara's response to Q2, she talks about students' approach and schemes depending on their school grade. This is also closely linked to her knowledge of the topics addressed at different points in the students' educational development and is therefore linked to knowledge of the curriculum.

Chiara: Students' approach depends on the class they are attending, [...] first graders who do not have strong mathematical tools at their disposal tend to reason. [...] the difficulty is mainly related to the numerical factor. [...] For second grade pupils, the difficulty is just setting the proportion correctly. [...] In third grade they will tend to solve this problem with equations, and the difficulty is just that.

Knowledge of Mathematics Learning Standards

In her answer to Q1, Chiara explains the steps she follows in the teaching of proportionality. These are aligned with the Italian national curriculum guidelines (MIUR, 2012).

Chiara: Indicatively, we start with ratios with the concept of ratio and then tackle proportions and then the whole part of direct and inverse proportionality

In her response to Q2, Chiara emphasizes the importance of proportionality in grade 7, viewing it as a key topic that is connected to numerous other topics intended for that specific school grade.

Chiara: In the second class (7 grade), the concept of proportionality is taken up several times [...] it is really a cross-curricular topic in the teaching of the topics that are planned especially in the second class...

Looking at the analysis as a whole, the identification of teachers' specialized knowledge related to the concept of proportionality and its teaching-learning process, highlights how the MK and PCK sub-domains are often interrelated and refer to each other, linked by the different dimensions of the concept. In fact, the teachers take into account specific reference situations, using a variety of schemes and multiple representations.

CONCLUSION

In light of the reflections shared by the two teachers during the interview, we attempt to formulate an answer to the research question earlier defined. The teachers discussed the concept of proportionality in its various representations (i.e., as a formula), while emphasizing the importance of contextualizing the concept in real-life situations (i.e., sun and shadows). This approach fosters a mathematical understanding of the concept by presenting it to students using different operational invariants. These reflections align with the development of a conceptual field as defined by Vergnaud (1990). The analysis also reveals how the two teachers examine the concept not only from a mathematical perspective (i.e., topics, connections, methods of procedure) but also from a teaching perspective (i.e., objectives, curriculum, challenges), highlighting strong connections between MK and PCK that complement each other (Carrillo et al., 2018).

The teachers also highlighted some strengths that were previously hypothesized in designing the teacher education program. In particular, the teachers find it interesting to explore the concept of proportionality through experiences grounded in real-world observation, specifically by investigating a natural phenomenon through an interdisciplinary dialogue between mathematics and science.

Although these findings are promising, they are limited to the perspectives of just two teachers. Therefore, in the coming months, we will expand on these reflections by interviewing additional teachers in our group. Furthermore, we will observe teachers designing instructional pathways on the topic of Sun and Shadows for their classes, as well as the implementation of these activities.

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THE IMPACT OF VIDEO-BASED PROFESSIONAL DEVELOPMENT ON MATHEMATICS TEACHERS' SUSTAINED IMPLEMENTATION OF ACCOUNTABLE TALK PRACTICES

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This paper reports on a study investigating the implementation of Accountable Talk (AT) practices among elementary school mathematics teachers who participated in a video-based professional development program. In particular, the study examined whether these practices were sustained a year later and how they were adapted. The analysis of four case studies helped to categorize and characterize the sustained implementation of AT practices in elementary mathematics classrooms.

INTRODUCTION

Over the past two decades, extensive research has highlighted the critical role of high-quality discourse in influencing various aspects of student learning, (e.g., Michaels et al., 2008). This body of research has been particularly prominent in the field of mathematics education (e.g., Heyd-Metzuyanim et al., 2019). While professional development (PD) programs have sought to familiarize teachers with practices to enhance discourse quality, maintaining high-quality discourse poses a considerable challenge for teachers (Schwarz & Baker, 2016). Moreover, sustaining the effects of the PD after its completion is far from being straightforward. Implementing and sustaining high-quality discourse practices is particularly complex due to the multifaceted nature of classroom interactions and the ongoing demands on teachers (Schwarz & Baker, 2016). Supporting teachers in adopting these practices requires not only a scrupulous PD design but also an explicit attention to ensuring that the practices are routinized. Research on the sustained impact of PD programs that foster high-quality discourse practices is scarce. In this study, we address this gap by examining the sustained impact of practices learned through a PD program focused on *Accountable Talk* (AT) - a structured discourse approach known to promote rigorous and equitable classroom discussions (Michaels et al., 2008). The PD program utilized video-based learning to model and discuss AT teaching moves and practices, whilst encouraging teachers to regularly implement these practices in their classrooms. The current report focuses on four case studies of primary mathematics teachers, exploring how they implemented AT practices and moves introduced in the PD, along the PD and a year after its completion. As background to the study, we provide a brief theoretical account on the concept of *sustainability* in the context of PD programs, and on *Accountable Talk*, the approach around which the PD was designed.

THEORETICAL BACKGROUND

Sustainability of PD programs

PD programs play a pivotal role in advancing teachers' instructional practices and content knowledge (Darling-Hammond et al., 2017), with sustainability emerging as a critical measure of PDs' effectiveness (Jarry-Shore et al., 2023). Herein we refer to sustainability as “the degree to which teachers use reform-related practices in high-quality ways after support for these practices has dissipated” (Coburn et al., 2012, p. 140). Research highlights that sustaining PD outcomes may be affected by several factors, including teachers' commitment and their clear understanding of the learned practices. Vaughn et al. (2004) identify four levels of sustainability: *proactive sustainability*, when practices are implemented much like they were taught and are highly valued by the teacher; *routine sustainability*, when practices become routinized and are used intermittently, either as taught or with minor adjustments; *modified sustainability*, when several elements of the practices are used, but others are eliminated or significantly modified; and *partial sustainability*, when some elements are used, but sporadically or even unawares. These four categories can be seen as indicating the degree of effectiveness of the PD program. There is increasing recognition that for a PD program to be effective, its design should align with proven principles (Prediger, 2022). In the design of the PD reported here, we considered three such principles: (1) The PD integrates teachers' experimentations with the learned practices and collaborative reflection thereof (Kohen & Borko, 2022); (2) The PD lasts long enough to allow for changes in practice (Darling-Hammond et al., 2017); and (3) The PD uses video-based artifacts to enable a microanalysis of teaching practices to be reflected upon (Coles et al., 2019). Each of these principles was mentioned in the literature as important for the effectiveness of PD programs in general, however we hypothesized that their conjunction would also support a sustained implementation.

Accountable Talk

Michaels et al. (2008) defined AT as an argumentative dialogic discourse that takes place within a collaborative learning environment through which students learn to reason in the context of a field of knowledge - in our case, mathematics. AT is characterized by three dimensions of accountability: (a) accountability to the learner community - the teacher acts to involve all students in the dialogue as agents; (b) accountability to reasoning - participants utilize logical moves or evidence for their claims; (c) accountability to knowledge - relying on knowledge already constructed and agreed upon. AT offers a set of talk moves such as “press for reasoning” or “add on”, which provide teachers with tangible means for eliciting students' reasoning, as well as for holding students accountable to making their thinking understandable to others (Heyd-Metzuyanim et al., 2019). AT integrates epistemic and social elements, requiring teachers to first experience it in supportive environments, as its adoption involves challenges often attributed to deeply-rooted traditional teaching methods (Schwarz & Baker, 2016). Considering the above we posed two research questions: (1)

To what extent were the learned AT practices sustained one year after the end of the program? (2) Following Vaughn et al.'s (2004) categories of sustainability, how can the sustainability of AT practices be characterized?

METHOD

Setting, participants and data sources

This study is part of a larger project, for which data was collected during 2016-2018. A 30-hour elective PD program, focusing on AT, was offered to primary mathematics teachers. The program spanned 10 sessions over a school year. Two groups with similar average teacher profiles were opened. In total, there were 45 participants with a teaching experience ranging from 3 to 23 years. Eight teachers were chosen for an in-depth study, through purposive sampling to represent diverse backgrounds. In a previous PME paper (Deri et al., 2024) we reported on *trajectories of change* in the AT implementation of two of these teachers, Jill and Eda (pseudonyms), representing very different teacher profiles: Jill had a mathematical background but only 3 years of experience, and Eda had 23 years of experience but no formal training in mathematics. In this paper we expand our look to two more teachers, this time focusing on the degree to which the AT implementation was *sustained*. The two other teachers are Shila, head of the mathematics department (HoD) in her school, with 11 years of experience, holding a minor in mathematics teaching, and Ann, HoD in her school with 15 years of experience, holding a minor in mathematics teaching and serving also as a language education teacher. Shila and Ann had previously participated in mathematics PD programs, whereas Jill and Eda had not. Shila and Ann thus represent different profiles than the first two, allowing sufficient variety for comparing and contrasting the cases.

Data collection included (1) video-documentation of all PD sessions, later fully transcribed; (2) videotaped lessons for each of the selected teachers, collected at four points of time - prior to the PD, halfway through the PD, at the end of the PD, and about one year after the PD concluded (total of 16 videotaped lessons); (3) post-lesson video-based stimulated-recall interviews (SRIs), held with each teacher (16 SRIs).

Data analysis

Two existing coding schemes were used for analyzing the implementation of AT in the teachers' classes: IQA (Instructional Quality Assessment; Boston, 2012), that rates the level of cognitive demand and the level of AT on a scale of 1-4, and ATC (Accountable Talk coding; Heyd-Metzuyanim et al., 2019), a tool for coding AT moves during a lesson, referring to the extent to which the teacher's talk moves help students clarify and share; become oriented to, and engaged with, the thinking of others; and deepen their reasoning. In addition, we created designated rubrics for coding the types of questions asked by the teacher, the type of student answers, and the amount of evaluative feedback. Due to space limitations, we present here only IQA and ATC results. These results were used for assessing the degree of sustained impact a year after the PD ended (RQ1) and for characterizing the nature of sustainability (RQ2).

FINDINGS

Implementation of AT along the PD and a year later

The IQA analysis for the four teachers in the four points of time is summarized in Figure 1.

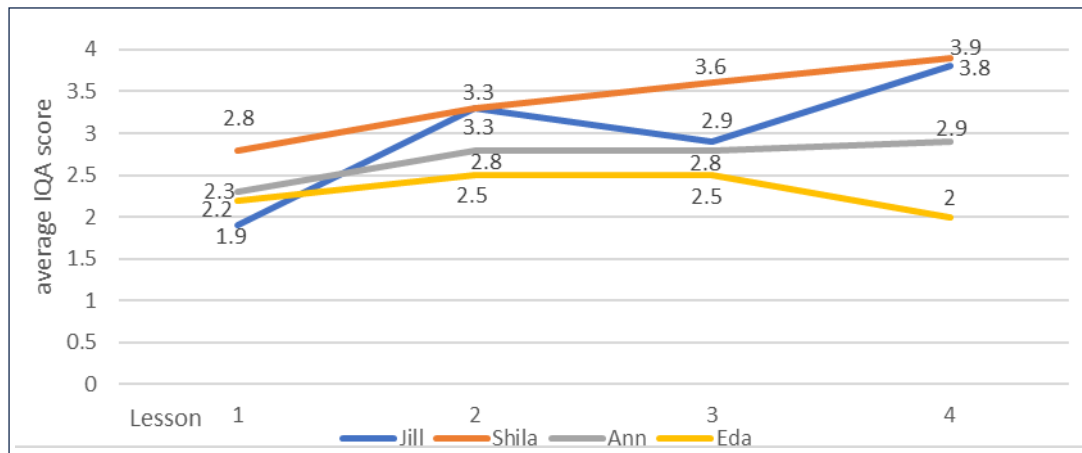


Figure 1: Average score of IQA components in four lessons of the four teachers.

As seen, for three teachers the IQA rating increased over time, compared to Lesson 1, with a consistent increasing trend for Ann and Shila (a moderate increase for Ann and a considerable increase for Shila), and a mixed trend for Jill. For these teachers a sustained impact was found, with high IQA ratings a year after the PD (Lesson 4). In contrast, Eda's IQA rating increased between Lessons 1 and 2, but this progress was not sustained.

Figure 2 presents the four teachers' use of AT moves according to accountability type.

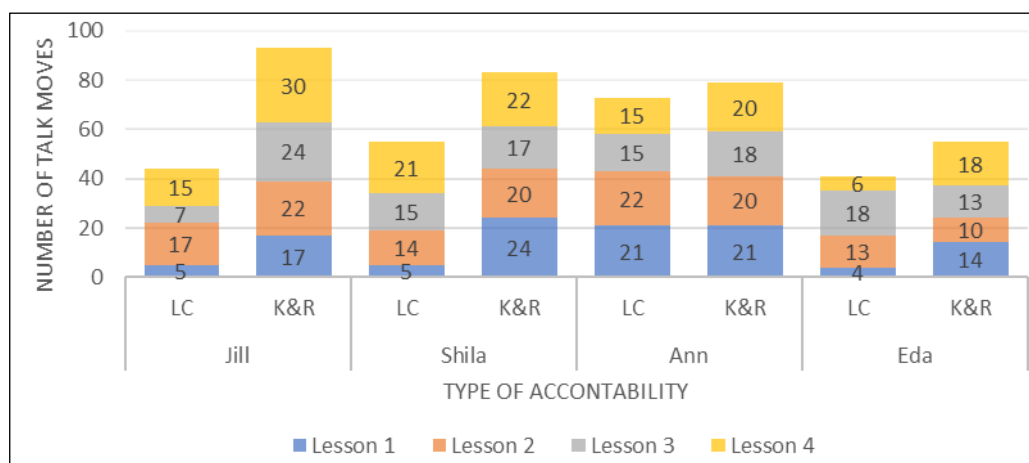


Figure 2: Number of talk moves in four lessons of the four teachers.

(LC = accountability to Learner Community; K&R = accountability to Knowledge and Reasoning)

Jill, Shila and Eda mostly showed an increase in their use of talk moves from Lesson 1 to Lesson 4 (although with different trend patterns), while Ann showed a high number of talk moves already from Lesson 1 (as she testified, she was acquainted with discursive practices from her training as a language education teacher).

Categories of sustainability identified in the teachers' implementing of AT

The analysis of the four teacher cases, with specific reference to Lesson 4, revealed that their level of sustaining AT implementation varied, and can be located on the 'spectrum' of Vaughn et al.'s (2004) categories of sustainability as shown in Figure 3.

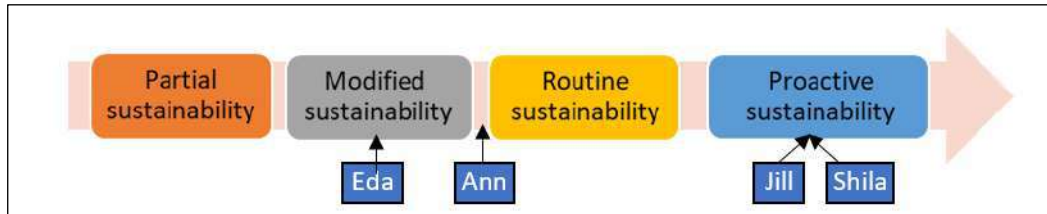


Figure 3: Cases mapped onto Vaughn et al.'s (2004) categories of sustainability.

Herein we demonstrate this mapping through three representative examples.

Proactive sustainability - an episode from Lesson 4 in Jill's classroom. In her 3rd grade class, Jill presented the following scenario: Grandma took her eight grandchildren to the museum, where tickets cost \$10 per child and \$24 for an adult. Four of the grandchildren suggested different calculation methods for the total price:

Ian: $8 + 8 + 10$; Sara $\frac{8 \times 10}{80 + 24}$; Maya: $8 \times 10 + 24$; Daina: $8 \times 10 - 24$.

Jill displayed these methods on the board, and students were given time to work in small groups to assess the accuracy of each method and provide justifications. The following dialogue unfolded during the whole-class discussion:

- 53 S1: Sara did 80 and 80, and she should have done eighty once and then added 24.
- 54 Jill: If I understand you correctly, you're saying she did eight times ten and that's good, but she wrote eighty twice, [...], and that's why it's not correct? Okay... Does anyone want to agree or disagree with S1's opinion or add to it? S2.
- 55 S2: I agree with S1, she [Sara] should have either not written the eight times ten, or not written the eighty.
- 56 Jill: Why?
- 57 S2: Because if she wrote it like that, it means she probably did eighty twice. And that's not correct.
- 58 Jill: [...] So, S2 agrees with S1, okay... I want to hear more children. Yes, S3.
- 59 S3: I don't agree with both of them, because eight times ten is eighty and then they wanted to show them the result. And then do eighty plus twenty-four.

The discussion following S1's remark was characterized by the participation of multiple students. The speakers based their reasoning on explanations and evidence drawn from the calculation methods presented (turns 53, 55, 59). Jill facilitated the discussion by asking questions that encouraged critical thinking and reasoning (turns 54, 56), reflecting where necessary (turn 54), and fostering arguments and connections between students (turns 54, 58), which promoted a dialogical argumentative discourse. This example reflects an AT implementation which is faithful to what was learned in the PD program.

In between routine and modified sustainability - an episode from Lesson 4 in Ann's classroom. This 6th-grade lesson focused on strategies for solving problems involving simple fractions and mixed numbers without performing calculations. Ann began by writing the main task on the board:

Dan and Tom ate 2 pizzas together. Dan ate $\frac{7}{8}$ of a pizza and Tom ate $1\frac{1}{4}$ of a pizza. Without calculating, did they eat the 2 pizzas? Explain.

The whole-class discussion focused on hypotheses and their justification:

- 20 S1: Dan ate almost all the first pizza. He left one piece for Tom, and Tom ate the second pizza.
- 21 Ann: So, you're saying that Dan ate $\frac{7}{8}$, right? And how much was left for him? You said he had one piece left. What fraction is that piece?
- 22 S1: $\frac{1}{8}$.
- 23 Ann: Do we all agree so far?
- 24 Chorus: Yes.
- 25 Ann: Who wants to repeat what S1 said?
- 26 S2: That Dan ate $\frac{7}{8}$ and $\frac{1}{8}$ was left for him.
- 27 Ann: Do we agree?
- 28 Chorus: Yes.

The discussion following S1's answer was characterized by the participation of more than one student and the use of chorus. During the discussion, Ann revoiced S1's words (turn 21) and used the *restate* move (turn 25). While *agree/disagree* moves may have been intended to ensure active listening, they did not promote the kind of dialectical discourse emphasized in the PD program. Students responded as chorus with yes/no answers, without justifying themselves (turns 24, 28). Ann occasionally used the talk moves as learned in the PD but also modified some to suit her purposes.

Modified sustainability - an episode from Lesson 4 in Eda's classroom. Eda's 6th-grade lesson focused on multiplying decimal numbers by 10, 100, and 1000. Eda displayed a procedural task on the board, which included several problems where students had to place the decimal point in the answer to achieve equality:

- 69 S1: So... 3.24 divided by 10, I move the decimal point one step to the left...
- 70 Eda: So instead of it being here, it will be here... is that what you're saying?
- 71 S1: Yes.
- 72 Eda: What else do I need to do?
- 73 S1: Add a zero [to the left side of the number]

74 Eda: [Adds the zero where needed] Because what happens to the number? Why do we need to add a zero here? No, I can't hear... sorry [...]

75 S1: It got smaller.

76 Eda: Smaller by how much?

The discussion following S1's answer was characterized by the participation of one student. Eda employed the talk moves *revoice* (turn 70) and *say more* (turns 72, 76), along with a closed question ("what happens to the number"; turn 74) and the *press for reasoning* move ("Why do we need to add a zero here?"; turn 74), which she did not follow up on. S1's responses were brief and centered on replicating learned procedures. It thus appears that Eda adapted the talk moves while maintaining traditional teaching.

SUMMARY AND CONCLUDING WORDS

The four teachers showed varied degrees of sustainability in implementing AT. While Jill and Shila demonstrated considerable improvements that appeared to be sustained, Ann's change was more moderate but was maintained a year after the PD. Eda's IQA rates increased during the PD but this was not sustained after its completion. In terms of Vaughn et al.'s (2004) categories of sustainability, the cases of Jill and Shila demonstrated proactive sustainability, as they both adapted AT practices to their goals with what seemed as deep understanding of AT's underlying rationales and purposes. Ann's case demonstrated routine to modified sustainability, with some integration of the AT practices. Eda's case demonstrated selectively applied practices in a way that aligned more with a traditional teaching orientation. These findings can be linked to the teachers' backgrounds and specifically their varying mathematical training (Deri et al., 2024), yet such links need further grounding.

A key observation arising from the findings is that increase in the number of talk moves, although reflecting efforts to promote accountability to both the learning community and to knowledge and reasoning, did not always align with discourse quality. In other words, teachers may use learned talk moves in ways that do not necessarily fully support students' high-quality dialogue. Nevertheless, the finding that for three out of the four teachers the average IQA scores were not only sustained but have also increased a year after the end of the PD, which seems a priori surprising, merits further explanation. Schwarz et al. (2017) described the sustained use of dialogic practices as "epidemic", meaning that once a person or a group sense that a certain dialogic behavior (e.g., a move, a strategy, an action) has a positive effect, this behavior tends to multiply, either in terms of time (repeating the behavior) or in terms of space (others adopt the behavior). In line with this description, we conjecture that when the teachers (and especially Shila and Jill) tried out the learned AT practices, they sensed their impact on students' involvement and thus it became their interest to maintain and even enhance this effect. A similar epidemic effect may have occurred with students' engagement. The SRIs seem to support this conjecture, however this analysis is beyond the scope of the present report and we intend to elaborate it in the future.

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LEXICOGRAMMATICAL STRUCTURE IN MATHEMATICS SCHOOL LEAVING EXAMS – AN EMPIRICAL STUDY USING NATURAL LANGUAGE PROCESSING

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In Germany a centralized common pool of mathematics tasks has been implemented for the final high school exam, the Abitur. Contextualized tasks form a significant part of that pool. Here we examine the familiarity of the vocabulary used as well as the level of syntactic complexity. The tasks are grouped by subject area, course level, level of demand and text function (informative or instructive). Natural Language Processing is used for the linguistic analysis. Differences between task groups are described and the implications for their lexicosyntactic complexity are discussed.

INTRODUCTION

Transferring mathematical skills to real-life problem situations has become one of the central intentions of recent developments in task design and, consequently, contextualized tasks are becoming increasingly important, see OECD (2013) and Watson and Ohtani (2015). In the German school leaving exams, the Abitur, contextualized tasks are the norm (KMK, 2012). As these require verbal or visual information about the real-world problem, this could lead to a higher demand on reading skills for their solution, while the mathematical aspects decrease in importance and complexity (e.g. Jahnke et al., 2013). The linguistic structure of mathematical tasks, and contextualized tasks in particular, contributes to their difficulty (Peng et al., 2020), which motivates further empirical analysis of their characteristics.

CONTEXTUALIZED TASKS IN GERMAN SCHOOL LEAVING EXAMS

In recent decades research on task design in mathematics has focused more on the application of mathematical concepts and procedures in real life. For example, as a framework for PISA 2012, the OECD (2013) introduced *mathematical literacy* as students' ability to transfer mathematical models to real-life contexts, which includes extracting the relevant information of non-exact situations from informative context sections. These tasks are different from word problems in the sense that they do not simply connect mathematical topics to real objects but use real-life problems as motivation for all mathematical activities (Tout, 2014).

In Germany, in response to the PISA and TIMSS performances and others, a greater focus on applied mathematical knowledge and mathematical competencies was introduced in the early 2000s leading to the idea of centralized curricula and exams prescribing this new approach for all 16 federal states. As a result, the Standing Conference of the Ministers of Education and Cultural Affairs (KMK) introduced

educational standards for the primary and lower secondary levels in 2004 and for the upper secondary level and the school leaving exams (Abitur) in 2012 (KMK, 2012). For the upper secondary level and the Abitur, the mathematical competencies for the examinations are *argumentation, problem solving, modelling, the use of representations, the use of symbolic, formal and technical elements* and *communication* (KMK, 2012). Tasks, in learning as well as examination situations, require three levels of demand in which the competencies need to be applied to problems of the subject areas of calculus, linear algebra/analytic geometry and stochastics, of which at least two need to be included in Abitur examinations (KMK, 2012). The further centralization of Abitur examinations in a common task pool by the IQB Institute in Berlin (2025) promises better comparability and standardization for the mechanism that offers access to tertiary education. However, the Abitur according to the KMK standards has also been subject of criticism, for example regarding a low and diverse level of cognitive potential (Frenken et al., 2024) or the lack of true application or mathematical depth (Jahnke et al., 2013).

LANGUAGE IN MATHEMATICS TASKS

Beyond its grammatical function verbal language is organized into registers, lexicogrammatical “logogenic patterns” (Halliday, 2004, p. 586) that identify a specific text type or communicational space. Schleppegrell (2001) described technical lexis, structured clauses with logical relations, condensation through nominalization and higher proportions of prepositions and relative clauses as characteristic for the context of schooling, the school register. The register of mathematics and mathematics tasks, aside from combining the semiotic resources verbal language, symbolic expressions and visualizations, has not been unanimously described, see Österholm and Bergqvist (2013). However, some characteristics or patterns have been identified such as compactness through an increased proportion of nouns instead of verbs, prepositional phrases instead of conjunctions, or impersonality (Lee, 2006; Österholm & Bergqvist, 2013). Furthermore, the vocabulary of mathematics tasks consists of mathematical technical terms as well as everyday language with or without new assignments of meaning (Dyrvold et al., 2015; Lee, 2006). Nevertheless, further analysis of contextual factors such as school year, subject area or real-life topic is needed to develop a suitable framework (Österholm & Bergqvist, 2013). For contextual tasks in particular the differentiation between informative text sections introducing the problem context and instructive subtasks may be necessary for a linguistic analysis as those represent two different speech functions, informative and demanding (Halliday, 2004).

Recently, the linguistic complexity of mathematics tasks and its influence on accuracy level has been discussed from a structural, lexical and syntactical perspective with some studies making use of Natural Language Processing, see e.g. Bednorz et al. (2024), Dyrvold et al. (2015) and Niederhaus et al. (2016). Elements of everyday language such as personal language, common words or less compact sentences might decrease linguistic complexity (Bednorz et al., 2024) while the opposite might be true

for everyday vocabulary non-typical for mathematical contexts (Dyrvold et al., 2015). Characteristics of the school register or academic language such as referential structures, compactness or uncommon vocabulary are considered factors for increased linguistic complexity (Bednorz et al., 2024; Dyrvold et al., 2015; Schleppegrell 2001).

In a previous analysis of lexical and syntactical differences between Abitur tasks of the Common Task Pool (IQB, 2025) for different subject areas a first description of these tasks was possible with Natural Language Processing (Ebel et al., 2025). In addition to the number of changes to other semiotic registers such as visualizations and symbolic sequences we measured concreteness, variance and register of the tasks' vocabulary as well as sentence length, dependency distance, entropy and referential word types (prepositions, conjunctions and pronouns). Calculus tasks can be characterized as diverse as for the semiotic resources implemented. They also show a higher proportion of prepositions and pronouns which might increase complexity through referential structures but also contributes to a more personal task structure. Additionally, the lower proportion of conjunctions indicates a communication that establishes references on a nominal or prepositional level leading to a denser syntax. Stochastics tasks consist mainly of verbal language which is closest to everyday language with a higher proportion of subordinating conjunctions indicating a lower complexity. In linear algebra and analytic geometry tasks the proportion of coordinating conjunctions is significantly higher which might also decrease syntactic density. Analytic geometry tasks, however, refer to visualizations more often. In conclusion, Abitur tasks of different subject areas differ in linguistic complexity which might also be the case for other curricular parameters. Furthermore, a more thorough analysis of the lexical structure is needed and to estimate complexity through syntactic density nominalizations need to be examined in addition to the prepositional structures.

RESEARCH QUESTIONS

The present study is adding to the previous findings regarding the lexical and syntactical complexity of mathematics exam tasks (Ebel et al., 2025). On the one hand, it is following up on curricular parameters as influential factors on linguistic complexity of mathematics tasks. Therefore, in addition to the subject areas, we examine the linguistic differences between tasks for different course levels (basic and advanced) as well as between subtasks of different levels of demand. Additionally, we contrast the linguistic structure of informative text sections introducing the task context with instructive task texts. On the other hand, the study combines the lexical level, the familiarity and concreteness of the vocabulary used, with the lexicogrammatical phenomenon of syntactic condensation through higher proportion of nominal and prepositional structures. Focusing on these two approaches the study is based on the following question: *In addition to differences between tasks of different subject areas in German school leaving exams, is there a difference in the lexical concreteness, the familiarity of the vocabulary and the syntactic condensation through nouns and prepositions for tasks of different course levels or different levels of demand?*

SAMPLE AND METHODS

The sample consists of $n = 562$ Abitur tasks (2721 subtasks and informative paragraphs) for the examination years 2017 to 2023 in the Common Pool of Abitur Tasks for the Federal States (IQB, 2025). First, the tasks were categorised according to the subject areas they address and the course level they were designed for (basic or advanced). For every subtask the level of demand was added, while level 0 was assigned to informative sections without instructions. Pictures and symbolic sequences were excluded from the analysis. All verbal elements per task (or subtask when grouping by level of demand) made up one unit of analysis. NLP was used to automatically measure the linguistic key figures. First, the corpus SUBTLEX-DE (Brysbaert, 2024) for German subtitles, a corpus established based on German Wikipedia articles (Zenodo, 2024) and a corpus of the Abitur tasks themselves were consulted to estimate the amount of everyday language, technical/academic language and the language that may be typical for mathematical tasks. Additionally, lexical concreteness was determined using the German list of extrapolated affective norms (GLEAN), which assigns empirically determined concreteness values to German lexemes (Lüdtke & Hugentobler, 2023). To determine the degree of syntactic condensation the mean proportion of nouns, verbs, prepositions as well as coordinated and subordinated conjunctions per sentence per task in percent was measured by making use of the German language model of the Python library *spaCy* (Honnibal et al., 2020). An ANOVA and pairwise comparisons were used to determine linguistic differences between tasks of different subject areas and course levels as well as between subtasks with different levels of demand. A Kruskal-Wallis test and a Dunn test were used in cases where the graphical inspection indicated a violation of the requirements.

RESULTS

Complementing the previous findings on linguistic differences between school leaving exam tasks of different subject areas described above, the mean proportion of nouns and verbs per sentence per task was compared for tasks in calculus, linear algebra, stochastics and analytic geometry. While there could not be observed any significant differences for the proportion of nouns, the use of verbal expressions was significantly more frequent in stochastics tasks than in all other tasks, see Table 1.

	Subject area					
	Calculus	Lin. Algebra	Stochastics	Anal. Geom.	Analysis of Variance	
Lexis	($n = 182$)	($n = 79$)	($n = 191$)	($n = 110$)	$F(3,558)$	p
Nouns ¹	21.2 (3.53)	21.6 (4.84)	20.8 (3.73)	20.6 (3.64)	1.372	.250
Verbs ¹	7.43 ^b (1.95)	7.57 ^b (1.98)	8.46 ^a (2.08)	7.76 ^b (1.79)	9.477	<.0001****

Table 1: Mean, standard deviation, variance analysis and pairwise comparison of proportion (in percent) of nouns and verbs per sentence per task for subject areas.

Lexis	Course level		Analysis of Variance	
	advanced (n = 325)	basic (n = 237)	<i>F</i> (1,560)	<i>p</i>
Concreteness	3.81 (0.27)	3.79 (0.28)	0.322	.571
SUBTLEX	1.40 (0.47)	1.31 (0.51)	4.785	.029*
Wikipedia	1.77 (0.22)	1.75 (0.22)	1.367	.243
Abitur	3.17 (0.30)	3.24 (0.32)	8.288	.004**
Prepositions ¹	10.4 (3.13)	9.99 (3.32)	2.559	.110
Coord. conjunctions ¹	2.06 (1.63)	2.19 (1.82)	0.810	.369
Subord. conjunctions ¹	1.39 (1.11)	1.24 (1.18)	0.338	.561
Nouns ¹	21.27 (3.78)	20.06 (3.87)	4.229	.040*
Verbs ¹	7.88 (0.02)	7.76 (0.02)	0.043	.835

Table 2: Mean, standard deviation, analysis of variance (t-test) of lexical characteristics according to course level (¹in percent).

Secondly, the tasks were grouped by the course level (basic or advanced) they were designed for, see Table 2. None of the two groups consists of significantly more concrete vocabulary than the other. While there was also no significant difference between the word frequencies of the task groups in the Wikipedia corpus indicating no difference in use of technical or academic language, the language in tasks for advanced level appears significantly closer to everyday language, here operationalized by German subtitles. Vice versa, tasks for the basic course level are significantly more typical for the Abitur task corpus itself. The two task groups mostly do not seem to differ in their use of word types. As the only exception the use of nouns is significantly elevated in tasks for the advanced course level indicating a tendency to condensation via nominalization although the proportion of verbs is not significantly lower.

Finally, the difference in lexical and syntactic density between informative (level 0) and instructive text sections as well as between instructive texts of different levels of demand was determined, see Table 3. Especially informative texts seem significantly different from instructive text sections. First, the vocabulary implemented in text sections of level 0 is significantly more concrete and particularly typical for the task corpus itself, supposedly because these tasks make up a large proportion of the sample of text sections. Furthermore, together with subtasks of the highest level of demand (level 3) the vocabulary in informative sections seems to be significantly closer to academic/technical language according to the higher frequency in the Wikipedia corpus. Level 3 tasks were the closest to the language of German subtitles followed by informative texts and level 2 tasks. As for the lexicosyntactic compactness we could observe a significantly lower proportion of prepositions for level 1 tasks indicating a lower use of prepositional phrases, while in level 3 tasks there could be detected the most prepositions per sentence per subtask.

Lexis	Level of demand				Analysis of	
	0 (n = 731)	1 (n = 615)	2 (n = 943)	3 (n = 432)	Variance <i>F</i> (3,2717)	<i>p</i>
Concreteness	3.92 ^a (0.38)	3.77 ^b (0.41)	3.73 ^b (0.37)	3.75 ^b (0.37)	36.41	<.0001****
SUBTLEX	1.37 ^b (0.65)	1.16 ^c (0.98)	1.36 ^b (0.87)	1.58 ^a (0.73)	22.23	<.0001****
Wikipedia	1.79 ^a (0.30)	1.68 ^b (0.44)	1.72 ^b (0.40)	1.79 ^a (0.32)	12.81	<.0001****
Abitur	3.24 ^a (0.53)	3.14 ^b (0.35)	3.17 ^b (0.36)	3.17 ^b (0.38)	7.429	<.0001****
Prepositions ¹	10.4 ^{ab} (4.90)	9.46 ^c (5.62)	10.2 ^b (5.22)	11.2 ^a (4.95)	9.237	<.0001****
Coord. Conj. ¹	2.38 ^a (2.63)	1.81 ^b (2.67)	1.59 ^b (2.47)	1.50 ^b (1.97)	17.26	<.0001****
Subord. Conj. ¹	0.48 ^c (1.12)	2.37 ^a (3.03)	1.85 ^b (2.56)	1.82 ^b (2.56)	79.44	<.0001****
Nouns ¹	22.0 ^a (5.76)	21.2 ^{ac} (7.18)	22.0 ^{ab} (6.16)	20.7 ^c (5.12)	6.594	<.001***
Verbs ¹	6.54 ^c (3.30)	7.69 ^b (4.95)	7.98 ^{ab} (3.98)	8.42 ^a (3.34)	26.58	<.0001****

Table 3: Mean, standard deviation, variance analysis and pairwise comparison of lexical characteristics according to level of demand (¹ in percent).

The proportion of conjunctions per sentence per text section depends on the conjunction type: In informative sections a significantly higher proportion of coordinating conjunctions but the lowest proportion of subordinating conjunctions were observed, while level 1 tasks showed the highest proportion of subordinating conjunctions. For both coordinating and subordinating conjunctions an additional Kruskal Wallis test was conducted which confirmed the differences between units of different levels of demand ($\chi^2_{nK}(3) = 71.83$, $p < .0001$; $\chi^2_{uK}(3) = 189.9$, $p < .0001$). The following Dunn test, however, only determines the proportion of subordinating conjunctions units with level 0 as significantly lower than the rest. Concerning the nominal structure of the analyzed tasks the highest proportion of nouns and the lowest proportion of verbs was observed for informative sections, while for level 3 subtasks the opposite is true. Therefore, informative text sections (level 0) show a more frequent use of nominal structures and appear more compact, syntactically dense and complex.

DISCUSSION AND OUTLOOK

As suspected, the lexical and lexicosyntactic structure of mathematics tasks depends, at least for this sample of contextual tasks in school leaving exams, on several curricular factors. First, tasks at the advanced course level seem to be slightly more complex. This is in accordance with the observation that these may be closer to everyday language and have a higher proportion of nouns per sentence. It is possible that a larger proportion of these tasks refers to real-life situations in everyday language while tasks for the basic course level make more use of recurring phrases typical of mathematics exam tasks. However, the most striking differences in vocabulary and word types were observed for items of different levels of demand. Especially texts with informative function (here level 0) show a particular linguistic structure: They are significantly more concrete, closest to academic language, here operationalized by the

Wikipedia corpus, and syntactically dense with higher proportions of prepositions and nouns. This leads to the conclusion that, here on a lexicosyntactic but possibly also on other linguistic levels, the language in contextual tasks in mathematics needs to be considered for two different sections: Instructional texts differ significantly from informative texts with the latter constituting a significant proportion of tasks that are based on real-life situations. Consequently, these different sections may require separate sets of reading and solving skills from the students so that all linguistic analysis of such tasks needs to consider them as a set of different text types with different speech functions (Halliday, 2004). As already discussed in the previous study (Ebel et al., 2025), tasks from different subject areas differ significantly in terms of their vocabulary with, for example, stochastics tasks appearing closest to everyday language and calculus tasks being significantly less concrete. In terms of syntactic density through elevated proportions of prepositions in comparison to conjunctions calculus tasks also seem to be more syntactically dense whereas stochastics tasks show a higher proportion of conjunctions and the lowest number of prepositions per sentence. Adding the proportion of nouns and verbs to the analysis, stochastics tasks also show the highest proportion of verbs per sentence per task which may indicate a lower tendency to compactness which supports the existing claim of stochastics tasks having the least dense syntactic structure. In general, the limitation to one linguistic model describing the genre of mathematical tasks does not seem to be sufficient as several lexical and syntactical parameters may depend on curricular factors. Furthermore, studies need to be conducted on interactions between those factors.

Additionally, contributing factors such as procedural competencies, e.g. modelling or problem-solving in contrast to arithmetic or algebraic tasks, as well as the topics of the real-life situations used need to be considered. The logogenic pattern (Halliday, 2004) of contextual mathematics tasks might also consist of specific phrases that either share structural characteristics or are repeated in their entirety as characteristic features of these tasks. For this it may be promising to add a qualitative approach in the research on linguistic features at the word and sentence level. Furthermore, this project until now has focused mainly on verbal expressions of written language with other semiotic resources such as symbols and visualizations, apart from counting their appearance (Ebel et al., 2025), being excluded from the analysis.

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DEDUCTIVE REASONING IN A SPATIAL TASK BY PITJANTJATJARA SPEAKING CHILDREN

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There is little research about primary aged children's deductive reasoning, especially in minority languages. We use an interactive spatial reasoning task to investigate how Pitjantjatjara-speaking children solve this task. A methodology was developed to track how locational and orientational information is combined by adults to make a spatial description, and how this information is used by children to make correct matches in a card matching task. This includes probabilistic models of reasoning and pragmatics. Pitjantjatjara speaking children make few errors, using spatial information to make logical and pragmatic inferences and resolve ambiguity.

INTRODUCTION AND CONTEXT

This paper reports on the novel use of an interactive spatial reasoning task to investigate deductive reasoning. While research on spatial reasoning continues to grow, there is little investigation of logical deductive processes in solving spatial tasks amongst primary aged children. We add the complexity of language to the mix by looking at children's problem solving in Pitjantjatjara, an Indigenous language of Australia. This provides rarely documented evidence of these children's cognitive capacities operating in their own languages, while adding a game-theoretic pragmatic approach to deductive reasoning. We used the Man and Tree elicitation task (Terrill & Burenhult, 2008) which is common in spatial language research. Such research is often descriptive, and focusses on what the speaker is doing; however we focus on the task as a very rich **interactional problem-solving** task.

The study was part of a project developing an early primary mathematics curriculum in Pitjantjatjara in collaboration with Areyonga School and community. Areyonga is a small community (under 250 residents) in Central Australia. Areyonga School provides English–Pitjantjatjara bilingual education but the majority of mathematics is taught in English. Pitjantjatjara is a Western Desert, Pama-Nyungan language. It has several thousand speakers and is still learnt by children as their first language. We were initially interested in the differences between child-directed and peer-directed speech by adults in this spatial reasoning task, as part of trying to understand normal development of Pitjantjatjara children's' spatial language and thinking. In piloting we found no notable differences between strategies used in adult-adult vs adult-child pairs. We therefore decided to concentrate only on the adult-child interactions. In this paper, we focus on how children solve the task, developing a method to track the information state of speaker utterance and available interpretations. Our research question is: How

do Pitjantjatjara speaking children comprehend and make use of spatial descriptions in a problem-solving task? We describe the methodology that we developed and present some examples of Pitjantjatjara speaking children's spatial reasoning.

SPATIAL REASONING AND ABORIGINAL CHILDREN

Spatial reasoning is increasingly understood to be related to mathematical proficiency, although the precise mechanisms and relationships are still under investigation (Lowrie et al., 2020). While many aspects of spatial reasoning can be investigated non-linguistically, there is evidence that language provides a “cognitive tool kit [...] ... symbolic systems that potentiate new ways to represent and reason about the world” (Gentner et al, p. 319, 2013). It is reasonable therefore to expect that differences between languages in spatial expression will lead to differences in spatial cognition between speakers of different languages.

There has been extensive study of differences in spatial language and cognition (Levinson & Wilkins, 2006; Palmer & Gaby 2022). There is less cross-cultural research on how children learn spatial language, although there is some evidence for language-specific developmental paths (Brown & Levinson 2000; Haun et al., 2006). There has been some research on visual spatial memory skills in Aboriginal desert children (Kearins, 1978; Butterworth et al., 2011; Lowrie et al., 2022) but little research that relates spatial cognition to the languages of those young speakers. This is important because many Aboriginal students who live in remote locations may not have access to the spatial language that is used in the school that they attend, where English is usually the medium of mathematics instruction (Lowrie et al. 2022), hence their development of logical thinking may not be accurately measured in school.

DEDUCTIVE REASONING

Research into the deductive reasoning of young children of young children is more limited than that focussing on older students and formal proof. The mental models theory (MMT), which proposes that rather than using formal rules of inference, reasoners construct visual-spatial mental models which they mentally manipulate including to generate counter-examples (Johnson-Laird & Byrne, 2002), has proved useful for the study of the development of reasoning. However there is increasing evidence for a dual strategy model which proposes that people can use either a counter-factual strategy or a probabilistic one in reasoning (Markovits et al, 2017; Verschueren et al., 2005). This has conceptual overlaps with game-theoretic approaches to pragmatics (Benz & Stevens, 2018; Franke & Jäger, 2016) that propose that people use probabilistic reasoning about interlocuter's intentions during communicative interactions which is useful for investigating logical reasoning in a paired problem-solving situation.

METHODOLOGY

Participants

We recorded 11 pairs of adult directors and child matchers. There were 11 children age 8;1 – 12;5 (mean = 10;7, 4 male, 7 female). There were 7 adults: 30s-70+, all female, who all worked in the school presently or previously. All participants were first language speakers of Pitjantjatjara, with varied levels of English.

Man and Tree Task

The Man and Tree task (Terrill & Burenhult, 2008) is a barrier card-matching task. A *director* describes each card, and a *matcher* finds the matching card from their own set, which is displayed on a vertical surface. The matcher can ask questions. We used the Senghas version comprising 16 photo cards of a small toy man and tree (Figure 1). The cards are named R_{xy} ; x refers to the *orientation* (ORI) of the man and y to his *location* (LOC) in relation to the tree. For example, R_{x1} refers to any card where the man is in the background of the photo, and the tree is in the foreground.

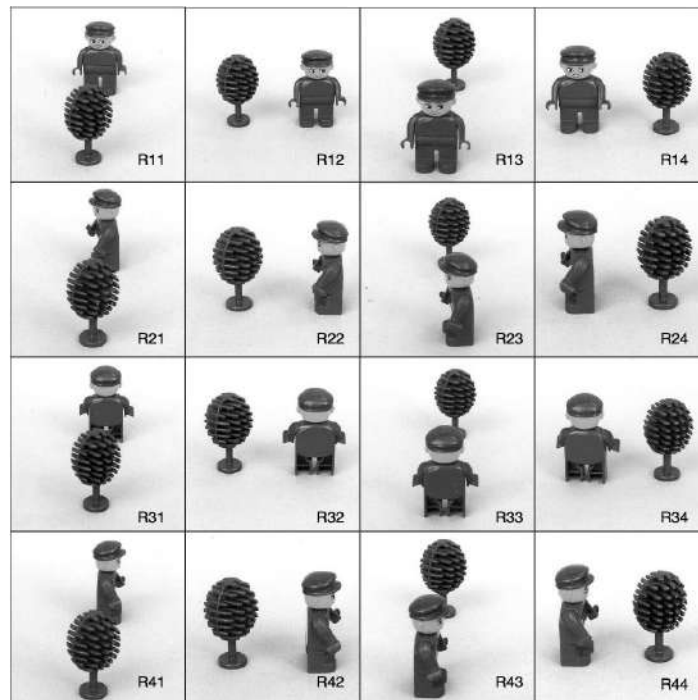


Figure 1: Man and Tree cards (Terrill and Burenhult, 2008).

We used a modified *OzSpace* protocol (Ennever et al., forthcoming) which specifies a fixed layout and order of cards. Each pair of participants sat side by side with a barrier between them, facing west to preserve the same geocentric orientation across pairs. The task was adapted for children by reducing the set of cards from hard (full set of 16 cards) to medium (12 cards) and to easy (8 cards). We started each round with the harder level that we thought that child was likely to be able to achieve, reducing the difficulty if required. Pairs also did a warmup activity using simpler topological relations pictures to learn the rules of the task. Sessions were video- and audio-recorded

with lapel mics on each participant. The recordings were later transcribed and translated by the second author using ELAN linguistic annotator software.

Coding

Each utterance had between 1-3 spatial expressions, each of which was coded for strategy, frame of reference, and function (whether the expression refers to orientation and/or location. For this paper we concentrate on how the combination of locational and orientational information is used to make a successful match. An example of a LOC expression is ‘The man is standing **on the east side**,’ an ORI expression could be ‘The man is **facing east**.’ An utterance such as ‘The man is standing **to the east**’ may refer to location, orientation or both. We also coded the matcher’s responses as touches (matcher deliberately touches a card but does not pick it up; $n=157$), selections (matcher picks up a card and shows it for confirmation; $n=302$), questions ($n=30$) and self-talk ($n=24$). Touches and selections were also coded as to whether they matched the target card’s location and orientation.

Information states

We also wanted to track what information was available to matchers: whether there was enough information to get the right card or potential ambiguities. For each expression, we therefore also coded whether the matcher could deduce that the expression referred to LOC and/or ORI (each with three values: Y/N/Potentially). At the utterance level, we also coded for *cumulative completeness* for both LOC and ORI (Y/N/Potentially). ‘Potentially’ includes where the set of potential interpretations is definitely restricted and potential restrictions where expressions may have variable readings yet to be clarified. For example, the allative suffix *-kutu* ‘towards’ can indicate either orientation or a vague location (Goddard, 1985, pp. 43–44); it was a source of many ambiguities. The coding allowed us to track how speakers combine strategies to provide different types of information, and how information states are updated interactionally.

RESULTS

We conducted 148 successful trials, coding 667 director utterances and 893 spatial expressions. Within-trial repetitions are excluded from our counts, that is, when the same strategy and frame of reference is used for the function. There were very few director errors (8 errors in 4 trials). The most frequent strategies were cardinal directions (50.8%), deictic directional prefixes (18.6 %) and a pair of terms meaning ‘on this side of X’/‘on the other side of X’ (*tjangati/munkara*) (10.5%). *Kuranyu/mala* ‘in front/behind’ were less frequent (5.0%) but there were interesting ambiguities that occurred with their use. While we had initially intended to compare child-directed and peer-directed speech by adults, we found no noticeable differences. There were also very few differences between strategy uses with the younger and older children in our sample. For each strategy we calculated the Pearson correlation between age and proportion that strategy was used. Only ‘local landmark’ strategy was significant at p

< 0.05 ($r(9) = -.74$, $p = .009$); that is adults were more likely to use utterances such as ‘He’s looking at you’ with younger children. The task was not successful with children as directors. Therefore we report on only the adult-directed sessions.

The children were usually correct on their second selection, making an average of 1.04 incorrect selections per card. There was no correlation between age and accuracy. Complete errors were very rare (4% of selections, 6% of touches). Most of the time either LOC or ORI was correct or on the correct axis (70% of selections, 74% of touches). 25% of trials were completed with incomplete or ambiguous descriptions.

Example 1: Ambiguity and deductive reasoning

Utterance	Intent	Matcher: Possible Cards			Cumul. Info. State
		LOC	ORI	LOC+ORI	
1 a. <i>wati ulparira-kutu</i> man south-toward	LOC	$Rx4$	$R2y$	$R24$	$Rx4UR2y$
1 b. <i>ma-ngaranyi</i> away-standing	ORI	n/a	$R3y$	n/a	$R34$
1 c. <i>wilurara-kutu</i> west-toward ‘The man is standing away to the south to the west.’	ORI	$Rx1$	$R3y$	$R31$	$R34$
2 . <i>punu alinytjara-tjaiti ngaranyi</i> tree north-side standing ‘The tree is standing on the north side.’	LOC	$Rx4$	n/a	n/a	$R34$

Table 4: Description of card $R34$

Table 4 shows a description of card $R34$. In 1a the director intends a LOC function for *-kutu* ‘toward,’ but the matcher has either LOC or ORI interpretations available. The LOC interpretations are those cards where the man is standing on the south side (cards $Rx4$). The ORI interpretation is those cards where the man is facing south ($R2y$). Hence possible interpretations are any card in the union LOC+ORI: $Rx4UR2y$. The intersection of those two possible sets ($Rx4 \cap R2y$) is card $R24$. 1b has only an orientational meaning: the deictic prefix *ma-* ‘away’ means away from the speaker. Hence 1b could refer any of the cards $R3y$ where the man is facing away. This in fact already disambiguates the card: The intersection of $R3y$ with $Rx4UR2y$ is $R34$. In 1c the director again uses the ambiguous *-kutu* ‘toward’. The LOC interpretations are all those cards where the man is standing on the west side (cards $Rx1$). The ORI interpretation applies to all those cards where the man is facing west ($R3y$). The intersection of those two possible sets is card $R31$. A complete description has been given in 1, but it contains dense information with two partial ambiguities communicated in a single utterance. The matcher, still processing this information, touches card $R24$ at this point and leaves their finger on the card. In 2, the director provides unambiguous LOC information ‘The tree is on the

north side’ which is the set *Rx4*. One interpretation of the possible deductive processes is that the provision of unambiguous locational informational information in 2 allows the matcher to process the ambiguities in 1, helping resolve a LOC interpretation of *wati ulparira-kutu* ‘man to the south’ and an ORI interpretation of *wilurara-kutu* ‘man to the west.’ The matcher selects *R34*, the correct card.

Example 2: Pragmatic inferences

	Utterance	Director Intention		Available Interpretation	
		LOC	ORI	LOC	ORI
1 DIR:	<i>Punungka mala ngaranyi anangu</i> tree-at behind standing person The person is standing behind the tree	Y	Y	Pot.	Pot.
2 MAT:	<i>Yaaltji-kutu kunyun wangkanyi?</i> where- towards you saying Towards where, did you say?				
3 DIR:	<i>Punungka mala ngaranyi</i> tree-at behind standing Standing behind the tree	Y	Y	Pot.	Pot.
4 MAT:	<i>Yaaltji-kutu face-amani?</i> where- towards facing? Towards where is he facing ?				
5 DIR:	<i>Nyuntunya nyanganyi</i> You(object) looking Looking at you	N	Y	Y	Y
6 MAT:	<i>Nyangatjanku!</i> This is the one!				

Table 5: Description of card *R11*

Table 5 shows a dialogic negotiation between the director and the matcher to find card *R11*. In line 1, *mala* ‘behind’ is used; this generally refers to an egocentric behind, with the man on the other side of the tree from the speaker and could be a LOC statement referring to cards *Rx1*. There is also a weaker intrinsic sense of *mala* ‘behind,’ (the man is behind the tree because the tree is in front of him). Taking both senses of ‘behind’ as true minimises the effort expended in the description to provide a pragmatically sufficient description (Benz & Stevens, 2018). However, the matcher appears to make only a LOC interpretation, asking in line 2 for clarification using the ambiguous *-kutu* ‘towards.’ The director simply repeats in line 3 ‘standing behind the tree’, perhaps interpreting the matcher’s use of *-kutu* as having a LOC function only. In 4 the matcher asks more specifically ‘Towards where is he facing?’ using the English loan word ‘face’. In 5 the director responds with the requested ORI-specific information ‘looking

at you,’ using a local landmark strategy. This allows the matcher to disambiguate the set *Rx1* to find *R11*, saying with confidence in 6: *Nyangatjanku* ‘This is the one!’.

DISCUSSION

Man and Tree is a pragmatically complex task. Matchers need to make deductive inferences and probabilistic guesses to solve the task. While our access to the children’s reasoning processes is not direct, the observation of their touches as well as selections of the cards gives us some insight into the processes of elimination that appears to occur. Our data shows us that Pitjantjatjara-speaking children use information in reasoned ways. They make logical and pragmatic inferences to resolve ambiguity. They ask for clarification, further refining their questions until they receive information that provides them confidence to make correct selections. Their errors are not random – they are usually within a contrastive class, for example mixing up ‘east’ and ‘west’ or ‘on this side’ and ‘on the other side.’

In adapting a tool developed for adult linguistic elicitation to investigate child reasoning, our methodology allows us to determine how linguistic expressions can be combined to create a complete description of a card as well as to track ambiguities in director’s expression. The specifics of what would make an ambiguous or non-ambiguous spatial expression will differ from language to language, but the tool and methodology should be possible to use with any language. This is because the administration of the task involves the participation of adult speakers of the language, rather than the translation of something designed in another (frequently European) language. Our study also provides openings to consider more deeply the relationship between language, spatial reasoning and deductive reasoning.

CONCLUSION

This study adds to the very limited research on the development of children’s spatial language and cognition in non-WEIRD (Western, Educated, Industrialized, Rich, and Democratic) settings. This is particularly important in an educational context where Aboriginal students are usually assessed and measured for mathematical cognition and attainment in English. Being able to observe Aboriginal language speaking children engaging in mathematical reasoning at their capacity in their own language should not be a novelty, however the opportunity for this is rarely provided in Australian schools.

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MAPPING OBJECTIFICATION IN EARLY ALGEBRAIC DISCOURSE

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This study is part of a larger study designed to produce tools for mapping students' arithmetic and algebraic discourse based on the commognitive theory. Commognition theorizes learning as a transition from ritual to explorative participation in the mathematical discourse, characterized by increased objectification. We propose a method that examines the level of objectification in 7-8 grade students' algebraic discourse, while identifying the objects about which the narratives in the students' discourse revolve. We demonstrate this method on one algebraic problem. Our findings show that our method succeeds in differentiating canonical as well as non-canonical solutions and in characterizing them on a continuum towards the objectification of a generalized number, which can be realized as a variable or an unknown.

Algebraic thinking is a critical milestone in mathematics learning. Successfully learning algebra in school opens the gate to the rich world of academic mathematics. Yet, the transition students experience in school from arithmetic to algebra is accompanied with some challenges that are not yet fully addressed by the research community (Kieran, 2022). Some of the efforts to overcome these challenges led mainstream studies to develop diagnostic tools that reflect snapshots of students' performance at particular points in time (e.g., Stacey et al., 2018; Klingbeil et al., 2024). These tools mainly focus on student's procedural challenges and misconceptions, yet they do not offer much insight into the developmental trajectory of algebra learning. Indeed, there is need for assessment tools that rely on a developmental theory of algebra learning, including the ways in which algebra learning builds on more basic, arithmetic, mathematical knowledge (Warren et al., 2016; Kieran, 2022). One such theory has been offered by Caspi and Sfard (2012), who suggested that algebra develops as a meta-discourse on arithmetic. So far, the commognitive theory has been useful for providing diagnostic tools for mapping students arithmetic discourse (Ben-Yehuda, 2003; Heyd-Metzuyanim et al., 2022). However, applying these tools to map students' algebraic discourse has remained a challenge, since a main part of the theory of de-ritualization, concerning objectification, has not been operationalized in terms of mastery of algebraic skills. Our aim is to address this gap, by offering a methodological tool that discerns different stages of objectification in the algebraic discourse.

THEORETICAL FRAMEWORK

The purpose of learning according to the commognitive paradigm (Sfard, 2008) is the learner's transition from ritual participation in discourse—focused on performing

procedures that imitate the performance of an expert—to explorative participation—focused on producing narratives about mathematical objects in discourse. A central aspect of this process is the objectification that occurs when the learner moves from using keywords or signs as “empty” signifiers (that do not denote anything but themselves) to those that denote objects (Lavie et al., 2019). The process of algebraic object formation germinates when students make their first steps in meta-arithmetic discourse (Caspi & Sfard 2012) and continues when students use symbols that signify algebraic objects. These will be called symbolically mediated objects e.g., $\frac{x}{7}$ that signifies a number divided by seven. Another aspect of explorative participation concerns the characterization of narratives in discourse. In explorative participation, the learner attempts to produce stories about mathematical objects in a consistent and logically connected manner (Baccaglini-Frank, 2021).

This study is part of a large-scale study designed to produce tools for mapping students' arithmetic discourse and algebraic discourse. In previous parts of the study, a tool was developed for mapping elementary school graduates' arithmetic discourse on the continuum of ritual and exploration (Heyd-Metzuyanim et al., 2022), as well as a tool for visually mapping aspects of objectification in algebraic discourse (Shahla Demirdjian, 2025). However, objectification in algebraic discourse remains difficult to map due to several challenges. First, there are several objects in algebraic discourse (variable, unknown, equation, etc.). Second, previous studies (Cohen, 2024; Shahla Demirdjian, 2025) have indicated a great deal of variation in the ways in which seventh- and eighth-grade students solve algebraic problems noncanonically, which has made it difficult to map the development of discourse along a single axis. This mapping is important for progress toward quantification, which will allow the application of commognitive theory to larger samples.

At the beginning of learning algebra, the key change required in students' discourse is a shift from arithmetic discourse, which deals with specific numbers, to discourse that deals with generalized (non-specific) numbers. Therefore, to characterize the degree of objectification in algebraic discourse, we ask: *To what extent do students' implicit and explicit stories in algebraic discourse revolve around generalized numbers?*

RESEARCH METHODOLOGY

The data are taken from a series of EADP (Early Algebraic Discourse Profile) interviews with 10 seventh grade students from different schools in Israel, and at different achievement levels. The EADP interview was conducted in a "think-aloud" format and contains 13 algebraic tasks taken from the seventh grade Ministry of Education exams and from Caspi (2014). To demonstrate the analysis method we developed, we focus on one of the questions - the "I thought of a number" problem:

I thought of a specific number. If I multiply it by seven and subtract from the product fifty-four, I will get the number I was thinking of. What is the number I was thinking of? Explain how you solved it.

We should emphasize that the problem is communicated within the meta-arithmetic discourse and is telling a story about a generalized unknown number (“I thought of a specific number” and “the number I was thinking of”). The interviews were video recorded using two cameras, one recording the writing and the other the student's face and were fully transcribed. To demonstrate the differential sensitivity of the analytical method, five students were selected (Liat, Gil, Tom, Mika and Alon – pseudonyms) but each of them used a different procedure and not all of which were canonical in the algebraic discourse. The analysis includes two stages, the first, detecting narratives in the students’ discourse, and the second, detecting the subject of each narrative, deciding whether it is a signifier of a mathematical object and if so, is it a generalised number. The narratives are sorted into three categories according to the mathematical object they are about: specific numbers, verbally generalized numbers, and symbolically mediated generalized numbers. Each narrative is marked as explicit if it was talked about directly or implicit if it was only implied by the student’s descriptions of his or her procedures.

FINDINGS

The analysis focuses on the solutions of five students, four of which produced non-canonical algebraic procedures: **Mika** focused solely on a trial-and-error procedure and found that the solution is the number nine. **Liat** declared at the beginning of the solution that she could solve the problem “in two ways”. She then tried to construct an algebraic expression but was unable to create an equation. She tried using arithmetic calculations, but this procedure was also unsuccessful, and she gave up. **Gil** also began solving the problem algebraically and, like Liat, encountered difficulty in creating an equation. However, when he reached a dead end with the equation, he switched to a trial-and-error procedure and like Mika, found that the number was nine. **Tom** began by declaring that he had no idea how to solve the problem, but with the interviewer’s encouragement, he created an equation. However, the equation was not canonical and the operations on it were not canonical, so he was unable to reach a solution. Finally, **Alon**, is the fifth student and the only one that formed an equation and solved it canonically.

Below is an analysis of the the five students’ solutions according to the explicitness of their narratives and the objects they revolve around.

In Mika’s discourse we found mostly narratives about specific numbers, yet we also found an implicit narrative about a generalize number. Mika’s solution is based solely on performing procedures on specific numbers. She starts her solution with:

237 Mika: ...say, well, I think about, say, (the number) two, I multiply it by seven, no, it doesn’t make sense, because it doesn’t get the...say I think of (the number) ten...no, this also doesn’t make sense, because if I think of ten... If I multiply by seven it equals seventy but, like, fifty-four minus seventy I already know by heart it cannot be two (*corrects to “ten” after interviewer asks, “why two?”*).

Mika clearly performs a procedure of trial and error by substituting the sequence of whole numbers: two and ten. Later she tries five and then nine, which leads her to concluding the number is nine.

Although Mika's procedures involve only specific numbers, her usage of a series of these numbers implies that she interprets "the number I thought of" appearing in the problem's text, as any number, and therefore, a generalized number.

In Liat's discourse, no narratives were found about generalized numbers, but only about specific numbers. For example, in the following passage:

272 Liat: Umm... x, I can do (it in) two ways, x times seven equals... uh...
to...something,

In this excerpt, although Liat writes a symbolic expression ($x \cdot 7 =$), the expression does not tell a story about mathematical objects. It talks only about actions that Liat can do. After the symbolically mediated attempt does not lead Liat to a satisfactory description of the situation, Liat moves, without realizing it, to trying to solve a problem similar to the given problem in which the resulting number is not "the number I was thinking of" (a generalized number), but the number seven. Such a problem would be indeed solvable using the reverse arithmetic operations. Liat performs on these reverse operations described, calculating the expression $\frac{7+54}{7}$ and then $\frac{61}{7}$. During this attempt, Liat's narratives revolve exclusively around specific numbers. After encountering difficulties with this attempt as well, she starts the procedure over again. During this process, she writes the expression $7 \cdot x - 54$. Regarding this procedure, Liat notes that "x is the specific number." This narrative ostensibly revolves around a generalized number, but it contains nothing more than a matching of signs ("x" to "the specific number" as worded in the task). Therefore, in Liat's discourse, there are no narratives whose object is a generalized number.

In Gil's discourse, one can find narratives that are mostly concerned with specific numbers. However, we can also find in his procedures implicit and explicit narratives about generalized numbers. In the following we explicate the different kinds of narratives we found in his discourse.

190 Gil: Okay, fine. So I'll do it x plus fifty four divided by seven.

Similarly to Liat's narrative [272], Gil's narrative in turn 190 describes a symbolic expression ($\frac{x+54}{7} =$) as something Gil does and not as a narrative about a generalized number. Gil also performs a trial-and-error procedure and finds the desired number. He says: "the number is nine" [194]. This utterance is indeed a narrative about a generalized number, as the signifier "the number" signifies the "the number I thought of" in the given problem, and the story told here regarding that number is that it is nine.

When depicting the procedure that yielded the canonical narrative Gil elaborates:

207 Gil: I tried to put in a number that was like a two-digit number and then I realized that it didn't make sense.

- 209 Gil: And then I tried... I added thirteen and then I realized that it didn't make sense as if it was too big a number, I tried the largest single digit, and I just got nine, I got nine.

Similarly to Mika, the fact Gil considers “a number” as a signifier of “any number” and his substitution procedure imply that the narrative given in the problem is about a generalized number. In addition, in this explanation, there are two verbal descriptions of generalized numbers: “a two-digit number” and “a single-digit number” which Gil tells the following two narratives about: When a two-digit number is substituted for “the number I thought of” it yields a number which is too big, and - the largest one-digit number is nine. To conclude, we found in Gil’s discourse symbolic signifiers detached from any object, narratives around specific numbers and implicit and explicit narratives (verbal, non-symbolic) around generalized numbers.

In Tom's discourse, one can find symbolically mediated narratives about generalized numbers.

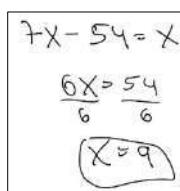
- 21 Tom: (Asking for clarification of the problem) It's as if we were to start with the same number, starting with one and ending with the same number, right?
- 24 Int: ...Do you have a way to begin with?
- 25 Tom: to go through every possible number and check it.

The verbal signifiers “the same number” and “every possible number” signify a generalized number. The first narrative [21] is embedded in what Tom thinks is given in the problem. This narrative implies the question: Is it true that we get the same number we started with? [21]. The second narrative is implied in the procedure Tom suggests for solving the problem. Like Gil and Mika, Tom uses the verbal signifier “the number I thought of” as a number which can potentially be any number, and therefore a generalized number.

As he continues to engage with the task, Tom formulates a non-canonical equation and at a certain point, he reaches the equation $\frac{x}{7} + 54 = \frac{x}{7}$ and says: “No, I just don't understand... How can something plus 54, be equal to the same thing?”. The verbal signifier “something” indicates $\frac{x}{7}$, which is used in the equation as some numerical quantity, that is, as a signifier of a generalized number. We conclude that the narratives in Tom’s discourse revolve around generalized numbers signified both verbally and symbolically.

Last, we present Alon’s discourse. Alon without uttering a word, reads the problem, writes an equation and solves it.

- 1 Alon: ((Alon reads the question silently and writes))
- 2 Alon:



$$7x - 54 = x$$

$$\frac{6x}{6} = \frac{54}{6}$$

$$x = 9$$

3 Alon: The number is nine.

We can see Alon's narrative about a generalized number, in turn [3] where he states that "the number", which is the number he was looking for, is indeed nine. This is an explicit verbal narrative about a generalized number.

When requested to provide an explanation to his solution, Alon explains

5 Alon: I gave the number an unknown which is x ,

Here we see a narrative about the same "number" which Alon previously recalled was "nine" and which is associated with the "number I thought of" in the problem text, yet is also symbolically mediated. Most of Alon's explanation onwards is devoted to describing his procedure of solving the equation. It includes descriptions of each of the steps Alon performed on the various symbolic signifiers. For example, "now I did... seven x ...to move it (the sign x on the right side of the equation [2]) to here (the left side)" is a description of moving the sign x from one side of the equation to the other. This description does not include any narrative about a generalized number. We conclude that in Alon's discourse there are explicit narratives about both verbal and symbolic generalized numbers, although his narratives about manipulating the equation refer to empty signifiers.

To summarize the findings, we use the following table (Table 1):

Table 1: Narratives and Canonical Procedures in Students' Discourse

student	Narrative about specific numbers	Narrative about verbal generalized numbers	Narrative about symbolic generalized numbers	Canonical symbolically mediated procedure
Mika	+	+/-	-	-
Liat	+	-	-	-
Gil	+	+	-	-
Tom	+	+	+	-
Alon	+	+	+	+

Note: + indicates explicit narrative, - indicates no narrative and +/- indicates implicit narrative. When both implicit and explicit narratives were found we marked +.

From Table 1 we can locate the five students on a continuum from those who authored no narratives about generalized numbers at all (Liat) to those who produced narratives about generalized numbers that were also connected to symbolic mediators (Tom, Alon). We note that the narratives in Alon's discourse are not only about generalized numbers (verbal and symbolic) but also canonical. In short, the placement on a continuum according to objectification of generalized numbers is Alon, Tom, Gil, Mika, and finally Liat.

DISCUSSION AND CONCLUSIONS

Our goal in this study was to suggest a methodological tool for differentiating levels of objectification in students' algebraic discourse, focusing on the objectification of generalized numbers, or variable/unknown. Applying this methodological tool to the discourse of five students, we found that it enabled us to place the students' solutions on a continuum, even though in terms of school mathematics, most of these students' solutions could not be considered as canonical in the algebraic discourse. In that sense, our analysis unveiled phenomena that might be at odds with common assessment practices. For example, in Liat's discourse we found various symbolic mediators, yet these symbols never functioned as labels of generalized numbers or parts of stories about. Her algebraic discourse was therefore purely syntactic, or ritual (Baccaglini-Frank, 2021). In contrast, we saw that students' discourse which, at first looked purely "arithmetic" (such as Mika's), could include implied narratives about generalized numbers, which at the least enabled the student to translate the story told in the problem's text, to a meaningful story about specific numbers. We thus conclude that despite some students not using any algebraic signifiers in their discourse, their participation may be more explorative than those who do use such signifiers, yet without any reference to generalized numbers.

We believe our suggested method has potential to open multiple future avenues for research, both in terms of developing additional parts of assessment and diagnostic tools, as well as assisting in characterizing students' algebraic discourse as they engage in school-based tasks. This potential stems from our method's alignment with a developmental theory of mathematical learning which does not relate just to beginning algebra. Rather, the theoretical tools of ritual-exploration and objectification can (and have been) applied to multiple other topics in mathematics. This method therefore strengthens the power of commognition to map students' algebraic discourse in its various stages. In the future, it is expected to enable mapping students' discourse along the ritual-explorative continuum in a way that will allow quantitative research that enables comparison of students' discourse over time and between students. This will enable combining micro-analytical lenses, which are necessary for understanding subtle changes in learning, along with broader comparisons that require quantitative tools.

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COMPUTATIONAL AND MATHEMATICAL THINKING AT KINDERGARTEN: MOVING *WITH* THE GRID

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This research study strives to investigate the ways in which 5 years old students can approach mathematical thinking through coding, taken as a first step towards the development of computational thinking. To this end, thirty kindergarten children were engaged in a series of activities involving the use of an educational robot, a Blue-Bot, with the aim of introducing the square as a geometric figure with spatial properties. The study focuses on video-taped episodes from the activities to reveal the importance of movement and bodily engagement with a grid—introduced to better structure the robot’s movement in space—in making sense of mathematical and computational concepts. In so doing, it wants to contribute to current discussions about the relevance and role of material practices in mathematics education.

COMPUTATIONAL THINKING & MATHEMATICS

Some decades ago, Papert (1980) pioneered a Logo-based educational approach, which envisioned a shift from teaching and learning mathematics through the manipulation of numbers, symbols and formulas by paper and pencil, to developing mathematical thinking by using computers as flexible and playful material. Since then, various Papert-inspired environments and microworlds have been taken up in the discourse of mathematics education, some in relation to the development of computational thinking, understood in the sense introduced by Wing (2006): that is, as a universally applicable attitude and skill across problem solving activities.

Some research has highlighted that computational thinking and mathematical practices intersect in educational contexts (e.g., Baldwin et al., 2013; Weintrop et al., 2016; Pei et al., 2018). Recent research investigates the potential challenges students experience in computational thinking-based mathematics environments (Cui & Ng, 2021) or examines computational thinking as a boundary object that can facilitate dialogue and transition between computer science and mathematics (Ng et al., 2023). Besides, the innovations of tools like Scratch or DGE—considered as a type of programming language—or other forms of computational thinking environments necessitate the exploration of new forms of *objects-to-think-with* (Papert, 1980), which extend the embodied and multimodal experience of learners with mathematical concepts (e.g., Sinclair & Patterson, 2018; Ferrara & Ferrari, 2022; Baccaglini-Frank et al., 2024). These technologies help promote a vision of mathematics learning as a creative and constructive process that directly involves learners in making and doing, in line with Papert’s vision. One reason for addressing computational thinking in mathematics education is linked to the role of problem solving and abstraction in mathematics and

the central position of computational thinking in interdisciplinary (STEM) practices. However, research on the relation between computational thinking and mathematical thinking as a powerful means to promote understanding in both domains is still scarce.

Recent studies have focused on computational thinking with educational robots at kindergarten, finding progress in learning computational thinking and spatial relations (Angeli & Valanides, 2020) or characterizing early computational thinking experiences in a guided-play environment (Hall & McCormick, 2022). Bers and colleagues (2019) have examined ways to incorporate computational thinking in STEAM teaching practices. On the other hand, Kotsopoulos and colleagues (2022) report about the spontaneous emergence of early computational thinking instances during free play in young children in a research context without any intentional intervention. This research shows the potential of computational thinking in early mathematics learning.

In this paper, coding is meant as a first step towards developing computational thinking and I want to focus on it as a type of material activity that can be introduced as early as kindergarten to build mathematical thinking.

MATERIAL PRACTICES IN MATHEMATICS EDUCATION

The relevance and role of material practices in mathematics education is a current topic of interest. De Freitas and Sinclair (2014) pointed out the need for rethinking the body in mathematics, expanding the embodied vision of mathematics learning to an inclusive materialist vision of mathematical activity which recognizes value both of the body and of the environment, both in thinking and in learning, and sees agency as distributed among humans, things and environment. Instead of isolating the human subject as separate from the environment and focusing more on the movements of the body alone, in a materialist perspective what matters are the movements of the body with/on things. In short, what matters is the body-things interaction, that is, the material aspects of mathematical activity. Movement plays a central role, since using a tool involves not only potentials and limits of the tool, but also structured ways of moving the body (*with* the tool). As movement becomes more and more structured, the environment also changes, becoming “more orderly and perhaps more mathematical” (Sinclair & Jablonka, 2024, p. 427), and what matters is the assemblage of body, environment (or tool) *and* mathematics, which is what moves the activity.

In previous work, I have studied different learning assemblages in different contexts, related to graphing motion, early algebraic thinking and ordinal aspects of number and involving both paper and pencil and digital tools (see de Freitas et al., 2019; Ferrara & Ferrari, 2017; Ferrara & Ferrari, 2022). In this study, a central point of concern is that doing mathematics with a tool (no matter whether physical or digital) is a kind of material practice where both the body (senses, bodily actions, gestures, ...) and the material surrounding (tool, surfaces, code, ...) have an import to mathematical sense making. What interests me are the ways in which tools can transform the computational image of mathematics (obtaining results based on procedures and formulas) into an expressive one (moving in particular ways through semiotic and material activity).

Indeed, I focus on how an educational robot can transform the teaching and learning of mathematics at kindergarten enabling a change in how children encounter both mathematical and computational concepts and in how the concepts themselves are vitalized or mobilized within the classroom. I also want to consider how body-based explorations of coding can create new opportunities both for learning and for teaching. In the next sections, I will first present the research context and methods, including the tool used in the study, then introduce the episodes and briefly discuss them.

CONTEXT AND METHODS

The protagonists of this study are 24 children aged 5 years old from a kindergarten in North-western Italy. The children learned about straight lines, direction, rotation and squares through a sequence of activities with the Blue-bot, an educational robot that I introduce below. Nine afternoon activities were carried out as part of a STEAM education project that the school had requested and that I was leading. The last activity involved the children's parents in the pursuit of one project goal: to change the view of mathematics from an abstract subject to a more concrete and accessible experience. The children were engaged in the activities divided into two groups of 12 children and 4 teachers were involved. The teachers participated in the activities as part of their probationary year at the school (new teachers have to be evaluated on a yearly project, which in this case involved teaching robotics and mathematics). The aim was to introduce the square as a geometric figure that satisfies spatial relations: congruence of the four sides and rightness of the four angles, through the movement of the robot.

Initially the activities only focused on children's bodily movement in space, with the aim of encountering the concepts of straight line and direction, as well as measure (activities 1 and 2). Subsequently, the robot was introduced to discover how it works, and the children were stimulated to learn how to move it between different locations on the floor (activities 3 and 4). Then, code was inserted as the programming language to move the robot, and the children were encouraged to discover how to compose movements (and pieces of code) to have L- and U-shaped trajectories, both through interpretation and prediction tasks (activities 5 and 6). From activity 4, a grid was used as a tool to interact with the robot and better structure its movement in space. The last activities were devoted to movement along a square-shaped trajectory, both using the code and imagining changes in the code according to the size of the square.

The activities were all video-taped using a mobile camera and written notes were taken by a researcher and a tutor (a teacher following the project to evaluate the work of the new teacher). The videos were watched a few times and class episodes were extracted for analysis. The data collected also included drawings created by the children during the activities. As far as the research methodology is concerned, the assemblage of the children, the Blue-bot and the objects constitutes my unit of analysis, and I will focus on how it changes and concepts emerge from it. I follow a qualitative analysis under a descriptive approach, using images taken from videos (drawing on the idea of re-

enactment of experience from Sinclair, 2024). The images are intended to invite the reader to simulate the activity while assuming the children's point of view.

The Blue-bot is a button-operated robot with a transparent body that allows you to see an electronic circuit inside and seven buttons on its back (Fig. 1). Four (orange) buttons show arrows for directions (forward/backward) and rotations (left/right). Two (blue) buttons are to stop or pause the robot's movement and the last (green) button ("GO") makes the robot move. Once a sequence of buttons ending with GO is pressed, the robot moves along a trajectory that depends on the sequence of commands.



Figure 1: The Blue-bot and the buttons on its back.

Programming the Blue-bot requires a code, i.e. a sequence of instructions, in this case given by arrows. The code is a new language for the robot's movement, which is more abstract in nature than physical movement, while retaining its structure. Working with the code involves the robot's body, its movement in space, the children's hands, the action of pressing buttons in certain orders, the perception of the robot's movement.

MOVING *WITH* THE GRID

The episodes have to do with the encounter of the children with the grid, in activity 4. The group I focus on only consists of 10 children (2 children were absent this day), one teacher and one tutor. A 4×4 square grid has been created on the floor with tape and consists of squares whose side is as long as two steps of the robot. Being it quite large, the children can all sit on the floor around it (Fig. 2a).

The grid is introduced as a tool to better explore the Blue-bot's movement along orthogonal directions. Up to this point, the children have already worked on straight and L-shaped trajectories, focusing on the (type and number of) instructions needed to make the Blue-bot move along them on the floor. They have begun to explore the number of instructions in relation to the length of the robot's step, and the task of making longer or shorter straight trajectories, which is difficult for the children without any reference other than the starting and finishing positions. The grid allows to return to these explorations by adding a reference system, implying the need for an exact number of steps of the robot, for example to realize the movement between two paper flowers (from red to blue, or vice versa) positioned at two intersections of the grid.

Longer or shorter trajectories

The first episode focuses on straight-line movement between two flowers one square apart, but the Blue-bot is initially positioned one square away from the first flower (Fig. 2a). The children are tasked to say aloud the instructions they want to give the

robot to perform the movement, before checking what they expect—an anticipation task that prompts the children to imagine the movement before it happens.

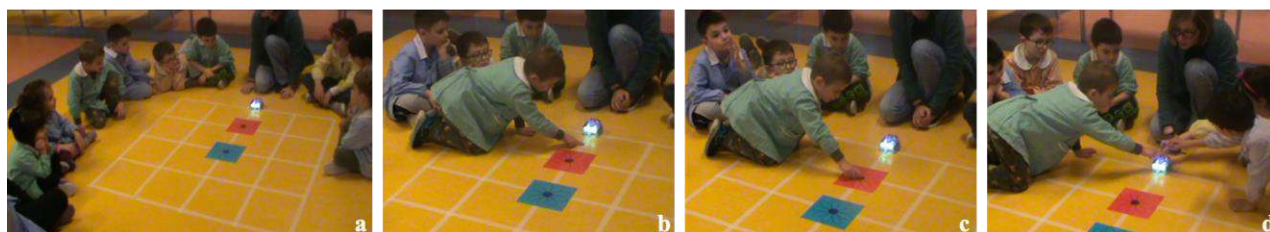


Figure 2: (a) Sitting around the grid; (b-c) moving forward; (d) pointing to GO.

Riccardo intervenes with the instruction to make the Blue-bot move from the initial location to the first flower, *‘Two buttons forward’* (instead of two steps forward). The teacher asks if she must press the forward arrow on the robot’s back twice (indicating it) and if all children agree: they agree in chorus. Riccardo is asked what he thought when he said twice and not once. The child suddenly moves inside the grid and affirms, *‘Because, because if it goes on the flower [points to the red flower] then it takes two, because one step is long up to here [moves the index finger of his right hand about halfway up the side of the square separating the robot from the flower; Fig. 2b] and then the other goes [moves his index finger to the red flower; Fig. 2c] on, on the flower. [points to the red flower]’*. Then the teacher suggests trying, presses the forward arrows twice and asks what to do next. Riccardo and other children reach out together towards the Blue-bot saying, *‘The green button’* and they all point at it (Fig. 2d).

The movement is realized, and the same strategy is used for the following movement from the red to the blue flower. As the teacher proposes the challenge of making the robot return to the red flower, Lorenzo, who sits on the opposite side than Riccardo, comes inside the grid and says, *‘Two steps backward,’* moving the index finger of his right hand two steps along the straight trajectory in opposite direction (Fig. 3a). He then verifies the movement with the Blue-bot, using the backward arrows.

A new challenge is introduced when the teacher moves the blue flower three squares away from the red flower to the last intersection of the grid, *‘If now I put the blue flower here, what does it happen?’* Beatrice rises her hand and says that more steps are needed: *‘More, because the robot, because the line is a little longer.’* Asked if more or fewer steps must be performed then, the children in chorus answer *‘More,’* and Beatrice highlights the change adding, *‘Because, if you do fewer it gets here. [reaches into the grid to indicate the previous position of the red flower; Fig. 3b]’*. Leonardo advances a conjecture about the number of steps, *‘You have to do eight [times] the one that goes forward,’* but Riccardo suspicious says, *‘I want to calculate first,’* and starts counting the three squares between the flowers and then the number of steps imagined for the robot (Fig. 3c). Riccardo takes four steps on each square after the first square, and ends up counting ten steps (instead of six). Only the realization of the movement after the Blue-bot’s programming, with the robot overcoming the blue flower and forcing Oceania to move back (Fig. 3d), makes him recognize the mistake.

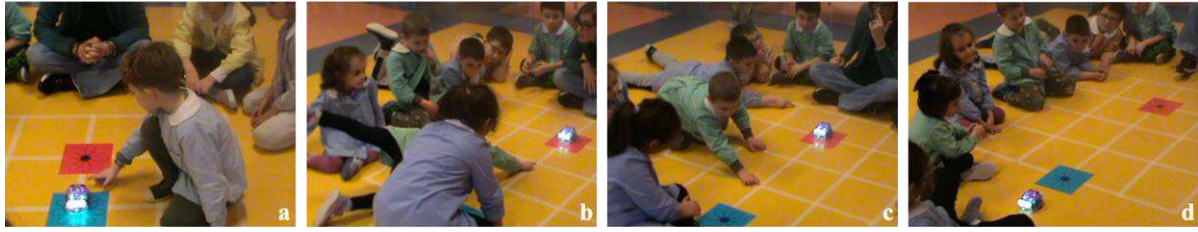


Figure 3: (a) Going backward; (b) highlighting changes; (c) counting steps; (d) the Blue-bot overcoming the blue flower.

The children understand that fewer steps are needed, and new hypotheses are made about the number of steps: eight, seven and six. The exploration brings forth first the question of how many steps cover the side of a square and then the rule of two steps per square as a strategy to conclude that six is the appropriate number of steps. Oceania programs the Blue-bot with a sequence of six forward arrows plus GO, and the children can check the forward movement. Instructions for the backward movement are easily anticipated by the children at this point in the activity.

Diagonal trajectories

The second episode focuses on movement between two flowers again positioned at two intersections of the grid but *not* on the same line. The situation is accidentally triggered by the teacher, when she is changing the movement configuration from a straight trajectory to a L-shaped one. She moves the red flower to a new position on the next line, saying ‘Now,’ and before she can place a third flower on the grid, Leonardo exclaims, ‘*Diagonally!*’, prompting an unexpected exploration. When asked to explain what he means, the child replies, ‘*As if it were an X*’ and moves the two flowers away from each other at two intersections on opposite sides of the grid (Fig. 4a).

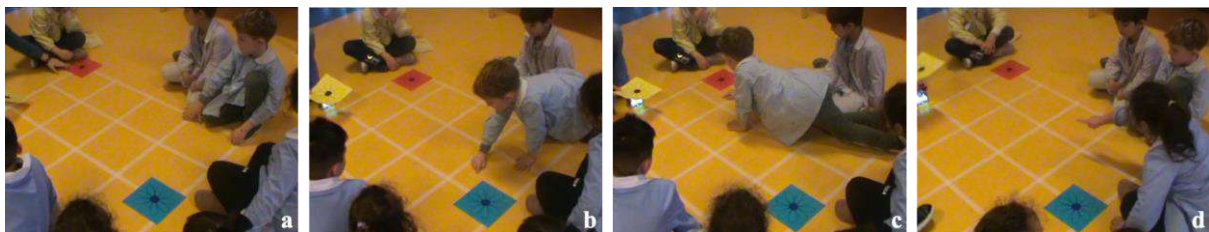


Figure 4: (a) Going diagonally; (b-c) moving diagonally; (d) stepping diagonally.

When asked, ‘*What path does the robot have to follow to get from one flower to the other? Show it*’, Leonardo stretches out his body across the grid, moving his right hand continuously from the blue flower to the red one (Fig. 4b and c). Then, when asked, ‘*How many steps should it take?*’, he stares silently at the grid for a few seconds, following the imaginary movement between flowers with his head and eyes, and says, ‘*Eight.*’ Oceania, sitting next to him, bounces the index finger of her right hand first on the grid, then in the air, making a movement towards the red flower as if to keep up with the robot’s steps (Fig. 4d). At the same time, when asked, ‘*Why eight steps?*’, Leonardo replies, ‘*Because the path is longer,*’ and as the teacher questions, ‘*So, shall*

we increase the number of?’, he says, ‘*Steps!*’. The children conclude that more steps (eight) are needed compared to the previous number of steps (six).

The discussion also raises the question of the orientation of the Blue-bot’s eyes in the direction of the red flower, which is significant for straight movement trajectories. On the other hand, increasing the number of steps is essential to understand that there are differences in length between straight paths on grid lines and straight paths on diagonal lines obtained simply by moving one end to a different position on the same grid line. Rational and irrational numbers are also involved, but that is beyond the scope of this paper. For the sake of space, I am interrupting the episode here.

CONCLUSIVE DISCUSSION

The episodes show the relevance and role of the material practice with the grid, which is introduced as a tool to better structure the Blue-bot’s movement. The grid is not inert. It is used not only to communicate (to the teachers and the other children) but also to operate on and with straight lines, directions, lengths (numbers of steps) and code (sequences of instructions/buttons). The children do not only move their bodies in space, but they move *with* the grid across imagined or real movements of the Blue-bot, straight or diagonal trajectories, possibilities of new lines or new distances, numbers of instructions or numbers of steps. The grid does not simply mediate knowledge: it makes pieces of knowledge emerge out of movement, like in the example of the Blue-bot moving out of the grid and forcing a child to move back, or in the example of the accidental shift of a flower to a different grid intersection prompting a new exploration. The children do not simply count the number of steps or give instructions, they relate steps and instructions to direction and length. I see the grid as an object to think with, which makes the environment increasingly mathematical *and* computational, not only in the sense that the children learn code and programming. The children are constantly challenged with new problems in the form of interpretation and prediction questions which involve the grid: both the code and the spatial relations are vitalized through locations and movement on the grid. The children move, smile, change, learn.

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INCREMENTAL AND AMBITIOUS PROFESSIONAL DEVELOPMENT FOR EARLY YEARS MATHEMATICS TEACHERS

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This article seeks to investigate the relationships between changes in the practice of teachers who teach mathematics in the early years and incremental and ambitious professional development, based on a formative process, with a focus on algebraic thinking. The methodology is qualitative-interpretive, and the data was collected from video recordings, questionnaires applied at the beginning and end of formative process and lesson observations of two teachers. The results indicate that incremental professional development could be the bridge on which the teacher can take safe steps towards ambitious practices. On the other hand, it is possible for professional development to work in an incremental and ambitious perspective in an integrated way, considering what the teacher already does in the classroom.

INTRODUCTION

Mathematics Professional Development (PD) aims to improve teachers' work and, in so doing, have an impact on student learning. In recent years, Mathematics Education has been investigating the results of the different PD that have been developed, which, considering the investment made (financial, time and resources), can be considered scarce (Otten et al., 2022). When I encountered the studies by Otten et al. (2022) and Star (2016), I was struck by a great concern: regarding the PD that I was planning and developing, was it enabling changes in the practice of the participating teachers? These authors question "what" should be prioritized in PD to help teachers who teach mathematics achieve "rapid success": PD that is ambitious or transformative, or it is possible to start with more modest goals? Considering this situation and seeking to understand the changes in teachers' practice when they participate in a PD, the purpose of this article is to explore the relationship between changes in the practice of teachers who teach mathematics in the early years and incremental and ambitious PD, based on a formative process. To this end, I aim to answer the following questions: (i) what changes have been reported or observed in the practices of early years teachers when they take part in a PD focusing on algebraic thinking? and (ii) what relationships can be observed between changes in the practice of these teachers and incremental and ambitious PD? If we want teachers to transform their practices to offer quality teaching, we need to understand how PD are helping them in this endeavor.

LITERATURE REVIEW

The field of Mathematics Education has been producing solid knowledge about the teaching of mathematics, presenting guiding principles of excellence to ensure that all

students are successful in their learning (NCTM, 2014). By taking these principles as objects of study, PD brings teachers the most innovative and powerful aspects of mathematics teaching. However, recent studies have shown that even though PD has been structured around these principles, there are few results in terms of student learning (Otten et al., 2022; Star, 2016). As an alternative to this problem, “some researchers have begun to argue for a more incremental improvement approach, one that develops bridges between teachers’ extant practices and desired innovations” (Litke, 2020, p. 2). Incremental change is characterized by small adjustments that Litke (2020) calls “modifications to teaching that are largely consistent with what teachers are already doing” (p. 9), focusing on the common, everyday pedagogical practices that teachers are comfortable with (Otten et al., 2022). PD that focuses on incremental change, by providing modest but important suggestions for student learning, provides “nudges” to the practices already carried out by teachers and therefore has a high probability of these changes being incorporated into teaching action, which can become new teaching habits (Otten et al., 2022). In the case of ambitious or transformational PD, teachers are expected to adopt pedagogies centered on students’ reasoning, supporting communication and argumentation of mathematical thinking, through cognitively demanding tasks, developing conceptual understanding and procedural fluency in an integrated way (Star, 2016).

A major change in mathematics teaching, and certainly a necessary one, involves moving from traditional teaching, in which the role of the teacher prevails, and the focus is on fluency in procedures, to pedagogies centered on the student. The difference between one approach and the other lies in the nature of the tasks and the way they are carried out (Ponte, 2005). In the direct and traditional approach, there is a prevalence of the teacher providing information, with closed tasks, in which “the teacher first demonstrates the method and then gives tasks for the student to practice” (Ponte et al., 2017, p. 3). In turn, the exploratory teaching approach, centered on the students, is a relevant way of working with algebraic thinking (Canavarro et al., 2012), fitting into an ambitious teaching of mathematics, as it is still far from the teacher’s practice. This type of teaching can be seen from three phases: (i) introduction of the mathematical task in which students understand what is proposed, ensuring their involvement, (ii) carrying out the task by students in pairs or small groups, and (iii) discussion and systematization of the task, also called plenary, which is the most challenging phase for the teacher, in which students present their resolutions under teacher orchestration (Stein et al., 2008).

With a focus on incremental improvement, it is necessary to have “a fine-grained description of current practices and an analysis of the kinds of small adjustments that could improve these practices.” (Litke, 2020, p. 2). Some authors indicate that the practice of teachers, who teach mathematics in the early years in Brazil, is based on textbooks, with a focus on numbers and the four operations (Passos & Nacarato, 2020), in addition to the teacher having a predominant role in teaching procedures (Ponte et al., 2012). Algebra is a mathematics topic that was recently included in the Brazilian

curriculum for the early years (Brasil, 2018), which indicates that teachers did not have access to this content when they were in basic education, nor did they have access to how to teach it. On the other hand, textbooks in Brazil have already adapted, presenting questions about generalized arithmetic and functional thinking.

METHODS SECTION

This research is based on a qualitative approach within the interpretive paradigm (Bogdan & Biklen, 1994). Data collection took place in two different contexts: (i) during the PD, through video and audio recordings of the meetings and sub-group discussions and through two questionnaires with open questions, one at the beginning of the meetings and the other at the end, and (ii) observation of lessons by the teachers, Moises and Adriana, after they had ended. The PD involved 14 teachers who taught 3rd, 4th and 5th grade in the same school network in São Paulo, Brazil, in 2019. It lasted 64 hours and the author of this article was the teacher educator.

The PD, designed by the author of this article and other researchers, based on the characteristics of high-quality PD (Desimone, 2009), aimed to help teachers understand what it means and how to develop work with algebraic thinking in early years classes, through Professional Learning Tasks (Ribeiro & Ponte, 2020). The TAPs were divided into: (i) TAPs for specific knowledge, which aimed to work on mathematical and didactic content of work with algebraic thinking; (ii) TAPs for planning, whose objective was for teachers to plan a lesson based on a set of intentions, and (iii) TAPs for reflection, in which teachers reflected on the lesson held with students, based on a pre-established script. While the TAPs for specific knowledge had as their main characteristic the focus on some mathematical and didactic knowledge, the TAPs for planning and reflection sought to operationalize this knowledge in the teacher's practice. Although the PD was not planned based on incremental goals, it carries one of its characteristics, when it seeks to work on algebra based on the arithmetic that teachers already work on.

Data analysis focused on changes in practice, described by teachers and/or observed in their classes, establishing relationships with incremental and/or ambitious PD. The results are divided into: (i) mathematical tasks divided into work with the textbook and work with the standard division algorithm, and (ii) the way in which these tasks were conducted, essentially considering exploratory teaching.

RESULTS

Working with the textbook: At the beginning of the PD, when asked how they planned their math lessons, the teachers indicated the textbook as their main planning tool. The EMAI textbook (used by all the participating teachers) was the subject of discussion during the PD, both as support for the moments when the teachers were asked to plan a lesson, and when they analyzed its potential for working with the regularities of numbers and operations. In one of these moments, the PD helped the teachers realize that the textbook contained questions that led the students to make conjectures about

mathematical regularities. As an example, when the multiplication table for 2 was presented, the students had to observe the results and answer questions about the regularities found. In this regard, Adriana took a position:

Adriana: The key point of this work is to promote reflection, working in groups, socializing... which I didn't do... I used to do the activity, that's it... now I'm going to correct it, at most I would tell the student to correct it on the blackboard.... but I hadn't done that. Often in the EMAI, it asks, "explain now how you arrived at this reasoning", and I'd skip it.

In this report, Adriana explained her way of working with mathematics prior to her participation in PD, which consisted of tasks that the students did, and the teacher corrected, a format related to traditional mathematics teaching. Adriana also pointed out that the textbook provides situations that involve reflection on the mathematical structure and asks students to socialize their reasoning, but that she didn't work on it because she didn't understand the work to be done. At the end, Paula testified to her perception of the textbook, based on the PD:

Paula: Before, I used to see the EMAI with those various questions about “what you saw or observed that was curious” or “what's similar” and found it “boring”. Now I see how important it is for them to argue, record and manage to organize their thinking and see patterns and regularities!

Considering that the textbook is something the teachers already work with, it's possible to consider that the PD gave a “nudge” to the practice. Those questions that were not worked on but were important for the development of students' algebraic thinking, became meaningful and part of the lessons. In this case, the formative process approached incremental PD by focusing on an instrument with which the teachers felt comfortable. In addition, when planning his lesson, Moisés was able to identify in an arithmetic task contained in the EMAI, among others in the textbook, the potential for working with algebraic thinking, seeking a generalization based on the associative property of the multiplicative field: understanding that dividing by 4 is the same thing as dividing by 2 and then dividing by 2 again.

Working with the division algorithm: In one of the moments of planning a mathematical task to be developed later with the students, the fourth-grade teachers, gathered in subgroups, tried to work on generalization, based on the standard division algorithm. After developing the task with the students, Debora said:

Debora: We thought we'd work on a difficulty in our class, which was the process of the operative technique of division. And within that, we systematized the concept that every number divided by one is equal to itself... When everyone had finished, I said: Now you have to find a division rule in these accounts... something that works for more than one situation. In four accounts you'll notice that they have similarities.

The task consisted of performing ten divisions in the form of an algorithm, four with a divisor equal to one, and the rest with other numbers in the divisor. This mathematical

task, planned and carried out by the teachers, presents a characteristic of incremental PD as the teachers use a teaching situation from their everyday lives, but it is also ambitious because the teachers were able to see a powerful algebraic task in arithmetic.

How the tasks were carried out: In the first meeting of the PD, the teachers were asked to describe how they used to develop their math classes:

Paula: I usually bring a problem situation to discuss and, based on that, I explore the content to be worked on. I then summarize the concept discussed and we do exercises on the topic.

Like Paula, almost half of the teachers described their classes as having characteristics very close to a traditional class in which the teacher explains how to solve a problem and asks the students to solve exercises. In two meetings during the PD, we discussed the importance and characteristics of the exploratory teaching approach. Towards the end of the PD, Paula reported changes in her teaching practice, in which it was possible to notice one of the phases of exploratory teaching.

Paula: I think the main change has been in the didactics of the class: I try to read with the children and explain or ask them to explain.

Regarding the introduction phase of the mathematical task, Paula reported that she tried to explain or ask students to explain what they understood from the problem, as an alternative to the common action of asking students to open the textbook to a certain page and do what is asked. In addition, other teachers have also incorporated the characteristics of introducing a mathematical task into their teaching practice:

Marina: Since you [the teacher educator] had commented on getting a student to speak, to explain, so I've been doing it on almost every occasion, even in other subjects and it's been really nice... one reads, the other explains.

Introducing the mathematical task according to the exploratory teaching approach seemed to be a simple matter, considering that in one of Adriana's classes, she invited several students to take part in this phase by asking questions such as "Who can explain to me what I have to do?" and intervening in such a way as to get the students to understand what was being asked of them. Her interventions showed her knowledge about achieving the objectives of the introductory phase: promoting adherence and ensuring understanding of the task. Regarding the task realization phase, in the final questionnaire, Elvira explained what had changed in her practice:

Elvira: Carrying out activities in productive groups, also in math classes, diversifying into pairs, groups of four students...

Although the phase of carrying out the task is not restricted to placing the students into groups, it was possible to notice an advance in their practice. Teachers in the early years have a habit of placing students into groups to carry out tasks, especially in Portuguese language classes, and they have started to incorporate this practice into math classes as well. Regarding the collective discussion phase, some teachers said

that they still didn't feel confident in carrying out this phase. When asked if the PD had contributed to a change in the way they teach mathematics, Estela replied:

Estela: Yes, especially regarding the types of tasks and how to organize them into phases. Although I haven't done the plenary [collective discussion] with my students, it's something that will enrich the work.

Estela understood the importance of collective discussions (plenary), but after three times when teachers were asked to plan and develop lessons based on exploratory teaching, she still hasn't carried out the collective discussion phase with her students.

Changing the way a class is taught is directly related to the way teachers see their role and the way they teach and learn mathematics, which is in line with ambitious teaching. Among the phases of exploratory teaching, it was possible to see that the introduction phase was better understood and put into practice by the teachers, because it requires small changes in the way the task is presented.

DISCUSSION

Although PD are well evaluated by teachers, research has shown that they have little impact on changing their practice (Otten et al., 2022). To understand this issue, this article explores the relationship between changes in the practice of teachers who teach mathematics in the early years and incremental and ambitious PD.

Considering that the mathematical tasks that teachers plan and develop with their students is one of the key elements of their professional practice (Ponte et al., 2012), reflecting on issues that deal with algebraic thinking present in the textbook that the teachers used in their daily lives was a factor in changing their practice. If, on one hand, the textbook is a tool widely used by teachers, on the other hand, working with algebraic thinking can be considered a distant and even absent topic in teacher practice. One of the principles adopted in the PD was that even though talking about algebra in the early years was something new and therefore far removed from teachers' practice, there was an intention to make a conceptual connection between algebra and the arithmetic that teachers already develop with students, in line with the idea that algebra in the early years is not something very different from what teachers already do (Blanton, 2008). The planning that the fourth-grade teachers and Moises did for a mathematical task, based on arithmetic and aimed at one of its regularities, can be considered evidence of the necessary relationship between what the teachers already do and ambitious practice for the success of the teachers' learning.

In addition to the choice of mathematical tasks, the way in which the teacher conducts them is equally important for the students' learning (Blanton, 2008). Exploratory teaching carries many of the characteristics of ambitious teaching, as it offers cognitively demanding tasks centered on the students' discourse. Working with these types of tasks requires in-depth mathematical knowledge, which requires understanding and acting with a focus on the students' reasoning. Teachers began to incorporate the characteristics of the task introduction phase into their classes, getting

students to understand what was to be done and seeking their involvement. However, in the other phases of exploratory teaching, which require teachers to make significant changes to their practice, there was not as much success. When carrying out the task, the teachers began to put the students into groups, which provides greater possibilities for interaction and exploration, supporting communication and mathematical argumentation. However, the task realization phase goes further, demanding a focus on productive ideas, asking questions and suggesting representations (Canavarro et al., 2012). The moment of collective discussion requires the teacher to identify and interpret the students' reasoning which, if on one hand is underpinned by mathematical knowledge, on the other is related to a habit of asking for explanations and justifications for the different resolutions. So, although knowledge and practice are linked, they are of a different nature. In other words, it is not possible to ensure that because a certain mathematical topic or way of teaching is known that it will be part of the teacher's practice, which was evidenced in Estela's response, when she spoke of the importance of plenary sessions, but that she had not yet developed them.

CONCLUSION

As a result of the PD, the teachers were able to look at the textbook through the eyes of algebra, making adaptations to the arithmetic they already developed with the aim of working on algebraic thinking. Although using the textbook to develop algebraic thinking was a small step in changing the teachers' practice, and therefore incremental, it can be considered a consistent step. The results indicate that PD should be a bridge for teachers to take safe steps towards ambitious math teaching practices, even if these steps have different measures for each teacher. Thus, the present study agrees with one of the hypotheses launched by Otten et al. (2022) that “incrementally improving conventional instruction will eventually increase the likelihood that teachers will respond positively to ambitious PD”. Furthermore, it is possible to work in an incremental and ambitious perspective in an integrated manner if the PD look at where the teacher's practice is and not where the practice “should be”. In this sense, this article contributes to the discussion about which practice should be present in PD that claim to be based on practice, “making sure that mathematics education research reaches the classroom”. In addition, further research needs to be carried out to understand how the design and actions of PD contribute to support incremental and/or ambitious professional learning opportunities, considering the change in teachers' practice.

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TEACHERS' TACIT ASSUMPTIONS IN THEIR ANALYSIS OF NON-OPTIMAL MODELLING SUPPORT: A VIGNETTE-BASED STUDY

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Encouraging students' active participation in mathematical modelling requires teachers' focus on their mathematising steps during classroom activities. However, despite its importance for students' learning, more empirical evidence is needed on pre-service teachers' noticing in such situation analyses. This study consequently focuses on analyses of situations represented in classroom cartoon vignettes of more than 200 pre-service teachers from three different universities. The evidence provides insight into pre-service teachers' tacit assumptions on solution pathways and the teacher role, highlights their detrimental role for teachers' analysis and suggests implications for improving learning opportunities in mathematics teacher education.

INTRODUCTION

Mathematical modelling has become a mandatory part of mathematics curricula from primary to secondary education in many countries worldwide (Borromeo Ferri et al., 2023; Kaiser et al., 2023). Authenticity, relevance to real-world contexts, and openness are considered fundamental criteria for quality modelling tasks (Maaß, 2010). When teachers react to students' individual mathematical modelling approaches, they can significantly influence learning outcomes related to modelling (Schukajlow et al., 2023): For instance, teachers can reinforce students' active engagement in individual modelling processes while fostering their critical evaluation of these approaches. However, despite its importance, empirical research on teachers' reactions to students' modelling attempts and teachers' underlying cognitive processes remains scarce. Hartmann et al. (2023) examined whether teacher trainees and practicing teachers were able to identify and analyse non-optimal teacher (re)actions in cartoon vignettes showing the work on modelling tasks in the classroom. The findings revealed that approximately half of the pre-service teachers (PSTs) failed to recognise that the vignette teacher's reactions were non-optimal, with some PSTs even evaluating these reactions positively. There is hence a need to find out about reasons why PSTs may evaluate such non-optimal reactions positively. Consequently, this study addresses this research need. For a more comprehensive reanalysis, the dataset was expanded, and the new analysis included novice PSTs as well as PSTs more advanced in their studies with the aim of identifying the range of potential tacit assumptions of the PSTs that may underlie their situation analysis and have influenced their evaluations of reactions to students' modelling approaches. In a first step and in order to control for PSTs' potentially more general difficulties in analysing modelling classroom situations, we

selected a focus sub-sample of PSTs for this study including those PSTs who evaluated a non-optimal teacher reaction in a classroom situation positively. An interpretive bottom-up analysis yielded categories, among which PSTs' task-related tacit assumptions, in particular related to preferences for specific single solution pathways of the corresponding modelling task, may have played a major role for their positive evaluation of non-optimal reactions. The findings thus cast light on the role of teachers' task-related views also for how they analyse modelling classroom situations.

THEORETICAL BACKGROUND

Processes by which information is transferred between the *rest of the world* and the sphere of *mathematics* form the core of mathematical modelling (Hartmann et al., 2023; Schukajlow et al., 2023). These transfer processes are often described using so-called modelling cycles (Blum & Leiß, 2007; Kaiser et al., 2023). However, comparatively less attention has been given to the characteristics of tasks in general and modelling tasks in particular as well as to the tacit assumptions teachers may hold about the tasks' characteristics and ways to deal with them in the modelling classroom.

Modelling tasks address real-world contexts, are open in terms of allowing multiple solution approaches, and focus on authentic and realistic questions (Maaß, 2010). Due to their openness, modelling tasks enable multiple solution approaches, in most cases at various levels of complexity (Borromeo Ferri et al., 2023). Whereas the advantages and disadvantages of various solution pathways are a significant part of current research in mathematics (Fischer, 2022) and being able to choose the best available model therefore is a long-term teaching goal of mathematics education, including multiple solution pathways in the mathematics classroom is an essential teaching element related to modelling tasks, as it is a prerequisite for reflecting on multiple modelling approaches (Borromeo Ferri et al., 2023). The extent to which a task allows for multiple solution pathways is closely linked to its degree of openness (Maaß, 2010). A task can be open in various ways; it is correspondingly insufficient to label a task as simply "open" or "open-ended": Yeo (2017) identified five aspects (goal, method, complexity, answer, and extension), each of which can be either closed or open (Yeo, 2017, p. 187). When implementing modelling tasks in the classroom, teachers should acknowledge these aspects of openness. Consequently, a prerequisite for optimally profiting from aspects of openness can be seen in corresponding tacit assumptions a teacher has on the task, its learning potentials, and how the task (or tasks in general) should be dealt with in the classroom. Inspired by Sherin et al. (2011, p. 28), by *tacit assumptions*, we understand the underlying beliefs that shape a teacher's approach to teaching and learning activities, including assumptions on aspects of tasks, prompts and behaviour towards students, among others.

Findings by Schukajlow et al. (2023) indicate that students' difficulties in recognizing the openness of a modelling task and the need to make modelling assumptions, which is a fundamental mathematical activity for modelling (Stylianides & Stylianides, 2024), are barriers to solving open modelling tasks, and that teacher prompts can help

students overcome these challenges. Consequently, the successful implementation of open tasks in classroom situations requires not only knowledge from the teacher on how to engage with students but also tacit assumptions which enable insight into how, when and which tasks to implement effectively in the classroom (Klein & Leikin, 2020). This highlights the need for modelling-specific research since teachers often face challenges due to perceived conflicts, for instance between the openness of a task and the possible tacit assumption that precision is a fundamental goal of mathematics instruction (Kuntze et al., 2018). Kuntze et al. (2018) underscore that teachers need to be able to analyse in a knowledge-based way how a task is implemented in the classroom, in order to be able to achieve a successful implementation of the task in their classroom: Teachers should not only be familiar with modelling as an overarching idea in mathematics and mathematics education, but also develop the ability to notice critical elements of classroom situations (Sherin et al., 2011): Especially from a mathematics education perspective, the key aspect of *noticing* is knowledge-based reasoning on aspects of classroom situations. Consequently, in noticing, teachers refer to their professional knowledge, including their instruction-related views (such as tacit assumptions about the task and possible solution pathways, for instance), which can also influence the noticing process and outcome (Kuntze et al., 2022).

Cartoon-style vignettes, i.e., representations of practice (Buchbinder & Kuntze, 2018; Skilling & Stylianides, 2020), can be used to investigate how PSTs' tacit assumptions may interdepend with their analysis of non-optimal reactions to students' modelling approaches. Vignettes can incorporate specific analysis demands in the context of modelling (Hartmann et al., 2023), facilitating research into PSTs' difficulties that may stem from their tacit assumptions, e.g., about aspects of modelling tasks.

RESEARCH QUESTION

Corresponding to the research need outlined above, the study focuses on the following research question: *Which tacit assumptions about modelling tasks and their preferred enactment can be reconstructed from PSTs' positive evaluations of non-optimal reactions to students' modelling approaches?*

METHODS AND SAMPLE

The new analysis of the extended data set presented in this paper focuses on PSTs' analyses of two vignettes from a larger vignette-based instrument (Hartmann et al., 2023) developed in the context of the project coReflect@maths (coreflect.eu). These vignettes show classroom situations in which mathematising activities play a central role. The tasks appearing in the vignettes are characterised by the absence of predefined models or mathematical procedures, leaving the modelling method open to exploration (Schukajlow et al., 2023; Yeo, 2017). One vignette features a task with significant modelling requirements (Figure 1), where a teacher approaches a student group. The students have developed a volume-based modelling approach, but the teacher, after giving rather unspecific positive feedback, expresses doubts about accuracy and suggests an alternative weight-based method. This reaction of the vignette teacher can

be considered as non-optimal, as it may discourage the vignette students from pursuing their own modelling ideas and as it leads them to a default solution pathway preferred by the vignette teacher, in contradiction with the benefits of the tasks' openness.

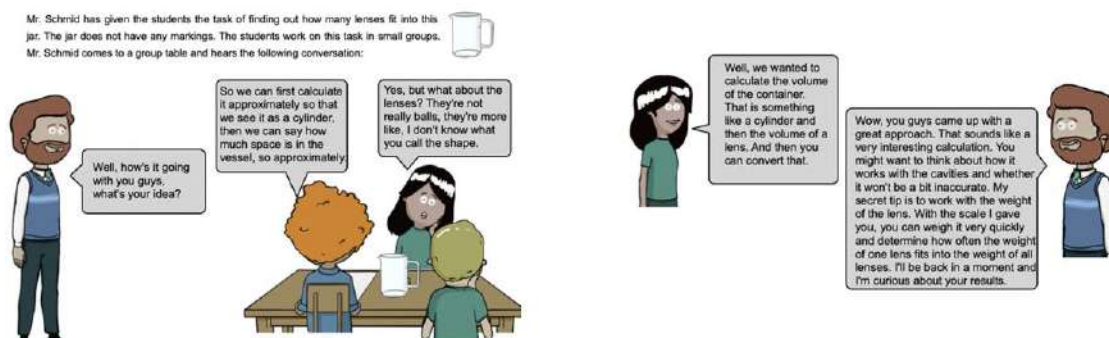


Figure 1: Vignette V1 from Hartmann et al. (2023).

The present study focuses on an analysis of data from PSTs from three German-speaking universities, involving both novice PSTs and PSTs nearing the completion of their Bachelor's degree, so that a range of variation in PSTs' backgrounds could be covered. The vignettes were administered as a paper-and-pencil questionnaire. The dataset (N=209) was double-coded using the category system from Hartmann et al. (2023). In cases of disagreement between the coders, the individual analyses were discussed to reach a consensus. Subsequently, the relevant cases were identified. In this study, analyses were considered relevant if participants evaluated the vignette teacher's non-optimal reaction positively in at least one of the two vignettes. These relevant cases were then reanalysed using a bottom-up approach to develop inductive categories directly from the data in order to reconstruct potential tacit assumptions about modelling tasks and their preferred enactment, which might have led the PSTs to their positive evaluation. According to the principles of inferential reasoning, a cognitive reconstruction was carried out during the data analysis. Based on the written expressions, inferences were drawn in a bottom-up approach (Mayring & Fenzl, 2019) regarding underlying assumptions and views on (modelling) tasks that may have influenced the respective individual's situation analysis. Upon repeated occurrence, these tacit assumptions were defined as inductive categories. This process led to the emergence of two frequent categories which will be presented along with their respective frequencies in the results section.

RESULTS

In order to address the research question, the dataset of 209 participants was first coded following the procedures described in Hartmann et al. (2023). 197 of these participants responded to both vignettes V1 and V2 regarding the respective vignette teacher's non-optimal response to the vignette students' mathematisation. Here is a PST's sample response (translated, [SI10WI]) to vignettes V1 and V2:

- V1: positive: He doesn't reveal the entire solution approach.
 negative: The students' approach wouldn't necessarily be wrong. It would

be instructive to give them the opportunity to pursue their approach, optimise it, and compare the different approaches of the groups at the end.

V2: Well done, as he helps the group optimise the method they have found without anticipating too much about the solution in advance.

Related to V1, the PST states both positive and negative aspects of the vignette teacher's handling of the vignette students' solution approach. According to the PST, the students should instead be encouraged to follow their own approach, refine it, so that it can be compared with other approaches. In contrast, related to the parallel vignette V2, the PST evaluated the vignette teacher's action only positively—by interpreting the vignette teacher reaction as “help by optimising their method”. As the vignette teacher intervenes in the students' mathematisation approach and aims at replacing it by an “optimised” (and thus presumed as better) approach, the PSTs evaluation reflects a tacit assumption that the vignette teacher's solution pathway is better and has to be preferred (the vignette teacher's method is implicitly considered as “optimal” by the PST), and the PST does not mention that the students in V2 may be discouraged from completing their own modelling approach by the vignette teacher's reaction. Another PST response (translated, [SU12HU]) reads:

V1: The introduction and the task setup are well done. The task is open and interactive. However, the guidance is a negative aspect.
It includes not only the method but also the complete calculation process. In my opinion, that is too much. A more restrained form of guidance would be better initially.

V2: The task setup and the teacher's guidance are generally well done. Only the last sentence of the guidance is too much; the students should be able to come up with this calculation method on their own.

The openness of the task formulation is praised in the case of V1, highlighting also its interactive nature. However, in V1, the teacher's guidance is criticised for being overly detailed, as it not only includes the solution approach but also the complete calculation process, which is perceived as excessive. The suggestion is made that a “more restrained form of guidance” would be preferable. The PST does not criticise that the “guidance” comes with a different mathematisation, but that it is too complete. Similarly, in V2, the final hint (“last sentence of the guidance”) is considered overly directive. It is suggested that students should be allowed to independently discover “this” solution approach, reinforcing the perception that the teacher's solution approach is implicitly regarded as *the* correct one. This highlights a tacit assumption that while “guidance” is necessary to find a best solution method, too much specificity in the teachers' reaction may undermine the students' own steps on “the” solution pathway. This example shows a combination of the tacit assumption T1 that there is a preferred (here even a unique) solution pathway on the one hand and the tacit assumption T2 that only minimal support should be provided. These two tacit assumptions have the potential to neglect the open and exploratory nature of the task.

A total of 59 participants positively evaluated the vignette teachers' non-optimal reactions to students' modelling approaches at least once. Below, we list the tacit assumptions of PSTs that influenced these positive evaluations.

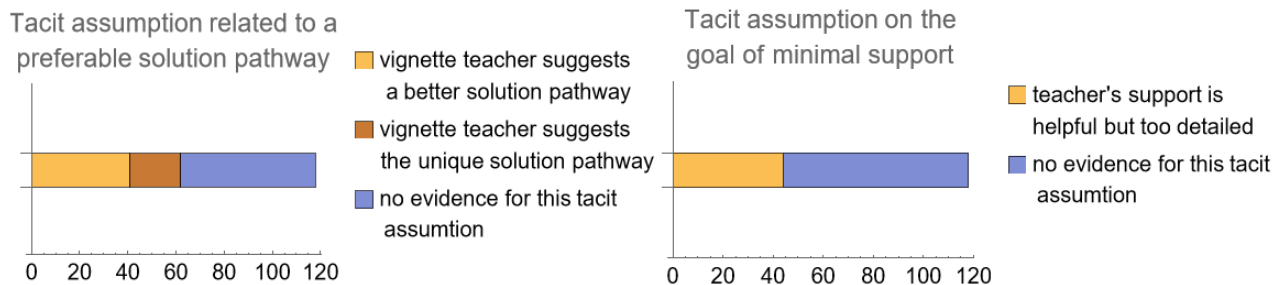


Figure 2: Number of cases with coded text segments related to PSTs' tacit assumptions T1 and T2.

A total of 53 participants made statements regarding the solution pathways. Of these, 44 participants (with 62 mentions, see Figure 2 left) showed evidence of an implicit assumption that the teacher's suggestion represented either a better approach ("Point out a more precise approach through tips," [DO01KL]) or the unique approach (e.g., "The solution approach has essentially already been 'revealed'," [IN11MI]). We distinguish statements evaluating the vignette teacher's suggested solution approach based on its level of detail, whether it is perceived as helpful, or whether it should be avoided. Most commonly, 33 teachers (44 mentions, Figure 2 right) assessed the suggested solution approach as "helpful but overly detailed" (e.g., [SU12HU] mentioned earlier), reflecting a tacit assumption about the appropriate extent of support, which may be related to the goal of minimal support.

T1/T2	evidence	no evidence	Total
evidence	28	16	44
no evidence	5	10	15
Total	33	26	59

Table 1: Out of the 59 PSTs, 49 exhibit at least one of the two tacit assumptions.

As shown in Table 1, these tacit assumptions appear together in nearly half of the cases (28 out of 59), while in ten cases, no tacit assumptions could be identified due to the lack of detailed feedback (e.g., "good: gives hints," [BI23TH]).

DISCUSSION

First of all, we would like to recall the limitations of this study: Given the small sub-sample, the results should be interpreted with caution, as the sample is in particular not representative of German-speaking PSTs and the findings are not independent from the two specific classroom situations shown in the vignettes. Nevertheless, the research question could be answered, providing insights into frequent tacit assumptions that led

the participating PSTs to positive evaluations of the vignette teacher's non-optimal reaction. Notably, almost three-quarters of the PSTs in the subset showed an assumed preference for the vignette teacher's solution pathway. This tacit assumption may stem from insufficient emphasis on methodologically open modelling tasks in their studies (Yeo, 2017) or could relate to the tacit assumption that precision as a key goal of mathematics instruction (Kuntze et al., 2018) might be in conflict with methodologically open modelling tasks. Another possible explanation is that participants identify themselves with the solution presented by the vignette teacher, analogous to a phenomenon well-documented in the literature that teachers often—consciously or unconsciously—favour their own solutions (Hartmann et al., 2023; Kaiser et al., 2023). The second inductive category of tacit assumptions points to a view concerning the students' active role in the sense that they have to discover or to generate solution steps on their own, but rather excluding that they may generate or choose their own mathematization approach. This tacit assumption mostly corresponds to a PST's focus on the teacher's solution instead of analysing students' modelling approaches and taking them seriously. Tacit assumptions from both inductive categories can be expected to limit the student-centred development of modelling approaches in the classroom and hinder the validation and comparison of the underlying modelling ideas (Kaiser et al., 2023).

For the practical training of PSTs, the findings underline the need of focused learning opportunities in teacher education. Beyond fostering knowledge about modelling competencies and the modelling cycle, particularly the methodological openness of modelling tasks (Maaß, 2010; Yeo, 2017) should be emphasised together with ways of implementing methodically open tasks (Klein & Leikin, 2020) in the classroom and the benefits of encouraging diverse solution approaches (Borromeo Ferri et al., 2023; Schukajlow et al., 2023). Work with classroom cartoon vignettes in teacher education can offer a spectrum of possibilities in seminars about modelling in the mathematics classroom and encourage PSTs' reflection on ways of reacting to students' mathematisation approaches. Follow-up research should focus on whether PSTs' tacit assumptions develop under such professional learning opportunities together with their situation analysis.

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PIVOTAL EXAMPLES IN GRAPHICAL DIFFERENTIATION – AN ANALYSIS OF SEMIOTIC AND THEORETIC CONTROL

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Graphical differentiation is an important topic in upper secondary education, but it should not become an end in itself. Rather, meaningful task design can help support the development of robust foundational concepts related to differentiation in connection with the slope of the tangent, both locally and globally, during the process of graphical differentiation. We present how this can be achieved through a series of examples consisting of pedagogically chosen examples that challenge students' superficial rule-based approaches and demonstrate how a pair of students navigates through their solution processes via semiotic and theoretical control.

INTRODUCTION AND THEORETICAL BACKGROUND

Graphical differentiation (GD) is an important topic in upper secondary education as it can provide input to deduce relationships in calculus and can serve as meaningful context for carrying out argumentation and reasoning on a theoretical and semiotic level (Fischer, under review; Arzarello & Sabena, 2011). However, while the exercise of GD cannot be categorized as purely procedural or conceptual (Sommerlade & Eichler, 2023), school often emphasizes a procedural method with focus on the outcome instead of leveraging the potential for developing conceptual understanding (ibid.). Also, examples chosen for implementation emphasize—or are often even restricted to—polynomial functions for which a sequence of steps is carried out along characteristic points of the function (Fischer, 2025). Such a ‘recipe’ can become an ‘empty rule’—a procedure or mnemonic applied to solve tasks without requiring underlying conceptual understanding. As a result, the treatment of GD—as well as other tasks—focuses on the product instead of the process, with relationships relevant for understanding—such as the one between the derivative and tangent lines—fading into the background. Mathematically, such ‘empty rules’ are incomplete at best and inconsistent at worst; pedagogically however they can bear potential for conceptual growth if challenged appropriately (Liljedahl et al., 2007; Zazkis & Chernoff, 2008).

This research report presents an approach to challenge such empty rules through a sequence of examples in graphical differentiation that is designed to “bridge from [a] learner’s initial (naïve, incorrect or incomplete conceptions) towards appropriate mathematical conceptions” (Zazkis & Chernoff, 2008, p. 197) of the relationship between the derivative of a function and its tangents. We will moreover analyze this bridge by identifying the development and pivots in the learners’ execution and development of semiotic and theoretic control (Arzarello & Sabena, 2011) while navigating through the sequence of examples of GD towards ‘filling the empty rule’.

Example spaces, pivoting examples and bridging examples

The intervention design builds on the concepts of *pivotal* and *bridging examples* (Zazkis & Chernoff, 2008) within the framework of example spaces proposed by Watson and Mason (2005). Watson and Mason distinguish three types of example spaces (ES): the *situated personal ES*, shaped by immediate tasks, cues, environment, and recent experiences; the *personal potential ES*, which draws on a person's broader past experiences, even those not explicitly remembered, but lacks clear structure for easy access; and the *conventional ES*, consisting of examples commonly recognized by mathematicians and found in textbooks, which teachers aim to introduce to students.

Pivotal examples challenge learners' current understanding or rules, creating a cognitive turning point by exposing gaps or inconsistencies in their approach (Zazkis & Chernoff, 2008). While such examples may initially trouble learners, they can also lead to the resolution of these conflicts, fostering deeper conceptual understanding. A specific type of pivotal example, the (*pivotal*) *bridging example*, helps learners transition from incomplete knowledge to a broader mathematical understanding, enabling generalizations beyond the specific task.

Zazkis and Chernoff (2008) highlight that the potential of an example to serve as pivotal or bridging depends on the individual learner's prior knowledge, experiences, and ability to engage with the task. While examples cannot inherently be designed to be pivotal or bridging, certain design considerations can enhance their likelihood of achieving this. Examples should be situated within the learner's personal potential example space but push its boundaries toward the conventional example space. Additionally, achieving these effects may require a "critical mass" of examples to effectively support a meaningful shift in understanding (ibid).

Semiotic and theoretic control

The task of GD heavily builds on identification of features of a function on a graphical level and deducing from those features the graph of the derivative. Such a deduction can stay on a mere semiotic level when staying within a limited example space that does not extend outside of the applicability of an empty rule. It might not raise a need to link the two concepts of tangent line and derivative as related to a given function and points of interest on this function on a more general, theoretical level. In the words of Arzarello and Sabena (2011), examples satisfying the rule-based execution of GD do not raise the need to execute theoretic control—to make a decision based on the selection and implementation of a more or less explicit theory or parts of it—but rely on decisions on a level of interpreting signs, that is on semiotic control (ibid., p. 191).

When solving a problem and students need to negotiate solutions and reason about them, semiotic and theoretic control can be identified in semiotic activities of "1) interpreting signs, 2) Identification of relationships between the interpreted signs, and checking with arguments; [and] 3) Elaboration of arguments that explain the 'why': towards a theory" (ibid., p. 193).

We will use this model of semiotic and theoretic control to trace students' engagement with a sequence of examples given to them. In particular, we ask the research question: *How do semiotic and theoretic control interrelate when examples become pivotal or help bridging towards a deeper relationship between a function and its derivative?*

A SEQUENCE OF ACTIVITIES OF GRAPHICAL DIFFERENTIATION

The task sequence consists of four examples of graphical representations of functions (Fig. 1a-d), with a prompt to sketch the respective derivative for each. Mathematically, the activity involves performing graphical differentiation on a series of functions in a sequential manner. Pedagogically, these examples are chosen to challenge students' conceptual basis for graphical differentiation by pushing the boundaries of the students' assumed situated personal example space to extend their personal potential example space, as will be outlined briefly in the following. Designing the intervention as a sequence of examples acknowledges a distance between personal example spaces, drawing on experience with polynomial functions, and a conventional example space aimed at fostering deeper mathematical understanding.

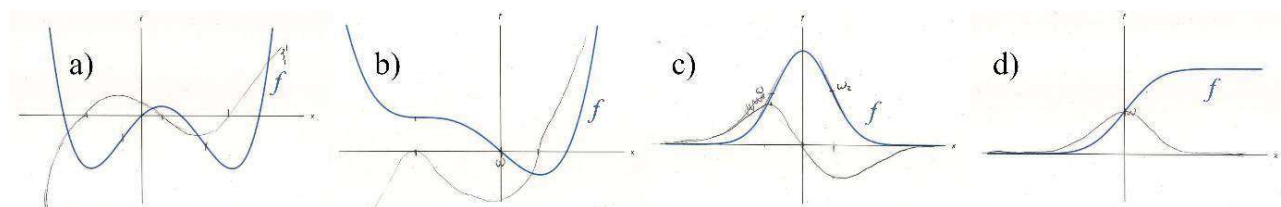


Figure 1: Sequence of four examples given to the students. The figures show the graphs of the functions in blue and the derivatives sketched by the students whose working process is presented in this report in grey.

The first example, a fourth-degree polynomial (Fig. 1a), allows students to apply a procedural recipe without facing conceptual difficulties. Extrema correspond to zeros of the derivative, points of inflection become extrema of f' , while the upward-opening shape and its apparent degree of the graph highlight features that influence the derivative's behavior. This familiar example serves as a typical introduction to graphical differentiation. The second example (Fig. 1b) introduces a fourth-degree polynomial with a terrace point, where a horizontal tangent—and thus a zero of the derivative—does not align with an extremum. This requires students to adjust their interpretation of these characteristics, prompting a conceptual shift in their reasoning about horizontal tangents and their connection to the derivative. In the third example, the normal distribution (Fig. 1c), the asymptotic behavior includes horizontal asymptotes, further extending the relationship between the function's behavior and its derivative. Comparing these characteristics with previous examples challenges students to refine and maintain consistency in their interpretive framework. The final example eliminates extrema altogether, compelling students to apply their reasoning to more abstract cases. This progression fosters the development of a deeper conceptual understanding of graphical differentiation, demonstrating the pedagogical depth of the sequence.

THE CASE OF MIA AND ELLA

Mia (M) and Ella (E) are two students at the beginning of their teacher training program. As part of a larger study conducted by the first author, they were invited to engage in the activity of graphical differentiation using the sequence of examples described in the previous section. Their working process was recorded and transcribed, and their dialogue was analyzed to identify *pivotal* and *bridging* moments emerging from their personal potential example space, as well as actions related to semiotic and theoretical control. The analyses in this report highlight key instances of *pivoting* and *bridging* as the students navigated through the sequence of examples.

Excerpt 1: A terrace point as turning point

After working straightforwardly through the first graph (Fig. 1a) using their recipe (lines 1-37) and subsequently repeating it, Mia and Ella move on to the second graph, displayed in Fig. 1b (lines 107-216). In this excerpt, we will see how the second example becomes *pivotal* in the students' understanding of graphical differentiation beyond their prior recipe as strategy, eventually serving as a *bridging example*.

107 E: Oh man.

108 M: Hmmm.

109 E: I see degree five here.

110 M: can't be five. It has to be something even.

Their initial reaction (107/108) suggests that they recognize some difference to the first example in terms of how to approach the task of graphical differentiation, potentially even expressing discomfort. They identify it within their *personal potential example spaces* related to polynomial functions (109/110)—even though Ella infers the degree incorrectly—and proceed to analyze the function.

131 E: f' must thus go the other way [...] then it's not a 2-polynomial. [...] if you have such a function, then you also have a spot where f' is 0.

While Ella focuses on comparing the general behavior of the function to graphs she is familiar with (131), Mia examines the graphs' specific properties, focusing on the terrace point. This is where the slope is acknowledged as significant for the first time:

143 M: that's a root anyway, because the slope is zero.

144 E: right. hm? If you take f' , you have the slope. Right? When deriving. And that is there (*points at local extremum*), roughly.

145 M: as said before. still, because this is the slope. And the slope is 0 here nevertheless (*points at the saddle point*).

146 E: Ah yes?

147 E: ehm. the slope here is, falling, falling (*pitch increasing*). End then here changes (*points at local extremum*). And now positive, positive, positive.

148 M: It falls, it falls, it falls. It falls consistently. means... So from here it falls the whole time. meaning, it has to (.) simply be negative here all of it, as long as it still falls (*traces the graph of the derivative by hand*).

Mia links the slope being zero—probably referring to the slope of the tangent line—to the root of the derivative (143). This leads to the slope becoming a mutual tool for deducing the behavior of the derivative more generally for monotony behavior (147-148). In continuing her analysis, Mia combines the steps of her recipe with the argument of the slope, stating that “it has to (.) simply be negative here [..] as long as it still falls” (148). She follows the derivative with her finger, focusing on the area around the terrace point. Ella then joins in, attempting to assimilate the feature of the terrace point into her *situated personal example space*:

163 E: these are two extrema. right? [...]

Mia then provides a more comprehensive explanation, testing her hypothesis against other graph properties, such as its curvature.

171 M: So I mean, the slope is zero there. Meaning, it has to be a root. But its not an extremum here. And (pause) but if, (pause) if it is an extremum here, then is (pause) no right, that works, because the curvature there is also zero then. Meaning, if the slope is zero too there, then the curvature is zero.

The analysis of the terrace point acts as a turning point in the students’ reasoning about how the graph of f' relates to the graph of f . It challenges their existing application of the recipe and its theoretical basis and makes them reconsider the relationship between slope and the derivative, making it a *pivotal example* in the students’ working process.

Related to *semiotic* and *theoretic control*, we can trace a development within the three semiotic actions (SE). SE1—interpretation of signs—concerns screening the graph for signs that might be potentially significant. Starting from the recipe that served them in the first example, the graph of the function is interpreted holistically as the graph representing a polynomial function (109). While Ella infers the degree incorrectly (109), Mia challenges her assumption, linking their recipe to the graphical representation by noting “it has to be something even” (110). Not going as far as elaborating on the ‘why’ of her conclusion, it can be seen as a first phase of SE2, an interpretation of relationships between the interpreted signs. This action is not completed but carried on in line 143, where she includes the slope for checking with arguments. Meanwhile, Ella stayed with SE1 (131). Her argument can be considered on the cusp of SE2 as she explores relationships between relevant signs. However, she focuses only on signs that align with her recipe, avoiding those that challenge it, and does not pursue further argumentation (131). When Mia extends her arguments on a preformal theoretical level, arguing that a root of the derivative does not necessarily indicate an extremum of f (143&145), Mia starts joining in acknowledging the relationships between slope and behavior of the graph (147). Mia’s argumentation, which connects relationships between signs to a mathematical theory (SE3), marks a *pivotal bridging* point where slope and monotonicity become central to her reasoning. When she identifies that the root of f' at the terrace point is also an extremum of f' , leading to an inflection point for f (line 185), she integrates this new argument into her reasoning and verifies it against other steps in her recipe.

Excerpt 2: The Normal Distribution—Anything but a normal example

Next, the students turn to the third example, the graph of the normal distribution. Its asymptotic behavior pushes the boundaries of their personal potential ES to serve as a *pivotal example*.

- 276 E: Oh man. Is that asymptotic?
 277 I: That's asymptotic, yes.
 278 E: Puh. so somehow with fraction, right? Is that always (*very quietly*). asymptotic with fraction.
 279 M: Hmmm
 280 E: well I always link them to fractions. [...] I always link them to the symbol [...] But as a second-degree, you can't see that either, right?
 289 M: Hmm. But I have to say, I never had to deal with a function looking like that in school.
 290 E: yes. So it can't take on values negative, (pausing) so v so f of x is smaller than 0. and it approaches the x -axis
 291 M: yes but f' of x can still be 0. [...]

Again, Ella's utterance in 276 suggests that, while the example falls within her personal example space, it pushes its boundaries, as the rule or recipe for polynomial functions no longer seems sufficient. Instead, her rather diffuse association to fractions ("somehow with fraction, right?", 278) indicates an attempt to align personal example spaces with Mia (278-280). Mia explicitly indicates that the example does not fall within her personal example space (289). However, she responds to Ella's attempt to infer properties of the graph of f' (291). Ella and Mia interpret the graph of the function (SE1), but do not manage to relate it to their existing recipe or their *personal potential example space* (278 & 280), not carrying out S2. In the third task, involving the normal distribution, the students struggle to align the example with their prior semiotic understanding.

Excerpt 3: A maximum as a moment of growth

Following a prompt from the interviewer to apply their previous steps, the students examine the maximum of the function's graph.

- 321 E: The slope. It's increasing or decreasing. And then it changes. Right?
 322 M: That means, I'd guess, there must be where the root is.
 323 E: Yes!
 324 M: If you can apply that.
 325 E: (almost simultaneously) But are you allowed to apply it without polynomial function?
 326 M: I'm not entirely sure. But the slope still always represents the instantaneous change. The first derivative... still always shows the instantaneous change. And if it's zero there...
 327 E: When you draw the tangent, it's straight...

- 328 M: ... it's still zero there. [...]
- 332 E: So let's imagine it's part of one of fourth-degree.
- 333 M: (simultaneously) Ah yes, increasing there. That means it's positive there and negative there, right?
- 334 E: So imagine it's one of fourth-degree, and then, wait, I am not allowed to. It's not a polynomial function.
- 335 M: So it must be something that comes from there...
- 336 E: Going down to there. (interrupting)
- 337 M: So something that goes from top left to bottom right.

Ella and Mia check (SE2) their new problem-solving approach drawing on the slope against the unknown function (325). Their previous realization (289) then leads to the third semiotic action SE3 described by Arzarello & Sabena (2011): seeking deductive explanations for relationships between signs within a mathematical framework. While Mia generalizes their recipe to accommodate a broader class of functions, Ella continues to interpret this local segment as part of a fourth-degree polynomial within her personal example space (lines 332&334). After adjusting her approach (334) with Mia's method, she aligns herself with Mia's generalization which can be seen as an extension of her example space based on the fundamental concept of instantaneous change, a *pivotal bridging example* for both students. Completing the example takes an additional eight minutes and requires another prompt from the interviewer, who suggests exploring inflection points. They then extend the concept of the tangent slope from local extrema to general points, ultimately addressing horizontal asymptotes. This becomes evident in Mia's remark just before completing the task:

- 491 M: And when the slope of the tangent approaches zero, the value of the first derivative also approaches zero.

The students' consistent reasoning about using the tangent slope of f as the value of its derivative ultimately enables them to solve the task correctly. Their work on tasks two and three leads to a *pivotal bridging* point in Ella and Mia's problem-solving approach. This progression is highlighted by their efficient handling of the subsequent task (Fig. 1d) which they complete in just two minutes and thirty seconds (lines 560-588).

SUMMARIZING DISCUSSION AND OUTLOOK

The goal of this paper was to understand better the relationships between semiotic and theoretic control when students' conceptual growth is prompted by examples that challenge their current understanding or/and help them extend their current understanding—serving as *pivotal* and *bridging examples*. Our focus was on the mathematical topic of graphical differentiation, a key task for developing conceptual understanding of relationships in calculus (Arzarello & Sabena, 2011; Fischer, under review). However, it is often approached in a result-oriented manner, relying on procedural recipes (Sommerlade & Eichler, 2023).

To create conditions for conceptual growth prompted by examples and to analyze it within the data, we adopted the framework by Zazkis and Chernoff (2008). A sequence of examples on graphical differentiation was designed to challenge students' reliance on 'empty rules'—rules followed procedurally without conceptual understanding. We examined how two students navigated this sequence, transitioning from their initial application of an 'empty rule' to adjusting it when faced with an example that resisted its application. This occurred when semiotic features of the graph, not addressed by the rule, had to be newly considered and interpreted. The resolution involved bridging their prior understanding of the relationship between slope and growth—previously confined to polynomials—to a broader, more general conceptual understanding.

The analyses presented here highlight how the framework of semiotic and theoretical control by Arzarello & Sabena (2011) can serve as a tool in task design for bridging procedural approaches to tasks with a deeper understanding of mathematical theory. The framework offers a theoretical foundation for designing effective mathematical tasks and refining them to enhance their quality further through “predictive analysis, trial, reflective analysis, and adjustment” (Liljedahl et al., 2007, p. 241), the latter stage still pending in this project. The next step in our research will therefore involve examining how students with varying levels of semiotic and theoretical control respond to our intervention. Furthermore, we will investigate whether our didactical intervention also functions effectively in the mathematics classroom as a pivotal bridging example, and validate our framework for a wider range of tasks and topics.

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ANALYZING STUDENTS' INFORMAL APPROACHES TO CREATING DECISION TREES IN THE CLASSROOM

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This study investigates how high school students construct decision trees for classification using the software CODAP (codap.concord.org). We report on different types of informal student approaches to creating multi-level decision trees after they had a brief introduction to decision trees and the software. We further address how the informal approaches can be utilized during subsequent teaching for introducing data-based algorithms, evaluation with test data, and overfitting.

INTRODUCTION

Data science is an emerging trend among statistics educators (Burrill & Pfannkuch, 2024), who are rethinking the content of traditional introductory statistics courses and studying how algorithmic models can be integrated (Fergusson & Pfannkuch, 2022; Fleischer et al., 2024; Zieffler et al., 2021). Creating algorithmic models is sometimes referred to as machine learning (ML) but also as statistical modeling, predictive modeling, or classification modeling. Zieffler et al. (2021) see the potential of algorithmic models like data-based decision trees (DTs) to be more accessible than other multivariate methods since the concepts are more intuitive. Erickson and Engel (2023) describe using the digital tool CODAP and its plugin Arbor, supporting students in building DTs manually, node by node, to acquire an understanding "not just for tree-building but for more advanced statistics, data science, and machine learning" (p. 112). However, they did not empirically study using Arbor with students, which we address in this paper by exploring students' informal approaches to creating DTs in class.

BACKGROUND

Studies about teaching decision trees

Ma et al. (2023) studied four middle school students using an unplugged intervention to classify pasta with DTs. The students successfully performed basic tasks like partitioning data, creating decision rules, and classifying items using DT diagrams. Podworny et al. (2022) examined 233 middle school students after an eight-lesson unit with unplugged data cards to construct DTs for classifying food items. Most students could use DT diagrams to classify items and reason accordingly. Fleischer et al. (2024) built on a teaching unit that used the same resources and interviewed fourteen students afterward. All students created DTs aligned with the strategies of data splits and label choice that were taught, with six of seven pairs creating optimal DTs. For DT evaluation, most used the intended formal strategies, but some relied on informal reasoning based on visual impressions of the data. Zieffler et al. (2021) found that in-

service teachers without prior knowledge of DTs could classify objects and create DTs using small datasets. Some created suboptimal DTs by prioritizing context knowledge over empirical data. However, using context knowledge is not necessarily a flaw but can enhance DT creation (Martignon et al., 2022). These studies demonstrate that data-based DTs are accessible from middle school onward. While students easily adopted basic tasks like performing data splits and classifying items, some developed alternative approaches with informal reasoning or reliance on context when creating or evaluating DTs. A gap in research is the use of CODAP to create DTs and the direct study of classroom activities to identify the origins of deviating informal approaches.

Pedagogical approaches

Sanusi (2023, p. 118) highlights participatory learning, rooted in constructionism, as a useful pedagogical approach for teaching ML. It builds on students' prior knowledge, collaborative learning, creating artifacts in open-ended tasks, encouraging autonomy, and valuing students' products and processes (Sanusi, 2023, pp. 51–53). This aligns with Fergusson and Pfannkuch's (2022) informal approach to introducing predictive modeling, where students engage with data representations without detailed instruction. They found this to provide an accessible entrance and a foundation for exploring more advanced concepts, a strategy also known in informal inference research in statistics education (Makar & Rubin, 2018). The teaching we present was designed with an informal approach to introducing DTs to foster participatory learning.

RESEARCH QUESTION

Our study examines students' informal approaches to creating and evaluating DTs in authentic classroom settings using the digital tool CODAP and an open-ended task. The aim is to integrate students' results into teaching, fostering participatory learning. Our research questions are: (1) What are the characteristics of students' DTs? What approaches do students describe for (2a) selecting predictors, and (2b) termination criteria? (3) How do students' approaches explain the characteristics of their DTs?

METHOD

Setting of the study and participants

The study took place in lesson 2 during a teaching module (6 lessons à 90 minutes) about ML with data-based DTs. Three student cohorts aged 16-18 took part: a regular math course in grade 11 (12 students), a regular computer science course in grade 11 (19 students), and a project course on data science in grade 12 (20 students).

Preceding teaching in lesson 1

In the first 90-minute lesson, basic knowledge was acquired about what a classification problem is, how a DT is used as a classifier, and basics of deriving a one-level DT from data. The context is recommender systems on online platforms, which recommend content based on predicted user interests (e. g. for online games). To start, we use a mini sample of the JIM-PB data (go.upb.de/jim-pb-data), shown in Figure 1 (A),

containing 14 cases and four binary variables. The classification problem is to classify a user as "frequently" or "rarely" playing online games. *OnlineGames* is the target variable that is to be predicted by using three predictors, namely *GameConsole* and *Computer* ownership and the frequency of *Instagram* use, coded binary as frequently (at least once a week) or rarely.

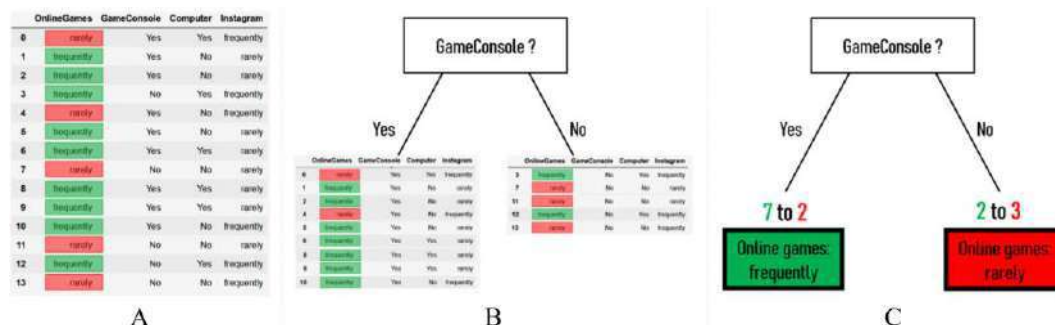


Figure 1: JIM-PB mini data sample (A), data split (B), and one-level decision tree (C)

To create, evaluate, and represent a one-level DT based on training data, four process components can be distinguished as Fleischer et al. (2024) described: First, a data split is performed by dividing the training data into two subsets based on the values of a predictor variable, as shown in Figure 1 (B). Second, a one-level DT is derived from a data split by using the majority label in each subset as a predicted label. In Figure 1 (C), this results in "If *GameConsole* = Yes, then *OnlineGames* = frequently" and "If *GameConsole* = No, then *OnlineGames* = rarely." Third, for evaluating the DT, the misclassification rate (MCR) regarding the training data is calculated, which is 4 of 14 (28.6%) here; the sum of all cases with minority labels in both subsets divided by the total number of cases. Fourth, the DT is represented as DT diagram (Figure 1 (C)) or as the "if, then" rule specified above. Both representations were used in class. The process shown in Figure 1 for the predictor *GameConsole* was performed in class for all three predictor variables. *Computer* and *Instagram* each resulted in 5 misclassifications, making the one-level DT with *GameConsole* the most accurate.

Preceding teaching in lesson 2

In the first 30 minutes of lesson 2, CODAP and Arbor were introduced with another sample of the JIM-PB data, represented as a data table in CODAP (see Figure 2).

Index	Playing OnlineGames	Own Computer	Own GameConsole	Own Tablet	Using Twitter	Using Snapchat	Using Instagram	Youtube MusicClips	Youtube LetsPlay	Youtube FunnyClips	Youtube SportsClips	Youtube FashionBeauty	Own Smartphone	Own E Reader	Sex
1	frequently	No	Yes	Yes	rarely	rarely	frequently	frequently	frequently	rarely	rarely	rarely	Yes	No	male
2	frequently	Yes	No	No	rarely	frequently	frequently	frequently	frequently	frequently	frequently	rarely	Yes	No	male
3	frequently	No	Yes	No	frequently	frequently	rarely	frequently	frequently	frequently	frequently	rarely	Yes	Yes	male
4	rarely	No	No	No	rarely	rarely	frequently	rarely	rarely	rarely	rarely	rarely	Yes	No	female

Figure 2: Excerpt of reduced JIM-PB data (53 cases, 15 binary variables)

The teacher introduced the following basic functions of Arbor. Again, the target variable is *OnlineGames* (frequently, rarely). First, the teacher shows how to perform a data split by dragging the predictor *GameConsole* from the table and dropping it into Arbor. The emerging subsets are evaluated automatically, as in Figure 3 (left). The one-level DT shows that of the 26 people who own a console, 20 play online games frequently, and 6 play rarely (20 to 6). Of those who do not own a console, 12

frequently play, and 15 rarely play (12 to 15). Second, it is demonstrated how predicted labels are chosen manually according to the majority, as in Figure 3. Third, the evaluation is demonstrated. One value of the target variable (here: frequently) is automatically represented as the positive outcome, which is shown in the second line above the DT. The DT's performance is represented by the displayed True Positives (TP), True Negatives (TN), False Positives (FP), and False Negatives (FN). The one-level DT misclassifies 18 cases (MCR: 0.34). After the demonstrations, students created, interpreted, and compared three one-level DTs, as in lesson 1, using self-chosen predictors. After discussing the results, the teacher showed how multi-level DTs are created by adding predictors to the end of a branch (Figure 3, right), motivated by improving performance. This study focuses on the subsequent open-ended task, where students created multi-level DTs choosing from 15 predictors.



Figure 3: Decision trees in CODAP with Arbor (tinyurl.com/CODAP-DT)

Data collection

The students worked in pairs in a CODAP document (Figure 3) for 30 minutes on the main task: *"Create a decision tree that classifies people according to whether they play online games frequently or rarely."* It was not specified how to do it in detail or what the ultimate goal is beyond reducing misclassifications as in the prior teaching. They got two additional questions: *"How do you select the predictor variables that you use for your decision tree?"* and *"How do you decide when to stop the construction process of your decision tree?"* We collected 25 screenshots of DTs from student pairs and 25 audio recordings (1-7 minutes) of students answering the extra questions. Presentations were held in class, enabling students in one cohort to hear and potentially influence each other, which, of course, is a realistic teaching scenario. To reduce this effect, students wrote down answers in advance, and feedback was only a neutral "thank you."

Data analysis

We did three analyses: (1) The collected DTs were examined for the number of predictors and misclassifications. More in-depth aspects, like specific predictors or predicted labels, are reserved for another longer paper. (2) Audio recordings were transcribed and analyzed for students' approaches to selecting predictors (2a) and deciding when to stop DT construction (2b). Using qualitative content analysis (Mayring, 2010), we inductively built categories for 2a and 2b. Finally, (3) typification (Kelle & Kluge, 2010) connected the prior analyses, examining whether the students' process approaches can explain the product characteristics of their DTs.

RESULTS

Characteristics of the DTs that students produced (1)

The students' DTs were very diverse. For a DT to be appropriate, it must include at least one predictor variable and specified predicted labels at each branch's end. One of the 25 DTs lacked predicted labels and was disregarded in the further analysis. The other 24 DTs were appropriate and had between 2 and 12 used predictor variables, resulting in numbers of misclassifications between 0 and 8. As shown in Figure 4, the students created very large DTs (A), mid-sized DTs (B), or small DTs (C).

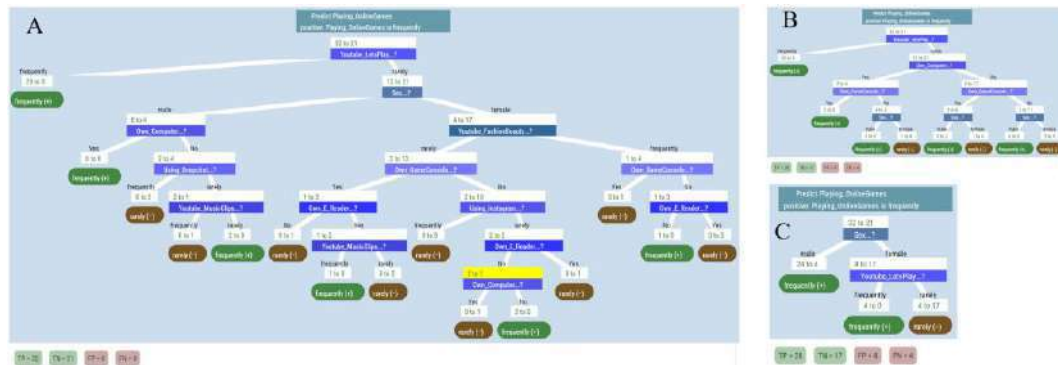


Figure 4: Examples of students' decision trees in CODAP (recreated in English)

Students' approaches for selecting predictor variables (2a)

In analyzing students' verbal presentations, we identified two broader types of selecting predictor variables: data-based argumentation (16 pairs), referring to frequency distributions in the Arbor DT or the data table, and context-based argumentation (8 pairs), using context knowledge to identify variables potentially related to playing *OnlineGames*. Some pairs combined both, pre-selecting variables with context knowledge and ordering them based on data; these were categorized as (dominantly) context-based. An example of data-based argumentation is the following excerpt:

Student: Then my goal was actually always, uh, I didn't really think about the names of the variables at all, but instead, I just went through all of them at each step and looked at how I could achieve a good ratio on both sides.

The student analyzed different data splits and the displayed ratios, disregarding their meaning, similar to professional DT algorithms. We identified subcategories of data-based predictor selection: the stepwise reduction of the MCR (9 pairs), aiming at reducing the MCR as much as possible in every step, the stepwise identification of "n to 0" (5 pairs), searching for a data splits with at least one pure subset without regarding MCR, and the keen observation of the data (2 pairs), where students closely looked at the data table to identify relationships between predictors and the target variable.

In contrast, context-based argumentation involves considerations that do not depend on the data. An example excerpt illustrating this type of argumentation is:

Student: I started by checking for devices. So first, whether a computer is available, then whether a console is available, then a tablet, and finally a smartphone.

Simply because I thought that it probably makes the most sense: if someone has a device, then they're likely to play on it as well.

The reasoning relies on context-based assumptions about the relevance of device ownership to predicting *OnlineGames* frequency, rather than on the available data.

Students' approaches for termination criteria (2b)

We identified four different types of termination criteria for students. One common data-based criterion was achieving zero misclassifications (12 pairs), as in this excerpt:

Student: The stopping criterion, in this case, was actually, uh, when there is a 0-to-n or n-to-0 ratio in each leaf, then I stop, as I no longer have any errors.

Here, the student stops the process when no misclassifications are remaining. Other students used the stagnation of the MCR (7 pairs):

Student: Then, uh, it turned out not to be very useful for us to add more because the error rate didn't decrease.

In this case, the student decided to stop when further additions to the DT did not lead to a reduction of the MCR. In a third type, students argued that a next data split would reduce the number of cases in a subset "too much" to still be "representative" (2 pairs):

Student: There were still 20 people who were assigned to it, and therefore, it was still relatively representative.

The word "representative" may not be accurate here, but it seems that the student was thinking of generalizability and classifying new cases beyond those represented in the data, anticipating the idea of overfitting. Then, we found a context-based termination criterion regarding the availability of variables they deemed as relevant (3 pairs):

Student: We actually only wanted to include what was relevant and truly related to playing computer games.

The approaches to termination criteria demonstrate different priorities, be it maximizing accuracy, ensuring context relevance, or avoiding overfitting.

Typification of relations between process approaches and DT characteristics (3)

To identify types, we used the students' approaches for predictor selection (2a) and termination criteria (2b) to explain the characteristics of their DTs (1). We formed subgroups by categories from 2a and 2b to examine the emerged DTs, and we identified three types (Figure 5). Our categories "data-based" and "context-based" predictor selection lead to a homogeneous subgroup in the case of context-based DTs (4-6 predictors, 4-8 misclassifications). The data-based DTs (2-12 pred., 0-8 misc.) have a high variability. A helpful subcategory for data-based DTs is the termination criterion – zero misclassifications vs. other criteria – resulting in two more homogeneous subgroups, which we call *data-based big DTs* aiming at zero misclassifications (5-12 pred., 0-5 miscl.) and *data-based small DTs* (2-3 pred., 6-8 miscl.). Figure 4 illustrates examples of a *context-based* (B), *data-based big* (A), and *data-based small* DT (C).

		selecting predictor variables					
		data-based			context-based		
termination criterion	zero misclassifications	data-based big trees (11 student pairs)			context-based trees (8 student pairs)		
		No. of predictor variables			No. of predictor variables		
		Min	Max	Ø	Min	Max	Ø
		5	12	8,45	4	6	5,13
		No. of misclassifications			No. of misclassifications		
		Min	Max	Ø	Min	Max	Ø
	others	0	5	1,91	4	8	5,88
		data-based small trees (5 student pairs)					
		No. of predictor variables					
		Min	Max	Ø			
		2	3	2,4			
		No. of misclassifications					
		Min	Max	Ø			
		6	8	7,6			

Figure 5: Three types of student approaches

Data-based big DTs include many predictors because students added variables until achieving zero misclassifications. In contrast, *data-based small DTs* emerged when students stopped early, unable to reduce misclassifications in the next step or considering generalizability. *Context-based DTs* are mid-sized, as students limited predictors to those they deemed relevant and tried to form a well-performing DT only with these. All three types were observed in each student cohort, as shown in Table 1.

	context-based trees	data-based big trees	data-based small trees
Cohort 1	2	2	2
Cohort 2	2	5	1
Cohort 3	4	4	2

Table 1: Frequencies of student approaches across cohorts

DISCUSSION

Implications for teaching

We identified three types of student approaches to creating DTs. Figure 5 helps teachers to anticipate and recognize these types. Each type appeared in all cohorts, and all three offer a valuable perspective for subsequent teaching. In classroom discussions, their relative advantages can be contrasted, so that two key questions for subsequent teaching arise: "Which approach is used in professional scenarios – context-based or data-based?" and "Which type of tree is better for classifying new cases – big or small trees?" The first connects to learning data-based ML algorithms, and the second to testing DTs with test data and addressing overfitting. Facilitating different approaches and building on them is an example of participatory learning because students autonomously create artifacts with informal approaches that can be valued and used afterward. The varying complexity of the approaches supports differentiated learning.

Conclusion

Our analysis suggests that starting with informal approaches to constructing DTs is accessible to secondary students. Except for one pair, all created appropriate results. Two perspectives on DTs are evident in students' approaches: a data-based and a context-based perspective. The majority used data-based reasoning, but a substantial number used context-based reasoning or mixed both, which aligns with the findings of

Zieffler et al. (2021), who found that while teaching in-service teachers. Similarly to Fergusson and Pfannkuch (2022), with an informal approach, some students laid a foundation for learning formal concepts. For example, among students who used the data-based perspective, some partially anticipated a DT algorithm, and some even had an intuition for advanced concepts like overfitting. Thus, our findings support Sanusi's (2023) suggestion that participatory learning is an effective strategy for teaching ML.

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EXPLORING THE RELATION BETWEEN MATHEMATICAL VALUES AND ACHIEVEMENT AMONG GIRLS: A COMPARATIVE ANALYSIS IN SINGLE-SEX VS. CO-EDUCATIONAL SETTINGS USING TIMSS 2019 NZ DATA

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Grounded in the Social Cognitive Career Theory, this study investigates the influence of values on girls' mathematics achievement across socio-economic status (SES) settings, contrasting single-sex and co-educational schools. ANOVA on 2019 TIMSS New Zealand data ($n = 2,898$) reveals that single-sex education is associated with enhanced girls' mathematical values in high-SES settings. However, its effect on translating these values into improved mathematics performance is relatively limited. Affluent learning resources are the more immediate factor in improving mathematics performance. Moreover, in low-SES environments, the relation between values and mathematics achievement exhibits a complex nonlinear pattern.

INTRODUCTION

In most OECD countries, including New Zealand, the proportion of female graduates in traditionally male-dominated fields of Science, Technology, Engineering, and Mathematics (STEM) in higher education is significantly lower than that of males, with an average of only 33% (OECD, 2023). This gender segregation trend extends to the occupational domain. For instance, a New Zealand government report indicates that only 27% of digital technology positions are held by women (Hindle & Muller, 2021). Academia faces similar challenges; according to data from the University of Auckland (NZ), the distribution of female faculty ranks in the Faculty of Science and Engineering is significantly lower than that of males, particularly in engineering, where the gap is most prominent (Brower & James, 2020). Without effective interventions, this gender imbalance is projected to persist until 2070 (Brower & James, 2020).

Girls in some Islamic countries exhibit a notable advantage in math achievement, values, and career aspirations for math-intensive careers, which stands in stark contrast to the gender difference trends observed in most Western regions (Michaelides et al., 2019). This phenomenon may be attributed to the unique social circumstances faced by women in these Islamic countries. Women's social status is relatively low in many Islamic nations, while careers in STEM fields often offer higher salaries. Consequently, girls may aspire to excel in mathematics and subsequently pursue high-paying math-related careers as a means to elevate their social standing (Stoet et al., 2018). This observation suggests that among the female population, there may be a significant association between value, mathematics performance, and the likelihood of

entering math-intensive fields. From this perspective, girls' values and achievements could serve as ideal entry points for investigating their propensity to pursue math-related careers.

Apart from social factors, the single-sex education widely adopted in these Islamic countries—where girls attend all-girls schools taught by female teachers and boys attend all-boys schools taught by male teachers (Michaelides et al., 2019)—may also contribute to girls' mathematics achievement to a certain extent. However, when examining the impact of single-sex education on girls' mathematics performance, it is crucial not to overlook the significant role of students' socioeconomic status (SES). SES is considered the most influential factor affecting students' mathematics achievement, accounting for more than 50% of the variance in girls' mathematics performance (Gao et al., 2025). Relevant research indicates that when SES is not controlled for, gender differences in mathematics achievement exist; however, when SES is controlled for, these gender differences significantly narrow or disappear (Clavel & Flannery, 2023). Therefore, to more accurately assess the effectiveness of single-sex education, it is necessary to consider and control for the SES. Smith and Evans' (2023) study in New Zealand serves as an excellent example. After controlling for students' SES, they found that girls from low SES families in single-sex schools outperformed girls attending co-educational schools in mathematics ($g = 0.90$), further supporting the potential advantages of single-sex education.

New Zealand's education system offers a distinctive setting for investigating the effects of single-sex schools. Among the 374 secondary schools in NZ, approximately 16% are single-sex girls' schools, with a striking 91% of these institutions being state or state-integrated schools (MoE, 2023). This contrasts sharply with the United States and Australia, where single-sex schooling is primarily concentrated in private or Catholic schools. Furthermore, New Zealand employs a decile system to measure the socioeconomic background of schools. This system ranks schools from 1 to 10, with decile 1 representing schools whose students come from the lowest 10% of families and decile 10 representing schools whose students come from the highest 10% of families.

THEORETICAL BACKGROUND

The performance model of the Social Cognitive Career Theory (SCCT) provides a comprehensive framework for understanding the interplay between individual motivation and environmental factors in shaping math-related performance (Lent & Brown, 1994). More specifically, the model centers on the core constructs of self-efficacy and outcome expectation, which dynamically interact with person factors, contextual influences, and learning experiences to collectively shape academic outcomes. Outcome expectations, as a center component, encompass the concept of values, reflecting individuals' preferences for the "reinforcers" of academic activities (Dawis & Lofquist, 1984). For instance, mathematical values are not only embodied in the interest in mathematics itself but also include the expectation that learning

mathematics can lead to career opportunities, social recognition, and economic rewards. Contextual and experiential factors act as antecedents to the formation of values or directly facilitate or hinder individuals' mathematics performance, with gender serving as a significant moderator in these relationships. For example, girls may encounter a higher prevalence of gender role stereotypes and negative evaluations, such as the pervasive notion that girls lack aptitude in mathematics, which can erode their outcome expectations and, consequently, impinge upon their mathematics performance.

RESEARCH QUESTIONS

This study addresses the relationship between school SES (e.g., decile), school gender (e.g., single-sex vs coeducational), mathematical values (e.g., self-reported values of mathematics), and mathematics achievement in TIMSS for Year 9 girls in New Zealand. Two questions were formulated:

1. Is there a disparity in girls' mathematical values between single-sex and co-ed schools?
2. What is the relationship between mathematics values and mathematics achievement, contrasting girls in single-sex and co-ed schools?

METHODS

The study comprised 2898 Year 9 girls in the first year of New Zealand secondary education from the 2019 TIMSS dataset. The 2023 TIMSS dataset was excluded due to its limitations: the impact of COVID-19-related disruptions, such as online learning, complicates disentangling the effects of school factors on mathematics achievement. Furthermore, access to key non-public variables, such as school gender and decile, was unavailable at the time of this research, further reducing the dataset's applicability for this study.

Decile and school gender are critical variables in this study. Decile bands are low = deciles 1-3, medium = deciles 4-7; high = deciles 8-10. Of these participants, 906 were enrolled in single-sex girls' schools, with 9.2%, 35.8%, and 55.1% across low-decile, medium-decile, and high-decile schools, respectively. Additionally, 1992 girls attended co-ed schools, distributed as 26.0%, 41.6%, and 32.5% across low-decile, medium-decile, and high-decile schools. Based on percentages, the distribution clearly indicates a substantial advantage of single-sex schools in terms of SES ($\chi^2 = 14.21, p < .001$).

The data were subjected to two-way ANOVA and three-way ANOVA to investigate the relationships between decile band (DB: Low, Medium, and High), school gender (SG: Single-sex and Co-ed schools), mathematical values (MV: Low, Medium, and High), and girls' mathematics achievement (MA).

Given the substantial disparity in girls' enrollment between co-ed and single-sex schools, potential bias in ANOVA was overcome by drawing 20 samples, mirroring

the enrollment figures of single-sex schools stratified by DB. Each sample comprised 232, 376, and 294 girls from low, medium, and high decile schools. The null hypothesis is rejected if only one of the 20 iterations proves statistically insignificant.

RESULTS

A two-way ANOVA was conducted to investigate the impact of school gender and decile settings on values in mathematics. A statistically significant interaction was observed between SG and DB on MV (Table 1: $F_{(4, 2858)} = 5.523, p = .004, \eta p^2 = .004$).

Source	<i>df</i>	<i>F</i>	Sig.	ηp^2
Corrected Model	5	3.113	0.008	0.005
Intercept	1	42743.54	<.001	0.937
SG	1	0.061	0.804	0.000
DB	2	2.975	0.051	0.002
SG * DB	2	5.523	0.004	0.004

Table 1: Interaction between gendered school and decile setting

Consequently, further analysis of simple main effects for SG and DB was performed, employing Bonferroni-adjusted statistical significance at the $p < .025$ level (Table 2). In terms of SG, a significant difference in mean "MV" scores for single-sex schools was found ($F_{(2, 2870)} = 4.471, p = .012$). A significant difference in mean "MV" scores between single-sex and co-ed schools for low-decile was observed ($F_{(1, 2870)} = 6.048, p = .014$). Among the 20 samples, only one was deemed insignificant, with the remaining 19 demonstrating statistical significance at $p < 0.05$.

DB	<i>df</i>	<i>F</i>	Sig.	SG	<i>df</i>	<i>F</i>	Sig.
Low-DB	1	6.048	0.014	Single	2	4.471	0.012
Medium-DB	1	3.196	0.074	Co-Ed	2	1.908	0.149
High-DB	1	5.221	0.022				

Table 2: Main effect of decile setting and gendered school

Pairwise comparisons were conducted within each main effect, with Bonferroni-adjusted confidence intervals and p-values. Girls scored significantly lower mean "MV" in single-sex than co-ed schools in low-decile settings (0.540; 95% CI [0.110, 0.971]; $p = .014$). Furthermore, girls had significantly lower mean "MV" scores in single-sex schools in low-decile compared to those in medium and high-decile bands.

In conclusion, a notable disparity in values between single-sex and co-ed schools was evident in low-decile bands, with the latter exhibiting significantly higher values. Moreover, across the low-to-high decile spectrum, the values of girls in single-sex

schools increased, while there was no significant difference in values for girls in co-ed schools.

A three-way ANOVA examined the nuanced impacts of school gender, decile band, and values on mathematics achievement. A statistically significant three-way interaction among these factors was identified ($F_{(4, 2858)} = 2.398, p = .048$), with a standardized effect size (r) of 0.064 (95% CI [0.0494, 0.0787]). Notably, Cohen's (1988) guideline designates this effect size as consistently ignorable (Table 3).

	<i>df</i>	<i>F</i>	Sig.	ηp^2
Intercept	1	52885.17	<.001	0.949
SG	1	39.557	<.001	0.014
DB	2	84.87	<.001	0.056
MV	2	21.201	<.001	0.015
SG * MV	2	1.343	0.261	0.001
SG * DB	2	11.604	<.001	0.008
DB * MV	4	4.931	<.001	0.007
SG * DB * MV	4	2.398	0.048	0.003

Table 3: Interaction between gendered school, decile setting, and values

Significant two-way interactions were found between SG and DB ($F_{(2, 2858)} = 11.604, p < .001$) and between MV and DB ($F_{(4, 2858)} = 4.931, p < .001$) (Table 4). Furthermore, the main effects of SG existed in low-decile bands and MV in high-decile bands. Pairwise comparisons showed that girls from single-sex schools had significantly higher mean "MA" scores than their co-ed counterparts in low-decile bands (61.659; 95% CI [41.713, 81.606]). High MV girls demonstrated markedly higher mean "MA" scores in high decile band compared to their counterparts with low and medium MV (Δ_{low} : 64.206; 95% CI [48.008, 80.403] and (Δ_{medium} : 35.446; 95% CI [23.187, 47.706]).

	SG				MV			
	<i>df</i>	<i>F</i>	Sig.	ηp^2	<i>df</i>	<i>F</i>	Sig.	ηp^2
Low-DB	1	36.739	<.001	0.013	2	1.43	0.239	0.001
Mid-DB	1	2.479	0.116	0.001	2	8.576	<.001	0.006
High-DB	1	3.524	0.061	0.001	2	49.482	<.001	0.033

Table 4: Main effect of gendered school and values

Consequently, a significant disparity in girls' mathematics achievement emerged between single-sex and co-ed schools in low-decile bands. Notably, values did not

significantly influence girls' math scores in low-decile and medium-decile bands. However, a positive relationship emerged in high-decile bands, indicating that higher MV corresponded to higher MA only among those girls.

DISCUSSION

Dual Necessities for Enhancing Girls' Mathematical Value: Resource Guarantee and Single-sex Education

As the decile level increases, the development of girls' values in co-educational schools remains relatively stagnant, whereas in single-sex schools, these values exhibit significant growth. This finding indicates that cultivating motivation, such as mathematical values, necessitates the availability of abundant resources as a foundational prerequisite. Within this context, single-sex education can effectively maximize its potential impact. The increase in decile levels implies that families and schools can provide more abundant resources, such as high-quality teaching equipment, diverse curriculum choices, and more extracurricular activity opportunities. Single-sex schools, compared to co-ed schools, can more effectively transform these resources into personalized support, helping girls break gender stereotype roles, such as "mathematics is not for girls," thereby promoting the development of girls' values (Smith & Evans, 2023). The initiatives of St Cuthbert's College for girls (decile 10) in New Zealand serve as a prime example. The college regularly invites women who have achieved outstanding accomplishments in STEM fields as role models to challenge prevalent gender stereotypes, thereby enhancing girls' perceptions of mathematics. In contrast, co-ed schools may struggle to provide targeted support and activities for girls. Therefore, the long-standing gender role expectations within co-ed schools are challenging to eliminate, weakening the promoting effect of affluent resources on girls' values in the co-ed environment.

Translating Mathematical Values into Achievements: The Significance of Learning Resources and the Limited Role of Single-Sex Education

In high decile schools, the positive impact of values on girls' mathematics achievement is prominently manifested. However, this positive impact does not appear to exhibit significant differences between single-sex and co-ed schools. This phenomenon suggests that the process of transforming motivation into academic achievement needs to be supported by sufficient resources, while gender-related educational influences are relatively limited. Nevertheless, it would be one-sided to completely deny the impact of single-sex education on math achievement based solely on this phenomenon. As a fundamental condition for educational quality, sufficient resources have a more direct and immediate effect on improving mathematics performance and can even explain 56% of the variance in mathematics achievement (Gao et al., 2025). Moreover, through the above analysis, it can be found that single-sex education can effectively promote the enhancement of girls' mathematical values, while the promoting effect of motivation on mathematics achievement has a certain lagging effect. The immediate effect of resources and the lagging effect of motivation also explain the research results

obtained by Smith and Evans (2023) in cross-sectional data: when both single-sex and co-educational schools have abundant resources, the performance differences of girls in mathematics and science are not significant. Therefore, considering the unique function of single-sex education in promoting girls' values, we cannot completely deny the role of single-sex education in girls' mathematics achievement. Further longitudinal analyses are imperative to elucidate this issue and provide a more comprehensive understanding of the intricate interplay between educational settings, values, and academic performance.

Complex Relations in Low-decile Band: The Nonlinear Influence between Mathematical Values and Achievement

In the low-decile band, the advantages of single-sex education in fostering girls' mathematical values appear to be diminished. In economically disadvantaged settings, the potential benefits of single-sex education in fostering values are inevitably overshadowed by systemic issues arising from resource scarcity. Low-decile schools often face challenges such as inadequate facilities, limited teaching resources, and weak socio-economic support networks. In such contexts, girls may not receive sufficient additional support and compensatory education. The targeted attention and support that single-sex education aims to provide may be weakened by the constraints imposed by resource scarcity, consequently affecting girls' development of values in mathematics.

Regarding math achievement, girls in single-sex schools surpass their counterparts in co-ed settings. Under resource-constrained conditions, single-sex schools appear to be more efficacious in enhancing girls' math performance. However, this does not necessarily imply that gender-specific support activities have a significant impact on math achievement. In fact, optimized resource allocation in single-sex schools may be the more direct factor influencing girls' math performance. In co-ed schools, girls' time and resources for math learning are often compromised due to boys' higher frequency of interaction with teachers. This imbalanced resource distribution undermines girls' mathematical confidence and engagement (Smith & Evans, 2024). Furthermore, the discrepancy between higher mathematical values and lower confidence in abilities in co-ed settings may elicit negative emotions, such as anxiety and helplessness (Pekrun, 2006). These emotions further exacerbate the detrimental effects of co-ed environments on math performance. In contrast, single-sex schools, by virtue of the absence of boys, can concentrate learning resources on girls. The direct impetus of these learning resource allocations on math achievement can effectively mitigate the negative impacts on academic performance caused by resource scarcity.

Notably, the relationship between values and mathematics performance exhibits complex non-linear patterns in low-decile environments. In other words, no significant correlation is observed between values and math performance in low-SES contexts. This suggests that economic disadvantage constrains the effective translation of mathematical value into achievement, challenging numerous academic success

theories, including SCCT. These theories typically assume the universal applicability of motivation to academic achievement, but their explanatory power may be primarily limited to academic contexts with resource-rich or high-achieving groups.

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STUDENT EXPLANATION IN MIDDLE AND SECONDARY MATHEMATICS AND STATISTICS: A SCOPING LITERATURE REVIEW

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This scoping review examines the literature on student explanation strategies in middle and secondary mathematics and statistics education from 2014 to 2024. Following the PRISMA protocol, we analyzed 41 studies that met the inclusion criteria. The findings classify student explanations into two types: self-explanation (SE) and peer explanation (PE). Both approaches enhance conceptual understanding and procedural knowledge, though each offers distinct benefits and challenges. SE is particularly effective when combined with worked examples and varies with prompting strategies, while PE significantly impacts students' affective development and social learning. The review identifies a significant gap in studies comparing the effectiveness of SE and PE, alongside an almost complete absence of research within statistics education.

1. INTRODUCTION

Over the past few decades, researchers have emphasized the importance of student-constructed explanations in mathematics and statistics education (Rittle-Johnson, 2024). Student explanation requires learners to articulate their understanding of mathematical concepts and reasoning processes (Rittle-Johnson & Loehr, 2017). Research shows this approach enhances both conceptual understanding and core competencies like logical reasoning and abstract thinking (Rittle-Johnson et al., 2017; Fiorella & Mayer, 2016a), reflecting modern mathematics education's emphasis on active learning practices (Fiorella & Mayer, 2016b).

Despite the recognized value of student explanations in mathematics learning, research lacks systematic organization regarding its effective implementation in educational settings. This study aims to systematically analyze existing literature to clarify the current state of research on this strategy.

2. THEORETICAL GROUNDING

Student explanations can be understood through multiple theoretical frameworks. Human cognitive architecture theory (Atkinson & Shiffrin, 1968) describes three key stages: sensory memory for receiving inputs, working memory for active processing, and long-term memory for storing information in schemas. Related theories, such as cognitive load theory (Sweller, 1988), further explain that working memory has limited capacity in processing new instructional content. Explanation generation supports schema construction by directing cognitive resources to integrate new knowledge with existing understanding (Mayer & Moreno, 2003). In mathematics and statistics

learning, this integration occurs through connecting concepts and applying principles to new contexts, fostering meaningful knowledge construction through cognitive processing and memory integration (Fiorella & Mayer, 2014; Lachner et al., 2021).

Social Presence Theory (Short et al., 1976) adds a social interaction perspective, emphasizing how learners adjust their explanations based on their audience's knowledge level—a process that deepens learning (Monrose Mills et al., 2020). When implemented through peer teaching and group problem-solving, these activities enhance social presence, motivation, and cognitive engagement (Nasir et al., 2023; Morris et al., 2023; Alegre et al., 2019), leading to more effective memory formation.

These frameworks complement each other in explaining student explanations' effectiveness: cognitive architecture theory addresses information processing and knowledge construction, while social presence theory focuses on social interaction benefits. Together, they reveal how student explanations enhance learning through the synergy of cognitive processing and social interaction.

3. METHODS

This review follows Arksey & O'Malley's (2005) five-stage framework: (1) identifying the research question, (2) identifying relevant studies, (3) study selection, (4) charting the data, and (5) collating and reporting results. This framework ensures a rigorous and replicable review process.

3.1 Identifying the initial research question

Our review focuses on examining the specific use of student explanations in middle and secondary mathematics and statistics education. Therefore, we propose the following research question: How can the literature on student explanation strategies in middle and secondary mathematics and statistics education be categorized to reveal current research patterns and trends?

3.2 Identifying relevant studies

All publications included in this review were published in English and indexed in three major academic databases: ERIC (Education Resources Information Center), Scopus, and Web of Science. To ensure comprehensive coverage, we searched major databases, supplemented with Google Scholar and Research Rabbit searches, and conducted forward and backward citation tracking. The search used systematic keywords from preliminary review and expert consultation, as shown in Table 1.

We reviewed literature from 2014-2024, focusing on "explanation" and related terms in middle and secondary education (Grades 6-13). While our search might have missed studies using terms like "articulation" or "justification," we mitigated this through broad initial screening. We included only English-language, peer-reviewed articles.

Search string
("Secondary Education" OR "Middle School" OR "Intermediate School" OR "High School" OR "Junior High" OR "Senior High" OR "Grade 7" OR "Grade 8" OR "Grade 9" OR "Grade 10" OR "Grade 11" OR "Grade 12" OR "Adolescents" OR "Teenagers") AND ("Mathematics" OR "Statistics" OR "STEM") AND ("Learning by explaining" OR "Explaining-based learning" OR "Explanation-driven learning" OR "Self-explanation" OR "Self-explaining" OR "Learning by teaching" OR "Learning-by-teaching" OR "Team-based-explanation" OR "Peer tutoring" OR "Peer instruction" OR "Peer assist" OR "Peer-led")

Table 1: Search string

3.3 Study selection

Following PRISMA protocol (Moher et al., 2009), our initial search retrieved 231 publications. After removing duplicates and screening titles and abstracts, 57 publications proceeded to full-text assessment. Further evaluation excluded 16 publications, leaving 41 studies for qualitative analysis (complete literature list is available on figshare: <https://doi.org/10.17608/k6.auckland.26933131>).

4. FINDINGS AND DISCUSSION

Based on the analysis and synthesis of the results of selected literature, the student explanation strategies can be categorized into two broad types: Self-explanation (SE) and Peer explanation (PE). Of the 41 studies reviewed, 16 focus on SE (16 on mathematics, 0 on statistics), while 25 discuss PE (24 on mathematics, and 1 on statistics).

4.1 Self-explanation

Self-explanation involves learners generating explanations to themselves (Hodds et al., 2014), actively connecting new information with existing knowledge (Rittle-Johnson et al., 2017). Students articulate their understanding of concepts, procedures, and problem-solving steps, helping identify knowledge gaps and build coherent mental models (Lachner et al., 2021; Rittle-Johnson & Loehr, 2017). In middle and secondary education, SE is implemented through written explanations in textbooks or student sheets, or through oral explanations via audio recordings or self-narration.

Six of the 16 studies examined combining SE with worked examples (e.g., Barbieri et al., 2021). This combination reduces errors, speeds up problem-solving, and decreases the need for teacher assistance (e.g., McGinn et al., 2015), while helping students overcome insufficient prior knowledge in algebra (e.g., Barbieri et al., 2023). However, students may become overly dependent on worked examples, potentially limiting knowledge transfer (Schalk et al., 2018).

Student achievement through SE manifests primarily in conceptual and procedural knowledge development (e.g., Özcan, 2024). For example, in mathematics learning, when students used diagrammatic SE as an anticipatory approach, they improved their procedural knowledge by better applying formal algebraic problem-solving strategies to transfer problems with negative numbers (Nagashima et al., 2021). On the conceptual side, SE through GeoGebra enhanced students' ability to visualize and connect different mathematical representations, demonstrating deeper conceptual understanding (Nordlander, 2022). Notably, females seem to benefit more from SE than males (Nguyen et al., 2022). This gender difference appears to stem from different engagement patterns: females tend to produce more thorough and thoughtful explanations, while males often provide quick, superficial responses, potentially missing the learning benefits that come from struggling with and articulating explanations (McLaren et al., 2022).

The self-explanation principle suggests that SE falls along a continuum: open-ended (students are simply prompted to self-explain, e.g., “Explain how you solved this problem”), focused (guiding students toward specific aspects, e.g., “Explain why you chose this mathematical operation”), scaffolded (providing support for explanation, e.g., “Using the given formula, explain each step of your solution”), resource-based (referring to specific materials, e.g., “Using the diagram provided, explain how you arrived at your answer”), and menu-based (selecting from predetermined explanations, e.g., “Choose from multiple explanation options”) (McLaren et al., 2022). Menu-based SE, which guides thinking through preset options, is more helpful for students with weak foundations (Wong et al., 2019). However, in terms of long-term effects, open-ended SE, which allows students to freely establish knowledge connections, shows better learning outcomes (McLaren et al., 2022).

However, SE has its limitations. In complex tasks, excessive SE may increase cognitive load (Hänze & Leiss, 2022) and may even lead to cognitive confusion and pseudo-confidence in students, affecting effective knowledge construction (Andres et al., 2023).

4.2 Peer explanation

Peer explanation (PE) has emerged as a key strategy in mathematics education's shift toward active learning (Alegre et al., 2019b). In PE, students work in pairs to explain mathematical concepts and problem-solving strategies (Moliner & Alegre, 2020b, 2022). Unlike simple peer tutoring, PE involves reciprocal knowledge construction where students alternate between explainer and listener roles, collaboratively building mathematical understanding (Moliner & Alegre, 2020a). This benefits both parties: explainers deepen understanding through articulation, while listeners gain new perspectives through engagement (Martí Arnándiz et al., 2022).

The 25 PE studies reveal two main positive impacts. First, PE improves students' affective domain, enhancing self-concept and reducing mathematics anxiety (e.g., Alegre & Moliner, 2017) by creating a safe, inclusive environment that encourages

participation (Arthur et al., 2022). Second, PE develops conceptual and procedural knowledge (Fukuda & Manalo, 2024) as students explain both how and why solutions work, motivating them to become role models (Roberts & Spangenberg, 2020). Same-age PE proves more effective than cross-age PE (Moliner & Alegre, 2020b, 2022), with female and younger students showing higher gains in self-efficacy (Arnándiz et al., 2022).

However, PE has several limitations: First, learning through teaching may lead students to spend considerable time on irrelevant activities, creating extraneous cognitive load and thus affecting learning outcomes (Fiorella et al., 2019). Second, PE may negatively impact explainers, leading them to develop a fixed mindset about intelligence and affecting their self-efficacy (Gandolfi et al., 2024).

Furthermore, PE's effectiveness is constrained by students' knowledge base and explanatory abilities, where weaker students' content knowledge may lead to misconceptions during transmission (Thomas et al., 2015). Additionally, PE's effectiveness largely depends on the paired students' knowledge levels. When knowledge differences between pairs are either too large or too small, PE's effectiveness may be compromised, with weaker students showing lower self-concept and participation willingness, while students with similar knowledge levels may benefit less from PE (Moliner & Alegre, 2020b).

5. CONCLUSION

This scoping review underscores the well-documented positive effects of self-explanation and peer explanation in mathematics education, where both strategies have been shown to enhance conceptual understanding and problem-solving skills. However, research on student explanations in statistics education is notably limited, with only one of 41 studies addressing PE in statistics learning, highlighting a significant research gap at the middle and secondary levels.

While self-explanation has been extensively studied for its cognitive benefits and peer explanation for its social and collaborative advantages, few studies have directly compared their relative impacts on learning outcomes. Future research should focus on comparative studies to identify the conditions and learner characteristics that influence the effectiveness of self-explanation and peer explanation. Such efforts are crucial to inform evidence-based teaching practices that optimize the use of these strategies to support student learning in both mathematics and statistics education.

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LEARNING, PARTICIPATION, AND SELF-DETERMINATION IN THE INCLUSIVE MATHEMATICS CLASSROOM

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This theoretical paper explores how inclusion in mathematics education can be operationalized starting from the dimensions of learning, participation, and self-determination. Grounded in Radford's theory of objectification, we propose specific indicators that bridge inclusion principles with observable processes in the mathematics classroom. A Grade 7 problem-solving activity serves as an example of how the defined indicators can be applied and their value in supporting inclusive practices in the mathematics classroom.

INTRODUCTION

In the international literature, the concept of inclusion is generally approached through two main categories of definitions: *narrow* and *broad* (Göransson & Nilholm, 2014). *Narrow* definitions focus on the integration of students with disabilities or special educational needs into mainstream education, often through targeted interventions. *Broad* definitions see inclusion as a process aimed at valuing the identity of each learner and ensuring equal access for all to both learning and meaningful participation (Göransson & Nilholm, 2014). The second category of definitions does not refer specifically to student with special educational needs. This theoretical paper adopts a broad definition of inclusion, consistently with international documents on inclusive education (e.g. UNESCO, 2005) and seeks to establish what this perspective implies in relation to mathematics education. By looking at documents and publications that adopt a broad definition of inclusion (e.g., UNESCO, 2005; Ainscow, 2020), we can identify three main categories characterizing inclusion in education: learning, participation, and the individual's capacity for choice and self-determination (Garzetti, 2023). Yet, these documents rarely provide explicit frameworks for translating such principles into discipline-specific classroom practices. Building on the broad vision of inclusion and focusing on mathematics education, this work proposes a set of operational indicators for participation and self-determination, informed by the literature on inclusive pedagogy and adapted to the nature of mathematical activity. Several literature reviews provide insights into how inclusion is conceptualised and practised, particularly in mathematics education. For example, Van Mieghem et al. (2020) examined systematic reviews on inclusive education: these studies highlight attitudes towards inclusion, teacher professional development, inclusive practices and student participation, but reveal a lack of research on what they define as academic participation, i.e. the learning dimension of the inclusion process in relation to specific disciplines. Within the field of mathematics education, Roos's literature review (2019)

distinguishes between what she defines *ideological* perspectives on inclusion, that focus on equity and access but are less specific about instructional methods, and research that focuses on classroom interventions without explicitly addressing the issue of defining inclusion at a broader level. There is a growing call to bridge ideological perspectives on inclusion with practical classroom design, and to examine how policies and values around diversity influence instructional choices (Roos, 2019). This work aims to bridge the gap highlighted by Roos (2019) clarifying how choices and perspectives on learning, participation and self-determination at the theoretical level profoundly influence the way inclusion is constituted and viewed in the classroom context. Specifically, we build discipline-specific inclusion indicators derived from chosen definition of learning, participation, and self-determination and we then illustrate how these indicators can be applied in an actual classroom context, demonstrating their potential for fostering a more inclusive approach to mathematics teaching.

THEORETICAL FRAMEWORK

Different learning paradigms in mathematics result in very different way of conceiving effective mathematical classrooms: they produce changes in the role of the student, the teacher, and mathematics (Radford, 2021). The choice of educational paradigm and definition of learning in mathematics also results in different ways of considering inclusion in the classroom. In this study, we refer to a socio-cultural paradigm of learning, and in particular to Radford's theory of objectification. The theory of objectification (Radford 2021) is a sociocultural theory, based on the idea that there is a deep intertwining between culture, and in particular mathematical culture, and individuals, which is realised in human activity. According to this perspective, learning is becoming progressively aware of culturally constituted forms of thought and action and positioning oneself critically in relation to them. This perspective on mathematics learning allows us, at the educational paradigm level, to emphasize the crucial balance at the heart of the inclusion process (Norwich, 2008): recognizing an individual's self-determined path while fostering their participation in the mathematical community and culture. This in turn brings to the fore the so-called 'dilemma of difference' (e.g. Norwich, 2008) in mathematics education: on the one hand, recognising children's differences to provide tailored support risks labelling and creating stigma; on the other hand, emphasising 'sameness' through uniform provision may fail to address the unique needs of individual children.

Radford address this tension by referring to the notion of learning as *objectification*, a process by which “knowledge in itself” becomes “knowledge for oneself.” Through tasks involving signs, gestures, physical artifacts, and linguistic expressions, termed *semiotic means of objectification*, students gradually take up and transform historically established mathematical forms of action and thought. Radford (2021) highlights that learners learn not just to do mathematics but also to perceive and interpret the world in culturally shaped, mathematically informed ways. In turn, this evolving awareness

allows learners to participate more fully in classroom dialogue and to develop critical insights into their own learning processes. A key methodological tool in this framework is the identification and analysis of *semiotic nodes*: segments of classroom activity in which different interpretations of a mathematical idea or procedure come to the surface and lead to the emergence of new semiotic means of objectification (Radford, & Sabena, 2015). During these moments, students propose strategies, discuss alternative viewpoints, or deploy new signs or artifacts to make sense of the content. By focusing on these highly interactive and meaning-rich instances, researchers can trace how students' ways of seeing and reasoning evolve. This perspective on learning offers a robust lens through which to capture the interplay between mathematical content and inclusive practices. In classroom observations, the semiotic nodes are identified through (1) changes in the use of signs or artifacts aligned with the activity's goals, (2) shifts in the interpretation of signs or artifacts consistent with the mathematical meanings at play and (3) the introduction of new signs or artifacts enabling problem-solving or advancing understanding (Garzetti, 2023).

We have clarified our approach to examining the learning dimension of inclusion. We now turn to participation and self-determination, which also emerge from Radford's definition of learning, but require specific indicators to become observable. Participation in this context is understood as active engagement in the classroom community, going beyond mere physical presence to include interactions that shape both the social climate and the mathematical processes. Specifically, in we adapted Ianes's (2021) levels of participation, which range from level -1 to level 3. Level -1 is defined as marginalization of the student by teacher or peers; level 0 is the physical absence of the student from the classroom; level 1 is physical presence without interaction; level 2a is reached when interaction is present in response to external stimuli, 2b is instead characterised by the initiation of interaction, verbal or non-verbal. The higher level, level 3, include forms of leadership that promote goal-oriented work (level 3a) or a supportive learning climate (level 3b).

The dimension of self-determination focuses on an individual's ability to make autonomous decisions, set goals, and act on personal interests (Wehmeyer et al., 2003). Self-determination in mathematics can be understood through the lens of the three dimensions of the causal agency framework (Shogren et al., 2015), each of which is closely linked to specific indicators that can be observed in students' behaviour. The first dimension, *volitional action*, emphasises autonomy and initiative, which is manifested when students engage in mathematical tasks independently. This includes completing activities without external prompting, actively choosing between tasks, and drawing on prior experience and strategies to guide their decisions in new contexts. The second dimension, *agentic action*, emphasises students' ability to plan, solve problems and collaborate effectively. In mathematics, this is evident when students explore and evaluate different approaches to a problem, justify their choices by linking them to resources and prior knowledge, and actively implement their strategies. Collaboration also plays a crucial role, as students work with peers and seek help when

faced with challenges, demonstrating a proactive and constructive approach to overcoming difficulties. Finally, *beliefs about control* focus on self-awareness and empowerment, which are reflected in how students perceive their own contributions and learning processes. In a mathematical setting, this involves students recognising how their actions affect the outcome of a task, reflecting on their strengths and limitations, and analysing moments of difficulty to understand how they successfully overcame them. This reflective practice builds confidence and helps students to take ownership of their learning journey.

Given the key theoretical references, we can specify the research question: How does use of inclusion indicators, grounded in the dimensions of learning, participation and empowerment, make the ideological vision of inclusion explicit and operational in the mathematics classrooms? In the following sections, we illustrate how these indicators work in practice, providing a lens through which to evaluate and refine instructional design in the mathematics classroom.

METHODS

To show how the indicators work, let us analyse the activity carried out by a group of Grade 7 students in relation to the following problem.

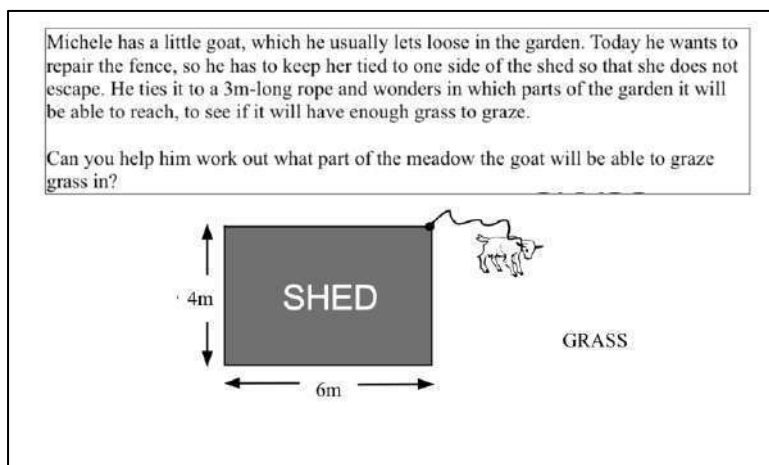


Figure n: The problem proposed to the students. Adapted from the teaching proposal of the Mathematics Assessment project (www.map.mathshell.org)

The aim of the problem is the objectification of circle and circumference and their mutual relationship. Specifically, in the given problem the goat traces a circumference only when the string is fully extended, with the radius of the circumference given by the length of the string. If, on the other hand, one considers all the points at which the goat can graze, one obtains a portion of the circle. The string, or its drawing, is the proposed semiotic means of objectification to objectify the circumference and the circle in the activity.

For data analysis, the group work is recorded and the protocols produced by the students are analysed. Six specific students in the classroom were taken as case studies for the analysis. The students were chosen in advance of the intervention to represent

the heterogeneity of the class. They are students with and without special educational needs, different attitude and results in mathematics. For the analysis presented here we focus on one of them, Paolo, a student with special educational needs with a specific learning disorder that affects his ability to complete a task and to maintain attention on the same activity or stimulus for long periods. Our analytical approach combines a two-step procedure: first, we identify and analyse the semiotic nodes of the group activity. Video recordings of small-group work are transcribed to identify semiotic nodes. After analysing the groupwork, we focused on understanding a specific student's contributions to semiotic nodes, participation, and self-determination, integrating various data sources to gain a comprehensive perspective. To examine contributions to identified semiotic nodes, we looked at how the student engaged with the identified nodes, such as introducing new reasoning, proposing novel approaches, or responding thoughtfully to the strategies shared by his peers. For participation, we segmented the video recordings into intervals of 3–5 minutes and assigned a level of participation (from -1 to 3) to the student during these segments. This analysis allowed us to track not only the frequency but also the quality of each student's involvement over time, providing a clearer picture of his overall engagement during the activity. In this paper we consider only the first two dimensions of self-determination, volitional and agentic actions, examined focusing on how the student produced autonomous outputs, made decisions independently, and interacted with his peers to negotiate strategies or solve problems. In this context, we do not analyse the dimension of beliefs about control for lack of space: this aspect can in fact be brought out by the analysis of the student's interview in relation to pathways that go beyond the single activity.

RESULTS AND DISCUSSION

The considered group is composed of Valeria (V), Paolo (P), Silvia (S), and Giorgio (G). We will only analyse the second semiotic node of the problem-solving activity, briefly describing the first semiotic node. In the first part of the activity, students begin modelling the circle that represents the goat's grazing area using a compass, only to discover that the goat's head appears outside the drawn circle. From the worksheet designer's perspective, the little goat was merely a point-like object, and its actual size was irrelevant to the problem. However, the students perceive the goat as a real figure: therefore, noticing that it extends beyond the circle and is not drawn to scale becomes significant for finding the solution. G and V decide to verify the circumference made with the compass by checking the 3 cm radius with the ruler. P experiences difficulty adjusting the compass, causing him to miss parts of the group's discussion. Nevertheless, once G demonstrates how to measure and rotate the ruler around the centre to confirm the constant distance, P re-engages by similarly measuring 3 cm on his sheet. The semiotic means of objectification that enables the students to conceive of the circumference in an embodied way, namely as the set of all points at a fixed distance from a given point, is not the compass, but rather the ruler. Because the ruler makes the distance both visible and controllable during rotation, it becomes essential to validating the answer to the question. At this point the teacher notices P's difficulty

with the compasses and intervenes to adjust the tips and show P how he can fix them. We report below part of the exchange that took place within the group after that:

P: In the meantime, I am trying to calculate these two areas here (he points to his drawing on the sheet) I do these ones here, plus this one and this one (he points to the quarters of a circle, indicating three out of four). That way it will be...If here...eh...if here is the sheep.

S: But it's 3 cm, no need to calculate.

[P comments on his drawing saying that it's not beautiful]

P: If she is tied here, and we have calculated this area here of 3 metres.

G: That's the area she grazes.

P: Exactly. But we also have to understand on this side here how much she grazes.

G: Yes, we've calculated.

V: 3 metres.

P: Yes, but behind here?

S: We've calculated it, look.

[V shows P that they have considered all the points where the goat can go by reproducing the circular movement of the rope with an eraser one end of which is tied to the corner of the hut].

P: But I'm saying this is the area, she grazes everything, here.

S: Yes, everything.

P: Yes.

V: It goes simply like this: this red dot is the goat, this one is the farm, if she is tied here she can reach here at most, so she can do like this... And so (continues miming in his model, indicating interior points of the circle with a pencil).

G: And everything inside is what she can eat.

P: Mh, ok. Then let's make it look like a cat (he transforms the circumference he drew into a cat...but I'm very good! It looks like a cat.

G: ...Yes. No, not the area, we haven't found it yet.

This segment is identified as a semiotic node because the difference between circle and circumference begins to emerge, which will lead in the next phase of the work to the definition of circle. It begins to emerge in the need to consider the points inside the circumference, although at this stage there is an overlap between the terms “area” and “circle”. While G and V discuss what to note on their worksheets, P successfully draws the circumference, subdivides it in quarters, and focuses on calculating “how much the goat can graze” referring to the “area inside”. Though his focus initially differs from his peers, who remain concerned with the circle’s radius and boundary. P’s insistence on exploring the interior region ultimately pushes everyone, and G in particular, to

recognize that the goat's grazing area differs from the defined circumference and its radius. By pointing to specific parts of the shape and encouraging others to model the problem differently, P initiates the semiotic node promoting the group's shift from focusing on the boundary to considering the area within it. He does this, perhaps unconsciously, by highlighting the goat's grazing area. The group, including P, had repeatedly measured the 3cm distance, which eventually highlighted the need to distinguish between a radius (the 3m measure), an area (how much grass is consumed) and the concept of a circle. The semiotic means that define this semiotic node are the gesture of P, who indicates the points of the circle, associating them with the need to calculate the area, the gesture of V, who in response to P creates a model of the problem and indicates with the pencil the points where the goat can graze, and the terms introduced by G: "everything inside" referring to the space of which the students have to find the area and that the goat can graze. In the first part of the group work, P is placed on level 1 of participation: he activates some Level I communicative competences listening and responding in a relevant manner to sentences of his peers, for instance drawing and measuring when prompted. In the transcribed phase, he intervenes at level 2 bringing to the second semiotic node. For what concern the first dimension of self-determination, volitional action, we can say that P engage in mathematical tasks independently, even if the chosen strategies are not always effective. Looking at agentic actions, P try to collaborate and propose his solutions and perspectives to peers, but he has more difficulties in listening and building on what his peers are doing. His attention shifts quickly, leading him not to listen to the explanations he receives.

CONCLUSIONS

Our analysis shows that P's objectification process was hindered by the artefacts used in the task, particularly the compass. This challenge affected his level of engagement as he did not always effectively seek help from his peers, although he did propose personal solutions and introduce new ideas. While P sometimes struggled to maintain focus on a single task, he demonstrated a genuine willingness to re-engage with and build on the work of his peers through multiple attempts. These observations provide valuable insights for supporting P's inclusion and for structuring a classroom environment that encourages more effective interaction. In answering our research question, the use of specific inclusion indicators highlights how Radford's definition of objectification, operationalised through semiotic nodes and individual contributions, helps us to understand P's evolving learning process and to identify areas where targeted support (e.g. in selecting or adapting artefacts) is most needed. Defining participation and self-determination also makes it possible to show how instruction for P should build on his ability to identify and plan strategies for achieving self-determined goals, while also promoting effective collaboration with peers. To achieve this, classroom interactions and communication need to be redesigned to include each pupil, and modifications introduced to address P's difficulty in sustaining attention for prolonged periods of time. Viewing inclusion through these three dimensions, learning,

participation and self-determination, paves the way for designing and evaluating educational contexts that combine an inclusive vision with practical classroom tools. This broad view of inclusion focuses on the individual characteristics of both the student and the context, and recognises inclusion itself as a dynamic, ongoing process.

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(HOW) DO STUDENTS' BELIEFS SHAPE THEIR PERCEPTION OF UNIVERSITY MATHEMATICS TASKS?

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Proof tasks are an important learning opportunity in mathematics undergraduate courses. Thus, understanding students' motivation regarding those tasks is relevant to support them during the transition from school to university. In this study we adopt an expectancy-value perspective of task motivation and analyse whether beliefs regarding school and university mathematics predict students' task-specific self-efficacy and intrinsic value. The results from N=120 student questionnaires show that beliefs regarding university mathematics are more predictive than those regarding school mathematics. Especially students that have stronger process and formalism beliefs regarding university mathematics report higher task-specific self-efficacy.

INTRODUCTION

The transition from school to university mathematics is challenging for many students, which is reflected in high dropout rates from mathematics undergraduate programs (e.g., Chen, 2013). Most first year mathematics lectures are accompanied by homework tasks (e.g., Pritchard, 2015), that are mainly proof tasks (Weber & Lindmeier, 2020). As students are not familiar with formal and rigorous proofs from school, these tasks are a major challenge during the transition. Although these tasks are important learning possibilities with regard to the exams, working on the tasks is often voluntary (Neuhaus-Eckhardt & Siller, 2024). Besides, many students avoid engaging with these challenging proof tasks by copying the solutions from other students or the internet (Liebendörfer & Göller, 2016). In order to support students in engaging with proof tasks, understanding their motivation regarding proof tasks is crucial.

In this study, an expectancy-value perspective of task motivation is adopted and linked to the theory of mathematical beliefs, as beliefs are considered to influence how tasks are approached and how much effort is put into them (Schoenfeld, 1985). In particular, we analyse whether students' beliefs regarding school and university mathematics influence their task-specific self-efficacy and intrinsic value.

THEORETICAL BACKGROUND

Mathematics tasks at school and university

Usually, undergraduate mathematics courses are accompanied by weekly tasks that students should solve alone or in small groups in the self-study phases (Pritchard, 2015). These tasks differ considerably from those which students are familiar with from school. In school, many tasks are rather procedural and can be solved via algorithms or schematic calculations (Engelbrecht, 2010). Moreover, applying learned

concepts to solve real-world tasks is relevant in school mathematics, whereas proof tasks are rather seldom (Gueudet, 2008). Thus, students have little experience with proof when entering university. Moreover, schematic calculations only play a minor role in tasks in undergraduate mathematics. Instead, most tasks require proofs and most of them are conceptual (Weber & Lindmeier, 2020). However, proof tasks can be posed either in a confirmatory way – so that students directly know that the statement to proof holds – or an explorative way. In the latter case, students first have to explore whether the questioned statement holds or not, before proving or rejecting it.

Expectancy-Value Theory and task-specific motivation

According to expectancy-value theory (Eccles & Wigfield, 2020) students' task-specific motivation is mainly determined by students' expectation of success (can I do this task?) and the value students attribute to the task (do I want to do this task?). A strong expectation of success and a high value are related to more engagement and effort towards a task and thus are related with higher learning outcomes. In contrast, low expectations of success and a low task value lead to less motivation and low learning outcomes (Eccles & Wigfield, 2020).

The expectation of success is usually conceptualised as self-efficacy and thus describes a person's conviction to be able to solve tasks with one's own abilities and competencies (Bandura, 1977). The task value comprises four facets (Eccles & Wigfield, 2020): intrinsic value (interest and enjoyment associated with the task), attainment value (personal importance of the task regarding one's identity), utility value (relevance of the task for present or future goals) and costs (negative associations with the task like negative emotions or necessary effort). In this paper, we focus on the intrinsic value which is conceptually closely linked to interest. Both, self-efficacy and task values are, among others, affected by students' beliefs (Eccles & Wigfield, 2020).

Beliefs concerning the nature of mathematics

Beliefs can be understood as “psychologically held understandings, premises, or propositions about the world that are thought to be true” (Philipp, 2007, p. 259). Thus, beliefs are more cognitive in nature than other affective constructs and can be seen as subjective knowledge. Beliefs are considered to have a strong influence on learning processes because they function as a filter concerning the information that is learned (Philipp, 2007). Information that fits well to one's beliefs is learned easier than information that seems to contradict one's beliefs. Moreover, beliefs influence how learners approach tasks and how much effort they put into working on a task (Schoenfeld, 1985). Thus, beliefs have an impact on students' perceptions of tasks.

Beliefs are always tailored towards a certain beliefs object (Grigutsch & Törner, 1998). In the current study, we focus on beliefs concerning the nature of mathematics. Grigutsch and Törner (1998) have proposed four aspects in which these beliefs can be structured. The *application aspect* emphasises possibilities of applying mathematics in other scientific domains as well as in everyday life. Thus, mathematics is seen as useful

in different situations. The *process aspect* highlights that mathematics is a vivid field of research, in which (at least subjective) new insights are possible. In this sense, the process of solving a task is more emphasized than the solution itself. The *schema aspect* stresses schematic and algorithmic facets of mathematics. Students with extreme *schema* beliefs see mathematics as a toolbox of facts, rules and algorithms that are often seen as unconnected. The *formalism aspect* instead focuses on abstract and logical thinking as important characteristics of mathematical work. Besides, the formal rigour and strength of mathematics as it appears for example in formal definitions and proofs is stressed. The proposed aspects have no normative character. They are rather independent dimensions and persons can agree (or disagree) to all these beliefs in varying degrees (Grigutsch & Törner, 1998).

The impact of students' beliefs during the transition from school to university

Given that most tasks at university require proofs and that many of them cannot be solved using schematic calculations or routine procedures, theoretically holding a strong *schema aspect* could be problematic. In contrast, *formalism* beliefs and *process* beliefs could be beneficial for solving those tasks (Geisler, 2023). Moreover, strong *application* beliefs could result in frustration, because university mathematics tasks only rarely contain application of mathematics to other domains or real-life contexts.

However, empirical results concerning the influence of students' beliefs during the transition are less clear. Especially studies relating beliefs to students' perceptions of proof tasks and their performance in these tasks are scarce. Geisler (2020) found no influence of students' beliefs on their performance in proof tasks. However, *schema* beliefs seem to be associated with rather superficial learning of students, whereas *process* beliefs are related to rather deep learning approaches (Crawford et al., 1994). Moreover, students with *schema* beliefs were less successful in their exams in this study. In contrast, Geisler (2023) found only weak to moderate effects of students' beliefs on achievement and dropout behaviour but moderate effects on students' satisfaction with their studies. Moreover, *process* and *application* beliefs are related with students' interest in mathematics (which is similar to intrinsic value) during the first term at university (Liebendörfer & Schukajlow, 2017).

One reason for these inconsistent results could be that most did not differentiate between beliefs regarding school and university mathematics. Given the clear differences between both kinds of mathematics, it is not clear whether students refer to school or university mathematics when answering such questionnaires during the transition. Therefore, Geisler (2024) adapted questionnaires that differentiated the four aspects proposed by Grigutsch and Törner (1998) with regard to school and university mathematics. Indeed, students clearly differentiated their beliefs and attributed *schema* and *application* beliefs stronger to school mathematics and *formalism* as well as *process* beliefs stronger to university mathematics (Geisler, 2024). Thus, it seems valuable to explicitly differentiate beliefs regarding school and university mathematics when analysing the impact of beliefs on the perception of tasks during the transition.

THE CURRENT STUDY

Research questions

The purpose of this study is to analyse the effects of students' beliefs concerning the nature of school and university mathematics on their perception of (exploratory and confirmatory) proof tasks in undergraduate mathematics programs. In particular we aim to answer the following research questions:

- 1) In which way do students' beliefs predict the task-related self-efficacy and the task-related intrinsic value? Are beliefs concerning school or concerning university mathematics more predictive?

Given that most proof tasks at university usually cannot be solved via schematic procedures and are not connected to real world contexts, we expect that the schema as well as the application aspect are rather negative related to students' intrinsic value and self-efficacy for the tasks (H1). In contrast, solving those tasks require problem solving and the incorporation of rigorous formalism. Thus, we expect that the formalism and the process aspect will be positively related to intrinsic value and self-efficacy (H2). We expect that beliefs concerning university mathematics will be stronger predictors than the beliefs concerning school mathematics (H3).

- 2) Does the type of proof task (exploratory vs. confirmatory) explain students' task related self-efficacy and intrinsic value beyond students' beliefs?

Exploratory tasks have a higher demand, because students have to check whether the statement holds before proving or rejecting it. Thus, students will have lower self-efficacy and intrinsic value for exploratory tasks than for confirmatory tasks (H4).

Methods

In this study $N=120$ students ($M(\text{age})=19$, 36% female, 8% divers) in first year mathematics courses for upper-secondary pre-service teachers and mathematics majors at three public universities in Germany participated voluntarily. All students answered a questionnaire concerning their beliefs during the lecture in the middle of the first term of their mathematics program. We used the scales from Geisler (2024) that differentiate *application*, *schema*, *process* and *formalism aspect* regarding school and university mathematics (see Table 1). All items were answered on a 6-point Likert scale (1=totally disagree, 6=totally agree) and the reliabilities of the scales were at least sufficient.

Task 1: Proof, that the difference of two consecutive square numbers is always odd.

Task 1: Investigate, whether the difference of two consecutive square numbers is always odd.

Figure 1: Confirmatory (left) and explorative (right) task used in the study.

After rating their beliefs, students were presented with a proof task and were asked to rate their self-efficacy ("I am confident that I can solve this task") and their intrinsic value ("I am interested in this task") regarding the task with single items adapted from Willems (2011). Students were randomly assigned to an explorative or confirmatory version of the proof task (see Figure 1). However, students were not required to solve

the tasks. Regression analysis in SPSS 29 with beliefs and type of task (explorative vs. confirmatory) as predictors were used to answer the research questions.

Beliefs Object:	University Math		School Math	
	#/ α	Example Item	#/ α	Example Item
Formalism	5 / .85	Of major importance for math, as it is done at university, is its logic rigour and precision.	5 / .87	Mathematical thinking in school is characterized by abstraction and logic.
Schema	4 / .64	Nearly all math problems in university can be solved by directly using known rules, formula and routines.	4 / .77	Math, as it is done in school, contains learning, remembering and application.
Application	5 / .82	Many aspects of university math have a practical benefit or direct applications.	5 / .87	In school math one works on tasks that have a practical use.
Process	5 / .86	Mathematical tasks and problems at university can be solved correctly in different ways.	5 / .79	Doing school math means understanding facts, seeing relations and having ideas.

Table 1: Overview of the scales with number of items and reliability (Cronbach's α)

RESULTS

Given the rather small sample, we performed separate linear regressions to analyse the influence of beliefs regarding school and regarding university mathematics. Table 2 gives an overview of the results with respect to beliefs regarding school mathematics.

Predictor	Self-Efficacy		Intrinsic Value	
	Model 1a	Model 2a	Model 1b	Model 2b
Application School	0.11	0.11	0.15	0.15
Schema School	0.19*	0.19*	0.20*	0.20*
Process School	0.10	0.10	-0.06	-0.06
Formalism School	-0.16	-0.17	0.03	0.02
Type of Task		0.02		0.03
R^2	.07	.07	.08	.11

Table 2: Results (standardized coefficients) of the regression models to predict self-efficacy and intrinsic value by beliefs regarding school mathematics, $N=120$, * $p<.05$.

Model 1a shows that only the *schema aspect* regarding school mathematics is positively related to the task-specific self-efficacy. Integrating the type of task (exploratory vs. confirmatory) in model 2a does not increase the explained variance in self-efficacy. Regarding the task-specific intrinsic value, model 1b again contains only the aspects regarding school mathematics as predictors. Again, only the *schema aspect* predicts higher intrinsic value. However, the explained variance is rather low and does not increase significantly when integrating the type of task as predictor (model 2b).

Table 3 gives an overview of the results concerning students' beliefs regarding university mathematics. Model 1c contains only the aspects regarding university mathematics as predictors to explain the task-specific self-efficacy. Both, the *process aspect* and the *formalism aspect* go along with higher self-efficacy. However, the *process aspect* is the stronger predictor. Model 1c can explain 29 % of the variance in self-efficacy. Integrating the type of task in model 2c does not increase the explained variance significantly, as the type of task is no significant predictor. In this model, the *process aspect* is still a highly significant predictor, while the effect of the *formalism aspect* is only weakly significant. The same models have been used to analyse the effects on the task-specific intrinsic value. In both models (1d without type of task and 2d with type of task as additional predictor) the *process aspect* is only a very weak predictor. Again, the type of task does not predict intrinsic value. The explained variance ($R^2=.06$) is rather low.

Predictor	Self-Efficacy		Intrinsic Value	
	Model 1c	Model 2c	Model 1d	Model 2d
Application University	0.02	0.02	-0.03	-0.03
Schema University	0.08	0.08	0.16	0.16
Process University	0.35**	0.36***	0.21 ^t	0.21 ^t
Formalism University	0.21*	0.20 ^t	-0.06	-0.06
Type of Task		-0.09		-0.02
R^2	.29	.33	.06	.06

Table 3: Results (standardized coefficients) of the regression models to predict self-efficacy and intrinsic value by beliefs regarding university mathematics, $N=120$,
^t $p<.1$ * $p<.05$ ** $p<.01$ *** $p<.001$

While *schema* and *application aspect* regarding university mathematics are not predictive for the task-specific self-efficacy and intrinsic value, a higher *schema aspect* regarding school mathematics is related to a higher self-efficacy and higher intrinsic value. Therefore, H1 has to be rejected. Higher *process* and *formalism aspect* regarding university mathematics leads to higher self-efficacy and slightly higher intrinsic value. However, both aspects regarding school mathematics were not predictive, confirming H2 only partly. As expected, the beliefs regarding university mathematics were

stronger predictors than those regarding school mathematics (H3). Against our expectations (H4), the type of task had no influence on students' perception of tasks.

DISCUSSION

According to Schoenfeld (1985) students' beliefs shape how they approach tasks. Thus, we expected that beliefs would influence students' perception of proof tasks in terms of the task-specific self-efficacy and intrinsic value. Our results show that only students' beliefs concerning the nature of university mathematics – more precisely the *formalism* and *process aspect* – had a considerable influence on self-efficacy. Given that formal and rigorous proof are required in most university tasks (Weber & Lindmeier, 2020), this is not surprising. Thus, these beliefs seem beneficial during the transition from school to university (cf. Crawford et al., 1994; Geisler, 2023). Moreover, as expected beliefs regarding school mathematics had only weak impacts on students' perceptions of the tasks, because university tasks differ clearly from school tasks (Gueudet, 2008). However, all beliefs were only weak predictors of students' task-specific intrinsic value. This result contradicts previous studies which found relations between students' beliefs and their interest in mathematics (Liebendörfer & Schukajlow, 2017) – which is conceptually similar to intrinsic value.

Limitations of the current study lie in the rather small sample as well as in the fact that students rated only one specific task concerning their self-efficacy and intrinsic value with single items. The task was chosen because it was possible to solve it by only applying prior knowledge from school, because students were in their first term of mathematics undergraduate programs. However, therefore the task was not closely aligned with the content of the lecture courses. Using more tasks that are more closely aligned with the lecture could lead to more robust results.

The results show that holding *formalism* and *process* beliefs regarding university mathematics is beneficial and thus should be fostered. Most lectures focus on rigorous proof and formal definition (Engelbrecht, 2010). Thus, the *formalism aspect* might be addressed already during lectures. In contrast, in many lectures only finished and polished proofs are presented instead of also presenting the process of proof generation (Pinto, 2017). Giving more insights in this process could foster students' *process* beliefs as well.

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MATHEMATICS EDUCATION FOR STEM AS PLACE

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What if we considered STEM a result of the interaction between land, languages, and humans? What is the role of mathematics education research in such a conceptualization? This paper will propose a research agenda that draws upon Indigenous knowledges and frameworks to explore the possibilities for developing research in mathematics education that contributes to a future STEM education that emphasizes critical thinking from multiple worldviews and perspectives, particularly Indigenous ways of knowing, to participate in responses to global crises.

INTRODUCTION

“The answers we seek... are literally all around us.... In the natural world, the definition of success is the continuity of life. The best ideas might not be ours. They might have already been invented.” (Benyus, 2015a, rearranged)

Biologist, educator, innovation consultant, and cofounder of the Biomimicry Institute Janine Benyus, believes “that the more people learn from nature’s mentors, the more they’ll want to protect them.” (Biomimicry Institute, 2021) Importantly, Benyus (e.g., 2015b) asserts that by “emulat[ing] nature’s designs and processes (e.g., solar cells that mimic leaves) [we can] create a healthier, more sustainable planet.” (Biomimicry Institute, 2021).

More than ever, the events we face today, locally, and worldwide— from geo-political unrest, to rising disinformation, to human rights movements, to increasing climate change— demand new perspectives and research for imagining new local and global futures. Such initiatives signal the recalibration of education to meet the ongoing needs, purposes, and innovations not only in science and technology (e.g., biomedical advances, smart buildings, the digital economy) focused on STEM (Science, Technology, Engineering, Mathematics) but also for our ever-changing society (e.g., Truth and Reconciliation in Canada / United Nations Declaration on the Rights of Indigenous Peoples, labour markets, physical and mental wellbeing). And yet, while education as both process and institution of society, “appear[s] to be under great strain—possibly approaching [a] breaking [point]” (PHC, 2019, p. 10), we see the ever-mounting tension as a tremendous opportunity for new mathematics education research. We take up Benyus’ call, which we see as cohering with Indigenous and ecological perspectives, as provocation to innovate ideas, meanings, purposes, and potentials in our proposed approach of: Mathematics Education for STEM *as Place* (Nicol et al., 2023a, 2023b). Given research focused on STEM *as Place* and Earth as

teacher is virtually nonexistent in the field of mathematics education, this paper offers an innovative vision for a research agenda in mathematics education.

We ask: How can (re)visioning and (re)generating Mathematics Education for STEM *as Place* contribute to humans learning to live with Earth in ways that are more and different? The focus of our proposed vision positions mathematics education similarly to Benyus' (2015a; Biomimicry Institute, 2021) approach of learning from and emulating Nature's designs and processes, as more than, and different from, merely a tool to support scientific and technologically based advances and societal change. We see Mathematics Education for STEM *as Place*, where Earth is mentor and teacher, to be a critical pivot point for such change to "address global challenges and to take transformative action for ensuring sustainable development [world-wide] (UNESCO, 2023). "STEM *as Place* inspires collaboration beyond human worlds, moving us towards being in the world in new ways, a kind of mutual flourishing and recognition of interdependence." (Nicol et al., 2023a). Our vision holds new potential for how conceptualizing and enacting mathematics education in this manner can give rise to both social insight and foresight concerning "how we relate to the land and ecosystems, how we make sense of our world, and how we connect to one another" (PHC, 2019, p. 6).

We question human relationships with planet Earth, specifically those that perpetuate and focus mathematics education on values and reductionist actions such as quantifying, maximizing efficiency, and exploiting Earth as commodity, resource, and stage for human activity, particularly, STEM. This theorizing marks a return to very old pedagogies of Earth such as rivers, forests, and corals that teach us how Mathematics Education for STEM *as Place* can give rise to recognizing intelligences which exceed those of humankind. Doing so not only recognizes Earth as teacher and mentor, but also promises insight into mutual flourishing through interdependence and value for collaborating with the natural world.

RELATED LITERATURE

Increasingly and especially within Indigenous and ecological discourses, there is need for STEM (education) and 21st century skills to be more than and different from solely a human endeavor aimed at supplying the demand for human capital (e.g., Bowers, 2016). Included in the literature is reconceptualizing STEM (education) and 21st century skills that actively engage ecological (i.e., social, cultural, and environmental) wellbeing (e.g., Barwell, 2018; Glanfield et al., 2020; Wolfmeyer et al., 2018). Such a concern is recognized internationally and includes school-aged children who see the need for such wellbeing to be a priority and focus of their education (e.g., Lynch, 2021). Further, work such as Chinn et al., (2023) advocate for future STEM education to emphasize critical thinking from multiple worldviews and perspectives, particularly Indigenous ways of knowing, to participate in responses to global crises.

Critiques of educating for STEM, including the M(athematics) in STEM are gaining attention in teacher education (Khan, 2020), curriculum (Wolfmeyer et al., 2018) and

communities (Thom, 2019; Wiseman et al., 2020); design (Glanfield et al., 2020), interdisciplinarity (Yaro, Amoah & Wagner, 2020), social justice (Davis & Renert, 2013; Wolfmeyer et al., 2018); and mathematical formatting (Barwell, 2018; Skovsmose, 2021).

Educators also raise issues such as equity and the need to rehumanize mathematics education (Guitérrez, 2018); to teach and assess in ways that build upon the strengths of students from marginalized groups and communities; and to recognize intersectionality (Gholson 2016). Other queries concern how mathematics education can reconnect humans ecologically through emphasizing place and multispecies freedom instead of continuing to reproduce logics of oppression, economics, and colonialism (Khan, 2020; Wolfmeyer et. al, 2018). Included here is the question, “How can the deeply human themes that drive us to do mathematics be channeled to build a more beautiful and just world in which all can truly flourish?” (Su, 2017, p. 489).

We fully support and are encouraged by the growing interest from researchers to identify and examine problematic issues as well as potentialities for M(athematics) in STEM, human needs, and the importance of place and land. Still, given most if not all global challenges today hinge upon human involvement with the natural world, there is even greater urgency for educators to explicate how teaching and learning mathematics for STEM can arise genuinely and practically *with* Earth.

Consider intelligence and particularly those knowledges, skills, and capabilities associated with STEM such as problem solving, innovation, engineering-design, and collaboration. More than ever researchers reveal these as not exclusively human attributes or breakthroughs, but in fact, intelligences demonstrated by Earth for 3.8 billion years. These intelligences include the mycelial network that all trees form to facilitate nutrient, carbon, water exchange, and defence signalling (Simard, 2021). Today’s corals, which carry forth 25 million years’ worth of evolution, are the largest architects and architectures on Earth— simultaneously creators and massive land structures which shape and protect the shorelines of continents. (National Geographic, 2019). “What about the lotus leaf whose rough-textured leaves trap air to create a waterproof cushion which causes droplets to bounce and roll off?” (Benyus, 2021) And the spatial geographical co-existence of genetic groups of grizzly bears with Indigenous language families in British Columbia Canada? (Henson et al., 2021) Or further still, the coinciding geographical patterns of language diversity and biodiversity— both significantly affected by our changing climate?

THEORETICAL-METHODOLOGICAL FRAMEWORK

While ideas and inventions such as these are gaining momentum in certain areas of STEM, severely under-examined within the field of education are the kinds of mathematics and mathematical ways of being that we can learn from Earth as teacher. How might we (re)vision and (re)generate Mathematics Education for STEM *as* Place which emphasizes the interdependencies between humans and the natural world? We suggest a theoretical-methodological framework that engages land philosophies

(Styres, 2017), Indigenous storywork (Archibald et al., 2019), and research approaches guided by theories of change.

Kanien'kehá:ka Mohawk scholar Sandra Styres' generative and fluid land-centric framework comprises elements of Vision (re)centring, Relationships (re)membering, Knowledge (re)cognizing, and Action (re)generating. Vision-(re)centring involves imagining possibilities and “opportunities for deep and profound insight and introspection [to] explore and examine the ways we engage with the natural, spiritual, and built worlds” (Styres, 2017, p. 5). To envision alternatives to educating mathematics for STEM that decentre humans' asymmetrical use of place, we suggest drawing upon the next element of Styres' framework, Relationships-(re)membering. Within this element land is alive with stories and thus, contrasts with colonial understandings of place as objects for human ownership and exploitation. Styres' third element, Knowledge-(re)cognizing, emphasizes opportunities to explore knowledge sources and perspectives within geographical, linguistic and culturally diverse settings. This element, brings forth awareness that knowledge can be storied, contextualized, and developed through live(d) experiences with the places we inhabit where language and land come together (Glanfield et al., 2020; Kimmerer, 2017). Action-(re)generating as the fourth element facilitates actualizing the Vision— for research— (re)visioning Mathematics Education for STEM *as Place*. Thus, Vision, Relationships, Knowledge, and Action enable (re)centring engagement with land; (re)membering community and inter-species relationships; and (re)cognizing land and language as connected through (re)generating mathematics education in ways that are more and different.

Conceptualizing and actualizing Mathematics Education for STEM *as Place* also necessitates being guided by ecological or Indigenous research methodologies such as Stó:lō scholar Jo-ann Archibald Q'um Q'um Xiiem's principles of Indigenous storywork as research methodology (Archibald et al., 2019). The principles of Respect, Reciprocity, Responsibility, and Reverence are essential to becoming “storywork ready,” in developing ethical relationships for respectful and responsive research. Such research could involve conducting interviews and discussions with STEM knowledge/wisdom keepers in their local contexts. Indigenous storywork principles can guide research that emphasizes the interdependencies between humans and the natural world.

PROPOSED RESEARCH AGENDA

We suggest research approaches grounded in philosophies of land and Indigenous storywork that allow for generative relationships between academy, community, and land; not only for potential individual benefits of each but for the mutual benefits of all. Therefore, we propose a cyclical research agenda that follows Styres' framework and involves four phases: Visioning (re)centring, Relationships (re)membering, Knowledge (re)cognizing, and Action (re)generating to engage in this unique research in the field of mathematics education.

Visioning/(re)centring and relationships/(re)membering

Guiding question: In what ways do STEM knowledge holders / wisdom keepers centre Earth as teacher to (re)member humans as an interdependent part of the natural world?

We propose that this phase involve formalizing relationships with members of communities so that work is guided by Respect, Reciprocity and Responsibility alongside community protocols and ceremonies. Through relationship building with these principles in mind researchers will begin to develop an understanding of community's desires and invite community members who are recognized as STEM knowledge holders / wisdom keepers to co-create approaches to teaching mathematics for STEM *as Place*.

Alongside the community work, and because conceptualizing Mathematics Education for STEM *as Place* is novel, we also suggest conducting an extensive systematic literature review (Gough & Thomas, 2016) of relevant contexts, possibilities, and theories for mathematics education with Earth. For example, as well as relationships between land and language, conduct searches and review research in biomimicry for earth-centric intelligences (e.g., fungi, corals, grizzly bears, viruses, forests). How does nature depend on local expertise, understand the constraints and opportunities of its place, depend on diversity, reward cooperation, upcycle everything, waste nothing, and avoid harming its home? (Benyus, 2021). This review will prepare researchers for engaging (e.g., conversations, storytelling, and interviews) with people recognized as STEM knowledge holders / wisdom keepers from across geographic regions who include, but are not limited to, Indigenous and non-Indigenous ecological practitioners, scientists, and mathematicians. These engagements provide opportunities to learn lessons told through millennia about, for example, corals, moss, lichen, trees, rivers, glaciers— across Indigenous languages and lands and oceans.

Knowledge/(re)cognizing

Guiding question: What principles are needed to develop meanings, purposes, and approaches to Mathematics Education for STEM *as Place*?

Research in this phase of Knowledge/(re)cognizing could include case stories (i.e. storywork) of conversations with STEM knowledge holders / wisdom keepers. We expect these case stories (e.g., shared stories of how coral live as both architects and architectures) with STEM knowledge holders / wisdom keepers have the potential to reveal and elucidate the ways in which Earth serves as teacher. Conducting a cross-case stories analysis with community, will lead to co-development of principles for meaning, purpose, and approaches to Mathematics Education for STEM *as Place* for understanding the interdependence of humans as part of the natural world.

Action/(re)generating

Guiding question: In what ways do the co-developed principles and co-created approaches provide opportunities for understanding Earth as teacher and the interdependence between humans and the natural world?

We suggest this phase involve action— implementing the co-developed principles to co-create approaches and co-researching the process with community. During this phase Indigenous storywork can be used to co-develop case stories and interactively analyze (Heath, 2004) how the approaches unfold as well as the kinds of opportunities and understandings of Earth as teacher and the interdependence between humans and the natural world arise in their contexts.

Vision/(re)centring and relationships/(re)membering

Guiding question: What new ideas, meanings, and purposes emerge for humans learning to live with Earth in ways that are more and different if mathematics education is (re)visioned and (re)generated as mathematics education for STEM *as Place*?

In this phase researchers and STEM knowledge holders / wisdom keepers could carry out a cross-case analysis of the co-developed case stories to respond to the overarching research question, re-visit the principles identified in the Knowledge/(re)cognizing phase, and co-develop methods for mobilizing the research including strategies for creating multimedia works (e.g. websites, composite case stories, short films, podcasts) and protocols for mathematics classrooms to inspire further action, research, and (re)envisioning.

CONCLUSION

This paper is meant to serve as an invitation to the mathematics education research community for imagining research agendas that position mathematics education and mathematics as vital to human and more-than-human flourishing. We are co-imagining these possibilities for developing a research agenda where Earth is teacher and the vibrant stories of land, and from land, are visible in mathematics education.

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USING STUDENT VOICE TO CONCEPTUALIZE AUTHENTIC MATHEMATICAL BELONGING

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We investigated how students who have experienced exclusion, especially those who are Black and Brown, conceptualize authentic belonging in mathematics classrooms. Our goal was to add nuance and detail about mathematical belonging from the students' perspective. The six students we interviewed had experienced exclusion to the point of moving to an alternative high school. They conceptualized authentic belonging in mathematics classrooms as: experiencing mistakes as important learning opportunities; participating in a collaborative, rather than competitive, learning environment; having authentic relationships; and seeing their peers' approaches to doing mathematics. A consistent theme throughout was the importance of the students being able to bring their whole selves to their mathematics learning.

INTRODUCTION

The importance of *belonging* to students' learning of mathematics is evidenced by the activities of professional organizations (e.g., NCTM, 2024) and by results from research (e.g., Barbieri & Miller-Cotto, 2021). Scholars in mathematics education, such as Louie et al. (2022), have called for *radical belonging*—belonging that involves the interconnected aspects of social belonging, academic belonging, and democratic belonging—as a way to move toward antiracist learning communities. Our work focuses on what it means to create inclusive classrooms, classrooms where radical belonging and equitable mathematics teaching are a way of being that is reflected in relationships between the teacher and each student, the students and each other, and the way that all of them treat the mathematics. We focus here on adding nuance, detail, and clarity about what it means to create such classrooms from the perspective of the students—those in the best position to provide this information. We do this by addressing the research question, *How do students who have experienced exclusion, especially those who are Black and Brown, conceptualize authentic belonging in mathematics classrooms?*

LITERATURE REVIEW

The concept of belonging has long been recognized as an innate human desire and psychological need, and has been studied within and across multiple disciplines (e.g., Osterman, 2000). Gray et al.'s (2014) work suggests that opportunities to belong significantly predict student engagement patterns, which is also related to student achievement. Matthews et al.'s (2024) critical review of three decades of literature

pertaining to school belonging among racially marginalized student populations identified the nascent nature of research on belonging, particularly for students of color. They concluded with recommendations for moving school belonging research forward that included “position[ing] the experiential knowledge of marginalized people groups as valid data” (p. 28) and “employing an intersectional lens in school belonging research” (p. 31).

In mathematics education, Barbieri and Miller-Cotto’s (2021) study found that a sense of belonging was a significant predictor of middle school students’ algebra learning. Id-Deen & Nalu (2024) surveyed mathematics teachers about their sense of belonging in schools and identified the role of collaboration, transparency, and empowerment in belonging. Maloney & Matthews (2020) found a relationship between teachers’ care practices and students’ connectedness in urban mathematics classrooms. Matthews et al. (2021) incorporated findings such as these, along with their own extensive investigation, into a construct called Belonging-Centered Instruction (BCI) and developed the BCI observational protocol. This protocol “allows observers (e.g., researchers, school practitioners) to evaluate the extent to which teachers’ actions facilitate belonging in mathematics classrooms through interpersonal and instructional processes” (p. 7). The BCI construct and observation protocol informed our framing and analysis, thus they are discussed further below.

THEORETICAL FRAMEWORK

We begin by describing our positionality and then discuss two frameworks that informed our work: Participatory Design Research (Bang & Vossoughi, 2016) and Belonging-Centered Instruction (Matthews et al., 2021).

Positionality

Fulfilling the entreaty by Milner et al. (2024), we have considered positionality “before (preactive), during (interactive) and after (postactive or reflective) classroom interaction” (p. 5). The authors belong to Empowerment in Equity (E²), a diverse team of mathematics education scholars representing various intersectionalities who are committed to addressing issues of equity and justice, including colonization and discrimination. E² was created during the preactive phase of the study specifically for the critical perspectives they would provide during the study. We intentionally flattened the authority structure and explicitly stated (and maintained) the stance of bringing our individual positionalities to all phases of the research as a core value of our team. Imani Goffney, a Black, US-born, Assistant Professor who orients her work in solidarity with other minoritized and oppressed groups, has experienced belonging in predominately Black universities and exclusion in predominately White ones. Offir Romero Castro, a Brown, Honduras-born, Ph.D. student with Cerebral Palsy, has experienced both instances of belonging and instances where teachers’ ableist/racist perspectives had been shown to overpower his bodymind and linguistic identity assets in class discussions. Shekira Edgar, a Black, St. Lucian-born, Ph.D. student, has experienced alternating instances of the presence and absence of a sense of belonging

as she has navigated schooling in different cultural and geographic contexts. Michelle Wilson-Banks, a Black, US-born, mathematics teacher, takes an anti-bias/anti-racist education (ABAR) approach to teaching and has never experienced belonging in a classroom setting as a student; only as a teacher at Phoenix High School, which has a whole-school focus on ABAR education goals. Laura Van Zoest, a US-born, White, Full Professor who approaches her work with a stance of curiosity and attention to when to lean in or step back to make space for others, has consistently experienced belonging in schools and universities. As we have described elsewhere (Goffney, et al., 2024), although our work is situated in the United States, we see it as part of addressing the colonization, anti-Black racism, and classism that results in harm and violence across the world.

Participatory Design Research

Drawing on participatory design research (PDR; Bang & Vossoughi, 2016), we attended to how “(a) critical historicity, (b) power, and (c) relational dynamics shape *processes of partnering* and the possible forms of learning that emerge in and through them” (p. 174; italics in original). Regarding *critical historicity*, we recognized that mathematics education takes place under a “white gaze”—the assumption that white perspectives, experiences, and values are the default or universal lens through which human experiences and creative expressions are understood. As a result, we attended to how this assumption emerged in our partnership and eradicated it to the extent possible. We critiqued existing traditional *power* differentials within our research team, such as professor-student, university-school, and researcher-teacher, and elevated the intersectional perspective, expertise, and skill set that each team member brought to the work. We attended to relational dynamics by listening carefully and investing time in each other until comprehension of each other’s perspective was achieved, even when that comprehension meant extending the research timeline and changing the nature of the work. The resulting *process of partnering* led us to focus on the student voice in authentic mathematical belonging.

Belonging-Centered Instruction

Matthews et al. (2021) conceptualized Belonging-Centered Instruction (BCI) as having two domains of teacher actions that support student belonging: *interpersonal supports* and *instructional supports*. They further broke those domains down into subdimensions and action indicators. Four subdimensions emerged as central in our study: communal orientation, social-emotional bridging, safety to be wrong, and mathematics to know myself and my world. Below, we elaborate on the action indicators that best captured the students’ perspectives on belonging in math class.

METHODS

To find robust answers to our research question, *How do students who have experienced exclusion, especially those who are Black and Brown, conceptualize*

authentic belonging in mathematics classrooms?, we engaged with students who have experienced being excluded on multiple levels and in significant ways that have impacted their learning and academic success to the point that they chose to seek out an alternative high school that was more welcoming.

Context & Participants

Our study took place during the first academic trimester in an anti-bias/anti-racist mathematics classroom in a mid-size alternative high school with similar numbers of male and female students, majority Black (50%), with White (25%) and Brown (12%) being the next largest groups. The philosophy of the school was consistent with the idea of *radical belonging* (Louie et al., 2022), and the mathematics teacher we worked with prioritized radical belonging in her classroom, creating a space and experiences where each student feels seen, and valued, and has the freedom to bring their authentic selves to class. In addition to supporting students who were not having success in their assigned school, it was also a "newcomers" school, where multilingual students who enrolled in the district and were new to the country were placed (30%). This context required building trust with students who have had traumatic experiences in general and, for other students, specifically with school and math class. Thus, before collecting data for the study, we joined the classroom community, supporting and informally interacting with students and the teacher while attending classes. The six students in our study reflected the school demographics, including two multilingual students, Alex and Eren. (Note that all students are referred to by self-selected pseudonyms.)

Data & Analysis

We audiotaped and transcribed interviews with the six students (range: 15 to 45 mins.). The semi-structured interview questions focused on: (1) finding connections between students' outside-of-school experiences and learning mathematics in the class, (2) students' use of home languages, lived experiences, and cultural ways of doing mathematics to collectively help them to solve math problems, and (3) students' participation in class discussions. For each transcribed interview, coding units were created to identify distinct conversational threads in the interviews. At least three members of the research team individually read and coded each transcript using the action indicators of the BCI observation protocol (Matthews et al., 2021, p. 7-9). Codes selected by at least two members were considered for this analysis. The actions to which students made the most references are the focus of this report.

RESULTS & DISCUSSION

There were a total of 69 coding units across the six interviewed students, with a mean of 11.5, and a range of 9 to 13. We provide illustrations for the most coded action indicator of each of the four subdimensions that emerged as central to our study: safety to be wrong (SBW), communal orientation (CO), social-emotional bridging (SEB), and decentering teacher authority (DTA).

Normalizing Difficulties (SBW; 6/6 students)

This action indicator is about the teacher showing that being wrong is a normal and expected part of the learning process. In our study, this was indicated by students expressing their comfort in making mistakes in class. For example, Jude said: “I don't have a problem with making mistakes. It's one of the ways, not exactly my favorite, but definitely a good way of learning.” In addition, students expressed that the way the teacher conducted class discussions was responsible for that comfort. Dennisee explained:

I feel very comfortable because, like the way she teaches it, she gets everybody's opinion on it. So, and then, like, if you make a mistake and another student talks, then you can fix your mistake, and like, won't be sad or anything about it, because another student can also make a mistake, and then y'all can see it.

Kors talked about the dynamic in the classroom:

She doesn't ever just judges you, and you might be wrong, but [the teacher] doesn't judge you. She just help you ... some people got to learn how other people are. You have to help them, in a way, to make them see that you're not trying to say that, no, you're stupid, you're dumb. You know, it's more like, I understand your thinking, but just look at it this way, and then this, this and that, that and that. And it just helps you learn more.

Each of the six students in this study pointed to teachers' promotion of mistakes as an expected part of the learning process, and not something to hide or be ashamed of, as important to their feeling of belonging in the class. Thus, contrary to some teachers' fear of students being harmed by their wrong answers being made public, not only can discussing student errors and misconceptions support learning (Tulis, 2013), it can also support students' sense of belonging in mathematics class.

Demanding Communalism (CO; 5/6 students)

In this action indicator, the teacher builds on the classroom community by explicitly and intentionally reinforcing and valuing collective work. To illustrate how this played out in our classroom and contributed to students' sense of belonging, we hear from Eren and Alex, who often worked together despite having different home languages. Eren, in the context of talking about using his home language to support his understanding, shared:

Classmates, that they understand...they kind, they will not laugh when you say something. They will focus on you and hear what, what you saying...we sharing ideas, when we sharing ideas we learn a lot. I learn more... sharing ideas with the groups. And no idea, this guy, what he think, and I take some from him, and he takes some from me.

In this, we see Eren acknowledging the sense of safety and security that has been established in the classroom environment. He identifies opportunities where he has both received and shared insightful information that contributed to the collective learning of the class.

Alex stated:

Of course, yes, yes, it's, it brings me much along together when we have the same idea, or we have the same problem, if we ask the same question, it helps a lot...it brings the, the knowledge and joy of the class to like, okay, you know, this is what the other persons think of this. Or maybe they found a better solution, or, or wait, you know, you're like, this is a better way to solve the problem.

This further illustrates the sense of community and shared learning that was created and maintained by the teacher. A core theme for this indicator was the sense of being in the learning process together and the students' recognition of how that communalism expanded their learning opportunities. This speaks to the importance of positioning students as collaborators, rather than competitors.

Applying Cultural Competence to Build Social Climate (SEB; 5/6 students)

This action indicator is about the teacher intentionally considering students' cultural backgrounds to strengthen relationships and attend to cultural nuances in the classroom. Esther described this as the "teacher allowing us to have a space where we can express ourselves respectfully, but express ourselves in our own ways." Kors provided additional insight:

I don't have to, you know, have to try to fit in with other people. I could just be myself, or how I was raised, or how my cultural people are and how they act... when it comes to just being comfortable in the classroom, being able to sit there and not worry about having me bullied or talked about or anything like that ... I mean, when I see everybody you know, we got people, we got like Arabs, we got, you know, different type of people, Africans, and different type of people, their culture, wearing what they wear, ... what they are used to wearing, and used to dressing like and acting like. They still bring that into school, into classes, and stuff.

The students' responses reflected the freedom and confidence to simply be. There was no need to try to "fit in with other people" by suppressing their cultural and linguistic identities, being their authentic and unapologetic self was enough. This highlights that an important aspect of belonging is having authentic relationships with one's teacher and classmates that allow students both to be seen and see others.

Honoring Multiple Ways of Knowing and Showing (DTA; 3/6 students)

In this action indicator, the teacher encourages students' mathematical ideas that illustrate multiple ways to solve problems. For example, Jude said:

... the way that [the teacher] does it is like we'll have the problem, then, if there's another question, then we'll break that down. Or if there's another way that someone solved it, we'll break it down. So we get to see how other people solve the problems and how we could adapt to that style.

Dennisee added specificity about how the teacher recognized students' ideas:

[The teacher] writes it out on the board, and then she'll ask us about it, and stuff like that. And then, like, she'll write everybody's out on the board, but we can see all the different ways people said or explained it. So I feel good about that.

In addition, Dennisee described “going back and forth” as a way the teacher used to honor students' ideas in class discussions:

... by going back and forth, it's like it helped me a little bit to see from another point than just mines, because mines might not always happen, and it might be better, but the other person's might also work for other people. So it's like it can work out both ways, but it's better to get both points of it.

These comments illustrate ways the students value hearing the thinking of their peers and how they feel it helps them to learn mathematics. The students' responses also suggest that including students as experts by respecting their approaches to doing mathematics contributes to students' sense of belonging in mathematics class.

CONCLUSION

This study used student voice to investigate authentic mathematical belonging. Our contribution to the field is to provide nuance and detail about what it actually means to create inclusive classrooms, classrooms where belonging and equitable mathematics teaching exist as a way of being by centering student thinking. We found that students who have experienced exclusion, especially those who are Black and Brown, conceptualized authentic belonging in mathematics classrooms as: experiencing mistakes as important learning opportunities; participating in a collaborative, rather than competitive, learning environment; having authentic relationships; and seeing their peers' approaches to doing mathematics. The students in our study revealed that belonging exists between the teacher and each student, the students and each other, and the way that all of them interact with the mathematics.

Our experience highlights the importance of, and need for, teachers and researchers to investigate classroom issues from the students' perspective and to use their feedback to measure success and to ensure authentic results and recommendations. Doing this work will support teachers to create classrooms where students move beyond surviving to thriving (Love, 2019).

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REVERSED CONGRUENCY AND GAP EFFECTS IN UNDERGRADUATE STUDENTS' FRACTION COMPARISON

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Fraction comparison tasks have been widely used to study children's and adults' fraction knowledge. Congruency and gap effects have often been found, but several studies' results conflict with theories of the congruency effect, as participants have shown a reversed congruency effect that has been conjectured to be due to gap thinking. Here, we asked how widespread reversed congruency and gap effects are and who is most affected by them. We report data from a fraction comparison task performed by 299 undergraduate students, revealing reversed congruency effects in the 70% of the sample that obtained the highest overall scores. Gap effects, instead, were strongest in a smaller subsample. Our findings contribute to the theoretical discussion on the reversed congruency effect, as gap thinking alone cannot explain it.

INTRODUCTION

Learning fractions requires not only a conceptual understanding of what fractions are but also a procedural understanding of how to solve typical fraction problems. These two types of understanding often influence each other: Learners who conceptualize fractions as nothing more than a pair of natural numbers written together may be more likely to declare that $5/17$ is larger than $2/3$ because the former has larger natural number components than the latter (e.g., Gómez & Dartnell, 2019; Stafylidou & Vosniadou, 2004). Likewise, learners who approach fractions as a representation of an additive (instead of a multiplicative) relation may be more likely to declare that $1/3$ is larger than $4/7$ because the former is “missing only pieces to complete the whole,” whereas the latter “is missing three pieces” (e.g., Clarke & Roche, 2009; González-Forte et al., 2020). In the literature, the first example is often called *natural number bias* or *congruency effect*, whereas the second is called *gap thinking*. These and other strategies that children and adults use to compare fractions have been extensively studied using qualitative and quantitative approaches (e.g., González-Forte et al., 2020; González-Forte et al., 2023; Morales et al., 2020; Obersteiner et al., 2013; Stafylidou & Vosniadou, 2004). However, research has shown some intriguing findings, such as a *reversed congruency effect* when high-achieving children or adults compare fractions that share no common numerator or denominator (Gómez & Dartnell, 2015; Morales et al., 2020; Obersteiner et al., 2013). Some authors have investigated whether this is due to participants' use of gap thinking (e.g., González-Forte et al., 2020), but no conclusive evidence has been found yet. In this study, we aimed to examine congruency effects and gap effects in a population of adults of diverse levels of

proficiency to determine at what levels the magnitude of the reversed congruency effect and the gap effect are evident and/or stronger.

THEORETICAL AND CONCEPTUAL FRAMEWORK

Fraction comparison tasks have been widely used to study children's and adults' strategies to work with fractions. The most basic version of this task consists of presenting participants with two fractions and asking which one of them is larger. Here, *larger* is taken to mean numerically larger or having a greater numerical magnitude. Research in mathematics education and cognitive psychology has discovered several item characteristics that systematically affect participants' strategies and responses to this task. Following Siegler and Araya (2005), we understand a *strategy* as any non-obligatory procedure aimed at the goal of solving a mathematical task, meaning that they are not necessarily consciously or rationally chosen. These authors proposed a model of strategy selection in which item characteristics play a key role in modifying a participant's likelihood of using one or another strategy. In the domain of fraction comparison, several studies have focused on what these item characteristics are and how they affect participants' strategy selection. Some of these characteristics may also affect a given strategy's applicability and success rate. In the following, we describe some item characteristics that are relevant to this work.

The presence of a common component

One of the most salient item characteristics in fraction comparison is the presence or absence of a common component (numerator or denominator) between the two fractions. Overall, fraction pairs that share a common component (e.g., $4/7$ vs. $4/9$ or $2/5$ vs. $3/5$) tend to be approached via component-based strategies, whereas fraction pairs sharing no common component (e.g., $4/7$ vs. $2/5$) tend to be compared via so-called holistic strategies or magnitude-based strategies. While the names “component-based” or “componential” are often used for the former type of strategy (e.g., Meert et al., 2009), it must be clarified that many of these studies refer to a specific kind of strategy, namely focusing on the non-shared component and selecting as larger the fraction with the larger numerator or smaller denominator. Whenever fractions share a common component, this strategy works quickly and accurately. In contrast, when the pair of fractions lacks a common component, this strategy is no longer applicable. Researchers have proposed that, in this case, participants resort to estimating the magnitude of both fractions and comparing these mental representations. These magnitude representations may be of varying levels of precision, ranging from number-line-like representations (Schneider & Siegler, 2010) to categorical distinctions like “above $1/2$ ” or “below $1/2$ ” (Obersteiner et al., 2020).

Congruency between fraction magnitude and natural number magnitude

Another characteristic shown to affect participants' strategy choice is the congruency relation that may appear between the magnitude of the natural numbers that compose a fraction, and that of the fraction itself. In this view, a fraction pair is called *congruent*

if the larger fraction has a larger numerator and denominator than the other one (e.g., $5/8 < 7/8$, $1/3 < 5/7$) and *incongruent* if the larger fraction has a smaller numerator and denominator than the other one (e.g., $1/3 > 1/8$, $2/3 > 4/9$). There are also *neutral* fraction pairs, where one fraction has a larger numerator while the other one has a larger denominator (e.g., $1/4 < 2/3$). The congruency of a fraction pair has been repeatedly shown to affect some children's performance in comparing fractions, as they tend to indicate fractions with larger components as being numerically larger (e.g., Gómez & Dartnell, 2019; Stafylidou & Vosniadou, 2004). In an extreme case, they may correctly answer all congruent items and incorrectly all incongruent items (e.g., cluster A in Gómez & Dartnell, 2019).

Some researchers have ascribed this behavior to a *natural number bias*, a tendency to overgeneralize properties of natural numbers (in this case, numerical order) to fractions (Ni & Zhou, 2005; Van Dooren et al., 2015). The most common theoretical interpretation of natural number bias in fraction comparison implies that congruent fraction pairs should be associated with higher accuracy rates than incongruent pairs. However, this is often not the case when the fractions to be compared have no common components (Gómez & Dartnell, 2015). Research so far has been unable to explain this reversed congruency effect satisfactorily.

Gap thinking: an additive solution to a multiplicative problem

Clarke and Roche (2009) documented a fraction comparison strategy named *gap thinking*, where children reasoned about fractions with a parts-of-a-whole model and focused on the number of parts that a fraction is missing to complete the whole. In this view, $2/3$ is missing one part and $4/9$ is missing five parts, from where it follows that $2/3$ should be larger than $4/9$. This strategy thus disregards the size of the parts, considering only their number. Another way of phrasing it is that gap thinking looks at the subtraction of numerators and denominators rather than their division. González-Forte et al. (2020) showed that the proportion of students who use gap thinking slowly increases throughout schooling, from about 15% in fifth grade to about 30% in tenth grade. An interesting fact about this strategy is that it can answer many fraction comparison items correctly. Morales et al. (2020) proved that this strategy can fail only when fractions to be compared constitute a congruent pair with no common components. In this specific subset of items, it may occur that gap thinking leads to the correct answer (e.g., $1/3$ vs. $6/7$), to the incorrect answer (e.g., $1/3$ vs. $4/7$), or it can be inconclusive because both fractions have the same gap (e.g., $1/3$ vs. $5/7$). Given this, gap thinking has been conjectured to explain the reversed congruency effect (e.g., Gómez & Dartnell, 2019; González-Forte et al., 2020).

The present study

In this work, we presented adult participants with a fraction comparison task and analyzed the differences in performance that emerged in relation to the items' congruency and gaps. In particular, we looked at (1) the extent to which congruency and gap thinking affect the responses in a fraction comparison task of adults of different

levels of proficiency, and (2) the relative magnitude of the reversed congruency effect and the gap effect in their responses, to see whether gap thinking can account for the reversed congruency effect.

METHODS

This quantitative study used an experimental, within-participant design. The research protocols were approved by the Scientific Ethics Committee of Universidad de O'Higgins.

Participants

Three hundred undergraduate students from different universities and programs were recruited for this study. We discarded data from one participant because the experimental program failed before completing the whole task.

Fraction comparison task

The task was conducted on a computer and programmed in OpenSesame (<https://osdoc.cogsci.nl/>). We used the same set of 180 fraction pairs as Morales et al. (2020). This set included fraction pairs with and without common components; congruent, incongruent, and neutral pairs; and, within congruent pairs without common components, pairs where gap thinking led to the correct answer, to the incorrect answer, or was inconclusive.

Procedure

Before starting the task, participants were informed about it and signed a consent form. Participants individually completed the task in a silent room. Fraction pairs were presented on the screen one at a time, for a maximum of 10 seconds. During this time, participants indicated using the keyboard whether the left side or the right side fraction was larger than the other (keys: 'Q' for left and 'P' for right). Items in which a participant did not answer during the allowed time were marked as omitted. The item set was split into three blocks of 60 items each, separated by self-paced pauses.

Data analysis

To analyze how congruency and gap effects change with overall performance, we separated participants into ten deciles based on their overall percentage of correct responses (there were between 28 and 31 participants in each decile). We then computed average accuracies for each item type and decile. Differences between item types for each decile were statistically tested using logistic mixed regressions, considering participants as a random factor. Since this required ten statistical tests per analysis, we used the Holm-Bonferroni procedure (Holm, 1979) to correct for multiple comparisons. For the sake of simplicity, however, we reported uncorrected *p* values.

RESULTS

Overall descriptive statistics showed that participants had an average of 72% correct responses ($SD = 25\%$). Figure 1 shows the distribution of overall performance for each

decile. It is worth noting that Q1 and Q2 correspond mostly to participants who showed an overall performance below the 50% chance level that someone would obtain answering at random.

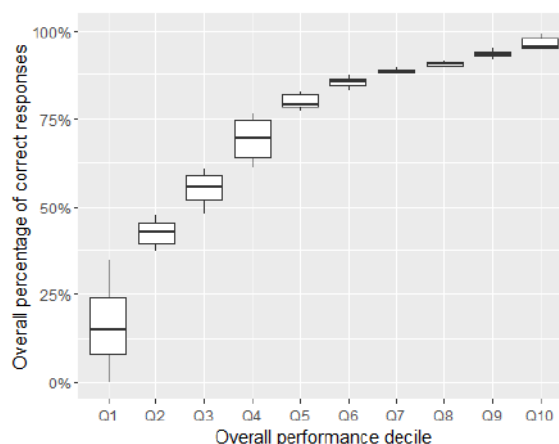


Figure 1: Performance of participants assigned to each overall performance decile.

We first looked at the effect of congruency across performance levels. Figure 2 presents average accuracies separately for each item type based on congruency (congruent, incongruent, neutral). Looking at fraction pairs with a common component, logistic mixed regressions revealed significant effects of congruency in deciles Q2 ($\chi^2(1) = 40.2, p < .0001$), Q3 ($\chi^2(1) = 5.9, p = .02$), and Q7 ($\chi^2(1) = 7.4, p = .007$). However, only the difference in Q2 remained significant after correcting for multiple comparisons. This showed that statistically significant congruency effects could only be found for participants with overall performance close to 50%.

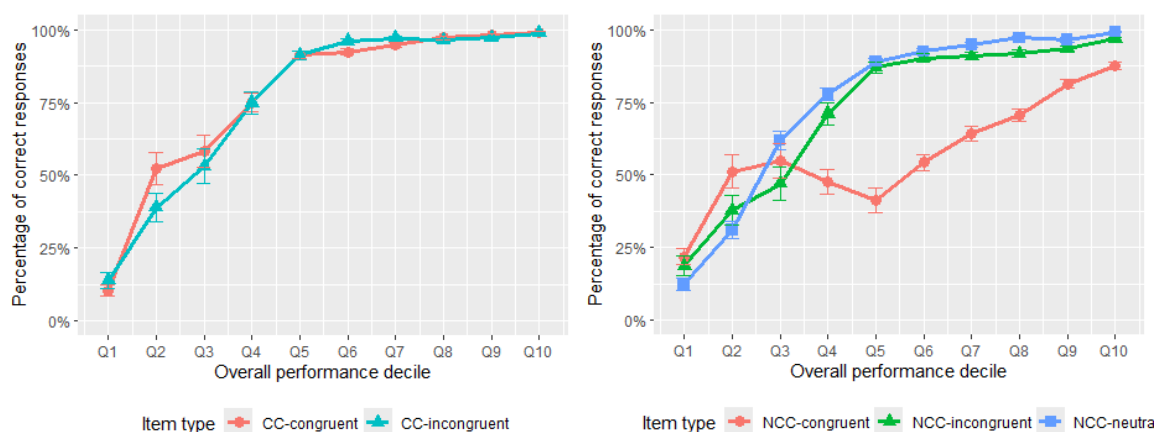


Figure 2: Average percentage of correct responses for participants in each performance decile and item type according to congruency. Left: Fraction pairs that share a common component. Right: Fraction pairs without common components.

In contrast, looking at fraction pairs without common components, logistic mixed regressions revealed significant effects of congruency in all deciles (all $\chi^2(2) > 35.4$, all $p < .0001$), even after applying the Holm-Bonferroni correction. The most salient congruency effect, in this case, is the distinct behavior that congruent items showed, in

comparison to incongruent and neutral items: for all groups with overall performance above about 60%, there is a significant reversed congruency effect where congruent fraction pairs are answered less accurately than the other two item types. The magnitude of this reversed effect is largest for participants in Q5, whose overall performance is around 80%.

To understand the extent to which this reversed congruency effect may be due to gap thinking, we repeated the previous analysis, separating congruent items into three groups: those that are answered correctly/incorrectly by gap-based reasoning and those where gap thinking is inconclusive (*gap favorable*, *gap against*, and *gap neutral*, respectively). As shown in Figure 3, there were significant differences between these three item types defined by gap thinking. Logistic mixed regressions contrasting the three gap-based item types revealed significant differences in five deciles (after correction for multiple comparisons): Q1 ($\chi^2(2) = 11.3, p = .004$), Q5 ($\chi^2(2) = 43.5, p < .0001$), Q6 ($\chi^2(2) = 59.0, p < .0001$), Q7 ($\chi^2(2) = 44.8, p < .0001$), and Q8 ($\chi^2(2) = 22.3, p < .0001$).

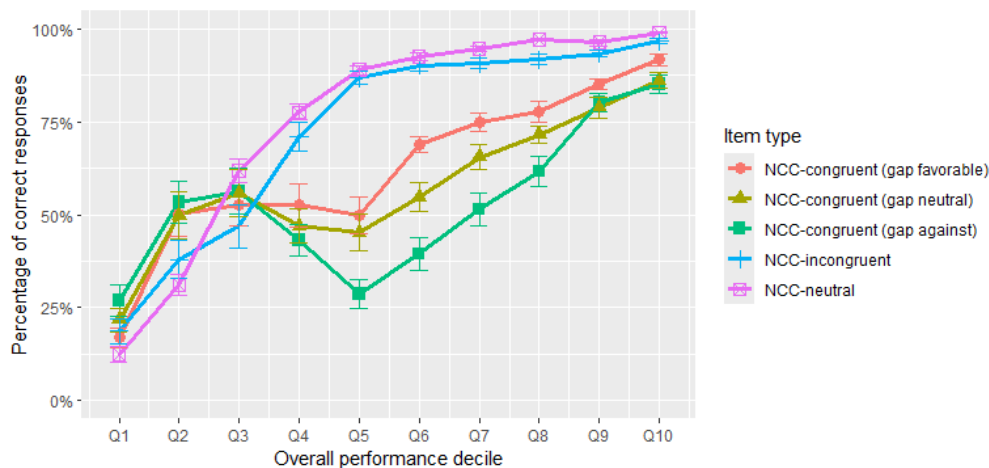


Figure 3: Average percentage of correct responses for participants in each performance decile and item type according to congruency and gap (considering only fraction pairs without common components).

While the differences in Q1 were difficult to explain because of its counterintuitive performance markedly below the chance level, the differences in deciles Q5-Q8 revealed a clear common pattern: fraction pairs where gap thinking would lead to the correct answer obtained higher scores, followed by pairs in which gap thinking would be inconclusive, and finally fraction pairs where gap thinking would lead to the incorrect answer with the lowest scores. This shows that gap effects were strongest in participants whose overall performance lies in the range of 75%-85%, particularly in Q5. However, Figure 3 also shows that gap effects do not fully account for the reversed congruency effect because congruent items where gap thinking would lead to the correct answer are not answered with similar accuracy to the other items types.

DISCUSSION

We used a fraction comparison task with items designed to contrast the effects of congruency and gap-based reasoning. Results showed that when fraction pairs lacked a common component (numerator or denominator), a reversed congruency effect emerged for about 70% of participants with the highest overall scores, where congruent items were answered less correctly than incongruent and neutral items. This aligns with previous analyses (Gómez & Dartnell, 2015) that suggested that the reversed congruency effect may be more prevalent than initially expected. Regarding its strength, our data suggests that this reversed effect may be strongest in participants whose overall performance was close to 80%.

Gap-related effects were evident in about 40% of the sample, with a similar profile to the reversed congruency effect (i.e., concentrated in relatively high-achieving participants). Regarding the relation between gap thinking and the reversed congruency effect, our findings aligned with previous results with expert participants (e.g., Morales et al., 2020), showing that gap-based reasoning was not able to explain the reversed congruency effect. We thus extended this outcome to a broader population of undergraduate students, observing it across a range of performance levels.

Limitations

While our quantitative approach facilitated collecting data from a large number of participants, it led to some limitations. Most prominently, it was not possible to know why some participants had extremely low scores (30% and lower). We conjecture that some of them might have misunderstood the instructions, for example, systematically selecting the lower fraction instead of the larger one. Deeper analyses are needed to check the likelihood of this hypothesis. Also, performance deciles were computed from the same fraction comparison task. Considering an additional, independent measurement of fraction knowledge might have led to different outcomes.

Conclusion

Our findings support and deepen previous observations that gap thinking cannot account for the reversed congruency effect. More theoretical work is thus needed to understand this effect and to predict its presence and extent in diverse populations.

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PRE-SERVICE TEACHERS' (IN)CONSISTENCY IN INCORRECTLY REASONING ABOUT FRACTIONS

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Primary and secondary school students, and even undergraduates often use incorrect ways of reasoning when comparing fractions. Well-documented examples are simple comparison of natural numbers (natural number bias), reverse bias, and gap thinking. However, empirical research addressing such reasoning in pre-service elementary teachers (PTs) is scarce. We investigated the existence and consistency of 93 PTs' incorrect reasoning through a fraction comparison test and a vignette. Many PTs' remained consistent with their reasoning when responding to the vignette, which seems to indicate that these incorrect ways of reasoning were stable and resistant to change. Most PTs who were inconsistent did not necessarily adapt their reasoning towards the better. The results indicate a need for addressing the topic in teacher training.

THEORETICAL AND EMPIRICAL BACKGROUND

Fractions are challenging not only for students, but also for teachers (Chinnappan & Forrester, 2014). This is concerning because, for obvious reasons, teachers need to possess a solid understanding of fractions. However, research has suggested that teachers and pre-service teachers (PTs) do not always have sufficient competence with fractions, some showing difficulty with fraction concepts that should have been conquered in elementary school (e.g., Thanheiser et al., 2016).

Understanding fraction magnitude is particularly important for students' mathematical development (Lamon, 2007). On the other hand, many studies have shown that primary and secondary school students, and even undergraduates often have difficulty in this regard, using incorrect reasoning when comparing the magnitudes of two fractions (e.g., Gómez & Dartnell, 2019, González-Forte et al., 2023). However, empirical research addressing *pre-service* elementary teachers' understanding of fraction magnitude is limited (for a review, see Olanoff et al., 2014). Given the key role that teachers' knowledge plays for student learning, the current study investigates PTs' understanding of and reasoning about fraction magnitudes.

Incorrect ways of reasoning about fraction magnitude

Fraction magnitude understanding has been predominantly measured through fraction comparison tasks (e.g., Fazio et al., 2016; Gómez & Dartnell, 2019). There are many ways to solve these tasks, such as obtaining common denominators, converting fractions into decimal numbers, or using benchmarks. However, previous research has shown that people often employ incorrect ways of reasoning when comparing fractions, such as natural number bias reasoning, reverse bias reasoning, and gap

thinking. Natural number bias reasoning (also known as NNB) refers to relying on the ordering of natural numbers to determine the magnitude of fractions, wrongly considering that the larger the numerator and denominator, the larger the fraction (Pearn & Stephens, 2004). This reasoning has been found in primary and secondary school students (González-Forte et al., 2020), and in undergraduates (DeWolf & Vosniadou, 2011). People applying natural number bias reasoning showed high accuracy in congruent items (those in which ordering of natural numbers leads to the correct response, e.g., $2/3$ vs. $7/9$) and low accuracy in incongruent items (those in which reasoning based on the ordering of natural numbers leads to the incorrect answer, e.g., $3/4$ vs. $5/9$). Qualitative evidence of this reasoning has also been identified through interviews with secondary school students (González-Forte et al., 2023) and through audio recordings with undergraduates (Fazio et al., 2016). This way of reasoning was also identified in Reeder and Utley (2017) from PTs' answers to a single item.

Reverse bias reasoning means considering that smaller denominators indicate the larger fraction (Pearn & Stephens, 2004). Previous research has found evidence of reverse bias reasoning, by identifying groups of primary and secondary school students who obtained high accuracy in incongruent items and low accuracy in congruent items (e.g., Gómez & Dartnell, 2019; González-Forte et al., 2020). Similar results have also been obtained with undergraduates (e.g., DeWolf & Vosniadou, 2011). Again, interviews and audio recordings have confirmed this way of reasoning (Fazio et al., 2016; González-Forte et al., 2023). Reverse bias reasoning has been also identified in some PTs' answers to a single item (Reeder & Utley, 2017) and through an interview (Whitacre & Nickerson, 2016).

Gap thinking is based on comparing the (absolute) difference between numerator and denominator (Pearn & Stephens, 2004). This reasoning leads people to consider that one fraction is larger if the difference between the numerator and the denominator is smaller, or both fractions are equal if the difference is equal. Previous research has found evidence of this reasoning in primary and secondary school students (González-Forte et al., 2020), and undergraduates (Gómez et al., 2017), by the identification of a group of students who obtained high accuracy in gap thinking correct items (those where gap thinking leads to the correct answer, e.g., $2/3$ vs. $5/9$), and low accuracy in gap thinking incorrect items (those where gap thinking leads to the incorrect answer, e.g., $1/3$ vs. $5/8$). Qualitative evidence of gap thinking has also been obtained from interviews with secondary school students (González-Forte et al., 2023). Reeder and Utley (2017) and Thanheiser et al. (2016) also identified gap thinking in some PTs' answers to a fraction item with the same gap between numerator and denominator.

Resistance of incorrect reasoning

The above-mentioned ways of reasoning have been characterised as “bias”, suggesting a systematic misjudgement. There is some debate in the literature about how resistant to change such biased reasoning is, or whether it is malleable, for example by

confronting people with alternative ways of reasoning. Some studies have confronted participants with alternative reasoning after they have solved fraction comparison problems. In González-Forte et al. (2023), 52 seventh graders were confronted in an interview with other kinds of alternative incorrect reasoning. Most of them did not change their answer when they saw other ways of reasoning, leading to the conclusion that these incorrect ways of thinking were stable and resistant to change. Similarly, in Fazio et al. (2016) undergraduates were confronted with three alternative strategies (the strategy the student originally used, a correct alternative, and an incorrect alternative). Those with lower fraction knowledge rarely switched from an original incorrect strategy to a correct alternative, leading to the conclusion that the use of incorrect reasoning largely demonstrates a lack of conceptual understanding, rather than difficulties in recalling or generating such strategies.

The present study

Some previous studies have identified natural number bias reasoning, reverse bias reasoning and gap thinking in PTs' answers to a single item (e.g., Reeder & Utley, 2017). However, studies using multiple items controlling congruency and gap conditions, aimed at explicitly investigating the systematic use of these ways of reasoning when comparing the magnitude of two fractions in PTs, are scarce. Furthermore, there is a lack of studies in which PTs have been confronted with these incorrect ways of reasoning, aimed at investigating if they agree with them or whether they are able to recognise them as incorrect reasoning.

The present study is part of a larger project investigating the existence of incorrect reasoning about fraction magnitude in 93 PTs. In the first part of the data collection, participants completed a test consisting of 16 fraction comparison items (one-digit numerator and denominator). The items were either congruent or incongruent with respect to the ordering of natural numbers and they were either correct, incorrect, or neutral with respect to gap thinking. A cluster analysis identified five clusters of PTs with similar response patterns: PTs in the *Correct* profile ($n = 39$) answered all or nearly all items correctly. *Gap Thinkers* ($n = 27$) correctly solved the items where gap thinking led to a correct answer, and incorrectly those where gap thinking led to an incorrect answer. These PTs seemed to solve the items based on the difference between the numerator and the denominator. PTs in the *Reverse Bias* profile ($n = 13$) correctly solved incongruent items and incorrectly solved the congruent ones. These PTs seemed to reason that the larger fraction is the one with the smallest denominator. PTs in the *NNB* profile ($n = 7$) incorrectly solved the incongruent items and correctly solved the congruent ones. These participants seemed to solve the items according to the natural number bias, based on the ordering of natural numbers. Participants in the profile *Other* ($n = 7$) solved the items without any identifiable pattern.

In the second part of the data collection, which is the focus of this research report, the same PTs were confronted with a vignette containing a fraction comparison item and the responses of three primary school students, each of whom demonstrated one of the

three ways of incorrect reasoning (natural number bias, reverse bias, and gap thinking). The PTs had to indicate whether they agreed with any student's response and to justify their answer. The results from the vignette will allow us i) to analyse the consistency between participants' test performance and their choices in the vignette, and ii) to obtain qualitative evidence of their reasoning. Measuring consistency allows us to draw conclusions about how deeply rooted potentially erroneous reasoning is in PTs (González-Forte et al., 2023). Furthermore, the vignettes require identifying school students' incorrect reasoning even though their solution as such may be correct. Therefore, the vignettes tap into PTs' assessment skills, which is fundamental for teachers.

Our specific research question is: *How consistent are PTs in their reasoning about fraction magnitude when confronted with (incorrect) reasoning?*

METHOD

The participants were 93 Spanish PTs (65 female, 28 male; $M = 18.62$ years, $SD = 1.84$ years) enrolled in their first year of the degree to become elementary school teachers. At the time of the data collection, participants had not yet received any instruction in mathematics or mathematics teaching.

The instrument consisted of a vignette showing a fraction comparison item ($3/5$ vs. $4/9$) and the answers from three primary school students that showed the three incorrect ways of reasoning described above (see Figure 1). Roberto answers that $3/5$ is larger, using gap thinking; Alicia answers that $4/9$ is larger, using natural number bias reasoning; and María answers that $3/5$ is larger, using reverse bias reasoning. The PTs had to indicate whether they agreed with any student's answer and to provide a reason for their response. If they did not agree with any answer, they had to justify why and explain their own reasoning.

The answers of 3 primary school students to a fraction comparison task are presented

Do you agree with any of the answers? Why?
If you do not agree with any of them, indicate why and write your own answer.

Figure 1: Vignette used. Graphical elements: DIVER (coReflect@maths, 2022)

We analysed PT's answers by coding which student the participants agreed with. Both authors of this paper coded the responses independently and reached high inter-coder reliability (Cohen's Kappa = .89). Cases of disagreement were discussed until a consensus was reached. Then, the consistency between the answers given in the vignette and the profiles previously identified in the test was analysed.

RESULTS

Table 1 shows the frequencies of PTs' answers to the vignette (which answer they agreed with), by the profiles they belonged according to the cluster analysis conducted in a previous analysis of their test performance (see above).

Choice \ Profile	<i>Correct</i> (<i>n</i> = 39)	<i>Gap Thinker</i> (<i>n</i> = 27)	<i>Reverse Bias</i> (<i>n</i> = 13)	<i>NNB</i> (<i>n</i> = 7)	<i>Other</i> (<i>n</i> = 7)
None	25	4	3	2	1
Roberto (Gap Thinker)	3	11	2	1	1
María (Reverse Bias)	9	10	8	2	3
Alicia (NNB)				1	1
Roberto and María	1	2			
Don't know	1			1	1

Table 1: Frequencies of PTs' choice in the vignette according to their profile

Of the 39 PTs who belonged to the *Correct* profile, 25 were consistent with their profile when they solved the vignette, stating that they did not agree with any of the three explanations given by the primary school students. This is notable because Roberto and María also chose the correct fraction, $\frac{3}{5}$, although they provided incorrect reasoning. The predominant ways of reasoning among these PTs were obtaining common denominators (11) and converting to decimal numbers (6). For example, PT41's answered: "*No, because they are not based on any valid theoretical foundation. It is $\frac{3}{5}$ because when the fractions are expanded to the common denominator, it is observed ($\frac{27}{45} > \frac{20}{45}$)*". On the other hand, 3 PTs agreed with Roberto's answer, and 9 PTs agreed with María's answer. This result may be because these PTs, despite having solved all the items of the fraction comparison test correctly, did not carefully evaluate students' incorrect reasoning in the vignette and simply went with the correct choice of the largest fraction. For example, PT3 said: "*The most logical answer, or the one that would be chosen without calculation, would be Maria's, because if you think about it, it makes sense, and I think that is how they taught it to me in class*".

Of the 27 PTs who belonged to the *Gap Thinker* profile, 11 were consistent with their profile, choosing Roberto's answer. That is, they not only answered the test following a faulty response pattern based on the difference between the numerator and denominator, but explicitly agreed with his way of reasoning when confronted with it in Roberto's answer. On the other hand, 4 PTs did not agree with any of the three explanations, all of them showing a correct reasoning. This result is surprising, since these PTs appeared to have used gap thinking in the test. One reason may be that when asked to justify their reasoning, these participants changed their initial strategy. The

most surprising result is that 10 PTs agreed with María's answer. This may be because María's answer corresponds to the choice they would make ($3/5$) and, in addition, the reasoning may have seemed more appropriate to them. For example, PT12 answered: *"I agree with María's answer because I think it is the correct one. I seem to remember that being the answer, although I am not 100% sure"*.

Eight out of the 13 PTs who belonged to the *Reverse Bias* profile remained consistent, considering that María's answer was correct. However, in the same way as in the previous profile, 3 PTs did not agree with any of the three explanations (all of them showed a correct reasoning) and 2 PTs agreed with Roberto's answer, despite having shown a response pattern in the test that seemed to show that they used reverse bias reasoning. In fact, PT86 assured to have used Roberto's way of reasoning in the test: *"Yes, with Roberto, since I have the same approach as him for resolving activities of the same type"*. This answer indicates that maybe this PT solved the fraction comparison items on the test without being truly aware of his way of reasoning used.

Finally, only one PT from the *NNB* profile was consistent with this profile, considering that Alicia's answer was correct, while 2 PTs did not agree with any of the three explanations. Although this profile consisted of only 7 participants so that we cannot draw general conclusions, it is striking that one PT agreed with Roberto's answer (PT82: *"I agree with Roberto's answer, since knowing which fraction is larger has nothing to do with the numbers that make it up but rather the difference between the two"*), and two PTs agreed with María's answer (PT22: *"I agree with María because when the denominator is smaller the parts have larger value"*). These responses are remarkable because these PTs' accuracy on incongruent items in the test was very low. Perhaps, participants in this profile did not have much confidence in their own reasoning, and when they observed other ways of reasoning (albeit incorrect ones), they lean towards.

DISCUSSION AND CONCLUSIONS

This study investigated in PTs the existence of incorrect ways of reasoning about fraction magnitude, similar to the ones previously found in primary, secondary and university students (e.g., Gómez & Dartnell, 2019, González-Forte et al., 2023). We focused on PTs' answers to one vignette, measuring the consistency between their answer and their test profile, and providing qualitative evidence of their reasoning. Within each profile, there were PTs that remained consistent with their reasoning when responding to the vignette. Thus, these PTs exhibited a way of reasoning not just implicitly but confirmed it explicitly when confronted with it in a student's response. These PTs' incorrect reasoning seems to be stable and resistant to change (González-Forte et al., 2023). This result seems problematic, considering that the participants were PTs of mathematics who will also teach fractions in their future professional lives.

However, not all PTs were consistent with their profile when responding to the vignette. Importantly, PTs who used incorrect reasoning in the test and were inconsistent in the vignette did not necessarily adapt their reasoning towards the better.

From the 27 PTs who were inconsistent within the *Gap Thinker*, *Reverse Bias* and *NNB* profiles, only 9 chose the correct response in the vignette (i.e., did not agree with any students' answer). Perhaps, these PTs were uncertain about their reasoning in the first place, making it likely to change their reasoning when being confronted with alternative ways. Another explanation may be that these PTs solved the fraction comparison items on the test without being truly aware of the way of reasoning they were using. Therefore, when they were required to note down their reasoning, this made them think differently than they had done in the test.

An interesting result is what happened with some PTs within the *Gap Thinker* profile. Their performance on the test showed a clear response pattern. However, 10 of them agreed with María's answer, suggesting a reverse bias. This may be because, when confronted with it, reverse bias reasoning may have made more sense to them. In fact, reverse bias reasoning shows a partial understanding of the fraction concept, since it is true that if the whole is divided into fewer pieces, the pieces will be larger. This result cannot be generalized since we only had one item ($3/5$ vs. $4/9$), which is incongruent and gap thinking correct (i.e., both profiles would answer $3/5$). To further explore this idea, it would be interesting to present these PTs with a congruent gap thinking correct item (e.g., $5/8$ vs. $2/7$; where the *Gap Thinker* profile would be expected to choose $5/8$ and the *Reverse Bias* profile would be expected to choose $2/7$). This would allow to see whether PTs who use gap thinking are more likely to agree with reverse bias reasoning (even if this means changing their choice of the larger fraction).

Finally, the results have implications for teaching in initial teacher training courses. Teachers and PTs need to be able to identify incorrect ways of reasoning in students' answers, even if students have given the correct response. Although we should emphasise that the participants in this study were PTs who were enrolled in their first year of the degree and had not yet received training in mathematics teaching, the result that fewer than half of the 93 PTs showed correct reasoning is concerning. PTs' own ability to solve fraction comparison tasks correctly appears not to be sufficient. Fourteen of the PTs in the *Correct* profile were not able to identify incorrect reasoning when being confronted with it in students' responses. It is therefore necessary to include information on the existence of these incorrect ways of reasoning in initial teacher training courses.

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LOGIC VS. LINGUISTICS? THE ROLE OF AGE AND CONTEXT WHEN INTERPRETING “OR” IN DIFFERENT SITUATIONS

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Boolean operators are fundamental in mathematics and information technology but the interpretation of the according linguistic equivalent (e.g. the word “or”) might be ambiguous (OR, XOR), potentially leading to confusion. We present data from a pilot study of 33 subjects aged 7 years up to adults. The participants solved four questions, each of them allowing an OR (inclusive or) or XOR (exclusive “either...or”) interpretation. In two questions the “or” was embedded in an extra-mathematical context from everyday life. In the other two questions the “or” was embedded in a simple mathematical context. We observed substantial interpretation differences associated with the context (preferred OR interpretation in mathematical context and nearly equal OR and XOR interpretation in extra-mathematical context).

Boolean operators and their compositions (e.g., AND, OR, NOR) have become more important in recent decades due to the development of computerized data storage and now also artificial intelligence. Without an understanding of Boolean operators, search queries on the Internet cannot be solved efficiently and take many times longer. Boolean logic operators and their compositions also play a foundational role in mathematics as a logical basis and the acquisition of related logical skills is relevant for the formation of mathematical competences. This applies in particular for logical argumentation and thereby formal proving (Hehner, 1996) as well as for a systematical understanding of mathematics as structural science. Empirical research from linguistics with children and adults suggests that age and native language of participants can play a role on the interpretation of Boolean operators and their compositions (e.g., Chierchia et al., 2004; Tieu et al., 2007; Zeijlstra, 2004). Usually, these studies use items with situations from an everyday context. The question of whether embedding Boolean operators in a mathematical context leads to the same results has not yet been examined. Since mathematics and its conventions are essentially learned in mathematics lessons, this could lead to a different interpretation of the Boolean operators. For an investigation, suitable tasks need to be developed. In a study with 33 participants described in this article, tasks with logical Boolean operators embedded in an extra- or intra-mathematical context were administered as written test followed by an interview. The participants had to interpret the operators and the aim of the study was to validate the tasks. This article deals with the results for items with the operator “or”, comparing the tasks with an extra- or intra-mathematical context.

THEORETICAL BACKGROUND

Boolean operators OR vs. XOR

A disjunctive sentence of the form "A or B" can, in principle, be interpreted in two ways. The first possibility is an inclusive interpretation in the sense of the Boolean operator OR which means that "A or B" is also accepted in situations in which both A and B are true. For example, the statement "Mira likes tulips or roses." would be interpreted inclusively as "Mira likes either tulips or roses or both.". However, there is a second possibility, namely interpreting "or" exclusively in the sense of the operator XOR. This means that "A or B" is only accepted in situations in which either A or B is true but not when both are true. This means for the previous example that "Mira likes either tulips or roses but not both.". In general, the interpretation of "or" in a given sentence depends on the context. In many contexts only one interpretation makes sense (e.g., "Jessica is thinking about whether or not to spend her vacation in Tokyo.").

Usually, the role of "or" is not explicitly analyzed in everyday situations, so individuals are not necessarily aware of the different interpretations. In many countries, a formal consideration of the "or" as operators OR and XOR – for example with the help of truth tables – first occurs at the beginning of university studies (mainly in STEM-related programs). Research on university students' understanding of OR and XOR indicates that many students have difficulties with both, the OR and XOR interpretation (Herman et. al, 2012).

The interpretation of "or" has also been studied intensively in linguistic research, in particular to analyze the role of age and also different native languages in the field of semantics (e.g., Ariel et al., 2024; Chierchia et al., 2004; Skordos et al., 2020; Tieu et al., 2007). For example, no interdependence on the native language was found in the interpretation of "or" (e.g. Tieu et al., 2007), although this was shown for other operations (e.g. double negation, Zeijlstra, 2004). However, it has been replicated several times that children and adults differ in the interpretation of "or" (Chierchia et al., 2004; Skordos et al., 2020; Tieu et al., 2007). While most children preferred to interpret the "or" inclusively (as a logical OR), adults interpreted the "or" almost always as an exclusive "either... or" (as a logical XOR). With younger children, it also happened that the "or" was interpreted as AND (about 17% of the children in the study of Skordos et al., 2020). One possible explanation for the inclusive interpretation of "or" in children is that it is challenging to rule out that both cases can be true, and that children learn this only gradually (see Ariel et al., 2024). An alternative explanation is that the previous findings were influenced by the respective study designs chosen in laboratory settings in which children had to give an answer in a decision situation (Truth Value Judgment Task). For example, Ariel et al. (2024) were able to show that children interpret the "or" like adults frequently exclusively in natural situations with spontaneous child-directed speech in dialogically relevant reactions to "or".

The role of extra- and intra-mathematical contexts

As already mentioned in the previous subsection, the context of the given “or” statement plays a role in the interpretation. Especially in everyday situations, the question is which intention is probably behind a statement. While the context in the statement “Jessica is thinking about whether or not to spend her vacation in Tokyo.” makes the exclusive interpretation highly probable, with other statements it is open as to what could be intended (e.g., “Mira likes tulips or roses.”). Irrespective of the meaning of the content of a given everyday situation, norms could also play a role. Mathematics is a particularly suitable context for research here, as mathematical (and logical) norms are acquired almost exclusively in the context of formal education in mathematics lessons (e.g., Yackel & Cobb, 1996, who stress the important role of mathematics teachers as representatives of the mathematical community to establish the socio-mathematical norms in the classroom). From a mathematics didactics perspective, it is therefore interesting to see to what extent there are differences between mathematical and extra-mathematical contexts in the interpretation of Boolean operators.

In our study presented below, we will accordingly compare the interpretation of “or” in mathematical and extra-mathematical situations. To ensure that the surface features of the tasks are not too different, no purely mathematical tasks disconnected to real context were used for the mathematical context. Instead, we described everyday situations in which mathematical properties of numbers or shapes occurred that were linked by “or”. This means that the “or” in tasks with a mathematical context suggest an interpretation against the background of mathematical norms, whereas the “or” in tasks with an extra-mathematical context could mainly interpreted based on individual experiences with similar situations from everyday life. Despite this difference, the described situations in both cases firstly had to be modeled and mathematized according to a modeling process (e.g., Verschaffel, Depaepe, & van Dooren, 2015).

Research questions

Based on the theoretical background, we investigated the role of mathematical and extra-mathematical contexts and the age of individuals when interpreting statements linked by an “or”. Our study was guided by the following research questions:

What is the relationship between the interpretation of the statements as inclusive disjunction (OR) or as exclusive disjunction (XOR) and...

1. ...the age of the interpreting individuals?
2. ...the context of the situations?

METHODS

The sample of our study comprised 33 German participants, 10 young students (7-9 years from grade 1-3), 12 students (10-12 years from grade 4-6) and 11 adults (at least 18 years). The data collection took about one hour (including breaks) and was

conducted in an individual session. In addition to the four tasks considered here with an “or” interpretation (see below), further tasks were set which have not yet been analyzed. The participants first completed the tasks in writing on a test sheet and were then interviewed about their solutions. The items were read aloud - if necessary multiple times - to the first and second graders as well as to children with reading difficulties. The interview was intended to validate the written solution. In addition, depending on the participants' “or” interpretation, they were asked whether they also considered the alternative interpretation to be plausible.

We have developed four tasks. Two tasks had an “or”-statement to be interpreted in an extra-mathematical context, and in two other tasks the “or” linked two properties of mathematical objects so that it should be interpreted in a mathematical context. The tasks were tested with a child and an adult in a pre-pilot phase and checked for difficulty and wording, among other things. When developing and revising the tasks, it was always a basic requirement that the “or” could be interpreted in both ways, inclusively and exclusively and that both interpretations are equally plausible.

Figure 1 shows one of the two tasks with an extra-mathematical context. The task describes a situation with a teddy bear collection and the child Emma gets to choose teddy bears. She wants “black or red teddy bears” and her sister agreed but reminds her to take as much as she can. In this situation, an inclusive interpretation of “or” comprises also the case that Emma takes black and red teddy bears, that is six teddy bears altogether, whereas an exclusive interpretation would result in the four red teddy bears. It is assumed that the participants will process this situation based on their interpretation of “or” which might be influenced by their experience with similar everyday situations or their goal to get as many teddy bears as possible.

Emma's uncle has a collection of small teddy bears. He has 2 black, 5 light brown, 7 dark brown and 4 red teddy bears. Emma is allowed to choose teddy bears. She says: “I take black or red teddy bears.”
Her sister says: "Agreed. Take as many teddy bears as you can."
How many small teddy bears can Emma take?

Fig. 1: Example of an extra-mathematical context task

Figure 2 shows a task in a mathematical context. There are geometric forms and the participant is asked, how many of these shapes are “round or yellow”. An inclusive interpretation of “or” comprises all circles and all yellow shapes, i.e. five shapes, whereas an exclusive interpretation would result in four shapes (either round or yellow shapes but not round and yellow shapes). In contrast to the task in Figure 1, there is no everyday context which might influence the participants' interpretation. Instead the task is similar to a math task from the first grades of elementary school.

How many of these geometric shapes are round or yellow? Circle all the matching shapes and tick the right number under the picture.

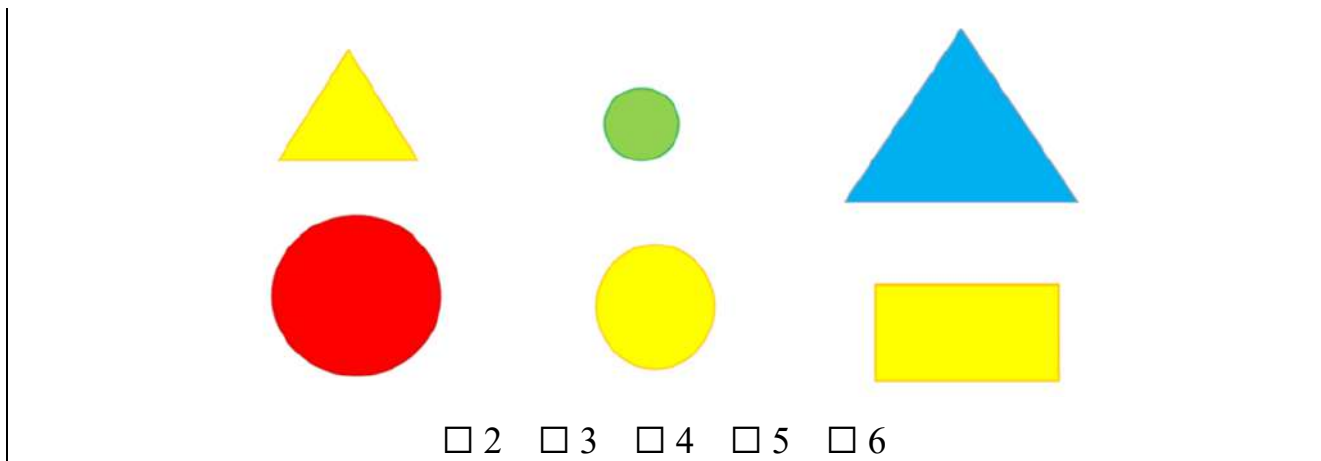


Fig. 2: Example of an intra-mathematical context task

When evaluating the four tasks, the written and verbal answers of each participant were taken into account. If these were inconsistent, the reasons given in the interview were decisive. For the data analysis, we first analyzed the frequency distributions of the XOR and OR interpretations descriptively as a function of age groups and the context of the task. A logistic regression was then carried out with the following independent variables: Age (in years), context (intra- and extra-mathematical) and the link between age and context. The XOR (= 0) and OR (= 1) interpretation was tested as the dependent variable.

RESULTS

The data collection was carried out as planned. All four tasks were understandable (even for the first graders) and the adults felt that they were taken seriously despite the low complexity of the situations. Inconsistencies between the written and verbal answers to a task occurred in approximately 11% of the cases.

Table 1 summarizes the number of the different interpretations of “or” for the two extra-mathematical tasks broken down by age group. The analogue findings for the two intra-mathematical tasks are presented in Table 2.

Age	7-9 years	10-12 years	Adults
OR (inclusive)	6 (35%)	10 (44%)	9 (43%)
XOR (exclusive)	7 (41%)	12 (52%)	10 (48%)
Other interpretation (e.g. AND)	4 (24%)	1 (4%)	2 (9%)
Total number of answers	17	23	21

Table 1: Number (percentage) of OR/XOR-interpretation by age group in an extra-mathematical context

Age	7-9 years	10-12 years	Adults
OR (inclusive)	14 (78%)	17 (74%)	14 (64%)

XOR (exclusive)	2 (11%)	1 (4%)	4 (18%)
Other interpretation (e.g. AND)	2 (11%)	5 (22%)	4 (18%)
Total number of answers	18	23	22

Table 2: Number (percentage) of OR/XOR-interpretation by age group in an intra-mathematical context

As already previously mentioned, the written and verbal answers of a person were consistent for a given task in most cases (about 89%). Rarely, however, there were cases where participants provided inconsistent interpretations. For the extra-mathematical problem in Figure 1, for example, one adult stated 4 (an exclusive interpretation) but changed her mind in the interview to the response 6 (an inclusive interpretation):

Interviewer: Yes, okay, in the task you wrote down four teddy bears. Which teddy bears are these?

Participant: (...) These are the red teddy bears.

Interviewer: Why them?

Participant: She had. Emma was allowed to choose teddy bears and said: "I'll have black or red teddy bears." And her sister said: "Take as many as you can." And then I thought well, then four. But I just realized that I then thought of the or as black or, either red teddy bears, so I thought of it more as "either or" and then that would be wrong. Then it would have, so then it would be right. But if it's normal "or", then it would, she could take black and red and then it would be six.

In this case, we used the revised answer for data analysis. The explanation of the adult shows that she became aware that the "or" interpretation is challenging. The first (exclusive) interpretation was influenced by the context. Afterwards she changed her mind and referred to the "normal 'or'" that should be applied here.

The percentages of OR and XOR interpretations shown in Tables 1 and 2 indicate that the age groups within a context do hardly differ from each other. With regard to research question 1, we therefore conclude that the age groups have no influence on the interpretation of "or".

For the second research question, it is striking that the inclusive "or" interpretation as logical OR was clearly preferred to the interpretation as XOR and other interpretations in the tasks with an intra-mathematical context. Other interpretations were chosen second most frequently with 11 out of 63 evaluable answers. This put them just ahead of the exclusive XOR interpretation with 7 cases. In the tasks with an extra-mathematical context, the "or" was interpreted inclusively (OR 25 times) and exclusively (XOR 29 times) about equally often. Other interpretations were chosen a total of 7 times. In the interview, some participants even mentioned that they drew on experiences from their everyday lives when solving these tasks.

The effect of the context could also be statistically confirmed. In the logistic regression analysis, only the context was significantly associated with the XOR or OR interpretation (*unstandardized beta* (B) = 2.83; p = .008).

DISCUSSION

Previous studies on the interpretation of logical operators – mainly from linguistics research – often administer items with situations in everyday contexts. It can be expected that the interpretation of the operators is influenced by the participants' experience with a given context. In the present study, we embedded the logical operators in extra- and intra-mathematical contexts in such a way that the reference frame for the interpretation in the intra-mathematical contexts relates to socio-mathematical norms. We took care that the “or” in each task could be interpreted meaningfully as either OR or XOR. Nevertheless, there was a considerable variation in terms of interpretation, depending on context but not on age.

Our finding that age has no effect is in contrast to results from some linguistic studies (Chierchia et al., 2004; Skordos et al., 2020; Tieu et al., 2007), according to which children interpret “or” usually inclusively as OR whereas adults prefer the exclusive interpretation as XOR. However, as Ariel et al. (2024) recently published, children frequently use the XOR interpretation if the item format is not restricted to decision situations (Truth Value Judgment Task) as it was the case in our study. Our findings confirm that there is hardly any difference for the interpretations between the age groups for tasks in extra-mathematical contexts. In contrast to other studies, however, “or” was interpreted almost equally frequently as OR and XOR in all age groups. A new feature of our study was the contrast between an intra- and extra-mathematical context for the “or” interpretation. Despite the small sample size, we found evidence that the participants of all age groups interpreted “or” in an intra-mathematical context predominantly in an inclusive way as OR. This suggests that the participants might link an inclusive interpretation to a mathematical context. This is expected, as it corresponds to the mathematical-logical interpretation of “or” as inclusive OR.

The study has some limitations. First of all, we have only a small sample size and used only four items. Hence, there is a strong need for replication of our findings. Second, some of the students were surveyed at school and others at home which might have an influence. Finally, the participants' reading competence might have an influence, although we read the word problems to the younger students.

In summary, we found different ways of interpreting “or” depending on an extra- or intra-mathematical context. There was a predominant inclusive OR interpretation in intra-mathematical tasks and an equally frequent OR/XOR interpretation in extra-mathematical tasks. Hence, it makes sense to further examine participants' interpretation by contrasting extra- or intra-mathematical context to get deeper insights into why children and adults follow a specific interpretation for tasks depending on the (mathematical) contexts.

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KIND OF WARRANTS AND PROOFS USED BY UPPER SECONDARY STUDENTS IN INTUITIVELY APPROACHING THE DENSITY OF \mathbb{Q} IN \mathbb{R}

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While research in Mathematics Education has mainly focused on the density of \mathbb{Q} , little is known about the conceptualization of the density of \mathbb{Q} in \mathbb{R} . This study investigates upper-secondary school students' warrants and proofs regarding the density of \mathbb{Q} in \mathbb{R} . A classroom activity focused on continued fractions, which provide a unified representation of rational and irrational numbers, was carried out. Through warrant- and proof-analysis, this study reveals how different kinds of warrant and proof shape the students' reasoning on the density of \mathbb{Q} in \mathbb{R} , providing insight into the level of mathematical depth reached and the reasoning effectiveness produced by students.

RATIONALE

The concept of density of \mathbb{Q} is a well-known obstacle in the transition from \mathbb{N} to \mathbb{Q} , and much research has focused on it (e.g., Hannula et al., 2012; Kim & Kwon, 2019; McMullen & Van Hoof, 2020), whereas the density of \mathbb{Q} in \mathbb{R} —i.e., the property that given two real numbers, there is always a rational number between them—, a key concept related to the transition from \mathbb{Q} to \mathbb{R} , has received little attention from research in Mathematics Education (Marmur et al., 2020). The transition from \mathbb{Q} to \mathbb{R} becomes crucial during the calculus courses at the university level when \mathbb{R} must be conceptualized as the domain necessary to investigate continuous phenomena and limits. Indeed, Marmur et al. (2020) investigate how first-year mathematics undergraduates grasp the concept of density of \mathbb{Q} in \mathbb{R} . Actually, students intuitively approach this concept already in upper-secondary school and acquire spontaneous conceptions—in the sense of intuitive beliefs based on experiences in various contexts, including everyday life (Fujii, 2014)—of it, on which the future institutionalized concepts will build. However, spontaneous conceptions are not directly accessible, and the individual is usually unaware of them. Investigating the type of warrants, i.e., the reasons expressed in an inferential step of an argument (Toulmin, 1958), and proofs—in the sense of Balacheff (1988)—that students use when they reason on the density of \mathbb{Q} in \mathbb{R} , or on concepts related to it, can be useful to understand how students spontaneously approach this concept. To this end, students' arguments used to support their reasoning in working on a task involving continued fractions are categorized and integrated, focusing on the following research question: What kind of warrants and

proofs do students use in facing a task that implicitly involves the concept of density of \mathbb{Q} in \mathbb{R} and how do these warrants and proofs shape their reasoning? The paper presents a part of the results of such warrant- and proofs-classification of data collected in two classes of Italian upper-secondary students (aged 17/18).

THEORETICAL FRAMEWORK

Toulmin (1958) defines the “skeleton of a pattern for analyzing arguments” (p. 99) as a structure consisting of data (the argument starting point, i.e., the evident facts), conclusion (the assertion that needs to be justified), and warrant (the logical link between data and conclusion). Even if both Toulmin and other authors added further elements to the original skeleton, the basic structure is sufficient for reconstructing the argumentation, i.e., the argumentative process in our context. We explicitly clarify that, in this paper, we use “argumentation” to indicate the overall argumentative process, whereas “argument” is for the single sentence examined during the analysis.

Rodd (2000) distinguishes between warrant, i.e., “that which secures knowledge” (p. 222), and justification, i.e., “rationale for a belief” (p. 222). If a justification warrants the truth of a belief, then it is a warrant, or better, a mathematical warrant; otherwise, it is not, and Rodd calls it simply justification. In our context, we do not make this distinction. Indeed, our research participants are in the initial phase of the intuitive approach with the mathematical object inquired—which still must be formally introduced—and we thus consider as a warrant also students’ arguments that are incomplete, incorrect, or expressed through registers unsuitable for mathematical formalization, e.g., the visual representations. Indeed, according to Rodd (2000), “visualization does not negate or detract from proofs of a deductive–symbolic character” (p. 236), and we accept it as a ground of a mathematical warrant.

We use an amount of seven warrant-categories to account, from an epistemic point of view, for the types of warrants students use during their reasoning. A first distinction was made between *pragmatic warrants* (related to utility and efficiency) and *theoretical warrants* (related to properties and definitions). The warrant-categories defined by Inglis et al. (2007)—*inductive warrant* (based on the use of a certain number of examples), *intuitive warrant* (based on the use of observations or examples that reflect some underlying cognitive structure), and *deductive warrant* (based on deduction from axioms, algebraic manipulations, and counterexamples)—are specific kinds of theoretical warrant. Subsequently, for each of the warrant-categories defined by Inglis et al. (2007), a distinction between *operational warrant* (related to a procedure) and *structural warrant* (related to properties and relationships between objects) was made. The first two distinctions—one between pragmatic and theoretical warrant and one between the three types of theoretical warrant—were made because of the participants’ reduced mathematical experience. Indeed, certain warrants (e.g., theoretical warrants) may rely on competencies (e.g., linguistic and cognitive skills) that students may not yet possess, whereas others (e.g., pragmatic warrants) are more accessible in this context. Instead, the latter distinction (between operational and

structural warrants) is a helpful descriptive tool in the warrant-analysis process, which distinguishes the type of warrant used in the argumentation more in detail. Defining the structural intuitive warrant, Inglis et al. (2007) implicitly refer to Rodd's (2000) idea of "visual arguments" (p. 5). Indeed, this kind of warrant is defined by the authors as "some kind of mental structure, be it *visual* or otherwise, that persuades them [the participants] of a conclusion" (Inglis et al., 2007, p. 11) (*italic introduced by the authors*).

To account, from a logical point of view, for the different kinds of proof that are not necessarily mathematical in Rodd's (2000) sense but are important from an epistemic perspective, we use Balacheff's (1988) classification of proofs. Balacheff distinguishes between *pragmatic proof* (based on actions) and *intellectual proof* (which employs verbalizations of the properties of objects and their relationships). This proof-classification accounts for the level of generality achieved by students.

Additionally, Balacheff identifies a taxonomy for the types of proofs used in Grisendi (2024). However, since we do not refer to these categories in this paper, we will not dwell on these further.

RESEARCH METHOD

To stimulate the emergence of spontaneous conceptions, an activity focused on continued fractions, which facilitates the transition from \mathbb{Q} to \mathbb{R} , was designed. Two classes for a total of 42 upper-secondary school students were involved. The activity, divided into two parts, was conducted under the lead of the first author in the presence of the classroom teachers but without their intervention. In the first part (two hours), students respond anonymously, working in small groups. This part involves a worksheet, which begins with a brief contextualized introduction to continued fractions, where a hypothetical student proposes a continued fraction as an approximation of an irrational solution of a second-grade equation; then, students are asked to argue for or against the statement (called Statement below): "Given any real number, can I always find a rational number that is as close to it as I want?" After reading the initial part of the worksheet, students complete two parts, called Assignment 1 and Assignment 2. In Assignment 1, students write down a points list of arguments for or against the Statement, while in Assignment 2, starting from the generated list, students are asked to write an argumentative text. These responses were used to track students' reasoning processes and identify the kinds of arguments they used to support their conceptions. Finally, students are interviewed by the researcher a few days later via semistructured group interviews, inviting them to deepen the argumentation presented by their group. In this sense, the collected data were written or spoken sentences produced by the students while arguing their position concerning the task claim.

We use qualitative content analysis (Mayring, 2015) and structure the analysis-outputs into three interconnected levels to explore the progression of each group's reasoning. Indeed, our data analysis approach draws inspiration from Toulmin's (1958) simile:

“An argument is like an organism. It has both a gross, anatomical structure and a finer, as-it-were physiological one” (p. 94). Firstly, the “Initial framing”-phase provides a holistic overview of the reasoning, presenting the group’s argumentation “anatomical structure.” Secondly, the “Detailed analysis and classification”-phase analyses each warrant and proof, describing the group’s argumentation “physiological structure.” Here, individual sentences are examined with the classification system explained in the Theoretical framework. Finally, after providing a holistic overview and a granular analysis of the argumentation, we explore connections and transitions between sentences, focusing on how the identified warrant and proof-categories reveal the characteristics of students’ reasoning as they approach the topic. In this sense, the third and final phase, “Concluding discussion”, bridges between the first two phases, synthesizing the findings and offering a global vision of the group’s arguments.

DATA ANALYSIS AND RESULTS

From the analysis of the twelve groups, only five of the seven warrant-categories defined in the Theoretical framework emerge. We focus only on four groups because these groups—analyzed more in-depth in Grisendi (2024)—cover all the warrant-categories that emerged from the data. For each warrant-category analyzed, we provide a representative example. The analysis of each group starts with the warrant used by the group; then, we present the proofs detected in reconstructing the group’s argumentative process. Presenting the results, we move from the least effective reasoning for conceptualizing the topic to the most effective.

Group 6 – Operational inductive warrant

Initial framing. From the responses provided by Group 6, it can be inferred that the legitimacy of the Statement is perceived as evident and does not require further consideration. The students focused on verifying the validity of the procedure involving continued fractions rather than providing an argumentation to support the proposed claim. **Detailed analysis and classification.** In Assignment 2, the students wrote: “[...] Given any real number, it is possible to obtain an irrational number arbitrarily close to the original. Starting from [the procedure using continued fractions], one can get increasingly closer to one of the results of the exercise. To demonstrate this, a list of different values was tested, including negative, positive, and decimal numbers.” Starting from the statement, “To demonstrate this, a list of different values was tested,” the argument can be classified as an operational inductive warrant because the students supported their position with a series of tested examples to demonstrate the validity of the procedure presented in the task. During the interview, students were asked to clarify what they meant by “negative, positive, and decimal numbers.” This revealed confusion regarding the nature of numerical sets and the relationships between them. **Concluding discussion.** The lack of prerequisites concerning numerical set distinctions and the students’ difficulty, in this context, in investigating the core concept, i.e., the density of \mathbb{Q} in \mathbb{R} , is exemplified through an operational inductive warrant.

Group 5 – Pragmatic warrant and pragmatic proof

Initial framing. Group 5 focused primarily on solving the equation without fully grasping the objective of the activity. They paid attention to aspects relevant because of the didactic contract, such as the need to specify an algebraic fraction's existence conditions and the Statement's domain of validity. **Detailed analysis and classification.** In Assignment 2, the students wrote: "The statement [...] is not universally correct because the [procedure with continued fractions] is correct [...] and allows one to get closer and closer to the solution [...], but it does not apply to all real numbers. Indeed, if I use the solutions of the equation, I do not obtain a number close to but equal to the solution." For Group 5, the Statement is legitimate because it is based on a useful procedure, as highlighted by the phrase "[the procedure] allows one to get closer and closer to the solution." In this sense, this argument is classified as a pragmatic warrant. However, the group also argued that the Statement is incorrect because, despite its basic legitimacy, it does not hold for "all real numbers." Because students show an example based on the act of substitution, "If I *use* the solution" (italic introduced by the authors), this argument falls under the category of pragmatic proof. **Concluding discussion.** Using a pragmatic warrant and a pragmatic proof likely indicates students' avoidance of effective reasoning in this context. Nevertheless, the substitution act that characterizes the pragmatic proof enables students to consider the issue of the Statement and approach the research topic.

Group 2 – Structural intuitive warrant

Initial framing. Students from Group 2 effectively framed the problem using a Venn diagram to represent numerical sets. Although this is only implicitly evident in their written work, the graphical tool enabled Group 2 to visualize the relationships between the various sets involved, reflecting an underlying cognitive structure. **Detailed analysis and classification.** The statement from Assignment 2, "Considering that real numbers are divided into rational and irrational numbers, we need to try to demonstrate that every number belonging to these two sets can be expressed as a fraction," is classified as an intuitive warrant because it represents an observation based on a mental structure that the students have constructed as they developed their initial reasoning. Additionally, this statement can be classified as a structural warrant, as the students refer to a "property" of rational and irrational numbers—namely, the property of representing these numbers in fractional form. **Concluding discussion.** The structural intuitive warrant is supported by a visual representation (Rodd, 2000; Inglis et al., 2007), which helped the students clarify their ideas about the relationships between numerical sets and propose a distinction of cases to base their reasoning on. Specifically, the students distinguish between irrational and rational numbers, referring to the distinctive "property" related to fractional notation. In this sense, the students recognize the role of continued fractions in this context, which bridges notation between the two categories of numbers and thus between \mathbb{Q} and \mathbb{R} .

Group 10 – Operational and structural deductive warrant and intellectual proof

Initial framing. Group 10 begins with the observation that irrational numbers are characterized by an infinite number of decimal digits. These characteristics form the foundation for constructing an iterative approximation method. **Detailed analysis and classification.** In Assignment 1, the students base their reasoning on the statement: “Given an irrational number, it will have infinite digits after the decimal point.” This represents a “property” of irrational numbers and is categorized as a structural deductive warrant. Subsequently, the students write: “We can truncate [the irrational number] at a point of our choice, but after doing so, we could decide to keep one more digit than before.” This statement outlines a procedure and is therefore classified as an operational deductive warrant. This reasoning leads the students to develop, in Assignment 2, an intellectual proof. The statement classified as an intellectual proof is the following: “We truncate the irrational number at any point, thus obtaining a rational number; however, at this point, we can keep one more digit than before, still resulting in a rational number.” We can observe that, while in the theoretical warrant, it is possible to distinguish between the theoretical concept and the operational procedure, in the intellectual proof, the two components produce a unified and coherent reasoning process. **Concluding discussion.** The warrant- and proof-categories used by Group 10 students demonstrate an original and well-focused approach to the topic under investigation, showing the ability to produce a general argumentation. The reasoning performed by Group 10 enabled the students to conceive the cardinality of the continuum of the set \mathbb{R} intuitively. This is evident during the interview when, referring to the arrangement of numbers on the real line, one of the students states: “There’s a little value in every spot,” followed by another student affirming: “There’s no space between one number and the next.”

DISCUSSION AND CONCLUSION

Thanks to the three levels of analysis shown before, we offer a comprehensive view of each group’s argumentative process, providing insight into the level of mathematical depth students reached and the reasoning effectiveness students produced. On one hand, in this context, “mathematical depth” refers to students’ ability to transcend specific examples or surface-level observations, uncovering broader structures, relationships, and implications in a mathematical framework. On the other hand, in this context, reasoning effectiveness indicates the objective—i.e., according to Rodd (2000), the “general, nonnegotiable, and reliable” (p. 242)—of the reasoning. Indeed, Group 6 students’ reasoning is less effective than that of others because it relies on a negotiable case system (“negative, positive, and decimal numbers”), which highlights students’ difficulties in distinguishing numerical sets. For this reason, it is unlikely to be reliable. We could say that there is a positive correlation between mathematical depth and reasoning effectiveness, in the sense that effective reasoning produced by a group may indicate abilities to achieve a higher level of mathematical depth regarding the concept because reaching a higher level of mathematical depth deserves general or generalizable arguments, which in turn are indicators of effective (mathematical)

reasoning. Figure 1 highlights this relationship: as the effectiveness of reasoning increases, so does the mathematical depth level. The schema illustrates the warrant- and proof-categories that emerged from the groups' argumentative processes. The groups are arranged from left to right, moving from the least effective reasoning to the most effective. Each group corresponds to a cell in the figure. The cell's height reflects metaphorically and in a relational way the level of mathematical depth of the group's reasoning—the taller the cell, the higher the level of mathematical depth.

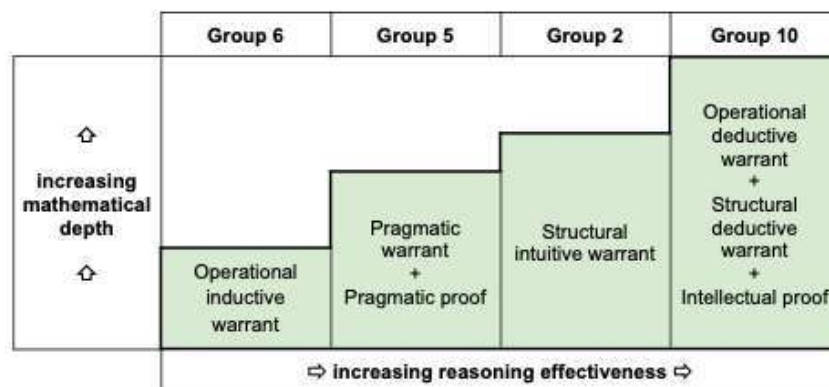


Figure 1: Schema of reasoning effectiveness and mathematical depth across groups

We provide a cross-sectional perspective across the four analyzed groups to address the second part of the research question. From the gaps of Group 6 to the advanced reasoning of Group 10, we highlight the higher and higher level of mathematical depth achieved. Group 6 students remain at a pre-conceptual stage, whereas Group 5 takes a modest step forward by approaching the core concept. However, both rely on practical observations rather than structured reasoning, which indicates only a superficial level of mathematical depth. By visualizing numerical set distinctions, Group 2 demonstrates a more significant conceptual leap, achieving an intermediate level of mathematical depth. Finally, Group 10 attains the highest level of mathematical depth, constructing a general argumentation, which develops into an intuitive conceptualization of the cardinality of \mathbb{R} .

This analysis shows that the level of mathematical depth and the effectiveness of the reasoning correspond to the types of warrant and proof-categories students rely upon and the interconnectedness of their arguments. Indeed, students whose reasoning demonstrates multiple and mutually reinforcing categories achieve a higher level of mathematical depth and more effective reasoning than those producing isolated and/or “basic”—i.e., not general—arguments.

Further research is needed to deepen the role of different kinds of warrants and their interplay with various types of proofs in conceptualization.

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INVESTIGATING STUDENTS' EXPECTATIONS AND EXPERIENCES WHEN VISITING AN OUT-OF-SCHOOL LAB FOR MATHEMATICS

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While students visit out-of-school labs (OSLs) for natural sciences with expectations in line with the aims of OSLs, they visit OSLs for social sciences and humanities with expectations not corresponding to the goals of OSLs and hindering their actual experiences. As our investigation of the initial expectations and actual experiences of 183 5th graders when visiting an OSL for mathematics reveals, students visit the OSL with expectations and experiences in line with the aims of OSLs. They especially expected and experienced a professional setting in the OSL and different learning activities compared to those in regular mathematics lessons. Based on observed, albeit not statistically significant, relations between students' initial expectations and actual experiences of the OSL visit, we derive implications for mathematical OSL visits.

INTRODUCTION

Out-of-school labs (OSLs) aim to trigger students' interest in and to enhance their knowledge about scientific ways of thinking and working by providing them with authentic experiences (e.g., Scharfenberg & Bogner, 2014). The extent to which these goals are achieved depends, among other things, on students' expectations when visiting an OSL (e.g., Nachtigall & Rummel, 2022). Previous research shows that students visit *OSLs for natural sciences* with expectations that are in line with these goals, for instance, to conduct experiments (as a typical scientific practice) or to have access to better laboratory equipment than in school (e.g., Garner & Eilks, 2015). In contrast, as the results of a study by Nachtigall and Rummel (2022) reveal, students visit *OSLs for social sciences and humanities* with rather unspecific and broad expectations that do not correspond to the goals of OSLs. Specifically, they found that students least expected to get authentic insights into scientific research, but rather expected to have a nice and free time with their classmates and without pressure to perform. Furthermore, the results indicate that students' initial expectations are related to their actual experiences and their situational interest in the project contents. For example, the initial expectation to get authentic insights into scientific research correlated positively with students' reported experience of having acquired content knowledge during their visit and their situational interest in the project contents. Contrary, the widespread initial expectation to have a nice and free time during the OSL visit only correlated positively with students' actual experience of the OSL visit as nice and free time and not with their situational interest in the project contents.

In light of the inconsistent expectations of students when visiting OSLs for natural sciences and OSLs for social sciences and humanities, which are likely to be related to their actual experiences of the OSL visit and, thus, affect the extent to which the goals of OSLs are achieved, the question arises as to what expectations students have when visiting an *OSL for mathematics*. Given that students often have limited conceptions about mathematics and the work of mathematical scientists, it is likely that students' expectations when visiting an OSL for mathematics do not correspond to the goals of OSLs. For example, students often consider mathematics merely as a body of facts and procedures (e.g., Østergaard, 2024; Schoenfeld, 1992) and believe that mathematical scientists do the same kind of mathematics as the students do in the classroom, namely calculating, but on a higher level (e.g., Hagenkötter et al., 2022). As a consequence, students' expectations when visiting an OSL for mathematics might hinder their actual experiences of the OSL visit.

Against this background, the present paper first investigates students' initial expectations when visiting an OSL for mathematics (research objective 1). We then examine students' actual experiences of the OSL visit (research objective 2). Finally, we explore how students' initial expectations are related to their actual experiences of the OSL visit (research objective 3).

METHOD

As part of the research project “MerLab” (Mathematik erleben im Lehr-Lern-Labor; experiencing mathematics in an OSL), funded by the Reinhard Frank-Stiftung, we developed a day-long project for an OSL for mathematics at a large German university. The OSL project entitiled “Experiencing proportions and fractions” aimed at providing students with first explorations of proportions and fractions. For this purpose, we implemented various hands-on tasks related to everyday life, which the students worked on in small groups of four or, in exceptional cases, due to class size constraints, three students. For example, they explored how to divide a pizza for varying numbers of people, dealt with mixing colours, or investigated the ratio of apple juice and water for the perfect apple spritzer.

Participants

Participants were 183 5th graders ($M_{\text{age}} = 10.98$ years, $SD = 0.67$; 43% female, 50% male) from six schools who visited the OSL as whole classes with their mathematics teachers to attend the day-long mathematical OSL project.

Measures

To achieve our research objectives, we analyzed data from two questionnaires which we administered to the students: The first questionnaire was completed by the students about a week before their visit to the OSL at school and assessed, among other things, their *expectations* related to the OSL visit. For this purpose, we used the following open-ended question adapted from Garner and Eilks (2015) as well as Nachtigall and Rummel (2022): What do you think, what are the differences between working in the

OSL and in regular mathematics lessons? The second questionnaire was completed by the students at the end of the OSL visit and measured, among other things, their *experiences* of the OSL visit by using the following slightly rephrased open-ended question of the first questionnaire: What differences did you experience today when working in the OSL compared to regular mathematics lessons? Please note that we also asked the students about expected and experienced similarities between working in the OSL and in regular mathematics lessons. However, due to space constraints, we only consider students' expected and experienced differences in the present paper.

We analyzed students' responses to the open-ended questions by using qualitative content analysis following Kuckartz (2018). In doing so, we used a deductive-inductive category system, in which the deductive categories arose from previous research on students' expectations when visiting both OSLs for natural sciences as well as OSLs for social sciences and humanities (Garner & Eilks, 2015; Nachtigall & Rummel, 2022). We used the same categories for students' expected as well as experienced differences between working in the OSL and in regular mathematics lessons. The categories are depicted in Table 1. A second rater coded 20% of the dataset with substantial strength of agreement (Cohen's $\kappa = .72$).

Category	Description and example
	Students expected/ experienced ... in the OSL (compared to regular mathematics lessons).
Different contents ^d	... different (mathematical) contents; e.g., "learning other things about math"
Different learning activities ^d	... different learning activities (e.g., experimenting, playing (mathematical) games); e.g., "do experiments"
Different materials	... to use different materials (e.g., hands-on material, material for research); e.g., "we can touch things and there are things that we don't have in math lessons"
Less difficulty, no pressure and grades ^d	... less difficulty and/ or more relaxed learning as they get no grades and have no pressure to perform; e.g., "I think it's easier in the OSL than in regular math lessons."
More collaboration ^d	... more collaboration; e.g., "that we work together much more"
More difficulty and intensive work	... more difficulty and/ or intensive work; e.g., "it's harder"
More interest, excitement, and fun ^d	... more interest, excitement and/ or fun; e.g., "it's much more interesting in the lab"
New (learning) experiences ^d	... to gain new experiences by trying new things and/ or learn something new; e.g., "we learned a lot of new things"

Professional setting ^d	... a professional setting (e.g., general laboratory equipment, university environment, more technology); e.g., “more space in the OSL”
No differences ^d	... no differences; e.g., “nothing”
No idea	The students have no idea about differences between working in the OSL and in regular mathematics lessons; e.g., “don’t know”
Other	The students provide an answer that does not refer to differences between working in the OSL and in regular mathematics lessons; “I thing math is good”

Table 1: Category system (^d deductive categories (adapted) from Garner and Eilks (2015) as well as Nachtigall and Rummel (2022))

RESULTS

Research objective 1: Students’ expectations

With regard to reseach objective 1 (i.e., students’ initial expectations when visiting an OSL for mathematics), the analysis of the student answers reveals that the students especially expected a professional setting in the OSL. For instance, many students expected a large laboratory with technology, as the following exemplary statements show: “more space in the OSL”, “large TVs, [...] large building”, or “perhaps there are many more apparatuses”. Moreover, the students often expected different learning activities in the OSL compared to those in regular mathematics lessons. In particular, they believed to “do experiments” or to “play games that have to do with math”. The results further show that the students thought that it is more difficult and the work is more intensive in the OSL compared to regular mathematics lessons. One student, for example, gave the following answer: “It may be more difficult in the OSL, it may be easier in math lessons.” Another student stated: “I believe that working in the OSL is even more intensive than in the classroom.” Figure 1 (dark grey) shows the descriptive statistics of the coded student answers.

Research objective 2: Students’ experiences

With respect to research objective 2 (i.e., students’ actual experiences of the OSL visit), the descriptive statistics (see Figure 1, light grey) show that the students particularly experienced different learning activities in the OSL compared to those in regular mathematics lessons. For instance, many students stated that they conducted “experiments”, “mixed colours”, or, more generally, “do different things in the OSL”. Furthermore, the students often emphasized the professional setting in the OSL, for example, by listing things that are different in the OSL compared to school: “Equipment, chairs, desks, sockets [...]“. In addition, the students experienced more interest, excitement, and fun in the OSL compared to regular mathematic lessons. They stated, for instance: “It was fun for the first time.”, “it’s more exciting in the OSL”, or

“there are very interesting tasks, not like at school”. The students also pointed out that they used different materials in the OSL compared to those in regular mathematics lessons, as the following exemplary statement shows: “to use more different objects to work with”.

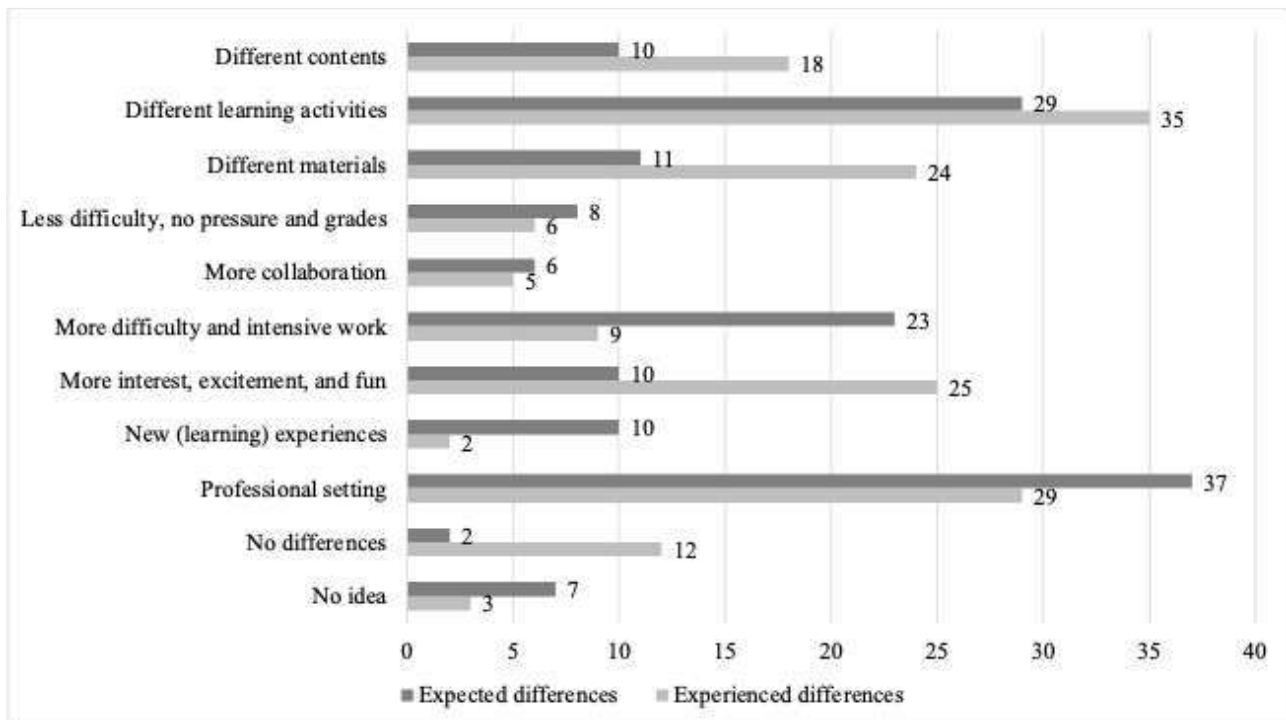


Figure 1: Absolute frequencies of students' coded answers on expected and experienced differences in the OSL compared to regular mathematics lessons

Research objective 3: Possible relations between students' expectations and experiences

To achieve research objective 3 (i.e., to explore how students' initial expectations are related to their actual experiences of the OSL visit), we used a cross table of students' expected and experienced differences between working in the OSL and in regular mathematics lessons (see Figure 2) and conducted a Pearson's chi-square test. However, as not all expected cell frequencies were greater than 5, we used a Monte Carlo simulation with 10.000 replicates. The results show no significant relation between students' initial expectations and actual experiences of the OSL visit, $\chi^2 = 13.07$, $p = .848$. Nevertheless, the descriptive statistics suggest the following tendencies: 1) Students who expected different learning activities in the OSL compared to those in regular mathematics lessons tended to also experience different learning activities during their OSL visit. 2) Students who expected a professional setting in the OSL tended to experience also a professional setting in the OSL, different learning activities or different materials compared to those in regular mathematics lessons.

		Experienced differences											Σ
		Different contents	Different learning activities	Different materials	Less difficulty, no pressure and grades	More collaboration	More difficulty and intensive work	More interest, excitement, and fun	New (learning) experiences	Professional setting	No differences	No idea	
Expected differences	Different contents	2	2	1	0	0	1	1	0	1	2	0	10
	Different learning activities	4	9	6	0	1	0	3	1	3	1	0	28
	Different materials	0	2	2	2	0	0	1	0	3	0	0	10
	Less difficulty, no pressure and grades	2	4	2	1	0	0	0	0	2	1	0	12
	More collaboration	0	2	1	0	0	1	0	0	0	1	0	5
	More difficulty and intensive work	4	5	2	0	0	2	3	1	5	1	0	23
	More interest, excitement, and fun	0	3	1	0	1	0	4	0	4	0	0	13
	New (learning) experiences	1	2	0	0	0	0	3	0	2	0	0	8
	Professional setting	2	8	7	1	0	4	5	0	9	3	0	39
	No differences	0	0	0	0	0	0	0	0	0	1	0	1
	No idea	1	0	0	1	0	0	1	0	1	0	0	4
Σ	16	37	22	5	2	8	21	2	30	10	0	153	

Figure 2: Cross table of students' expected and experienced differences between working in the OSL and in regular mathematics lessons

(Please note that there may be deviations from the absolute frequencies of students' coded answers in Figure 1 due to possible multiple answers and the fact that here only the answers of the students who answered both questions are considered.)

DISCUSSION

The findings of the present paper regarding students' expectations when visiting an OSL for mathematics (research objective 1) show that students especially expect a professional setting, different learning activities, such as experimentation, or more difficulty and intensive work in the OSL compared to regular mathematics lessons. Thus, contrary to students' expectations when visiting OSLs for social sciences and humanities (e.g., Nachtigall & Rummel, 2022) and in line with students' expectations when visiting OSLs for natural sciences (e.g., Garner & Eilks, 2015), the expectations of students when visiting an OSL for mathematics partly reflect the goals of OSLs (e.g., Scharfenberg & Bogner, 2014). In light of students' limited conceptions about mathematics (e.g., Schoenfeld, 1992) and the work of mathematical scientists (e.g., Hagenkötter et al., 2022), it may be surprising that students also expected to conduct experiments when visiting the OSL for mathematics. One possible reason for this expectation could be that the students, when answering the question about their expectations, may not explicitly referred to the mathematical OSL, but rather to OSLs for natural sciences or laboratories in general. This could be caused by the fact that the term "science" is often associated primarily with natural sciences (e.g., Ziegler et al., 2018) and that students often believe that scientists carry out dangerous experiments in a laboratory (see, e.g., Finson, 2002). Moreover, in contrast to the results of the study by Nachtigall and Rummel (2022) on students' expectations when visiting an OSL for

social sciences, the students of the present study do not primarily expect to have a nice and free time with their classmates and without pressure to perform when visiting the OSL for mathematics. Instead, they often expect more difficulty and intensive work compared to regular mathematics lessons. Our results further reveal that students' actual experiences of the OSL visit (research objective 2) are also highly consistent with the goals of OSLs (e.g., Scharfenberg & Bogner, 2014). Particularly the aim of triggering students' interest is reflected in students' actual experiences of the OSL visit, as many students especially experienced more interest, excitement, and fun in the OSL compared to regular mathematics lessons. In addition, even though we did not find a statistically significant relation, certain tendencies between students' initial expectations and actual experiences (research objective 3) can be identified. For example, we observed that many students actually experienced during the OSL visit what they initially expected: Students who expected different learning activities in the OSL often also experienced different learning activities and students who expected a professional setting in the OSL often also experienced a professional setting. In addition, we found that students who initially expected a professional setting in the OSL tended to experience different learning activities or different materials compared to those in regular mathematics lessons. This observation may also be attributed to the assumption that the students possibly did not explicitly refer to the mathematical OSL when formulating their expectations, but rather to OSLs for natural sciences or laboratories in general. The often very generally formulated expectation of a professional setting in the OSL was then fleshed out during the OSL visit so that the students were afterwards able to give more concrete answers regarding the experienced differences between working in the OSL compared to regular mathematics lessons. In doing so, they often referred to the specific experiences they had gained during the OSL project, as the aforementioned examples show.

In summary, in light of our findings, it seems worthwhile to not only visit mathematical OSLs as a one-off experience, but to implement repeated visits. This might enable (even more) students to form realistic expectations in line with the goals of OSLs, which could lead to these goals being achieved increasingly. In addition, repeated attendance at various mathematical OSL projects might enable students to gain diverse insights into the role and importance of mathematics and mathematical work. Taking into account several OSL visits from a research perspective may also counteract the potential limitation of the present paper of only considering one specific mathematical OSL project, which could have influenced the results. In future studies, it would also be useful to include further dependent variables. For example, given that students' limited conceptions about mathematics and the work of mathematical scientists are probably largely based on their experiences in mathematics lessons (see, e.g., Hagenkötter et al., 2022; Schoenfeld, 1992) and in light of students' experience of different learning activities in the OSL compared to those in regular mathematics lessons, it would be beneficial to also investigate the effects of mathematical OSL visits on students' conceptions about mathematics and mathematical work.

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MATHEMATICS TEACHER SUPPORT ON PERCEIVING MATHEMATICS CHALLENGES: THE ROLE OF MATHEMATICAL SELF-CONCEPT AND VALUE OF MATHEMATICS

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This study, framed with the Expectancy-Value Theory, examines the relationship among each type of mathematics teacher support (MTS), mathematical self-concept (MSC), and the Value and Struggle facets of mathematical resilience. The data were taken from self-reports of 863 Indonesian students. Maximum Likelihood Structural Equation Modelling with robust standard errors was used to analyze the relationship. The results show that MSC indirectly affects Struggle via Value, emotional MTS indirectly affects Struggle via Value, instrumental MTS indirectly affects Value via MSC, and instrumental MTS indirectly affects Struggle through MSC and Value. These results highlight the mechanisms of MTS in developing students' mathematical resilience. They also emphasize the unique role of MSC and Value in the relationship.

INTRODUCTION

Recent efforts to address student disengagement and low performance in mathematics have focused on developing positive attitudes rather than merely addressing negative ones. Johnston-Wilder and Lee (2024) suggest the development of mathematical resilience as a set of positive attitudes to help students manage negative emotions when facing mathematical challenges, enabling them to progress in their learning. Kookken et al. (2016) differentiate between three subjective appraisal-related facets of mathematical resilience: Value, Struggle, and Growth, while Johnston-Wilder and Lee (2024) add Support as a fourth facet. In a previous investigation of classroom support, we found that mathematics teacher support (MTS) was positively related to both Value and Struggle (Hamidy et al., accepted). This finding parallels the Expectancy-Value theory (EVT), which posits that subjective task values are influenced by socializers' behavior, for instance, MTS. However, EVT proposes that the effect of the social environment on an individual's subjective task values is indirect and facilitated by their self-concept of abilities. Therefore, we assume that mathematical self-concept (MSC) mediates the relationship between MTS and Value and Struggle. Furthermore, Federici and Skaalvik (2014) found that a high positive value for physical education could counteract its perceived cost, indicating that perceived cost and other positive task

values are unlikely to be equal. By viewing Struggle as a positive view of perceived cost, this finding may also apply to the mathematical resilience context, suggesting that high Value can increase Struggle. Consequently, we can assume that MSC and Value mediate the relationship between MTS and Struggle. However, these assumptions remain untested. Therefore, using the EVT framework, we aim to investigate the relationships between MTS, MSC, Value, and Struggle. We further differentiate teacher support into two types: emotional and instrumental support (Federici & Skaalvik, 2013). Our contributions should illuminate the detailed development of mathematical resilience through its facets, clarify how teacher support promotes mathematical resilience, demonstrate the EVT framework's applicability in new contexts, and highlight the possibility of distinguishing perceived costs from other positive task values.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

Enabling students to be ready and successfully adapt to the challenges in mathematics, mathematical resilience is developed from four facets: Value, Struggle, Growth, and Support (Johnston-Wilder & Lee, 2024; Kooken et al., 2016). Value refers to a positive appraisal that mathematics is important for future success, while Struggle refers to a belief that challenges are a normal part of learning in mathematics (Kooken et al., 2016). Together with Growth and Support, they are essential to make students stay engaged and progress in mathematics despite the challenges. Regarding students' decision to be resilient and engaged (or not) in a learning context, the EVT by Eccles & Wigfield (2020) proposes that one's achievement-related choice and performance are determined by expectancies for success and subjective task values. Both are connected to several constructs that are also influential for mathematical resilience. For instance, in the EVT, the utility value of subjective task values is defined as how well an activity or task aligns with an individual's current and future goals (Eccles & Wigfield, 2020), which is similar to Value of mathematical resilience. Subjective task values also include a negative aspect known as perceived cost. This refers to the negative perceived consequences of engaging in an activity or task that may lead individuals to avoid it (Flake et al., 2015). Eccles and Wigfield (1995) operationalized cost in their mathematics study as task difficulties and required effort of a task, which is related to Struggle of mathematical resilience. That is, difficulties and effort could be perceived as a cost to disengage from learning, while Struggle embraces difficulties and effort as a normal part to keep engaged in learning. Considering these connections, we use the EVT as a framework for understanding how students develop and maintain resilient behaviors in mathematics.

The subjective task values serve as a benefit-cost analysis for deciding whether to participate in an activity or task (Eccles et al., 1983). This means that the higher the cost of an activity, the lower its perceived value tends to be. On the other hand, a strong perceived value of a task can buffer the perceived cost that may arise. The last relationship aligns with the study of Flake et al. (2015), which found that students'

appraisal of challenges and required effort varies across motivation levels. Highly motivated students perceive challenges and effort positively, while less motivated students perceive them negatively. This suggests that positive values could influence perceived cost, so it may differ from them, as proposed by Barron and Hulleman (2015). Empirically, the assumption is supported by Federici and Skaalvik (2014), which found that a high positive value could counteract perceived cost. Therefore, Struggle, as a positive value in facing challenges and effort, seems to be positively influenced by Value in the context of developing mathematical resilience. Furthermore, the EVT framework suggests that the subjective values are related to one's goals and self-beliefs, such as self-concept (Eccles & Wigfield, 2020). Self-concept in mathematics can be understood as a part of academic self-concept, which refers to students' awareness and beliefs about their performance (Bong & Skaalvik, 2003). While the study by Agtarap and Miranda (2022) suggests that MSC is a key predictor of developing mathematical resilience, it does not specify the particular facets influenced by MSC. Nonetheless, Pintrich and Schunk (2002) provide insight into the relationship between self-concept and Value, indicating that when students recognize their abilities in a specific area, they begin to perceive the importance of succeeding in that area. Although the previous explanation indicates that MSC seems to be positively related to Struggle, we could not find supporting studies for this claim. Instead, we found studies indicating negative relationships between self-concept and both positive attitudes toward solving mathematics problems and expending effort on them, with this relationship being associated with other positive beliefs (e.g. enjoyment), cultural, and environmental factors (Julius, 2022; Pinxten et al., 2014). In the EVT framework, environmental factors are also considered by proposing that socializers' beliefs and behaviors affect individuals' developing self-concept and, in turn, their subjective task values (Eccles & Wigfield, 2020). This aligns with the development of mathematical resilience, which emphasizes a psychologically safe mathematics learning environment indicated by ensuring "...that the relationships between teachers and students are nurturing, respectful, and warm" (Johnston-Wilder & Lee, 2024, p. 3). The teacher-student relationship can be viewed as a give-and-take support, which is divided into emotional and instrumental support (Federici & Skaalvik, 2013). Emotional support involves nurturing relationships characterized by empathy, encouragement, esteem, and friendliness, while instrumental support is action-oriented, enhancing students' understanding through clarification, correction, elaboration, and demonstration (Federici & Skaalvik, 2013; Semmer et al., 2008). In our previous study, we did not differentiate the type of support and found that mathematics teacher support is positively related to Value and Struggle (Hamidy et al., accepted). Furthermore, Lee and Simpkins (2021) found that MTS is crucial for developing students' MSC. Additionally, the study by Han et al. (2022) highlights the importance of emotional teacher support on students' utility value, while Federici and Skaalvik (2014) suggest that utility value mediates the relationship between instrumental teacher support and perceived cost. These studies point to the role of MSC and Value in mediating the relationship between MTS and Struggle, and suggest that the different types of teacher

support may have varying effects on this relationship. However, these assumptions have not been extensively examined within a single structural model. Therefore, our research questions are: How is the relationship between MTS, MSC, Value, and Struggle in the development of mathematical resilience? In particular, to what extent does each of MTS influence Value through mediation by MSC, and how does it influence Struggle through mediation by MSC and Value?

METHODS

To address the research question, we conducted a correlational study using data collected from 5-point Likert scales adapted from previous studies. We used the MTS scale from Federici and Skaalvik (2013), which includes 6-item emotional teacher support and 6-item instrumental support. Students' MSC was measured by the 5-item mathematical self-concept from PISA 2012. Additionally, we incorporated the 8-item Value subscale and 9-item Struggle subscale of mathematical resilience developed by Kookan et al. (2016). Since the mathematical resilience scale was originally designed for university students, we adjusted some items in the Value subscale to better align with the educational level of our study. Subsequently, we translated all items into Indonesian. The translation was then reviewed by an experienced researcher fluent in both English and Indonesian, along with several Indonesian mathematics teachers. All items were designed using a 5-point Likert scale, with higher values indicating higher levels of agreement. Afterward, we employed the final scales to collect data from 9th and 10th-grade Indonesian students through an online survey. They joined the survey anonymously and voluntarily. 863 (60.8 % female, 2.9% did not answer) respondents were included in this study (average age: 15;2). As one item of the mathematical self-concept scale was negatively worded, it was recoded before analysis. For our data analysis, we began by conducting confirmatory factor analyses (CFA) and estimating omega reliability. We then perform structural equation modeling (SEM) with the model displayed in Figure 1. For estimation, we applied maximum likelihood with robust standard errors (MLR) to account for missing values. The lavaan package (Rosseel, 2012) in R (R Core Team, 2023) was used for the analysis.

RESULTS

The CFA results indicate that the scales meet the fit criteria, except for the negative item of mathematical self-concept, which has a low factor loading; therefore, we exclude this item from further analysis. The reliability of each scale is acceptable ($\omega > 0.7$). Furthermore, the model indicates that the model fits the data well ($\chi^2(485) = 1359.313$; CFI = 0.938; TLI = 0.932; RMSEA = 0.050; SRMR = 0.042). Estimates of the standardized regression weights, which indicate the direct effects, are shown in Figure 1. It reveals that emotional MTS is positively related to Value but is not significant for MSC and Struggle. Conversely, instrumental MTS is a significant predictor for Struggle and MSC but is not significant for Value. Furthermore, MSC is positively related to Value but is negatively related to Struggle. In addition, Value has a positive relationship with Struggle.

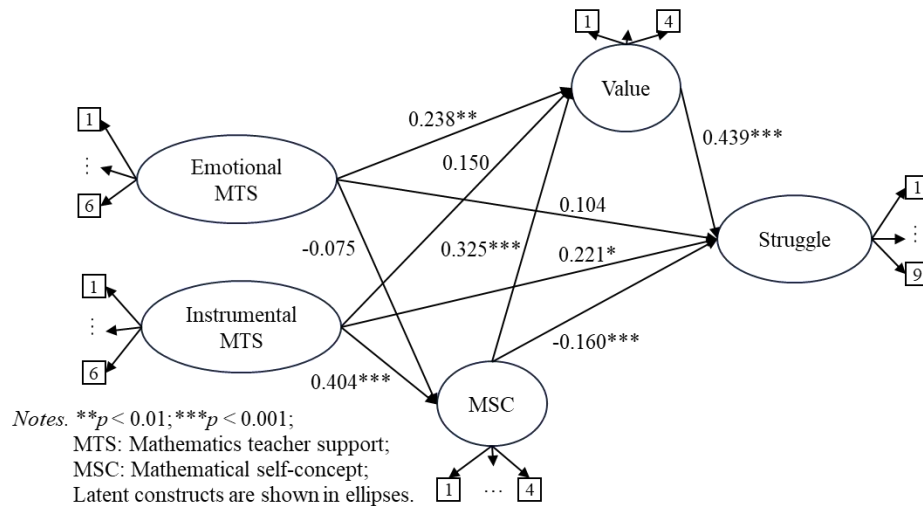


Figure 1: The structural model and its measured relationships

The indirect effects shown in Table 2 reveal that MSC positively affects Struggle through Value. The indirect effect of emotional MTS is not significant on Value when mediated by MSC, nor on Struggle when mediated by MSC and Value. However, it is significant on Struggle when only mediated by Value. The indirect effect of instrumental MTS on Value is positively significant through MSC, while it has a negative and significant effect on Struggle with the same mediator. When both MSC and Value are included as mediators, the effect of instrumental MTS on Struggle becomes positive and significant.

Path	Est. std.	Path	Est. std.
MSC → Value → Struggle	0.143***		
EMTS → MSC → Value	-0.025	IMTS → MSC → Value	0.132***
EMTS → MSC → Struggle	0.012	IMTS → MSC → Struggle	-0.065*
EMTS → Value → Struggle	0.104**	IMTS → Value → Struggle	0.066
EMTS → MSC → Value → Struggle	-0.011	IMTS → MSC → Value → Struggle	0.058**

Table 1: The standardized regression weights, standard errors, and p -values of the indirect effects (* $p < 0.05$; ** $p < 0.01$; *** $p < 0.001$)

DISCUSSION AND CONCLUSION

In line with our assumptions, the results show that the higher perceived futuristic value of mathematics increases individuals' tolerance of challenges and efforts in the subject. The finding supports the relationship between the utility value and perceived cost found by Federici and Skaalvik (2014), indicating that both constructs can be considered separately (Barron & Hulleman, 2015). Additionally, this emphasizes the dynamic between the facets of mathematical resilience (Kookan et al., 2016). The effect of self-concept is also observed in both Value and Struggle. The MSC has a

positive relationship with Value and a negative relationship with Struggle, which aligns with prior studies (Julius, 2022; Pintrich & Schunk, 2002; Pinxten et al., 2014). However, the effect of MSC on Struggle turns positive when mediated by Value. In other words, although the direct relationship between MSC and Struggle is negative, it turns positive when Value is taken into account. This indicates that Value should be considered when examining the relationship between MSC and Struggle. This also highlights Value's role in transforming the perception of struggle into a growth mindset challenge that fosters persistence and engagement. Since positive values are linked to students' motivation, the results also align with the study by Flake et al. (2015), which suggests that students' motivation levels influence whether they perceive challenges and efforts positively or negatively. Furthermore, the results show that the influence of emotional and instrumental MTS on Value and Struggle varies, enriching our previous study (Hamidy et al., accepted). Emotional support from mathematics teachers is closely linked to students' strong belief in the utility value of mathematics. Since emotional support emphasizes nurturing teacher-student relationships to keep engaged in mathematics learning (Federici & Skaalvik, 2013; Semmer et al., 2008), this leads students to recognize the importance of what they are learning, thereby enhancing their belief in the benefits of mathematics (Han et al., 2022). However, this support does not contribute to the development of students' confidence in their abilities, as well as their perception of the challenges and effort they put into mathematics. Nonetheless, the effect of emotional MTS on Struggle becomes significant when mediated by Value, which highlights utility value's key role in shaping students' perception of challenges and effort in mathematics. This result also extends the work of Federici and Skaalvik (2014), which examined the relationship between teacher support and perceived cost mediated by utility value. On the other hand, instrumental MTS is directly related to MSC and Struggle. We believe this connection arises from the nature of instrumental support, which focuses on understanding (Federici & Skaalvik, 2013; Semmer et al., 2008). Therefore, unlike emotional support, instrumental support is more aligned with students' confidence in their abilities, as well as their perceptions of challenges and the effort they put into mathematics. In addition, while the direct effect of instrumental MTS on Value is not significant, its indirect effect becomes significant when mediated by MSC. This suggests that support related to understanding can help students value the benefits of mathematics, as it highlights their abilities and helps them recognize their competence. Besides the direct effect, the relationship between instrumental MTS and Struggle is also mediated by MSC and Value. The results highlight the important role of both MSC in strengthening the effect of instrumental MTS and Value in reframing challenges and efforts in mathematics. The results further clarify and extend the relationship between MTS and MSC examined by Lee and Simpkins (2021). However, the relationship becomes insignificant when only mediated by Value, which likely contrasts with the previous related study (Federici & Skaalvik, 2014). The relationship also becomes negative when only mediated by MSC. We argue that these latter two results arise from the absence of MSC in mediating between instrumental

MTS and Value, as well as Value in mediating between MSC and Struggle, which we previously discussed.

In summary, the results suggest that types of MTS influence the development of mathematical resilience through different mechanisms. While emotional MTS is simply related to Value and Struggle, instrumental MTS involves MSC to explain its effect on Value and Struggle. The effect of instrumental MTS on Value and Struggle then confirms the EVT framework of Eccles and Wigfield (2020). Ultimately, the results should guide mathematics teachers in providing support that enhances students' mathematical resilience. However, we consider several limitations of this study. Besides a cross-sectional setting that could not provide sufficient support for causality, other longitudinal studies on EVT also suggest that there is a reciprocal relationship between subjective values and self-concept. Therefore, longitudinal studies of the same model should be considered to capture a more detailed mechanism of mathematical resilience development by mathematics teacher support. In addition, as the concept of perceived cost becomes more complex, a deeper examination may be required to determine which types of costs are associated with the struggle for mathematical resilience. Yet, our study sheds first light on the detailed development of mathematical resilience in relation to teacher support and MSC. In addition, it shows relationships between the EVT framework and mathematical resilience that should be explored in more detail in future studies.

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CASSETTE EFFECT AS THE CONDITIONS AND CONSTRAINTS IN NON-WESTERN MATHEMATICS EDUCATION: A CASE STUDY OF JAPAN

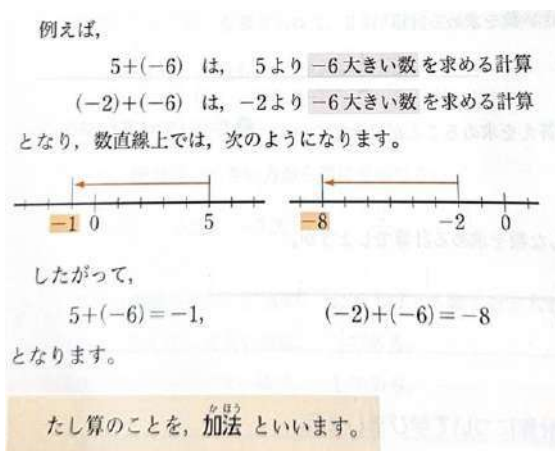
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In non-Western countries, the words of mathematics (education) differ from their native languages. In our study, we do not focus on the difference between the two, but on the fact that mathematical words are translated. The theoretical framework is the cassette effect (Yanabu, 1976, 2009), in which translated words have unique values and thus behave in a particular way. We present examples of Japanese textbooks perceived as incongruous by non-Japanese speakers and analyze them using this theoretical framework. The results show that the use of the translated language affects learning through unique conditions and constraints.

BACKGROUNDS

Fig. 1 shows a citation from a Japanese mathematics textbook authorized by the Ministry of Education for Grade 7.



Our translation:

For instance,

$5 + (-6)$ is a calculation to find the number -6 greater than 5

$(-2) + (-6)$ is a calculation to find the number -6 greater than -2

Using number lines, we can express them as follows.

Thus,

$5 + (-6) = -1$

$(-2) + (-6) = -8$

We found them.

We say addition is addition.

Figure 1: Learning addition of negative numbers (Okamoto et. al., 2020, p. 22)

This is the first opportunity for learners to learn the addition of negative numbers. The last line says “たし算” is “加法” in Japanese; however, in English, both “たし算” and “加法” mean addition. “たし算” is the name of addition used in elementary school, whereas “加法” is the name of addition used after 7th grade. For non-Japanese native speakers, this tautological statement is unclear in terms of its meaning and intention. By contrast, Japanese native speakers, including the author and Japanese 7th-grade students, find nothing unusual about it.

Currently, students learn Western mathematics not only in Japan but worldwide. As a result, mathematics terms differ considerably from those in native (not Western) languages in many areas globally. Various studies have examined these differences in discipline of mathematics education. Our interest lies in phenomena related to language and words. However, this study focuses specifically on words are translated. To the best of our knowledge, no previous study has addressed the fact that these words are translated. In conclusion, these translated words are used in non-Western cultures (at least in Japan), which have accepted Western civilization through translation, as a kind of conditions and constraints (Bosch and Gascón, 2006). These are the hypothesis and conclusion of this study. We will clarify them using the notion of the “*cassette effect*” (Yanabu, 1976, 2009).

Here, “Western” is used in the sense of civilization centering in Europe. This is because mathematics, which is the basis of mathematics education, was primarily developed in European countries. As discussed later, many countries (including Japan, the subject of this paper) have accepted Western mathematics and educational systems. However, during this process, a problem arose in that the words used in Western mathematics (education) could not be expressed in the language of the host country.

The following are the effects of such translation difficulties in Japan: First, we briefly describe the history and character of the Japanese language, which is necessary for understanding this study. Next, we describe our theoretical framework “*cassette effect*” proposed by Yanabu (1976, 2009). Next, we analyze Figure 1 within this framework.

SHORT BRIEF OF JAPANESE HISTORY AND LANGUAGE

As Japanese is not the native language of many readers, we briefly introduce its history and characteristics. Historically, Japan has been strongly influenced by Chinese civilization. Therefore, the Japanese uses a combination of Chinese characters (*kanji*), *hiragana*, and *katakana*. *Kanji* are ideograms, whereas the rest are phonograms.

Many *kanji* have two types of pronunciation. For instance, “森 [woods]¹” can be pronounced *mori* or *shin*. The former is called “*kun-yomi*”; its origin is derived from the pronunciation of the indigenous Japanese language (*Yamato-kotoba*) before it was influenced by Chinese. On the other hand, the latter is called “*on-yomi*”; it originates from Chinese pronunciation. For example, “森林 [woods, grove]” is pronounced “*shin-rin*” in modern Japanese and almost the same, “*sēnlín*,” in modern Chinese.

The pronunciations of words of Chinese origin exhibit certain characteristics and tendencies. For this reason, Japanese native speakers can intuitively identify the “*on-yomi*” of words they see or hear for the first time, having the impression that they are formal words. In particular, many nouns with two *kanji* characters, such as the aforementioned “森林,” are pronounced phonetically and are often translated from other language (of course, there are not a few exceptions). The words that are “*on-yomi*” were originally unique to Chinese civilization. The Japanese language is characterized by the use of Chinese characters to represent these words and by retaining the

pronunciation of Chinese-derived words, thereby preserving their heterogeneity while incorporating them into the Japanese language (Yanabu, 1976, 2009). Therefore, many formal words in Japanese, such as academic and legal terms, are written in Chinese characters (often in two letters).

Based on this background, the incorporation of various elements of Western civilization, including mathematics, was promoted at a rapid pace from around 1860 due to various historical circumstances. At that time, many words from Western civilization, including mathematics (education), did not exist in Japanese. For example, the word “liberty” was proposed in several translations, including “自主 [oneself, main]” and “自在 [oneself, exit],” and finally settled on the translation “自由 [oneself, wherefore]” (Yanabu, 1976, p. 111). Similarly, many words in mathematics have been translated into Chinese characters, such as “加法 [add, method]” for addition, “証明 [sign, bright]” for proof, “面積 [surface, volume]” for area, and so on.

THEORETICAL FRAMEWORK: CASSETTE EFFECT

Nida’s translation theory

In general, when translating a word, the source word may not correspond to the target language. For example, the English word “water” means almost the same thing as “水 [water]” in Japanese, “*(l’eau)*” in French, and “*(el) agua*” in Spanish. Therefore, these translations are not difficult. On the other hand, as mentioned above, the English word “liberty” had no counterpart when it was introduced to Japan. In such cases, it is necessary to translate the word before translation, taking into account the cultural context of the source language so that the translation matches the language and cultural context of the target language. This translation is called a translation with dynamic equivalence by Nida (1964, p. 232), as illustrated in Fig. 2 (the original figure shows translations between three languages, but Yanabu (1976) only extracts translations between two languages).

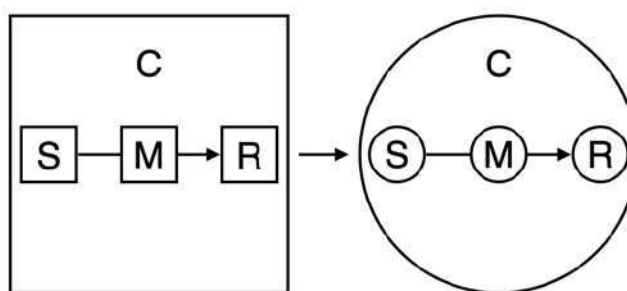


Figure 2: Translation with dynamic equivalence

(Yanabu, 1976, p. 34; based on Nida, 1964, p. 147)

In Fig. 2, S, R, and M represent the source, receptor, and message, respectively. C, with a large square or circle, represents the cultural context in which the message is transmitted. In Fig. 2, the translator is both square R and circle S. Changing square M to circle M, which can convey meaning in a different circle-context C, is a translation

with dynamic equivalence. For instance, in the case of the aforementioned translation from liberty to “自由,” the square C represents Western culture and the round C represents Japanese culture. The square C, “liberty,” is translated into the round C, “自由” in Nida’s (1964) interpretation.

Translated word with the cassette effect

By contrast, Yanabu (1976) points out cases in which a translation, such as Fig.1, does not occur. As mentioned above, “liberty” is usually translated as “自由”; this word existed before the word “liberty” was introduced to Japan. For example, the following description is found in *Tsurezuregusa* (Essays in idleness in English), a famous Japanese essay written around 1330; “この僧都...よろづ自由にして、おほかた人に従ふといふことなし (Kenko, 1995, passage.60; underlined by author)”; in translation, it is “However, he was an eccentric who placed little value on the world, went his own way in all things and never deferred to others (Mckinney, 2014, p. 42).” Clearly, “自由” is a word with a negative connotation, similar to “selfish” in English. Such translation is shown in Fig. 3.

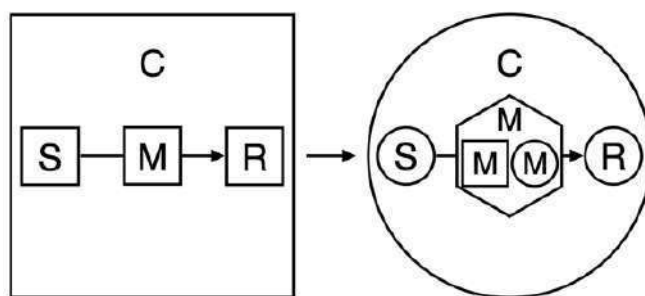


Figure 3: Translation with the cassette effect

The words chosen for translation were rarely words with no meaning. Thus, the chosen words contained their meaning before translation (M in a square) and as they existed in the target culture (M in a circle). However, at the same time, it functions as a word that is neither of these. These words are called translated words with the *cassette* effect (hereafter CW; Yanabu, 2009, p. 14). The term *cassette* means a jewel box in French. A jewel box, even if its content is unknown or invisible, is accepted as having something good inside. Accordingly, CW becomes a “jewel box” that contains (unknown and/or invisible) original and translated words, and the value is assigned to the word masking its meaning. The cassette effect masks the inadequacy or inconsistency of the meaning of a translated word and makes it behave as if it were sufficiently meaningful and valuable. CWs are considered to have a contextually valuable (positive or negative) meaning, rather than simply being words whose meanings are unintelligible. Clearly, Yanabu (1976) assumes that translated words are perceived as having value. The validity of this assumption is supported by the modality of using *kanji* in Japan (Yanabu, 2009) and, in our opinion, the fact that Latin words (without translation) such as “*cogito, ergo sum*” are still used in Western civilization.

Thus, CW is given a (tentative) meaning consistent with context. However, because it is not truly consistent with the context, the meaning of CW is updated, and at the same time, the context in which CW is placed is updated (Yanabu, 1976, p. 127). Such interaction continues not only at the moment the newly introduced word is translated but it continues because the word is still heterogeneous (i.e., it is different from both ○ and □ in Fig. 3). Taking “liberty” as an example, the negative word “自由” appears in a positive context; thus, there is a positive change in the value of “自由” and a new meaning different from “liberty” in the original meaning. Consequently, “In some way, the cassette effect is useful for the acceptance of foreign languages, cultural elements and objects. It has helped the Japanese to accept foreignness even if they do not understand it (Yanabu, 2009, p. 19).”

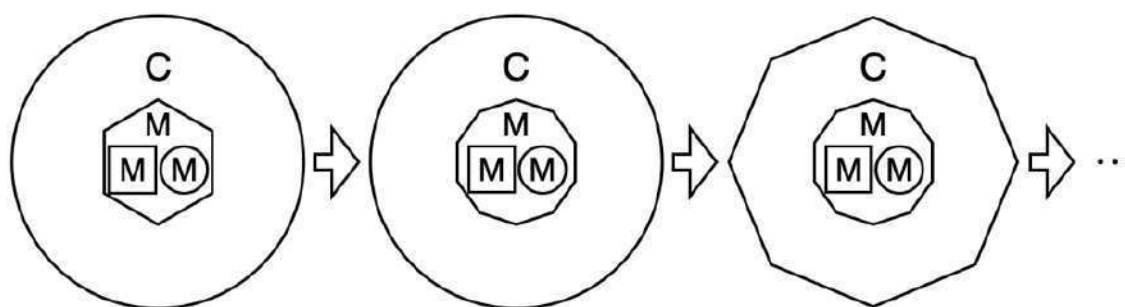
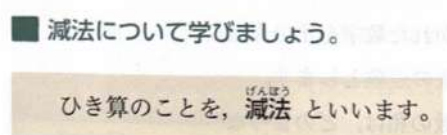


Figure 4: Examples of interaction between CW and contexts

DISCUSSION

Now, let us consider Fig. 1 again based on the notion of CW. In the Japanese curriculum, elementary school students learn natural numbers (including 0), (decimal) fractions in the positive range (rational number concept is not studied) and their four arithmetic operations. For students at this stage, addition is learned in relation to quantities, such as “increase,” and named “たし算.” Similarly, subtraction is learned in relation to quantity, such as “decreasing,” and named “ひき算.” On the other hand, in the first grade of junior high school (7th grade), negative numbers are learned. Students learn negative numbers smaller than 0. As shown in Fig. 1, students learn the addition of negative numbers; immediately after it, it is declared that “たし算” is “加法 [add, method].” After learning negative number addition, students learn about the subtraction of negative numbers. At the beginning, without any preamble, it is declared that “ひき算” is “減法 [reduce, method].”



Our translation

Let's learn about 減法 [subtraction].

We say subtraction is subtraction.

Figure 5: First part of negative number's subtraction (Okamoto et. al., 2020, p. 26)

After learning about the subtraction of negative numbers, students learn that all expressions that are a mixture of addition and subtraction can be expressed by addition only. Here, the intention is to connect arithmetic addition and subtraction to those in algebra (cf. Selter et. al., 2011; Greer, 2011, p. 433).

The features of the above process are as follows:

- i) There is no explanation of the reason for changing the names of addition and subtraction (especially, introducing “減法”);
- ii) The declaration of renaming is placed at the beginning of content learning;
- iii) The difference between addition and subtraction, which is learned before elementary school, is not clearly explained.

Here, the pronunciation of “加法” is “*ka-hou*” and that of “減法” is “*gen-pou*,” and all the pronunciations are “*on-yomi*” (Chinese-derived reading). On the other hand, the terms addition and subtraction in elementary schools are read in “*kun-yomi*” except for the “算.” That is, “加法” and “減法” are CW. In fact, when addition is mentioned in daily communication in Japan, “加法” is rarely used. In the author’s personal experience, the same is true in everyday life, even in communication among mathematicians. Thus, these CWs are understood by the learner as valuable and different; however, their concrete values are unknown and/or invisible. By introducing CW, students could easily separate the meanings of addition and subtraction from the existing ones. As shown in Fig. 4, they then learn algebraic addition and subtraction (cf. Selter, et. al., 2011) through the interaction between CW and the context in which it is used. This learning method is unique to cultures characterized by translated languages and words. Whether separation by the CW has a positive or negative effect was not the subject of our study. However, in the author's opinion, this probably has a positive effect on learning.

CONCLUDING REMARKS

In this study, we focus on the notion of CW in non-Western cultures, using Japan as a case study, and describe the learning process of addition and subtraction based on CW. Although this is a single case study, similar phenomena have been observed in various areas of mathematics education in Japan. These are conditions and constraints peculiar to the linguistic civilization that deals with translated languages and words. A subsequent task is to determine other effects of CW. Readers may pay attention to the fact that, in many civilizations, translations are performed using so-called substitute characters (that is, transliterations); in Japan, translations using *katakana*, a phonetic character, can also be found in mathematics. According to Yanabu (1976), these are also CW; however, this is a future task. It is our limitation that we do not distinguish between languages with different roots, such as English and French, in Europe, and take them as “Western civilization.”

Another future task is to examine the impact of the cassette effect outside of Japan. According to Yanabu (2009), the cassette effect is also observed in languages other than Japanese. As many countries in Asia and other regions have accepted mathematics as a translated language, CW is probably used in many countries. Moreover, CWs may be found in Western mathematics as well. For example, the words “QED” and “algebra” are translated words in Western mathematics. However, this influence may be more pronounced in Asian countries, where language and cultural structures are completely different.

^{1 2} Hereafter, the meanings of *kanji* are referred to based on the web dictionary *tangorin.com* (Preston, n.d.) in principle so that English speakers can easily access them, and the meanings of individual *kanji* are given in brackets []. As *kanji* has multiple meanings and the first meaning is not always used, the author chose the most appropriate one on *tangorin.com* based on the context before and after. On *tangorin.com*, “*kun-yomi*” is represented by small alphabet letters and “*on-yomi*” by capital letters.

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TRACKING TEACHERS' LEARNING TO SELECT STUDENT WORK FOR CLASSROOM DISCUSSION IN A SIMULATION

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We examine data from a simulation of teaching where teachers noticed similarities and differences among pieces of student work done in a problem-based lesson where they are selecting work to be discussed. In this methodological contribution, we illustrate the articulation of multidimensional scaling and Mantel correlations to examine the presence of considerations of normativity and serviceability—categories of perception that matter for a teacher's management of problem-based instruction.

SIMULATIONS FOR NOTICING ASPECTS OF STUDENT WORK

The work of practice-based mathematics teacher development can benefit from the increasing ease to develop virtual teaching simulations (Mikeska et al., 2021). In this context, the question arises of how to use the traces teachers leave in the technology (i.e., click data) to track learning through participation in simulation activities (Nickl et al., 2024). As many big data problems, this requires cobbling techniques that have been used in diverse research fields. This paper describes and illustrates a methodological approach for examining teachers' performance in one of a series of simulations designed to teach skills needed to manage problem-based, discussion-engaged lessons: Specifically, a simulation designed to support teachers learning to notice aspects in student work (SW) important to bring up in classroom discussion.

Stein et al. (2008) highlighted the importance of selecting students' work on a problem in preparation for a classroom discussion about the problem. This selection is key in a teacher's capacity to steer the discussion toward the instructional goal of a lesson. Stein et al. (2008) suggested that in selecting SW teachers need to notice its mathematical sophistication. Scholars who work on noticing and on discussions have been interested in better understanding what goes into selecting students' work for discussion (Ayalon & Rubel, 2022; Dunning, 2023). In the context of practical rationality theory, Herbst et al. (2023) named four categories of perception that identify different aspects of sophistication in students' work on a problem whose recognition might impact whether teachers select it for classroom discussion. They include (1) SW's correctness; (2) SW's responsiveness to the problem; (3) SW's normativity, or whether the SW abides by the norms of the instructional situation used to frame the problem (e.g., the norms of construction if it is a construction problem); and (4) SW's serviceability, or whether the SW stages ideas that advance the work toward the instructional goal.

We designed a set of simulations based on the tangent circle problem ("Suppose that you have two lines that intersect each other and two distinct points, one on each line.

How can you find a circle that is tangent to those lines at those points?”) which is posed in a high school geometry class the day after introducing tangents to circles and the theorem that characterizes tangents as perpendicular to the radius at the point of tangency. The problem is meant to lead to the tangent segments’ theorem (i.e., the points of tangency are equidistant from the intersection of the tangent lines). The simulations include what Beaubien and Baker (2004) call *full-flight* simulations (i.e., opportunities to teach the full lesson virtually) as well as *part-task* simulations (i.e., opportunities to practice selected aspects of the lesson, such as observing and selecting SW). Schwarts et al. (2024) showed that in comparing participants’ performance in initial and final full-flight simulations, one indicator of improved performance was participants’ increased selection of SW which was serviceable for the goal of the lesson, while attention to the normativity of SW remained unchanged. This raises the question of whether performance in the part-task simulation “Anticipating Student Work,” in which teachers learn to observe and select SW, could help trace how such recognition developed. This simulation situates participants at the time after they presented the problem to the class, asks them to look over students’ shoulders and sort students’ work into bins, and then name the bins for future reference. They do four variations of this task before being asked to select and sequence four pieces of students’ work. These four tasks vary in the number of pieces of student work to sort and the number of bins available: 1) 6 SW in 2 bins; 2) 6 SW in 3-5 bins (participants choose); 3) 18 SW in the same chosen number of bins; 4) 18 SW in up to 8 bins (they choose again). In between sorts, participants see examples of how other people sorted the work. Those examples were not chosen to highlight serviceability or any other of the categories of perception; instead, they relied on pilot data where participants had named their bins after specific mathematical ideas used in SW (e.g., using angle bisector).

Our research question is methodological: Given these activities of sorting students’ work on a problem, what are the data structures, variables of interest, and analytic procedures that can be used to account for the performances that can be observed as a sample of teachers goes through the Anticipating Students’ Work simulation? How could we track the development of awareness of serviceability and normativity?

METHOD

The simulations were completed by 29 practicing high school mathematics teachers with an average of 16 years of total experience teaching mathematics and an average of 11 years of experience teaching the high school geometry course in the US, who were recruited using a national database. Their experience teaching geometry suggests that they are good representatives of the population that could both benefit from this treatment and might have prior knowledge that could also make it challenging for them to recognize serviceable SW in spite of its normativity or correctness status.

To visualize the data collected we first created similarity matrices per activity. For each activity k ($k = 1, 2, 3, 4$), we defined a matrix $S_k = (S_{ij})$, which has as many rows and columns as the number of pieces of SW, $\{W_n\}_{n=1}^m$ involved in that activity ($m = 6$

for $k = 1, 2$ $m = 18$ for $k = 3, 4$). S_{ij} counts how many participants grouped W_i and W_j in the same bin for one activity. The question for us here is how to transform these similarity counts into estimates of how similar the group of participants considered two pieces of student work to be. To address this, we used multidimensional scaling (MDS), a technique used in psychology and marketing research (Valentin et al., 2018), which provides a continuous distance metric. MDS represents pairwise dissimilarities between pieces of student work in a reduced-dimensional space. Consider the fourth sorting activity ($k = 4$) where there are 18 pieces of student work. The process begins with a similarity matrix, where each cell, S_{ij} , is defined as above. The matrix is normalized by dividing cell counts by the total number of participants. The similarity values are transformed into dissimilarities by subtracting each similarity value from 1. The MDS algorithm then maps these dissimilarities into a reduced-dimensional space (a 2D space here), where the relative distances between points reflect the original similarity (or dissimilarity; Geron, 2022) between pieces of SW. The distances between points in a MDS visualization (see Figure 1) correspond to abstract dimensions that optimize the mapping of these dissimilarities into a 2D space, where the axes don't have a physical or semantic interpretation. This transformation allows MDS to capture the relative similarities between student work pieces, with closer points indicating higher similarity and farther points indicating greater dissimilarity (Hout et al., 2013).

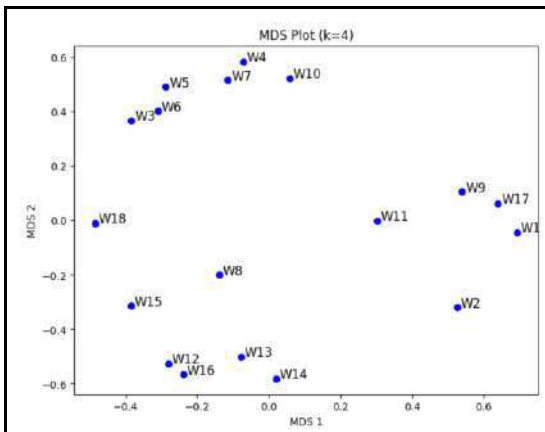


Figure 1. MDS Visualization of Similarity Among 18 Pieces of Student Work from the Fourth Sorting Task

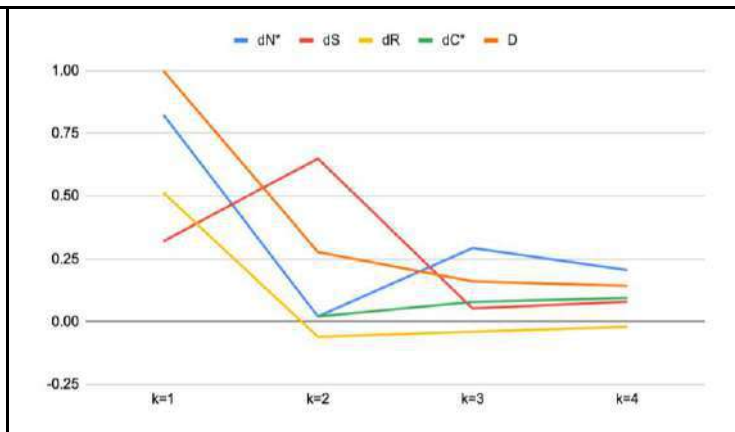


Figure 2. Correlations between MDS and D, and MDS and the four discrete metrics for each of the four activities.*

*: dN and dC correlations overlap at $k=1, 2$

As the research questions have to do with the extent to which the sorting activity plays a role in participants' increasing their recognition of the serviceability of SW, it seemed important to compare the distances found empirically via MDS with distances that could be predicted using the apriori coding of each piece of SW (which had been done by consensus for the design of the simulation). Thus, we created four unidimensional discrete distances, one for each of the four categories of perception (Correctness, Normativity, Serviceability, Responsiveness). The distance dM for each dimension M ,

was defined as $dM(W_i, W_j) = 0$ if W_i and W_j apriori share the property described by M (e.g., if M is serviceability, whether W_i and W_j had been coded either both serviceable or both not serviceable) and $dM(W_i, W_j) = 1$ if they don't share the property M (e.g., if M is normativity, either W_i is normative and W_j is not, or W_j is normative and W_i is not). Thus, each dM is a discrete metric represented by a binary matrix of 0s and 1s, for each dimension. Finally, considering that each W_j could be represented as a vector (C_j, N_j, S_j, R_j) , using all four dimensions to indicate whether W_j has (1) or does not have (0) each property, we defined a continuous metric D as the Euclidean distance between the vectors representing each piece of student work. In metric D , the distance between two pieces of student work is the length of the straight-line distance between their corresponding vectors in the 4-dimensional space.

The questions that arise are how similar these theoretical distances are to the empirical distance determined by MDS: To what extent could we say that the way participants sorted the pieces of students' work resembles the sorting that could be done by considering the dimensions in question? We are particularly interested in how well normativity or serviceability may account for how participants sorted the students' work in the different activities. Thus, we looked for how to compare distances and found the Mantel correlation, of use in ecology and information science, as a suitable measure for this purpose (Schneider & Borlund, 2007). The Mantel correlation is a nonparametric test that calculates the similarity between two distance matrices X and Y defined over the same set of objects W_i by comparing the Pearson correlation between (X_{ij}) and (Y_{ij}) or observed r , against a sample of Pearson correlations $\{r_z\}$ that can be obtained between (X_{ij}) and a set of matrices $\{(Y_{uv})_z, z:1,...,1000\}$ where each $(Y_{uv})_z$ is generated by permutations of rows and columns of (Y_{ij}) .

Mantel correlations can be calculated to compare the dissimilarity distance generated by MDS with the continuous metric D and any of the discrete dM metrics (e.g., serviceability or normativity) for each activity. The variability over time (i.e., across activities) in the Mantel correlations with D could be seen as a series of four background Mantel correlations describing how D accounts for the variability over time that could be a proxy for learning. Similarly, a Mantel Normativity correlation (viz., a Mantel Serviceability correlation) for each activity is defined as the Mantel correlation between the MDS distance and dN (viz., dS), obtaining two other series of four Mantel correlations. Figure 2 offers plots of all Mantel correlations across activities. Ideally, if the activities supported increasing noticing of serviceability, we should see the Mantel Serviceability correlation increasing over time. Also, we would expect attention to normativity would not increase, which would be visible by a Mantel Normativity correlation being stable or decreasing.

While the Mantel correlation is useful for comparing distance matrices, there are key limitations to consider. One limitation is the small number of objects involved in the distance calculations, which can reduce statistical power and the sensitivity of the test. Additionally, comparing matrices with different structures (e.g., binary vs. continuous)

can affect the robustness of the correlation, as the assumptions underlying the test may not be fully met (Legendre & Fortin, 2010). The sample size also impacts the generalizability of findings, with smaller sample sizes potentially resulting in higher variability and p-values, making it difficult to discern meaningful relationships (Anderson & Walsh, 2013).

RESULTS

We computed Mantel's r correlations to compare the empirically found distances generated from MDS with those derived from the four dimensions: normativity, serviceability, and the continuous metric, across all four activities. Table 1 specifically reports the Mantel's r correlations between MDS and the normativity (dN), serviceability (dS), and background (D) metric dimensions across the four activities. These correlations were calculated separately for each activity to examine how normativity and serviceability contribute to explaining the variation in the background correlations over time.

k	Comparison	Mantel's r	p -value	k	Comparison	Mantel's r	p -value
1	MDS vs. dN	0.825	0.030*	3	MDS vs. dN	0.294	0.005*
	MDS vs. dS	0.321	0.141		MDS vs. dS	0.053	0.515
	MDS vs. D	1	0.009*		MDS vs. D	0.161	0.032*
2	MDS vs. dN	0.021	0.92	4	MDS vs. dN	0.206	0.022*
	MDS vs. dS	0.649	0.048*		MDS vs. dS	0.08	0.245
	MDS vs. D	0.278	0.195		MDS vs. D	0.143	0.046*

* p -value < 0.05

Table 1. Mantel's r Correlations Between MDS and Normativity (dN), Serviceability (dS), and Background (D) Metrics across Four Activities

Figure 2 presents a plot of the Mantel's r coefficients across the four activities. Table 1 offers the numerical data that fed the plots for the correlations between MDS and dN (Normativity), dS (Serviceability), and background D across four activities. We note that a few of these coefficients are significantly different from 0. Some conjectures are proposed below to interpret these figures and provide feedback for the next iteration of simulation design. However, the more important observation to make is that the combination of applying multidimensional scaling and Mantel correlations allows us to obtain indicators that can track the progress of teachers' attention to the serviceability of students work in the simulation, based exclusively on click data.

DISCUSSION

The foregoing shows how the data could be examined to track whether any of the activities done when sorting students' work could have impacted recognition of serviceability. We see no evidence that as activities progressed, the distance D could

account for the change in participants' noticing. Rather, correlations between the theoretical distance D and the MDS distance decreased monotonically over the four tasks. This raises questions for us regarding the four categories of perception that form the distance D , which uses the Euclidean norm with all four categories assumed to be linearly independent and of equal weights. We conjecture that if any learning is happening as participants go through these activities, such way of modelling D does not account for such learning. By design, the sorting tasks included more bins and more pieces of work over time to challenge the efficacy of correctness as the sole way of differentiating students' work—we had hypothesized that correctness would be initially strong. Indeed, the dC plot in Figure 2 shows that considerations of correctness were very strong in Task 1, and became much smaller after, even when 12 more pieces of SW could elicit some increase in Task 3. Along those lines, we suspect that alternative models for calculating D might better explain how the four dimensions of interest explain the background learning that took place as participants went through the four activities considering that some distinctions (correctness, normativity) might be more available to participants initially than others, namely serviceability.

The distances dN and dS are ones that we were especially interested in tracking given the results offered by Schwarts et al. (2024). The dN plot shows lack of monotonicity across the activities. Though normativity decreases when more bins are made available, that is, between tasks 1 and 2 and between tasks 3 and 4, as more pieces of student work are included (i.e., between tasks 2 and 3), participants increased consideration of normativity. The notion that as more bins were made available, participants decreased considerations of normativity is auspicious: As participants were the ones requesting more bins potentially to make more distinctions among the same pieces of SW that could not be made with the bins originally available, it seems that those new distinctions did not prioritize normativity more than participants originally had. The two initial measures of attention to normativity expressed in the correlation of dN and the empirical distance for tasks 1 and 3 apply to sets of SW of different size, however, with the second corresponding to a much larger set of SW. The data tells us that this correlation was higher with fewer pieces of students' work but as more students' work was available, the participants attended less to normativity. Thus we argue that the lack of monotonicity across activities is less important; the important observations to make are between Tasks 1 and 2, Tasks 1 and 3, and Tasks 3 and 4. In all of them, the correlations between dN and the MDS distance decrease, suggesting that Normativity becomes less important a consideration for the participants as time goes on, though important enough for it to show an effect when more pieces of student work are added.

As regards the correlations between dS and the MDS distance we also observe lack of monotonicity, with this correlation increasing between tasks 1 and 2 and slightly increasing between tasks 3 and 4. This suggests that as more bins were available, the new distinctions and resemblances participants could make were more informed by serviceability considerations. There is a decrease in the correlation between these two distances from task 2 to task 3, where the number of bins is the same and the number

of pieces of student work increases and this correlation is even smaller than the initial one in Task 1. This would suggest that the organization of an increased variety of pieces of student work in Task 3 may not have been associated with increased considerations of serviceability, though having more bins was.

A possible explanation for the lack of consideration of serviceability could be in the nature of the feedback provided after each sorting. As described by Schwartz et al. (2024), one decision we made in the design of the simulation was to add as feedback examples of how other educators had sorted the pieces of student work, including naming the bins into which they had sorted pieces of work. Our design decision was not to make the feedback didactic, hence we did not infuse an explicit consideration of serviceability or of any of the other categories of perception (responsiveness, normativity, or correctness) with that feedback. Instead, the feedback provided focused on strategies students appeared to have used (e.g., perpendiculars vs. angle bisectors), as can be inferred from how prior participants had named the bins—this identification of strategies might be described as noticing students’ mathematical thinking without much consideration of the teacher’s agenda for the lesson. Furthermore, everybody received the same feedback. It is possible that what we surmised not to emphasize any of those categories of perception under consideration was still perceived as didactic by the participants who surmised they should be focusing on student strategies, which dampened the impact that serviceability could have had when more diverse work is shown. This suggests that in a future design cycles, we should consider and compare various feedback options.

CONCLUSION

The analytic toolset for representing and examining data collected in simulations of mathematics teaching is in its beginnings and can benefit from a variety of techniques available in diverse fields. Theoretical concepts homegrown in mathematics education, such as normativity and serviceability, are needed to help direct the search for methods.

Additional information

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WHICH CATEGORIES DO PROSPECTIVE TEACHERS APPLY WHEN SELECTING TASKS FOR THE SCALAR PRODUCT?

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Selecting tasks is a typical job in everyday teaching that requires teachers to have content-specific expertise, including categories to think about and perceive conceptual understanding. This contribution investigates the activation of prospective teachers' categories when solving tasks on the scalar product and evaluating their relevance for teaching in upper secondary school. To further investigate, whether input on content-related aspects could be helpful to promote a professional justification for the selection of tasks, two groups with and without input are to be differentiated. Different ways of dealing with underlying concept elements and evaluating the relevance of tasks are discussed by examples.

INTRODUCTION AND THEORETICAL BACKGROUND

To select tasks for lessons professionally, teachers acquire content-specific expertise, which can be specified as “the personal capacity to cope with situational demands called jobs” (Prediger, 2019, p. 371). Content-specific categories are part of this expertise and are defined as “conceptual [...] knowledge elements that filter and focus the categorical perception and the thinking of the teachers” (Prediger, 2019, p. 370). In the presented study, the job of selecting tasks for initiating development of conceptual understanding of the scalar product is to be examined. The scalar product is a content of vectoral analytical geometry in German upper secondary school.

Aspects teachers focus on while considering a task can regard different elements of instruction like the content or the goal of the task, or more specific information about the lesson context (González et al., 2020). As various studies such as PISA and TIMSS show that conceptual understanding is often neglected in Germany, the principle of conceptual focus, which underlines the understanding of both concepts and procedures (Prediger et al., 2022), is placed at the centre here. In the following, prospective teachers' categories relating to a conceptual focus are specified to describe how they select tasks for teaching the scalar product. An example of using categories to investigate in content-specific jobs can be found in Prediger and Zindel (2017).

Categories referring to conceptual understanding of the scalar product

From the perspective of cognitive theory, conceptual understanding can be defined as the construction of flexible and transparent structures, which is the theoretical assumption for a model of conceptual understanding according to Drollinger-Vetter

(2011). In this model, *underlying concept elements* form the basis for conceptual understanding and must be explicitly addressed in a comprehensible lesson to enable students to deal with different types of representation. Then, understanding a concept manifests itself in the *unfolding and condensing* of concept elements, representations and the concept itself, which is also linked to other concepts. Fig. 1 shows the model, which needs to be specified in terms of the scalar product (Herrmann, 2025). Elements of the three different layers can then be identified as content-specific categories, marked with $\langle \dots \rangle$.

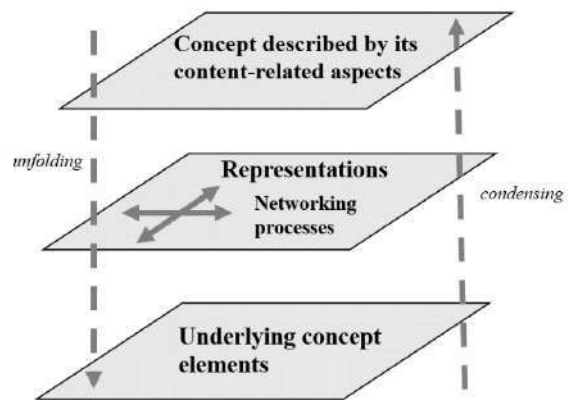


Fig. 1. Model of conceptual understanding based on Drollinger-Vetter (2011, p.190)

An initial point of reference for the didactic reconstruction of mathematical content is provided by **content-related aspects** that can be found at the top layer in Fig. 1. In the case of the scalar product, the following four aspects (Hoffmann et al., in prep.) are found to be relevant to teaching: a) The aspect of $\langle \text{product operation} \rangle$ describes the scalar product as an operation between two vectors. b) The aspect of $\langle \text{monotonicity in angle and length changes} \rangle$ relates to the dependence of the scalar product on elementary geometric quantities. c) The aspect of $\langle \text{static angle} \rangle$ is based on qualitative and quantitative statements about the angles involved. Finally, d) the aspect of $\langle \text{equal scalar products} \rangle$ makes a geometric statement about vectors with equal scalar products.

To make the scalar product understandable by having those aspects in mind, content-specific representations are required, which can be found on the middle layer in Fig. 1. In the area of vectorial analytical geometry, geometrical and algebraical representations are usually distinguished, whereby both can be used with or without vectors. This results in four **forms of representations**: $\langle \text{elementary geometry} \rangle$, $\langle \text{arrow geometry} \rangle$, $\langle \text{elementary algebra} \rangle$ and $\langle \text{vector algebra} \rangle$ (Herrmann, et al., in prep.). These representations can be networked with each other through various processes. For example, to convert a geometric representation into an algebraic representation, an $\langle \text{algebraisation} \rangle$ is required, which is a result of applying “the power of algebra to geometry” (Dorier, 2000, p.13). Other kinds of those **networking processes** in the field of analytical geometry are $\langle \text{geometrisation} \rangle$, as well as $\langle \text{vectorisation} \rangle$ and $\langle \text{elementary interpretation} \rangle$ (Herrmann, et al., in prep.).

Unfolding these representations, content-related **underlying concept elements** can be identified and categorised in the bottom layer in Fig. 1. Examples for these elements, which relate to different kinds of representations, are *angles*, *lengths*, *arrows*, *projections*, *cosinus* and *n-tuples*. While *angles* and *lengths* are an example of $\langle \text{elementary geometric elements} \rangle$, *arrows* can be described as $\langle \text{arrow geometrical} \rangle$

elements>, trigonometric concepts such as *cosinus* as <*elementary algebraic elements*> and *n-tuples* as <*vector algebraic elements*>.

Insightful handling of categories

To successfully condense and unfold the category elements on the three different layers, it is necessary to deal with them in such a way that they have a meaning for the concept of the scalar product addressed in the tasks. According to Lotz (2021), there is a difference between a naïve and an insightful approach to these elements. For example, an angle between two vectors could be determined naively by looking up a formula, plugging in the known variables and calculating the magnitude of the angle. An *insightful handling* of underlying concept elements, representations or content-related aspects is characterised by the fact that a person sees it, what they are supposed to see in it in relation to the concept of the scalar product (Lotz, 2022, p. 199).

RESEARCH GAP AND RESEARCH QUESTIONS

Focussing on the selection of tasks as a typical job for teaching mathematics, it is known that task-centric approaches improve teachers' ability to select cognitively challenging tasks (Boston & Smith, 2011). It is not yet clear whether an input on content-related aspects also makes a difference to the categories teachers activate. Regarding teachers' expertise, Prediger and Zindel (2017) found out that an intervention on certain categories has an influence on how teachers deal with content-specific jobs, but no such study had been carried out for the selection of tasks on the scalar product.

This contribution therefore examines the extent to which prospective teachers activate content-specific categories for conceptual understanding when selecting tasks on the scalar product. Therefore, a deeper look is taken at how prospective teachers solve tasks themselves and discuss their relevance for students afterwards.

(RQ1) Which content-specific categories can be reconstructed when prospective teachers solve tasks and assess their relevance for teaching the scalar product depending on whether they have received input on certain categories.

(RQ2) How do prospective teachers justify the selection of tasks for teaching the scalar product depending on whether they have received input on certain categories?

Based on Prediger and Zindel (2017, p. 235) it can be conjectured that the categories mentioned in the input are activated more often than it would be the case without input.

METHODS

Data collection: The data on which this study is based was collected in two classes for prospective secondary teachers at Paderborn University. At the beginning of the course, participants of one seminar received a 15-minute input on the scalar product (*intervention group*), while those in the other seminar did not (*control group*). The input is located in the top and middle layer of the model of conceptual understanding, as the content-related aspects (<*product operation*>, <*monotonicity in angle and*

length changes>, <*static angle*> and <*equal scalar products*>) are given. To visualise and describe the aspects, different representations are used, e.g. a geometric representation of arrows with equal scalar products (Fig 2.). Underlying concept elements and general didactical principles haven't been addressed explicitly. Then, both classes were given 30 minutes time to solve the three different tasks and evaluate their relevance for teaching the scalar product: The first task deals with the aspect of orthogonality. The aspect of equal scalar products is addressed in task 2 by drawing different arrows with the same scalar product in relation to \vec{v} . In the third task, a three-dimensional angle must be computed using the scalar product as a measure of geometric quantities. All three task can be found here: go.upb.de/tasks

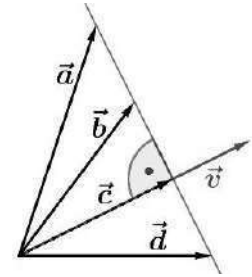


Fig. 2. Geometric representation of equal scalar products

Sample: Data was collected from a total of $n=26$ participants, which had been divided into six break-out groups (each 3 groups with and without intervention), consisting of 3-6 participants. A total of 180 minutes of video material was produced and completely transcribed. The written results are also available. All of the participants study in a master's programme to become mathematics teachers for secondary school. Most of them are in their third or fourth master's semester and have already completed most of the courses as well as a six-month school practical training. Overall, it is assumed that the participants have the same formal prerequisites. The courses have been put together at random so that natural heterogeneity is created.

Data analysis: As part of an evaluative qualitative content analysis (Mayring, 2014), the content-specific categories of the model of conceptual understanding were identified and coded in the transcripts in three steps. In a first step, all statements in the transcripts were categorised according to whether they relate to the solution of the task or to the evaluation of relevance. Then, the above-mentioned categories were coded in relation to both the task solution and the evaluation. A compilation and comparison of all coded elements shows that a further differentiation must be made in the coding system regarding *how* the prospective teachers deal with the various coded elements. Therefore, all coded elements were analysed in a third step to determine whether they had been used insightfully. The result was a document in which only the category elements that could be identified as insightful were coded. All elements that were used naively or could not be clearly assigned were omitted from the second coding cycle. The inter-coder reliability of the coded elements has been ensured by an open discussion. About 5% of the coded material of the first coder have been discussed with different second coders (Mayring, 2014, p.114). By comparing the analysed data, both quantitative and qualitative phenomena can be outlined descriptively.

RESULTS

While participants of the intervention group spent around 59% of their 30 minutes time on solving the tasks, the average in the control group was 72%. The greatest difference

in time was found when working on task 2, where arrows with the same scalar product in relation to \vec{v} had to be found. The following analyses provide a more detailed insight into the prospective teachers' solution and assessment processes for task 2 and contrast participants in the intervention and control group.

Insightfully activated categories when solving and assessing tasks

The following transcript excerpt comes from participants of the control group and shows a discussion that arises during their process of solving task 2. They tried to find equal scalar products by drawing perpendicular arrows to \vec{v} before but recognised that they can't find three arrows with different directions that fulfil these requirements.

- 1 Anne: Isn't the scalar product the angle between those [arrows]. Or has anything to do with // angles?
- 2 Mick: It's an area.
- 3 Theo: If it's orthogonal, it is 0.
- 4 Mick: Yes, but let's think about it – three different directions
- 5 Anne: Then we can make it with // three different angles.

<Elementary geometric elements> like *angle* and *area* and <arrow geometric elements> like *direction* can be identified. Besides, Anne deals with the content-related aspect of <orthogonality> to confirm Theo's statement that the scalar product has something to do with angles, at least in special cases. The idea of linking directions of arrows with angles can be described as <vectorisation>. What makes this transcript section remarkable is that none of the coded categories is used *insightfully* to solve the given task. As far as the model of conceptual understanding is concerned, the group isn't able to condense underlying conceptual elements into meaningful representations.

Since the intervention group previously got to know a geometrical representation of equal scalar products (fig. 2), all break-out groups were able to reconstruct the sketch in relation to the arrow \vec{v} and evaluate it as the correct answer. The following excerpt shows a discussion of such a break-out group on how students might deal with the task.

- 1 Lea: I think a student can only solve the task, if this sketch is known. And then it doesn't make sense, does it?
- 2 Tom: Yes, but if you know that you are projecting orthogonally from one vector onto the other... and then take the length of the part, where the orthogonal projection occurs. So, you could probably work it out, but you need to understand how the scalar product works with the orthogonal projection and not just how to calculate it.

Again, various <geometric elements> can be identified, such as *projection*, *orthogonal* and *length*, but unlike the other group, they are used *insightfully*, as they are meaningfully linked to the given representation. Tom obtains these concept elements from *unfolding* the given representation, which is remarkable, because the concept elements were not explicitly given in the input.

The two transcripts are shown as examples for the failure of *condensing* on the one hand and successful *unfolding* on the other, which turned out to be a difference between the control and intervention group working on task 2. For the control group, the aspect of *<orthogonality>* is the first point of reference for the scalar product, which works for task 1, but does not apply to task 2. Although many more *<networking processes>* have been coded in transcripts of the control group, the fewest of them are used *insightfully* to solve the problem.

Adding up the number of coded underlying concept elements both in relation to task solving and evaluating the relevance of task 2, the control group used 28% *insightfully*, while the intervention group achieved a percentage of 45%. This can be partly explained by the given input, but it is worth mentioning that the underlying concept elements were not explicitly addressed in the input. Table 1 shows the absolute numbers of all coded elements and the number of those used insightfully.

Underlying concept elements	Intervention group		Control group	
	all	insightful	all	insightful
<i><Elementary geometric elements></i>	19	5	22	4
<i><Elementary algebraic elements></i>	2	2	30	14
<i><Arrow geometric elements></i>	28	10	64	13
<i><Vector algebraic elements></i>	1	1	13	5

Table 1: Coded underlying concept elements for task 2

In addition, the data in table 1 shows that the control group used lots of *<elementary algebraic elements>* and *<vector algebraic elements>* when working on task 2. Their strategy to solve the task, coded as an *<algebraisation>*, is mostly to read off the coordinates of the given arrow \vec{v} and try to solve the task algebraically by setting up a system of equations to find vectors with the same scalar product. Although this type of strategy does not require geometric knowledge of equal scalar products, only one out of three break-out control groups succeeded in finding different vectors with the same scalar product. The group that found different vectors algebraically was not even able to specify an *<arrow geometric representation>* of equal scalar products, which is why not a single one of those was coded in the control groups' transcripts. The intervention group, on the other hand, did not need this type of *<algebraisation>*, as they had become familiar with an *<arrow geometric representation>* of equal scalar products in the input.

Prospective teachers' justification of selected tasks for teaching

In the above transcript excerpt, Tom (intervention group) points out that he associates conceptual understanding of the scalar product with projections and distinguishes it from mere computation. For solving task 2, he, therefore, emphasises the need for

conceptual understanding, which he underpins with *<Elementary geometric elements>* and *<arrow geometric elements>*. Anne, Theo, Mia and Mick (control group) noticed the conceptual focus of task 2 as well, but used it on a *surface-level* (Lotz, 2022, p.120):

- 1 Anne: How relevant do you find the tasks? [...]
- 2 Theo: 12 from 10 [points]. All of them are relevant for the final examination.
- 3 Mia: For conceptual understanding, I guess?
- 4 Mick: Task 2 is best for that, isn't it? [...] But you don't understand that the scalar product is an area. That doesn't even happen in school lessons.

It should be emphasised here that, although Mia and Mick are aware of the conceptual focus of task 2, they are not able to work it out in a content-specific way. Mick actually brings in some sort of conceptual understanding that he assumes to be relevant to the scalar product, but he does not notice a connection to the task, like Tom from the intervention group does in the transcript section above. That is why Mia's and Mick's contribution is not rated *insightfully*. Such phenomena can also be observed in other parts of the data, particularly in relation to the other two tasks.

CONCLUSION AND OUTLOOK

To summarise and answer RQ1, the aspect of *<equal scalar products>* with the associated *<arrow geometric representation>* and underlying *<elementary geometric elements>* and *<arrow geometric elements>* could be descriptively reconstructed in solving and evaluation processes of prospective teachers from the intervention group. It was found that prospective teachers of the control group use fewer categories related to the conceptual understanding model in an *insightful* way and that, in addition, there is an emphasis on *<elementary algebraic elements>* and *<vector algebraic elements>* instead. For both solving and evaluating a task for teaching the scalar product, it is noticeable that participants of the intervention group seem to be more successful in *unfolding and condensing* in the sense of the model of conceptual understanding.

About the justification of selected tasks for teaching, the analysis of various transcript excerpts shows that prospective teachers with an input on content-related aspects may be able to describe the potential for conceptual focus more precisely and do not remain on a surface-level, which strengthens Baumert and Kunters' (2006, p. 496) assertion that "content is the basis on which subject-specific didactic flexibility can develop."

This ends up in consequences regarding the content quality of prospective teachers' development programmes. Input on content-related aspects in case of the scalar product could help prospective teachers to unfold underlying conceptual elements that are crucial for teaching, especially for selecting tasks. In some constellations, didactic assumptions can already be described as fruitful for teaching the scalar product due to the activated categories of conceptual understanding. Nevertheless, not all prospective teachers who received input automatically activate didactic categories in a targeted manner. For this reason, professional development programmes that deal with the mathematical content of scalar product must also develop content-related didactic

categories. Further research is needed to specify the inductive categories that prospective teachers activate when selecting tasks, as well as to find deductive didactic categories that are compatible with condensing and unfolding elements related to the model of conceptual understanding for the scalar product.

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SYSTEMATIC ERRORS WHEN INTERPRETING STATISTICAL GRAPHS – A SYSTEMATIC LITERATURE REVIEW

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Interpreting statistical graphs is a crucial component of the 21st-century skillset known as statistical literacy. Despite their importance for statistical literacy, research reports that systematic errors often occur in the interpretation. As no consolidated overview of these errors existed to date, a systematic literature review was conducted. This review identified systematic errors in the interpretation of graphical representations of the distribution of one variable, such as histograms, dotplots, boxplots, and case-value plots. These errors were then categorized into four overarching themes, to provide a structured framework for understanding and addressing them.

INTRODUCTION

In today's world, the ability to analyze and interpret data, including its distribution, is considered a fundamental competence (Engel, 2017). These skills are collectively called statistical literacy (Friedrich et al., 2024). Wallman (1993, p.1) provided an early and influential definition of statistical literacy, describing it as:

“the ability to understand and critically evaluate statistical results that permeate our daily lives—coupled with the ability to appreciate the contributions that statistical thinking can make in public and private, professional and personal decisions.”

Watson and Callingham (2003) proposed a notable hierarchical framework for statistical literacy outlining six levels of statistical understanding. Each level reflects a progression in the depth and complexity of this understanding. One component that spans across multiple levels of this hierarchy is the ability to interpret statistical graphs and their characteristics. The skill of understanding graphical representations of data forms a cornerstone of statistical literacy. Proficiency in this area enables individuals to extract meaningful insights from visual data for informed decision-making in both professional and everyday contexts (Engel, 2017).

Systematic errors when interpreting statistical graphs

Although critical to statistical literacy, the interpretation of statistical graphs frequently presents conceptual difficulties, leading to systematic errors. Systematic errors describe recurring errors in a particular type of task (Cox, 1975). As opposed to (random) mistakes, they are based on concepts and strategic considerations, allowing

persons who make a systematic error to explain how they come to their erroneous solution.

Recent research has examined such systematic errors in the interpretation of the statistical graphs that depict the distribution of a single variable: histogram, boxplot, dotplot, and case-value plot (e.g., Boels et al., 2019b, Lem et al., 2013, Schreiter & Vogel, 2023). However, there is only a limited overview of systematic errors that occur when interpreting these graphical representations, and to our knowledge, no comprehensive synthesis addresses such errors across all four graph types examined. On this basis, we posted these two research questions:

1. What systematic errors are reported in the interpretation of statistical graphs depicting single-variable distributions?
2. How can systematic errors in the interpretation of statistical graphs depicting single-variable distributions be meaningfully categorized?

This research is part of the overarching "Eye-teach stats" project, which aims to design and evaluate innovative learning modules to support teachers in diagnosing and addressing systematic errors in statistical graph interpretation. Within this broader aim, this study focuses on providing a comprehensive synthesis of systematic errors in the interpretation of statistical graphs. By identifying and categorizing these errors, this study establishes a foundation for developing effective educational interventions as part of the larger project.

METHOD

To address the research questions, we conducted a systematic literature search in December 2023, using equivalent search strings on PsychInfo, ERIC, and Web of Science. The systematic review adhered to the PRISMA 2020 guidelines (Page et al., 2021) for planning, conducting, and reporting the research. As depicted in Figure 1, the search string was developed based on the PICO scheme (Stern et al., 2014)., with terms from the three areas connected by the Boolean operator AND. This wide search string was used to screen titles, abstracts, and keywords.

Population	People in the context of education	learner* or student* or pupil* or grader* or child* or teacher* or educator*
Interest	Statistical misinterpretation	challeng* or difficult* or misconception* or misinterpret* or hurdle* or obstacle* or mistake* or bias* or error* or fail* or overgeneral* or heuristic*
Context	Specific types of graphs	boxplot* or (box near/5 plot*) or dotplot* or (dot near/5 plot*) or histogram* or (line near/5 plot*) or lineplot* or (case near/0 value near/5 plot*) or (case-value near/5 plot*)

Figure 1: PICO scheme for constructing the search string (Stern et al., 2014)

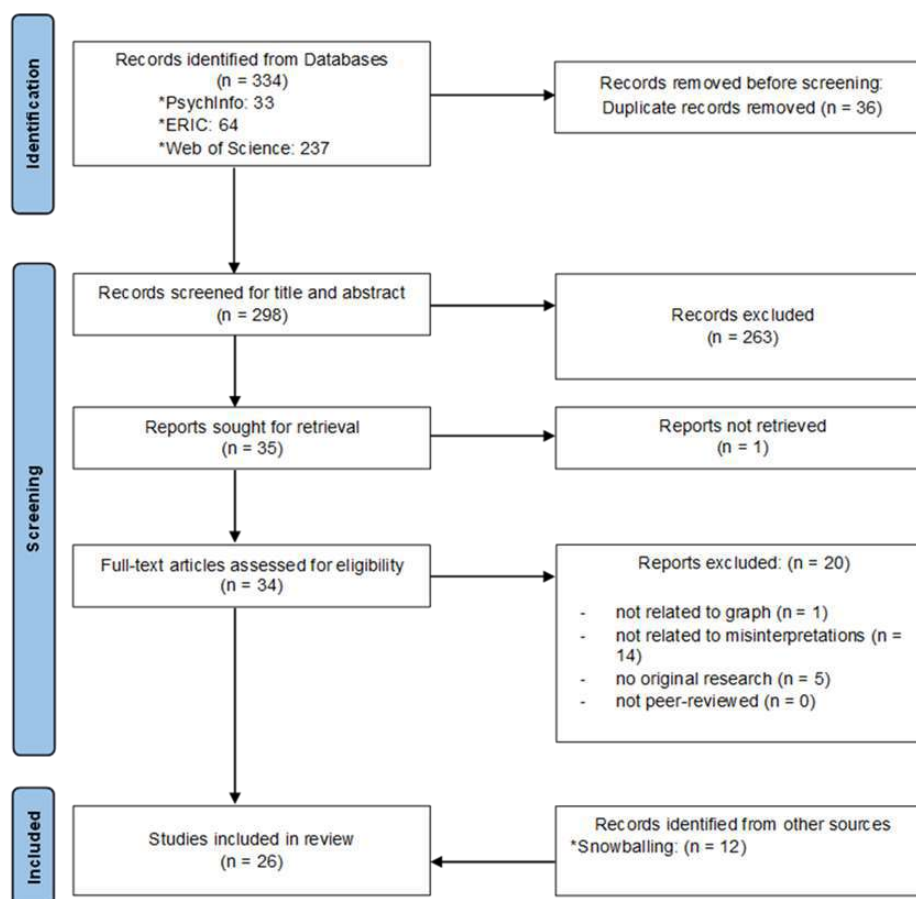


Figure 2: Flowchart for the process of identifying studies included (adapted from Page et al., 2021)

After duplicates were removed, the remaining studies were examined against the following eligibility criteria: (1) English abstract, (2) original research (in particular no review studies), (3) peer-reviewed research, (4) related to graph types, and (5) related to misinterpretation. The title-abstract screening was conducted in duplicate, achieving a high interrater agreement (Cohen's $\kappa = 0.92$).

To broaden the dataset and capture additional high-quality studies, a supplementary snowball search was conducted (Figure 2). The included 26 studies were uploaded into the software MAXQDA (VERBI Software, 2021). To answer the first question, two researchers analysed the studies independently. To answer the second research question, the systematic errors found were uniformly named and summarised into categories through collaboration among all project team members. These results informed the development of a framework, which was subsequently applied to reanalyse all included studies for systematic errors in the interpretation of statistical graphs. This analysis was carried out by two raters; Rater 1 was also involved in the first phase and the development of the framework while Rater 2 was not. The interrater reliability for all studies was very good (Cohen's $\kappa = 0.88$). The studies in which the raters disagreed were presented to a third person—the second rater from the first phase—who made the final decision.

RESULTS

In total, 18 systematic errors were found. They could be grouped into four meaningful categories; *variability shape confusion*, *graph mechanics representation confusion*, *not reasoning about data as a set*, and *looking at a wrong parameter for measures of central tendency*. Due to space limitations, here, only the first category *variability – shape confusion* and its systematic errors are presented. This category includes cases where conclusions about the variability of data are incorrectly drawn based on the perceived shape of the graph.

Eleven of the 26 studies examined report systematic errors in the interpretation of the graphs, which we have summarized under the category of *variability – shape confusion*. This category includes four different systematic errors. An overview of the studies documenting these errors can be found in Table 1. Notably, these errors were observed across all sample groups: secondary school students, university students, teachers, and even experts. Experts, in this context, are defined as people who professionally engage with these types of graphs, such as university professors. For all the different graph types analyzed—histogram, dotplot, boxplot, and case-value plot—systematic errors were reported that fall into this category.

Study	Bar height differences	Flat shape – less variability	Flat shape – more variability	Symmetric shape – less variability
Boels et al., 2019a				X
Chaphalkar, 2014	X	X	X	X
Cooper & Shore, 2008	X	X		
Cooper, 2018	X	X	X	
Dabos, 2014	X	X	X	X
delMas et al., 2007		X		
González, 2021	X			X
Kaplan et al., 2014	X	X		X
Lee & Meletiou-Mavrotheris, 2003	X			
Lem et al., 2013a		X		X
Meletiou-Mavrotheris & Lee, 2005	X	X	X	X

Table 1: overview of variability – shape confusion

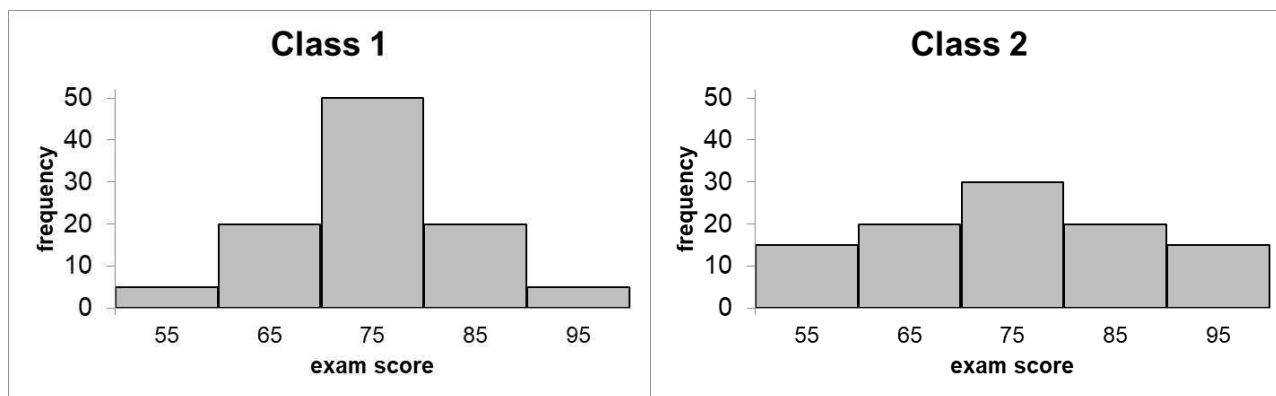


Figure 3: example item for *flat shape – less variability* (adapted from Cooper & Shore, 2008)

The first identified systematic error in interpreting statistical graphs is referred to as *bar height differences*. This error commonly occurs in the analysis of histograms and dotplots, where individuals focus primarily on the height differences between neighboring bars when assessing the variability of data distributions. Thereby the shape of the shown distribution is described as “irregular” or “bumpy” (Kaplan et al., 2014). More variability is often attributed to the bumpier distribution (Meletiou-Mavrotheris & Lee, 2005). This systematic error has been observed across students, teachers, and experts. It occurs in particular when no clearly flat graph is shown and when the span of the bar height is approximately the same, but the heights of the bars change unevenly. This distinguishes the systematic error *bar height differences* from the following systematic error.

The systematic error *flat shape – less variability* is closely tied to the previously discussed error of bar height differences. This error is particularly prominent in the interpretation of histograms, where individuals erroneously associate a flat graph with low variability in the underlying data. Flat graphs are characterized by minimal differences in the heights of their bars. In particular, data that is uniformly distributed is often mistakenly attributed the lowest or even no variability at all (Chaphalkar, 2014; Kaplan et al., 2014). Additionally, data that exhibits a uniform increase or decrease in bar heights is similarly perceived as having less variability, (Meletiou-Mavrotheris & Lee, 2005). Figure 3 shows a possible case in which this error occurs. When asked which class shows more variability in the exam scores, about half of the students surveyed incorrectly answered that the exam scores of Class 1 have more variability than Class 2. In subsequent interviews, interviewees stated that they chose Class 1 because Class 2 shows a flatter graph (Cooper & Shore, 2008). This systematic error is reported among both students and teachers.

On the other hand, the systematic error *flat shape – more variability* represents the reverse of the previous error. While it is similarly based on interpreting the flatness of a graph, this error involves the incorrect assumption that a flat graph signifies greater variability in the data distribution. This interpretation is observed in the context of histograms and case-value plots, where individuals misjudge uniform bar heights or

patterns as indicative of more variability. As with the previous error, this misinterpretation has been documented among students and teachers (Cooper, 2018).

The fourth and final systematic error categorized under *variability – shape confusion* is referred to as *symmetric shape – less variability*. This error stems from the mistaken assumption that symmetrically distributed data inherently exhibits less variability. This systematic error is particularly common when interpreting data that is normally distributed (Dabos, 2014; Kaplan et al., 2014). This systematic error arises when interpreting histograms and boxplots, and it is observed among individuals across various levels of expertise, including students, teachers, and experts.

It is important to emphasize that not every approach—by definition—leads to an incorrect answer. Systematic errors typically arise when a particular idea or strategy is repeatedly applied across different contexts. While correct in some contexts, these strategies are also used in contexts where they do not yield correct solutions. For example, some approaches also reflect strategies that are valid for other graph types. Conceptual difficulties underlying systematic errors can often only be recognized by accompanying comments from the respondents. Furthermore, the review reveals that study designs explicitly focus on certain statistical concepts such as variability, so that this review reveals several systematic errors in this area. The occurrence of systematic errors is closely tied to the respective design of the study. Therefore, this review cannot conclusively determine the frequency of certain systematic errors for a specific graph type. Nevertheless, patterns in the systematic errors can be identified. These patterns enable systematic errors to be categorized, forming a basis for further exploration.

CONCLUSION

The systematic literature review highlights the challenges encountered when interpreting statistical graphs, a crucial component of statistical literacy. It revealed that systematic errors in the interpretation of the statistical graphs – such as histogram, boxplot, dotplot, and case-value plot – are noticeable across different populations and different graph types. A comprehensive analysis led to the identification of 18 systematic errors, which were summarized into four overarching categories. This preliminary presentation focuses on such systematic errors, where the shape of the graph cannot easily be interpreted as the variability of the underlying data.

This systematic literature review serves as a first step towards enhancing the comprehension of statistical graphs, a key component of statistical literacy. Building on these insights, the “Eye-teach stats” project aims to design and evaluate learning modules designed to support teachers. These modules will enable teachers to effectively diagnose and address these systematic errors. The study makes an important contribution to reducing systematic errors among students and teachers while enhancing educators’ mathematical knowledge for teaching statistics.

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EXPLORING TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE FOR TEACHING LENGTH ESTIMATION

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This study was a small scale cross-country comparison of elementary school teachers' pedagogical content knowledge for teaching length estimation. A total of 35 in-service elementary school teachers (22 Taiwanese and 13 German teachers) participated in one-on-one semi-structured interviews. Results showed that the majority of teachers in both groups appreciated the value of the guess-and-check procedures for teaching length estimation. There were differences in the two groups' knowledge of the use of estimation strategies for teaching students to make estimations. No differences existed in the two groups' knowledge of students' estimation skills. Suggestions for improving teacher's knowledge for teaching length estimation are suggested.

INTRODUCTION

Developing students' skills of length estimation has been highlighted in elementary mathematics curriculum guidelines in many countries (Andrews et al., 2022). Length estimation is also an essential subject-specific topic in elementary mathematics curricula in Taiwan and Germany (Ruwisch & Huang, 2018). Hoth et al. (2023) conducted a cross-national study comparing Taiwanese and German elementary school students' skills of estimating length measurements. It revealed that the characteristics of the various estimation situations involving discrepancies in object size and accessibility resulted in differences in the two groups' estimation performance. Hoth's study suggests the value of further cross-country inspections of length estimation instruction in different educational contexts to better understand factors leading to differences in students' length estimation performance.

In the context of teaching mathematics, pedagogical content knowledge (PCK) for teaching is an integrated understanding that bridges subject matter knowledge and teaching practice (Ball et al., 2008). Hill et al. (2008) and Ball et al. (2008) suggested that the content of pedagogical knowledge for teaching includes teachers' habitual ways of thinking about how to teach various mathematics topics and knowing how students learn mathematics. For teaching measurement estimation, the literature suggests conducting "guess-and-check" procedures and using of strategies (Joram, 2003; Joram et al., 2005; Subramaniam, 2014). Generally, teachers should provide guidance for the following estimation approaches: operating unit iterations through touching to-be-estimated objects (TBEOs), constructing various daily measurement

references (benchmarks) by memorizing, and using these benchmarks as mental rulers.

The importance of measurement estimation has been widely acknowledged in mathematics education (Andrews et al., 2022). However, very little is known about teachers' PCK of teaching length estimation or their awareness of students' skills of estimating length measurements. Specifically, the later aspect is part of the knowledge integrating understanding of students' mathematical thinking and knowing about mathematics content of specific topics (Ball et al., 2008). Moreover, it remains unclear whether discrepancies in teachers' PCK of measurement estimation instruction exist in groups of teachers from different educational contexts, which may influence teacher education and pedagogical views on classroom practices (Hoth et al., 2023).

Considering that teachers' mathematics pedagogical knowledge for teaching has influence on students' mathematical learning and achievement (Hill et al., 2005), one purpose of our study was to explore elementary school teachers' PCK of teaching length estimation. This study may contribute to the understanding of teachers' PCK for teaching length estimation, and may enrich understanding of the types of length estimation lessons common in mathematics classroom.

This study focused on common classroom practices for teaching the various length estimation strategies suggested in the previously cited literature. Additionally, differences in the PCK for teaching length estimation of a Taiwanese teacher group (T-group) and a German teacher group (G-group) were examined. Three main research questions were addressed: 1. To what extent do the two groups of teachers conduct guess-and-check procedures for teaching length estimation? 2. What are the differences in the two groups of teachers' knowledge for making use of various estimation strategies in their classroom teaching? This included teachers' knowledge of the uses of various estimation strategies: operation and touching TBEOs, memorizing specific benchmarks, and using benchmarks as mental rulers. 3. What are the differences in the two groups' knowledge of students' estimation skills?

THEORETICAL FRAMEWORK

Length Estimation in Elementary Mathematics Curriculum and Instruction

Length estimation means determining a quantitative value of the length of an object without the use of measurement tools (Joram, 2003). Length estimation with reasonable accuracy in the context of a given problem is a core competence, with applications in mathematics, science, and everyday life functions (Tretter et al., 2006). Research on mathematics curriculum and instruction has increasingly highlighted the importance of length estimation-related opportunities provided for students, including the content of the content and how to teach and learn length estimation, (Andrews et al., 2022).

Although estimators tend to use a variety of strategies for estimating length measurements, Joram et al. (2005) suggested that using unit iteration and comparing benchmarks with TBEOs while incorporating calculations are common strategies.

Benchmarks are constructed from measure units that are salient to estimators, for example, a hand span is about 15-16 cm, which is about three to four times the length of a small paper clip (3.5 cm). Multiplying the known length of one's own body part may reduce the need to keep track of the total number of units iterated, thus facilitating the accuracy of estimated measurements. Accordingly, benchmarks may make measurement units more meaningful (Joram et al., 2005), and are easily retrieved from estimators' mental repertoire of length-estimating benchmarks.

Teachers' Pedagogical Content Knowledge for Teaching Length Estimation

To conduct meaning-oriented activities for mathematics classroom instruction, PCK for teaching is a fundamental base of teacher professional knowledge. For the dimension of teaching practice, Subramaniam (2014) proposed that the essential content of pedagogical knowledge for teaching is how to transform specialized knowledge for a mathematics topic from one symbolic representation into another through dynamic networks involving teacher, students, teacher-student interaction, and pedagogical knowledge. Teachers' PCK is considered a predominately support for teacher, student, and teach-student interactions in mathematics teaching (Ball et al., 2008; Hill et al., 2008; Subramaniam, 2014).

The components of teachers' PCK for teaching mathematics not only include the mathematical knowledge that is common to individuals working in various professions, but also mathematics content knowledge that supports teaching specific topic, such as why and how particular mathematical procedures work for solving problems. For example, for the PCK for teaching length estimation, teachers need benchmark knowledge that can be used as measurement units, and knowledge of how to guide students to construct and apply benchmarks.

A Brief Overview of Teachers' PCK for Teaching Length Estimation

Length estimation skills are teachable (Joram et al., 2005). However, Joram et al. (2005) argued that the teaching of length estimation remains quite open because guessing and obscurity commonly exist in estimation instruction. For example, teachers tend to simply ask student to guess the length of a given object, instead of teaching explicitly ways for making estimations or how to construct benchmarks.

Furthermore, Hoth et al. (2023) suggested that disparate educational traditions which exhibit in different educational contexts (e.g., curricula, teacher education, and teachers' instruction perspectives) may have a significant impact on teaching practices. For example, in length estimation activities conducted in Taiwanese elementary schools, a measurement process using non-standard units (e.g., body parts) and finding the number of iterated non-standard units for estimating the TBEO given prevail. In such situations, viewing and touching TBEOs is allowed (Huang, 2015). In contrast, length estimation in German elementary schools is considered a purely mental process (Subramaniam, 2014). Having students memorize the length of specific objects as mental reference points (benchmarks) for 1 cm, 10 cm, and 1 m, which can be retrieved

and serve as mental rulers, is common practice in German schools. Extending Hoth et al.'s (2023) cross-country study, the PCK of elementary school teachers from Taiwan and Germany on teaching length estimation were taken as noteworthy Western and East Asian cultural contexts in this study.

Considering the above research findings, this study proposed the two hypotheses. Hypothesis 1: There are differences in Taiwanese and German teachers' PCK for teaching estimation strategies. Hypothesis 2: There are differences in teachers' knowledge of students' estimation skills.

METHODOLOGY

We conducted a small preliminary study on teachers' PCK for teaching length measurement and estimation, based on teaching practices. The teachers' teaching practices were collected via one-on-one interviews, conducted face-to-face or in a video meeting. Interviews were audio recorded and transcribed for analysis. For this study, the focus was on teachers' practices of teaching length estimation.

Participants, Instruments, and Scoring

The sample consisted of 35 (22 Taiwanese, 13 German) in-service elementary school teachers with experiences of teaching length measurement and estimation. The teaching experience of the T-Group ranged from 4 to 30 years ($M = 18.27$, $SD = 6.83$), and for the G-group, from 1 to 27 years ($M = 9.85$, $SD = 8.67$).

Data of teacher's PCK for teaching practices were collected using semi-structured questions developed with reference to previous studies (Ball et al., 2008; Hill et al., 2008; Huang, 2015). The questions collected teachers' PCK on teaching the estimation process, the types of estimation strategies they used, and their knowledge of students' estimation skills. Each aspect is described as follows.

For length estimation instruction, there were four open-ended questions, including one asking about the extent to which teachers conduct guess-and-check procedures for teaching estimation, and three about teachers' knowledge for using various approaches to teach length estimation (e.g., operating unit iteration through touching TBEOs, constructing benchmarks by memorizing the lengths of certain items, and using benchmarks as mental rulers without touching TBEOs). An example question is: "What are your opinions on teaching students estimate lengths through touching to-be-estimated objects and calculating the estimated answers? Why?" The interviewees' responses to the three estimation strategy items were analyzed and categorized into three levels of agreement, namely, disagree (1), partially agree (2), and agree (3), where "agree" represented "support," "disagree" represented "dissent" from the viewpoint stated, and "partially agree" meant that students' experiences of length measurement and capability of estimation should be taken into account.

For knowledge of students' estimation skills, interviewees were asked to rate the level of accuracy of students' estimated measurements from 1 to 4 based on their awareness of students' estimation skills: "In comparison to physical measurement, how well do

students estimate lengths? (Mostly [over 80%]; partially [60~80%]; slightly [30~59%]; hardly [below 30%]) accurate.” The higher the score, the better the estimated student performance.

To ensure interrater agreement on scoring and categorizing the participants’ responses to the interview questions, two raters independently coded 22 participants’ responses based on the scoring scheme. The Kappa coefficient of agreement was .74. Considering that the data did not meet the normal distribution assumptions, the non-parametric Mann-Whitney U test was performed on all dimensions of the scores to compare the two groups.

RESULTS AND DISCUSSION

Regarding teachers conducting guess-and-check procedures for student estimation, results showed that about 82% (18/22) of the T-group did it frequently, and about 18% (4/22) did not particularly adopt such procedures. In contrast, all (100%) of the G-group reported teaching guess-and-check. Based on the interview data, the main reason that the Taiwanese teachers do not use the guess-and-check process was their consideration of students’ limited capability of making length measurement estimations, especially first and second graders who are beginning to learn length measurement. Results implied that most teacher in both groups appreciated the value of guess-and-check, as suggested by research, to nurture students’ measurement sense of object size (Joram et al., 2005).

Table 1 shows the frequency of teachers’ knowledge for teaching estimation strategies by group. All the T-group agreed with operation by touching TBEOs, whereas about 50% of the T-group agreed partially with memorizing benchmarks and with using mental rulers. In such cases, the Taiwanese teachers who expressed partial agreement tended to believe that knowing one or two benchmarks (e.g., body parts) would be helpful for making length comparisons. They suggest that whether the two strategies are adopted depends on students’ knowledge and experience of measuring length measurement. Moreover, some of the Taiwanese teachers did not even consider the need for students in the first to fourth grades to memorize benchmarks (18%) or use mental rulers (50%).

On the basis of Table 1 and the interview data, after integrating the frequencies of agreement and partial agreement showed in the German group, almost the whole G-group agreed with memorizing benchmarks (100%) and using mental rulers (92%), as long as students have some knowledge and experience of length measurement. The German teachers would allow students to operate measurement units by touching the TBEOs if students are less able to make estimations. Furthermore, Table 1 also presents that both groups of teachers agreed with the necessity of operation with touching objects for developing students’ ability to length estimation. Particularly those students who are at the beginning of learning length measurement and estimation.

Knowledge	Agreement	T-Group (<i>n</i> = 22)	G-Group (<i>n</i> = 13)
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		<i>f</i>	%	<i>f</i>	%
Memorizing benchmarks	agree	7	32%	1	8%
	Partially agree	11	50%	12	92%
	disagree	4	18%	0	0%
Using mental ruler	agree	0	0%	1	8%
	Partially agree	11	50%	11	84%
	disagree	11	50%	1	8%
Operation by touching	agree	22	100%	1	8%
	Partially agree	0	0%	12	92%
	disagree	0	0%	0	0%

Table 1: Frequency of teachers' knowledge for teaching estimation strategies by group.

Table 2 presents the results of the Mann-Whitney U tests performed on the two aspects of teachers' knowledge, including the use of estimation strategies and knowledge of students' estimation skills. The *z*-values showed significant differences in the two groups' levels of agreement with the three aspects of using estimation strategies. The *z*-values of memorizing benchmarks, using mental rulers, and operation by touching TBEOs were $z = -4.04, p < .001$; $z = -2.67, p < .05$; and $z = -4.04, p < .001$, respectively. Hypothesis 1 was therefore supported.

Knowledge	Mean Rank		Sum of Ranks		z	p
	T-Group	G-Group	T-Group	G-Group		
Estimation strategy						
a. Memorizing benchmarks	13.02	26.42	286.50	343.50	-4.04	< .001
b. Using mental ruler	15.00	23.08	330.00	300.00	-2.67	< .05
c. Operations by touching	24.00	7.85	528.00	102.00	-5.48	< .001
Students' estimation skills	19.75	15.04	434.50	195.50	-1.51	.132

Table 2: The results of Mann-Whitney Test on the two aspects of knowledge.

Results suggest that the two groups of teachers had different PCK for teaching estimation strategies. The T-group believed that using measure units (i.e., body parts and daily objects) as tools through touching TBEOs helps students produce a feeling of object size, which in turn help build up mental rulers. The case that Taiwanese teachers were inclined to consider operation and touching TBEOs as important process for estimating measurements may be impacted by textbook materials in which iterating measurement units via touching TBEOs is commonly illustrated on figures or photos

(Hoth et al., 2023). In contrast, the G-group teachers tended to agree/ partial agree with operation by touching only if students who were less able to measure and estimate length measurements. These results concur with Subramaniam's (2014) findings that prospective German secondary school teachers considered benchmarks as important representations of units that may enable students to estimate length measurements.

Our findings suggest a possible reason for differences in Taiwanese and German students' length estimation performance reported in Hoth et al. (2023) study which revealed that Taiwanese students' estimation performance in the situation of the given objects being not-small (≥ 15 cm) and not-touchable was inferior to that of German students. Taiwanese students' weakness in this estimation situation might be due to a lack of benchmarks and less experience of using benchmarks to mentally estimate length.

With respect to teachers' knowledge of students' estimation skills by rating the level of accuracy of students' estimated length measurements, the mean scores of the T-group ($M = 2.95$, $SD = .65$) were close to those of G-group ($M = 2.62$, $SD = .51$). There were no differences in the two groups of teachers' knowledge, $z = -1.51$, $p = .13$. Therefore, Hypothesis 2 was not supported. Findings suggested that the two groups of teachers shared a common viewpoint that estimating length measurement is a challenge for elementary school students. The two groups of teachers evaluated students' length estimation as being only slightly [30-59%] accurate. This result is similar to Huang's (2016) finding that fifth and sixth graders' pass rate for estimating length measurements was about 45-56%.

IMPLICATION FOR MATHEMATICS EDUCATION

Findings revealed that Taiwanese and German elementary school teachers supported using guess-and-check procedures, and perceived students' challenge in estimating length measurement. The two groups had different knowledge for using estimation strategies. Although operation by touching objects is important for producing a feeling of object size, memorizing some specific measurement units for constructing available benchmarks (e.g., 10 cm and 1 m) should be considered to develop students' sense of measurement and scale concepts (Joram, 2003; Tretter et al., 2006). The study findings suggested that providing teachers with more professional development opportunities with respect to teaching measurement estimation may enhance their PCK and power for teaching length estimation (Ball et al., 2008). Finally, it should be noted that the data of this study were collected from a small sample and a limited number of interview questions. The generalization of the study results is therefore limited.

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EXAMINING MIDDLE SCHOOL STUDENTS' NOTICING OF ALGEBRAIC STRUCTURES WITHIN EQUIVALENT EQUATIONS

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Developing algebraic reasoning with primary and middle years students has received increasing attention in policy documents and research studies. A key aspect in this area is facilitating students to notice mathematical structure in the context of number systems. In this research, we explore solutions generated by 168 students aged 11 years to 13 years to a free-response mathematical assessment task related to algebraic structures within equivalent equations. We use a coding framework to investigate the properties that students identified as well as the level of explanation and generalisation. Findings indicate that most students had difficulties in identifying the general structure and instead focused on calculations. Students were more readily able to recognise equivalent equations connected to the associative property of addition.

INTRODUCTION

Internationally, there has been increasing recognition of the importance of students in primary and middle years developing algebraic reasoning. This has included a focus on growing patterns and the development of functional thinking, the unification of arithmetic and algebra as an integrated curricular strand and supporting student capability to engage in the generalisation process including making conjectures and justifying reasoning (Chimoni et al., 2018; Fonger et al., 2018; Hunter & Miller, 2021; Kieran, 2022; Ministry of Education, 2007). A key factor related to these differing aspects of early algebra is positioning students to notice mathematical structure to support them to make sense of mathematics, understand operations as mathematical objects, and both engage in algebraic transformational activity and make sense of transformations (Kieran, 2018; Schifter, 2018). Schifter (2018, p. 310) defines mathematical structure as “behaviors, characteristics or properties that remain constant across specific instances”. Positioning students to notice and identify structure supports students’ capability to work flexibly with numbers and equations and develop generalisations, another key aspect of algebraic reasoning. In this paper, we explore solutions generated by students aged 11 years to 13 years to a free-response mathematical assessment task which asked students to identify, explain, and develop generalisations related to algebraic structures within equivalent equations. We investigate the properties that students identified as well as the level of explanation and generalisation by presenting a coding framework that we developed to examine the responses.

EMPIRICAL AND THEORETICAL BACKGROUND

Central to doing mathematics is the ability to notice and identify structure across a range of mathematical strands including algebra (Mason et al., 2009; Schifter, 2018). Engaging in structural thinking requires both awareness and the use of relationships but extends beyond this to include the use of properties to transform mathematical expressions (Carpenter et al., 2003; Mason et al., 2009). Specifically in the domain of early algebra, arithmetic equations, and equivalence, Slavit (1999) contends that understanding structure requires attention to differing components of operations including aspects related to properties, relationships, and applications. For number systems and early algebra, recognising and using structure refers to the properties of arithmetic including commutativity, associativity, and distributivity properties, identity law, and the inverse relationships between operations (Kieran et al., 2016). A definition of structure in number systems by Warren (2003) includes: (i) relationships between quantities; (ii) group properties of operations; (iii) relationships between the operations; and (iv) relationships across quantities (p. 123). However, in the teaching of arithmetic, it appears that students are rarely given the opportunities to focus on and build an appreciation of these structures (Arcavi et al., 2017).

In both primary and middle school years, considerable attention in mathematics lessons is given to providing students with opportunities to engage with a range of equations across number systems. However, this typically focuses on teaching procedures of how to calculate in contrast to attending to the structure of equations (Schifter, 2018). As a result, often students do not notice the structural differences between each of the equations and as Schifter argues a “consequence of such absence is the lack of salience of the operations in students’ minds. The operations are interpreted as instructions to perform a set of steps rather than as objects, each with its own set of characteristics and properties” (p. 325). Shifting from a procedures/operations focus on arithmetic, to instead examining the properties of operations and numbers, gives space for these to be stand-alone mathematical objects (Kieran, 1989; Slavit, 1999). This then allows opportunities for students to demonstrate reasoning and understanding about relationships (Schifter, 2018). However, earlier research (e.g., Arcavi et al., 2017), highlights how student compulsion to calculate numerical answers presents a barrier to recognising patterns and mathematical structures. Consequently, there is a challenge in shifting student attention from “calculating” to noticing the underlying structures of operations and seeing these as mathematical objects, which is essential for algebra.

To date, there appears to be little research that focuses on how middle years students describe and generalise the structures and relationships they notice when considering pairs of equivalent equations connected to the properties of arithmetic. This research paper addresses the gap in the field by asking, how do middle years students notice, explain and generalise the properties of arithmetic related to structure in the context of number systems?

METHODOLOGY

Participants and data collection

In this study, 168 middle school students at Year 7 – 8 aged between 11 years to 13 years from one low socio-economic school participated in completing a free-response task developed by the first author and aligned with the New Zealand Curriculum elaboration for Level Four which relates to Year 7 and 8 student expectations (Ministry of Education, 2007). The task (see Figure 1) included a set of 12 equivalent equations (6 matching pairs) with follow-up prompts aiming to facilitate students to notice, describe, explain, and generalise the structural properties without calculating, and two blank pages for the student response.

$76 \times 15 =$	$37 + 43 + 40 + 36 =$	$99 \div 3 \div 3 =$
$7 \times 86 =$	$99 \div 9 =$	
$6^3 =$	$(70 \times 5) + (70 \times 10) + (6 \times 10) + (6 \times 5) =$	
$37 + 40 + 36 + 43 =$	$12 \times 22 =$	$6 \times 6 \times 6 =$
$(7 \times 90) - (7 \times 4) =$	$4 \times 66 =$	
<p>Look at the number sentences above:</p> <ul style="list-style-type: none"> Describe what patterns you can find Why do your patterns work? Do they work with other numbers? 		

Figure 1. Number sentence task

The structural properties and relationships within the equivalent equations are shown in Table One.

Set	Equivalent equations	Property
1	76×15	Distributive
	$(70 \times 5) + (70 \times 10) + (6 \times 10) + (6 \times 5)$	
2	$37 + 43 + 40 + 36$	Associative
	$37 + 40 + 36 + 43$	
3	$99 \div 3$	Associative
	$99 \div 3 \div 3$	
4	7×86	Distributive
	$(7 \times 90) - (7 \times 4)$	
5	6^3	Exponents
	$6 \times 6 \times 6$	
6	12×22	Associative

Set	Equivalent equations	Property
	4×66	

Table 1: Equivalent equation pairs in the task and the related property

Administration of the task was undertaken by the classroom teacher with all students completing the task both independently and individually during their regular mathematics lesson. Students were provided with adequate time to complete the task and then the responses on the two blank pages were collected and scanned for analysis by the research team.

Data analysis

All student responses were coded using a coding framework that was developed both through an inductive and deductive method by the research team. The framework was constructed from previous research studies and frameworks in early algebraic research (e.g., Ellis, 2007; Shifter & Russell, 2022; Stephens et al., 2021) and an examination of the student responses in relation to the identification of structure.

A framework for coding to investigate what students noticed and their level of explanation and generalisation was developed as shown in Table One.

Thinking	Levels	Explanation	Illustration
No relationships identified	Level 1A	No response or individual equations solved with no indication of connections or relationships.	$6 \times 6 \times 6 = 216$
Operational relationships identified	Level 2A	A calculation and indication of a relationship between equivalent equations	$99 \div 3 \div 3 = 11$ $99 \div 9 = 11$
	Level 2B	Indication of a relationship between equivalent equations with no calculation.	$37 + 40 + 36 + 43$ $37 + 43 + 36 + 40$


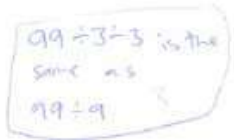
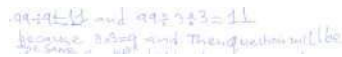


Thinking	Levels	Explanation	Illustration
Demonstrating structural awareness	Level 3A	Calculation and explanation of equivalent equations (no justification)	
	Level 3B	Explanation of equivalent equations with no calculation (no justification)	
	Level 3C	Equation re-written to show equivalent relationship and support justification.	
Generalising	Level 4A	Specific generalisation. Generalisation provided to another specific instance.	
	Level 4B	Global generalisation with the property described in words (including informal language)	
	Level 4C	Global generalisation including an alphanumeric (symbolic) generalisation	

Table 2: Description of the coding framework with corresponding examples of student responses

For each of the six pairs of equivalent equations, the response was coded at the highest level displayed from the coding framework. All student responses were coded independently by two of the research team with an inter-rater reliability of 95.24% overall for the items.

RESULTS AND DISCUSSION

Analysis of student responses using the coding framework is presented in Table Three. For each set of paired equations, the frequency is reported to show how students identified the structure and what level of explanation was provided (see Table 2 for descriptors).

	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
Level 1A	160	126	137	163	119	153
	95.2%	75%	81.5%	97%	70.8%	91.1%
Level 2A	1	9	7	2	6	7

	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
	0.6%	5.4%	4.2%	1.2%	3.6%	4.2%
Level 2B	4	9	10	1	14	6
	2.4%	5.4%	6.0%	0.6%	8.3%	3.6%
Level 3A	2	7	6	1	8	0
	1.2%	4.2%	3.6%	0.6%	4.8%	0%
Level 3B	0	2	3	0	11	1
	0%	1.2%	1.8%	0%	6.5%	0.6%
Level 3C	1	0	5	1	3	1
	0.6%	0%	3%	0.6%	1.8%	0.6%
Level 4A	0	0	0	0	6	0
	0%	0%	0%	0%	3.6%	0%
Level 4B	0	15	0	0	1	0
	0%	8.9%	0%	0%	0.6%	0%
Level 4C	0	0	0	0	0	0
	0%	0%	0%	0%	0%	0%

Table 3: Frequency of student responses to paired sets aligned with framework

The results highlight the difficulties that many students had in noticing, explaining, and generalising the properties of arithmetic related to structure in the context of number systems. Across each set of items, the most common student response was to treat the task as a calculational activity with no connection identified between the equivalent equation. These students recorded calculations and solution strategies in the form of equations without identifying the structural properties. This supports the contention by Arcavi and colleagues (2017), that a focus on calculating answers can present a barrier to students recognising mathematical structure.

Overall, there was relatively little evidence of students developing generalisations about the structure and properties of arithmetic. No students developed and provided a global generalisation using alphanumeric generalisation. Students ($n = 15$) were able to provide a global generalisation describing the property in words for the associative property related to addition. It should be noted that the majority of students provided the generalisation and justification for this using informal mathematical language, for example, “the numbers are the same, but they don’t have to be in order” or “the numbers can just be swapped over”. In relation to the exponents, a small group of students ($n = 6$) were able to develop a specific generalisation to another instance. For

example, one student wrote $6 \times 6 \times 6$ is the same as 6^3 , works for $2 \times 2 \times 2$ is the same as 2^3 .

Furthermore, it appears that the most challenging tasks for students to identify any structures were set 1 ($76 \times 15 = (70 \times 5) + (70 \times 10) + (6 \times 10) + (6 \times 5)$) and set 4 ($7 \times 86 = (7 \times 90) - (7 \times 4)$) which both focus on identifying the distributive property. It could be argued that, in order to be able to see the structural relationship between these equations students need to recognise a familiar structure of one equation (e.g., 76×15) and transform it to the other (e.g., $(70 \times 5) + (70 \times 10) + (6 \times 10) + (6 \times 5)$). This requires the students to understand at a deeper level the operations involved with 76×15 and develop a level of flexibility with this equation to see the relationship with the match equation. Once students are more flexible in their thinking for the familiar structure, they can then transform this structure and abstract to other models. Schüler-Meyer (2017) has suggested for this to occur, there is a need spend time deconstructing the familiar structure, which in their research was referred to as the compound term. By unpacking the underlying structure of this familiar operation, it will lead to new structures being noticed. It is argued that through this students will build relational understanding of the structures of the matched equations.

CONCLUSION AND IMPLICATIONS

Despite the increasing focus on early algebra in both policy documents (e.g., MoE, 2007) and research studies (Chimoni et al., 2018; Fonger et al., 2018; Schifter, 2018), we illustrate that the middle school students in this study largely focused on calculational strategies and did not identify or generalise the arithmetic properties evident in the equivalent equations. This potentially could be related to the task including an equal sign with each equation and students interpreting this as a direction to find an answer, however, it also aligns with earlier work highlighting that students have difficulty in seeing equations as mathematical objects (Kieran, 1989; Slavit, 1999; Schifter, 2017).

A key implication from the study is the need for further research to understand how students can be supported to notice mathematical structure and identify and generalise arithmetic properties in number and algebra within the mathematics classroom. Central to this is developing teachers' capability to notice how their students are seeing structure so that their practices can assist with developing students to move beyond a focus on calculations or simplified explanations of the structures of equations.

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CREATIVITY IN DIFFERENT FACES: HOW CONVERGENT AND DIVERGENT THINKING SHAPE CREATIVE PROCESSES

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Divergent and convergent thinking have played a central role in research on mathematical creativity, especially since the 21st century skills have focused on abilities such as creativity. However, there is only little research on how these two sides of the same medal interact. The aim of the study is to investigate the interplay of divergence and convergence and contribute to a better understanding of creative processes as well as a characterization of different processes. Fifteen students solved a geometric multiple solution task, the processes of which were analyzed quantitatively with regard to divergent and convergent actions. We found five clusters in which the corresponding processes vary in their level of divergent and convergent thinking, thus describing different qualities of creative processes. The results contribute to a deeper insight on how mathematically creative processes can be characterized.

INTRODUCTION

As a reaction to recent challenges—for example changing technological requirements or globalization (Binkley et al., 2012)—the so-called 21st century skills were published. These skills are essential to meet future demands. Alongside critical thinking and collaboration, creativity is emphasized as vital, since the society needs to be able to deal flexibly, appropriately in new ways with future challenges. Fostering these skills is increasingly important in STEM, especially in mathematics, where research into creativity has grown significantly (Maass et al., 2019; Gontijo et al., 2023).

There are different notions of creativity, referring to different theoretical conceptualizations and perspectives, which are common in mathematics education research (Joklitschke et al., 2022): One notion goes back to Guilford (1967) who set the basis for creativity research by describing divergent and convergent thinking as important factors for creativity. Both concepts vary in their characteristics. For divergent thinking, it matters whether someone can combine ideas to generate a range of new ones (indicating a high degree of divergence) or finds it more challenging to develop new ideas that differ (indicating low divergence). With regard to convergent thinking, major differences in its quality exist, such as between a well-founded selection and successful solution of approaches (high convergence) and an unfocused switching between ideas or unsuccessful work on an idea (low convergence).

Current research focuses on creativity and its relation to phenomena like giftedness or achievement (e.g., Vink et al., 2022), often analyzing *products* to infer creativity (Joklitschke et al., 2022). In addition, there is still ongoing basic research aiming to

better understand creativity (e. g., Bruhn, 2024; Liljedahl, 2022). For this aim, Bruhn's and Liljedahl's studies, for example, examined *processes*. Nevertheless, research into creative *processes* is comparatively underrepresented (Joklitschke et al., 2022).

We see a research gap in the investigation of mathematically creative *processes* with an emphasis on divergent *and* convergent thinking. The aim of the study is to investigate the interplay of divergence and convergence and contribute to a better understanding of creative processes as well as a characterization of different processes.

THEORETICAL BACKGROUND

Joklitschke et al. (2021) conducted a systematic literature review of the theoretical conceptualizations of mathematical creativity in the current research literature. They identified six different notions of creativity, one of which traces back to Hadamard's (1945) work. In his studies on how famous mathematicians work, Hadamard conceptualized creative processes as being characterized by typical phases: preparation, incubation, illumination, and verification. This fundamental work has also influenced today's mathematics education research. Liljedahl (2013) conducted research on preservice teachers and mathematicians working on complex and challenging tasks and found that illuminations had a distinctive affective component and underlines that "illumination is at the heart of the creative experience" (p. 256). In an exploratory study, Schindler and Lilienthal (2020) characterized the individual phases at student level and emphasized that these phases are by no means linear, but complement each other in an interplay.

Another notion of creativity in mathematics education research originates from the psychologist Guilford and his research on intelligence. Guilford (1967) conceptualized creativity as a factor of intelligence, with divergent thinking being a central part of creativity. Divergent thinking "is often conceived as the ability to explore in multiple and diverse directions a thought space, from an initial problem or reference point" (Vries & Lubart, 2019, p. 147). Later, Guilford emphasizes more strongly that not only divergent thinking is essential for creativity, but that convergent thinking is also important in creative thinking. Convergence involves selecting ideas and elaborate on them or choosing the best solution. Current work, however, criticizes that convergence has been lost sight of and that creativity is frequently synonymous with divergence (Cropley, 2006). Both are seen as important: "although [divergent thinking is] necessary, it is not sufficient on its own [...] Convergent thinking is necessary, too" (p. 398). The fascinating field of the interplay between divergence and convergence has not yet been researched so well, and following this criticism, some more recent research in STEM education pays attention to investigating this balance.

In STEM education research, Vries and Lubart (2019) investigated the correlation between convergent and divergent thinking and social backgrounds. For this purpose, they used a test for scientific creativity. Using questions from scientific fields, divergence was determined based on the number and uniqueness of the answers given. Convergent thinking was evaluated by the number of concepts used and the originality

of these answers. The evaluation of $N=118$ students aged 7–10 years showed that convergence and divergence correlate only weakly with each other and are described as “relatively distinctive [...], and that the relation is more complex than commonly assumed” (Vries & Lubart, 2019, p. 151). Hence, it is necessary to conduct research that takes a more in-depth look at this interaction.

In mathematics education, the study by Vink et al. (2022) is one of very few studies to examine both divergence and convergence. In this study, $N=229$ fifth-grade students took part in several tests, including divergent thinking, convergent thinking, and a mathematical creativity test (as MSTs, see below), which were evaluated quantitatively. Among other results, it was shown that the higher the visual divergent and convergent thinking, the better the performance in mathematical creativity tests. For verbal thinking, this positive correlation with creative performance could only be shown for convergent thinking. The authors emphasize that “divergent thinking and convergent thinking complement each other” (Vink et al., 2022, p. 496). They conclude, that “more fine-grained (qualitative) research that focuses on how divergent and convergent thinking alternate each other in problem-solving could shed light on subtle differences between the two” (p. 497)

Taken together, only little is known about divergent and convergent thinking in STEM education research, and especially in mathematics education. Previous research has shown that both divergent and convergent thinking are important for creativity and that both thinking operations correlate with creativity (Vink et al., 2022; Vries & Lubart, 2019). But there is still a need for research to understand how processes differ qualitatively depending on how divergent and convergent thinking are involved in these processes. Examining these two thinking operations and their interplay in a mathematical context is promising, as they offer useful insights and approaches to mathematical creativity. Based on the theoretical background of mathematical creativity as a complex interplay of divergent thinking and convergent thinking and the research desiderata identified, we pursue the research question: *In what ways are divergent and convergent thinking involved and interrelated in students’ creative processes when solving a mathematical creativity problem?*

METHODS

To answer the research question, this study takes an in-depth look at students’ creative processes. Fifteen upper secondary school students from Germany, aged between 14 and 16 years, took part in the study. For the study, they worked on a so-called Multiple Solution Task (MST) (see Figure 1) which are specifically designed to assess creativity. The format of the task requires the participants to find several solutions to a given problem. Originally, MSTs were used

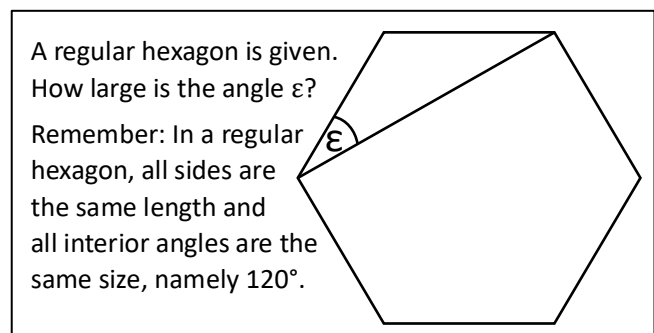


Figure 1: MST “hexagon”

to analyze creative *products* by quantifying the solutions in terms of *fluency* (i.e., the number of solutions produced), *flexibility* (i.e., the diversity of solutions), and *originality* (i.e., the rarity of solutions) (Leikin & Lev, 2007). However, it has also been shown that this type of task is suitable for investigating creative *processes* (Schindler & Lilienthal, 2017). students wore eye-tracking glasses during the task, the (Tobii Pro Glasses 2, infrared, binocular, 100 Hz) that recorded their eye movements. After that, we produced individual eye-tracking videos using Tobii Pro Controller software, in which the gaze was visualized as semi-transparent point. This individual video then served as a prompt in a subsequent stimulated recall interview. During this interview, the individual students and the researcher (first author) watched the video and the student was asked to comment on how they worked on the task as much as they can. The interview was then transcribed and evaluated using qualitative content analysis (Kuckartz & Rädiker, 2023). For that purpose, based on Hadamard's phases, a deductive-inductive coding system was developed.

In order to present the reconstructed, creative process in its entirety, flowcharts were created based on the coded transcript (see Figure 2). In these flowcharts, vertically the chronological course of the process is shown: The further down a box is, the later it is in the process. Mathematically distinctive approaches are shown in different columns: When a new approach begins, the corresponding box in the diagram appears in the next column. The vast majority of these corresponding boxes are comments on mathematical properties that were decisive in generating a solution that the student had not yet worked out. The flowcharts also reveal how approaches are interacting with each other: In Fig. 2, it can be seen that approach 2 is interrupted and that approach 2 is taken up again after approach 3 is finalized, and then completed to achieve a final solution. It can also be deduced that the preparation phase in approach 3 was initiated from approach 1. The highly condensed structure of the flowcharts is the basis for the analyses in this study.

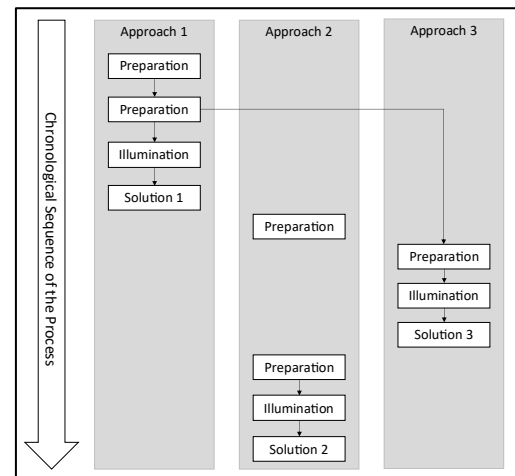


Figure 2: Flowchart of the reconstructed process

It can be seen that approach 2 is interrupted and that approach 2 is taken up again after approach 3 is finalized, and then completed to achieve a final solution. It can also be deduced that the preparation phase in approach 3 was initiated from approach 1. The highly condensed structure of the flowcharts is the basis for the analyses in this study.

In light of the preliminary theoretical considerations and the research question, it is essential for this analysis to take a more in-depth look at the characteristics of divergent and convergent thinking. For an initial analysis that meets the quality standards, a dichotomous operationalization for both convergent and divergent thinking was therefore chosen. The specific criteria have been derived inductively from theoretical descriptions of divergent and convergent thinking and have been supplemented and further elaborated through joint discussions and the consideration of individual processes. The processes were coded as follows with respect to their divergent and convergent characteristics:

Simple divergent actions (div–) are coded if

- new approaches are developed almost entirely in isolation, or
- neither many nor diverse approaches can be identified.

Complex divergent actions (div+) are coded if

- a new approach is developed based on another, single existing approach,
- a new approach is developed based on several existing approaches,
- several new approaches are developed in parallel based on a single existing approach, and
- both many and diverse approaches can be identified.

Non-productive convergent actions (conv–) are coded if

- initiated approaches are not successfully completed,
- initiated approaches pass through many alternating phases before they are (eventually successfully) finished (also including wild goose chase),
- interrupted approaches are not revisited, although they could be completed to an appropriate solution, or
- interrupted approaches are revisited even though they cannot be completed to a solution with the available resources.

Productive convergent actions (conv+) are coded if

- the majority of started approaches are successfully completed,
- suspended approaches are successfully revisited, and
- suspended approaches are not revisited with appropriate mathematical justification and are therefore reasonably discarded.

RESULTS

The following results do not focus on the generation of a single solution, but look at the entire process in which several solutions were generated by each student. All processes could be evaluated according to the different levels of divergent and convergent thinking. The two levels in each case resulted in a 2x2 matrix (see Table 1), into which the 15 processes could be mapped. An analysis of the processes in each individual cell shows that the fields also differ qualitatively from each other and that the mapping has created separable clusters. However, a closer look at the individual processes shows that the processes in cluster 3 (div–/conv+) are not homogeneous. Two qualitatively separate groupings (3a and 3b) emerge here.

All five clusters are briefly outlined below by presenting a process of a student (with pseudonym) in each case. Please note, that the processes within a cluster were naturally different. Every cluster forms a group of certain common characteristics, but of course, individual differences are thoroughly evident and also reasonably to be expected.

action	simple div. (div−)	complex div. (div+)
non-productive conv. (conv−)	cluster 1: undirected	cluster 2: flexible
productive conv. (conv+)	cluster 3a: adhered cluster 3b: associative	cluster 4: flexible-integrative

Table 1: Overview of the found clusters for creative processes with a focus on divergent and convergent actions.

(div−/conv−): Charlotte's process, which can be described as *undirected*, is assigned to cluster 1. Although Charlotte shows several approaches, these are primarily created in isolation from each other and that they also have limited diversity, which is why they can be described as *simple divergent actions* (div−). She also clearly shows that she does not complete many approaches and sometimes jumps back and forth between approaches without progress, so that she goes through many phases until (in some cases) she arrives at a finished solution—*non-productive convergent actions* (conv−).

(div+/conv−): In Thorsten's *flexible* process, it can be seen that he describes many and varied approaches that also emerge from one another (div+). He also has some approaches that he brings to a successful solution, which indicates good convergent actions at first, but the first part of the process is characterized by the repetitive revisiting of an approach that cannot be continued with appropriate solutions (conv−). As a result, he can't focus on other approaches; for example, he describes the emergence and further development of an approach, but didn't write down the solution.

(div−/conv+ a): Benedict's process could be assigned to cluster 3a *adhered*. His approaches largely arise in isolation from one another. But there is also the case in which he generates two further approaches from one approach, although they are almost identical to each other, so that *simple divergent actions* (div−) could be assessed here. The individual approaches are mostly successful and very short, and he sometimes skips phases of the Hadamard process. The discarding and revisiting of approaches is always appropriately justified, so that overall this can be referred to as *productive convergent actions* (conv+).

(div−/conv+ b): The second cluster in cell 3 reflects an *associative* process (cluster 3b). Markus, who was assigned to this cluster, shows many approaches, which, however, all emerge in isolation from one another (div−). He shows an extremely high level of *productive convergent actions* (conv+). He starts and completes one approach after the next almost effortlessly, and can quickly overcome the few hurdles that arise without interrupting approaches in order to continue working on other approaches.

(div+/conv+): Alex's flexible-integrative process could be assigned to cluster 4. His process clearly shows that ideas develop from existing approaches or that existing approaches are combined to generate a further approach so that complex divergent actions can be deduced (div+). In terms of convergence, Alex is able to develop his approaches into solutions without any problems and can successfully overcome any

hurdles that arise. He balances between approaches and only takes up those that seem promising to him, which speaks for good convergent actions (conv+).

DISCUSSION

The analysis of creative mathematical processes based on the theory of divergence and convergence (Cropley, 2006; Guilford, 1967) shows that creative processes differ qualitatively and can be classified into clusters with different characteristics. This finding underlines the importance of divergence and convergence as central categories for describing creative processes in mathematics. The empirical identification of an additional fifth cluster (3b, “associative”) expands the theoretical model and enables a more differentiated description of the processes. The qualitatively different characteristics of the clusters show the potential that the analysis of divergence and convergence in mathematical processes holds, how it makes sense and thus extends the research of Vries and Lubart (2019) and Vink et al. (2022).

However, the study has methodological limitations. For example, Markus’ process probably remained incomplete due to his limited communication, which points to possible undiscovered connections between approaches. The supplementary analysis of eye-tracking data could provide additional insights into divergent actions, but requires a substantial investment of time and effort. In addition, the case of Thorsten, who has characteristics of cluster 4, shows that there are borderline cases that call into question the homogeneity of the clusters. Further validation and analysis of the heterogeneity between the clusters are therefore necessary. In the future, the development of ideal types could represent a valuable theoretical extension that would strengthen the influence of the results on mathematics education research. The development and systematic-empirical identification of clusters offers a new approach to describing mathematically creative processes in a differentiated way through the interplay of divergent and convergent actions. These findings can form a basis for more in-depth research, but also create the opportunity to meet students more precisely in their creative activities and to support them adequately.

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SOUND IN THE CLASSROOM: EXPLORING COLLECTIVE LEARNING OF DURATION AS A QUANTITY

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This study explores how sound-based activities can foster collective learning of duration as a measurable quantity. Centered on a second-grade classroom in Paris as part of the Maths-Musique project, we draw on the theory of objectification to analyze students' multimodal semiotic activity during a session comparing the durations of two animal sound recordings. Our findings reveal how students develop an initial awareness of duration by integrating sensory modalities such as sound, touch, and vision. Notably, we emphasize the pivotal role of sound in facilitating collective adjustments to sensory perception through shared creative experiences. These findings underscore the value of further research into students' early sensory intuitions to enhance pedagogical practices.

INTRODUCTION

Reasoning with quantities is a cornerstone of elementary mathematics curricula worldwide. Students engage with various quantities—length, area, volume, weight, and duration—that shape the conceptualization of quantity and measurement as fundamental mathematical ideas. Measurement holds historical significance in mathematics and can play a pivotal role in supporting the learning of calculus in elementary education (Chambris & Subramaniam, 2023) while fostering algebraic and geometric reasoning (Thompson, 2011). Its ubiquity in daily life underscores its critical importance in early education.

From Thompson's perspective (Thompson, 2011), reasoning with quantities involves a dialectical relationship between a quantity and its quantification, intertwining matter and measurement (de Freitas & Sinclair, 2020). At its core, a quantity is an attribute that facilitates comparison, allowing us to discern how small or large something is. This foundational comparison represents an initial form of quantification that applies across diverse contexts. Deeper engagement with the concept of quantity involves encounters with sensory, bodily, and culturally embedded measurement practices. While these practices may incorporate symbolic systems such as ruler marks or numerical representations, they are fundamentally rooted in sensory experiences, "tied directly to a body's first attempts to 'sense' the world autonomously, through movement, touch, sound, vision, and even taste" (de Freitas & Sinclair, 2020, p. 1). According to these researchers, measurement practices emerge more fundamentally from dynamic, specific correlations between two or more material processes, embodying the relational proportionality of the material world.

Despite its significance, measurement is often taught in a procedural and disconnected manner, reduced to tasks such as using rulers or protractors to assign numbers to quantities. This separation of measurement practice from reflexion and imagination risks promoting superficial learning (Lehrer & Schauler, 2023). While curricula outline progressive goals—from direct comparisons to the use of arbitrary and standardized units—they frequently fail to address the nuanced conceptual progression involved in the learning process. Furthermore, measurement in schools is often narrowly framed as the act of "covering" spatial objects with standardized units (de Freitas & Sinclair, 2020; Lehrer & Schauler, 2023). In contrast, these researchers propose alternative measurement practices rooted in the dynamic correlation of materialities. For example, spanning a distance can foster an embodied sense of its extent, and support a distinctive way to conceptualise length (Lehrer & Schauler, 2023). This perspective invites further inquiry into how emerging correlations between materialities might shape students' early understandings of quantity.

Among the magnitudes introduced in primary education, duration stands out as particularly underexplored, likely due to—and despite—its distinctive nature. Unlike other magnitudes tied to tangible, physical objects, duration is intrinsically linked to temporal phenomena such as motion and sound. Historically, vision has dominated research and pedagogical practices, often sidelining sensory modalities like sound. For instance, the well-established field of visualization in mathematics education contrasts sharply with the scarcity of research into auditory modalities.

Given that measurement concepts arise through sensory encounters with quantities, it is crucial to examine how different sensory modalities—particularly sound—shape students' learning of duration and its measurement in the early years of primary education. From a sociocultural perspective, social interactions in the classroom are not merely inputs for individual learning but are consubstantial to the learning process itself. This raises the question: how might auditory sensory experiences contribute to the collective learning of duration? We are particularly interested in the sensory and bodily ways students approach duration when the concept of a unit of measurement has not yet been formalized.

In the following sections, we outline a theoretical framework that supports a collective view of learning and provides tools for analyzing classroom interactions and interpreting our findings.

THEORETICAL FRAMEWORK

The theory of objectification (TO), developed by Radford (2021), offers a socio-cultural perspective on learning, conceptualizing it as a social, bodily, material, and symbolic process structured along two dimensions: knowing and being. Regarding knowledge, learning is understood as the progressive engagement with, or awareness of, historically and culturally constituted forms of action, and thought—a process referred to as objectification. Simultaneously, this awareness involves a transformation

of the self, termed subjectivation, whereby teachers and students position themselves socially in relation to knowledge and explore ways of being within mathematical practices.

The TO emphasizes the collective nature of learning and identifies specific conditions that facilitate it (Radford, 2023). First, a common object of learning must be established, progressively revealed through the joint efforts of students and teachers. Second, social relations should be embedded within a community ethic, which encompasses (1) a commitment to active participation, including attentive listening; (2) the responsibility to recognize and value others; and (3) care for others, demonstrated through thoughtful and engaged interaction.

Certain sensory and perceptual qualities of sound resonate strongly with these aspects of collective learning. Sound emerges simultaneously for all participants, fostering shared attention and requiring care (e.g., maintaining silence to ensure clarity). Moreover, sound production inherently supports coordination, allowing multiple participants to contribute fluently to a unified auditory experience. These qualities of sound invite us to consider their potential role in shaping collective ways of learning duration. By analyzing how sound-based activities create opportunities for joint exploration, we aim to better understand how sensory and bodily experiences might contribute to students' progressive awareness of duration as a quantity.

METHODOLOGY

We employed a qualitative approach to investigate multimodal semiotic activity in a CE1 classroom in Paris, comprising 13 students aged 7–8. This study incorporates elements of design-based research and forms part of the Maths-Musique (Maths-Music, in English) project, which aims to foster learning about multiplication and measurement through musical and sound-based activities. While the project includes activities related to multiplication, these are beyond the scope of this report.

The classroom teacher voluntarily participated in the study during the first semester of the 2024–2025 school year, implementing approximately one activity per week during mathematics instruction periods. These activities were designed by the project researcher, who collaborated weekly with the teacher to evaluate the achievement of prior learning objectives and plan subsequent sessions.

Central to the project is the concept of pulse, defined as a series of uniformly spaced beats. In the initial phase of the sequence, students were tasked with identifying and following the pulse in various songs. They learned to recognize the beat as a percussive sound with regularity and constant speed. The data presented in this study were collected during the second of three consecutive sessions in the second phase of the sequence. This phase focused on comparing durations, and exploring the relationships between duration, the time intervals marked by the pulse, and the speed of pulsations. In the analyzed session, students worked collaboratively in small groups of 2–3 to compare the durations of pre-recorded animal sounds.

Data Analysis

We adopted a microgenetic approach (Radford, 2021) to analyze the classroom's multimodal semiotic activity, focusing on the detailed reconstruction of learning processes over time. Data collection included video and audio recordings, conducted in full compliance with ethical guidelines, including school authorization and parental consent.

Video recordings from the session were partially transcribed, centering on a 25-minute segment of the 1-hour session. Participants' use of various semiotic resources—such as actions, gestures, interactions with objects, written symbols, postures, and intonation—was annotated. To enhance the analysis, key moments were documented with screen captures accompanying the transcripts, providing visual context for the semiotic productions.

ANALYSIS

Twenty minutes into the session, the teacher introduced a task requiring students to compare the durations of two animal sounds: the frog's croaking (10 seconds) and the cricket's stridulation (5 seconds). The sounds were played sequentially for the whole class, and students worked in small groups to determine which was longer and justify their answers.

Most students correctly identified the frog's sound as longer, with only one student differing. The predominant strategy among groups was counting the discrete noises made by the animals. For instance, the frog's croak, croak was perceived as two distinct sounds. This approach reflects a common inclination to use counting as a comparative strategy without establishing a consistent unit of measurement, as noted in prior research (Clements & Stephan, 2004).

Our analysis focuses on the portion of the session preceding the review of student group strategies. During this time, the teacher facilitated a collective exploration of the limitations of counting as a strategy for comparing durations. We discuss moments that we consider key to collective learning about duration.

A challenge on the horizon

To prompt reflection, the teacher challenged the counting strategy by highlighting variability in the frog's croaks, contrasting long and short durations:

Teacher (P): Look, if I croak, croak (with rhythmic, evenly spaced beats), how many noises do I make?

Students (C): Two.

P: Two (raising two fingers on the left hand). But if I croak (imitating a long croak, drawn out over 5 seconds, with subtly ascending intonation, causing laughter), how do we count?

C: One only.

P: One only (raising one finger on the right hand). And which is longer?

C: The second one.

P: It's the second one! But in the second one, I make only one noise (raising one finger on the right hand). And in the first, I make two (raising two fingers on the left hand, with a challenging expression).

From a conventional perspective, the teacher's exaggerated sounds and lively delivery might be seen as peripheral to the learning process. However, within the framework of the TO, these exaggerated guttural sounds, combined with intonation and rhythm, emerge as semiotic resources that contribute to the meaning of a long duration, sensuously coordinating with the time of pronunciation. This performance also functions as an aesthetic proposition, inseparable from the materiality of the sound produced. The students' laughter reflects their fascination and engagement—a form of aesthetic enjoyment underscoring the poetic and sensitive character of learning.

In terms of subjectification, the teacher reasserts her role as a subject within the collective practice, adopting a theatrical orientation that positions her uniquely in relation to the students. By embracing this performative approach, the teacher affirms her commitment to collective learning, exposing herself to the students' reactions and fostering a shared mathematical experience. At the conclusion of her intervention, the teacher presents a challenge: the longest sound is not the one associated with the largest number when counting the noises. Through her facial and bodily expressions (see Figure 1), she intensifies the challenge, inviting students—and herself—into a deeper exploration of the task. This sets a shared object on the collective horizon, guiding the joint work that follows.



Figure 1: Facial expressions of the teacher to set a challenge.

Collective Exploration of duration

Gradually, students began to spontaneously and collectively imitate the long croak whenever the teacher referenced it (see Figure 2). These collective repetitions transcend mere mimicry. The sound vibrates not only in the air but also in the students' bodies, creating a shared experiential space. By reproducing the sound together with the teacher, students sensorially experience it in a way that orients them differently to both knowledge and the collective. First, this activity provides an opportunity to refine

their bodily and sensory perception of duration in search of an object that has not yet been fully revealed to their consciousness.



Figure 2: Collective production of the long sound.

Second, through this shared exploration, students reaffirm their commitment to learning and strengthen their sense of belonging within the classroom community. This emerging “symphony of voices,” enriched by individual singularities in a shared creative experience, highlights the close interrelationship between subjectification and objectification, revealing a process of collective learning.

Counting Long Sounds

On two occasions, a student produced rapid hand gestures that simulated counting, sequentially raising the fingers of each hand several times within a couple of seconds (see Figure 3a). The first instance occurred during a discussion immediately after the teacher first produced the long croak. The second instance took place during the teacher's subsequent imitation of the same sound (see Figure 2), where the student concluded the gesture by quickly and simultaneously opening and closing both hands twice. Following a collective repetition, another student independently performed a similar gesture while intoning the long croak (see Figure 3b).

3a



3b



Figure 3: Sequence of captures of fast counting gesture, just after the teacher produced the long croak (3a); and while pronouncing the long croak (3b).

These rapid gestures do not appear to aim for precise counting. Instead, they suggest an emerging form of sensorial coupling between counting and duration, reflecting a sensory-perceptual adjustment to the initial discrete-noise counting strategy. Detached from verbal enumeration, the counting gestures function primarily as tactile, bodily, rhythmic, and visual processes. Students explore sensorially the interplay between the material act of counting and the auditory experience, which varies across cases, encompassing recall from memory, active listening, and vocal production.

The gestures thus reflect a rhythmic and bodily way of perceiving long sounds as durations potentially associated with “many counts.” While this activity does not involve consistent measurement with a standard unit, it signals an early attempt to relate duration to rhythmically delimited pseudo-units that are tactile, temporal, and sensory.

CONCLUSIONS

This study investigated the potential of sound-based activities to foster collective learning of duration, focusing on the initial perceptual forms that emerge during this process. Using the lens of the TO, we analyzed a session in a second-grade classroom to examine how sound’s unique qualities contribute to collective learning. Crucially, sound explorations alone do not spontaneously create the conditions for collective learning; these conditions arise from the sustained, joint efforts of teachers and students.

Our findings highlight specific ways in which an awareness of duration develops and how social relations are framed within a community ethic during sound-based activities. The teacher and students’ joint efforts to reinterpret sounds and their durations become integral to this collective learning process. The materiality of sound leaves its imprint on mathematical activity in the classroom, offering fluidity and versatility to sustain collective attention and foster shared creative experiences. These conditions support the collective refinement of sensory perceptions of duration, grounded in ethical engagement and collaborative inquiry. Sound, therefore, emerges as a potentially powerful medium for fostering collaborative learning.

The processes of signifying sound duration unfold through the dynamic interplay of sensory and semiotic modalities. As evidenced in our analysis, modalities such as intonation, posture, touch, gestures, objects, displacement, speed, rhythm, sound, and words sensuously interrelate to convey meanings of long and short durations. Just as spatial and geometric reasoning supports the learning of length, area, and angles (Clements & Stephan, 2004), we propose that rhythm and tone could play a central role in understanding duration, particularly in auditory contexts. Rhythm facilitates the segmentation of sound intervals and accentuates their relative durations, while tone contributes to signifying long durations. Encouraging diverse sensory and experiential practices with quantities may broaden students’ understanding of measurement.

The convergence of multiple semiotic modalities not only fosters the creative and sensitive engagement of teachers and students but also actively contributes to the process of making sense of mathematical meanings. Bodily and sensory intuitions, as demonstrated in our analysis, play a crucial role in reasoning processes that connect duration, measurement, and sound. Students appear to begin sensing relationships between long-duration sounds and shorter intervals through the interweaving of various materialities. Although these intervals are not yet formalized as units of measurement, they represent early attempts at quantification.

If the goal is to nurture students' recognition of more quantitative ways of perceiving sound, disregarding their perceptual intuitions would be a significant oversight. Instead, we advocate for an approach that values these intuitions, fostering a sensory and perceptual foundation that can be progressively refined through reflection and discussion. Further research into the opportunities and challenges posed by sound-based activities in duration learning holds substantial promise for enriching our understanding of these processes and enhancing pedagogical practices.

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