PME 49
Helsinki, Finland
2026

Sensing and making sense of mathematics

Proceedings

of the 49th Conference of the International Group for the Psychology of Mathematics Education

July 27 to August 1, 2026

Helsinki, Finland

Editors

Heidi Krzywacki, Markus Hähkiöniemi, Eeva Haataja, Markku S. Hannula and Peter Hästö

General Contributions

Plenary Lectures, Plenary Panel, Research Forums, Working Groups, Seminars, Publication Reports, Oral Communications, Poster Presentations

49th Conference of the International Group for the Psychology of Mathematics Education (PME)

July 27 to August 1, 2026

Helsinki, Finland

Website: https://www.helsinki.fi/en/conferences/pme49

Cite as:

Krzywacki, H., Hähkiöniemi, M., Haataja, E., Hannula, M.S., and Hästö, P. (Eds.) (2026). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*. PME.

Proceedings are also available on the IGPME website:

https://www.igpme.org/

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ISSN 3081-0833

Logo designed by Noora Isoeskeli

PREFACE

We are pleased to provide the Proceedings of the 49th Annual Conference of the International Group for the Psychology of Mathematics Education. PME is one of the most important international conferences in mathematics education and it brings together educators, researchers, and mathematicians from all over the world. This year, we had exceptionally high number of participants, 789 participants attending the conference, coming from 51 different countries, presenting 648 contributions.

PME 49 was co-organized by University of Helsinki (Helsinki, Finland). The conference was hosted by the Faculty of Educational Sciences and Faculty of Science of University of Helsinki. The PME 21 was organised in Lahti, Finland in 1997 and it took almost three decades to have an honour to host PME conference in Finland again.

The theme of PME 49 was "Sensing and making sense of mathematics". This theme was chosen to emphasize the embodied perspectives on mathematics learning, which have been gaining increasing interest in this millennium. Through our senses we interact with mathematical representations and the mathematical sense making happens in and through our body. The four plenary lectures examined embodiment from the perspectives of brain, movement, vision, and interaction with technology. The plenary panel debated on the fundamental nature of mathematical sense making. We believe that these talks and other presentations in PME49 have given an opportunity for members to engage in inspiring exchange of ideas and to gain new insights.

We had the honour of receiving plenary speakers and panellists from different countries, representing a diversity of educational contexts and theoretical perspectives, reflecting the diversity of the PME community. The PME 49 scientific program includes the traditional PME individual presentation formats (Research Reports, Oral Communications, and Poster Presentations) and group presentation formats (Research Forums and Working Groups). This year, a recently introduced individual presentation format, Publication Report, seems to be found by the PME community and there were as many as 19 Publication Reports in the programme.

Moreover, you might have noticed that, starting from PME 48, the PME Proceedings has changed because Research Reports are indexed in Scopus. This year adds one more change as only digital proceedings will be available and there are no printed proceedings. Accordingly, this Proceedings comprises of two parts: the first part contains all the general contributions, and the second part all the Research Reports published in a digital format.

The conference was organized at the University of Helsinki that provided facilities for a successful PME 49 conference. We also thank the City of Helsinki for hosting and sponsoring a welcome reception at Helsinki City Hall and The Federation of Finnish Learned Societies for a significant support to the scientific program.

PME 49 Local Organizing Committee comprised of scholars from several Finnish universities, University of Jyväskylä, Åbo Akademi University, University of Eastern Finland, University of Helsinki and Tampere University. This collaboration was essential for preparing one of the biggest PME conferences ever and being successful in providing a fruitful and inspiring experience for all participants. PME has its own spirit that is based on high quality scientific programme that is based on stimulating contributions presented in this proceedings and social programme that was to enhance networking and the spirit of PME community.

Greetings from Helsinki!

Markku S. Hannula Heidi Krzywacki

University of Helsinki

PME 49 co-chairs

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

HISTORY OF PME

The International Group for the Psychology of Mathematics Education (IGPME) is an autonomous body, governed as provided for in the constitution. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI) and came into existence at the Third International Congress on Mathematics Education (ICME 3) held in Karlsruhe, Germany in 1976. Its former presidents have been:

Efraim Fischbein, Israel	Fou-Lai Lin, Taiwan
Stephen Lerman, UK	Nicolas Balacheff, France
Richard R. Skemp, UK	João Filipe Matos, Portugal
Gilah Leder, Australia	Kathleen Hart, UK
Gerard Vergnaud, France	Barbara Jaworski, UK
Rina Hershkowitz, Israel	Carolyn Kieran, Canada
Kevin F. Collis, Australia	Peter Liljedahl, Canada
Chris Breen, South Africa	Markku Hannula, Finland
Pearla Nesher, Israel	Wim Van Dooren, Belgium

The current President is is Oh Nam Kwon (Korea).

THE CONSTITUTION OF PME

The constitution of PME was adopted by the Annual General Meeting (AGM) on August 17, 1980 and changed at the AGM on July 24, 1987, on August 10, 1992, on August 2, 1994, on July 18, 1997, on July 14, 2005, and on July 21, 2012. PME decided to seek charitable organization status under UK law and the new constitution related to this change was accepted by the Annual General Meetings in 2018 and 2019. The name of the Charitable Incorporated Organisation ("the CIO") is the international Group for the Psychology of Mathematics Education. The objects of the CIO are to advance the field of mathematics education for the public benefit by:

- Promoting and stimulating research.
- Organizing regular educational conferences around the world.
- Supporting regional workshops around the world in general, and in underrepresented regions of the world in particular.
- Collaborating with organizations with similar aims.
- Facilitating cross-disciplinary discussion and the sharing of information and research with an international emphasis.

- Helping scholars from different parts of the world establish collaborative networks to further our collective understanding of how to improve mathematics education in their respective countries.
- Providing grants to help bring scholars from under-represented regions of the world to our annual meeting.
- Disseminating our research for the benefit of improving mathematics education at the classroom, school, district, and national levels.
- Disseminating our research for the benefit of improving students' experiences with mathematics education at the classroom, school, district, and national levels.
- Disseminating our research for the benefit of improving the preparation of mathematics teachers at the university level around the world.
- Providing access to our research publications to the public.

All information concerning PME and its constitution can be found at the PME website: www.igpme.org

PME MEMBERSHIP AND OTHER INFORMATION

Membership is open to people involved in active research consistent with the aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between 700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other during working groups, poster sessions and many other activities. Every year the conference is held in a different country.

There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter can be found on the PME website. Occasionally PME issues a scientific publication, for example the result of research done in group activities.

WEBSITE OF PME

All information concerning PME, its constitution, and past conferences can be found at the PME website: www.igpme.org

PROCEEDINGS OF PREVIOUS PME CONFERENCES

Previous PME proceedings can be found in digital form at the PME website: www.igpme.org

HONORARY MEMBERS OF PME

Efraim Fischbein (Deceased)

Hans Freudenthal (Deceased)

Joop Van Dormolen (Deceased)

PME ADMINISTRATIVE MANAGER

The administration of PME is coordinated by the Administrative Manager who is currently Dr. Peter Gonscherowski.

email: info@igpme.org

BOARD OF TRUSTEES OF PME

Members of the Board of Trustees (BoT), formerly known as the PME International Committee, are elected for four years. Every year, four members retire and four new members are elected. The BoT is responsible for decisions concerning organizational and scientific aspects of PME. Decisions about topics of major importance are made at the Annual General Meeting (AGM) during the conference. The BoT work is led by the PME President, who is elected by PME members for three years. Before becoming President, the President-Elect works together with the past President and the Board of Trustees for one year.

President

Oh Nam Kwon (Korea)

Board of Trustees

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* Portfolio Group leaders

THE PME 49 CONFERENCE

Two committees are responsible for the organization and execution of the PME 49 Conference: The International Program Committee (IPC) and the Local Organizing Committee (LOC).

In addition, we thank all the reviewers and the IPC for the detailed review work that has led to the selection of the contents of these proceedings.

THE INTERNATIONAL PROGRAM COMMITTEE (IPC)

LOC representatives

Markku Hannula	University of Helsinki, Finland PME49 Co-Chair
Heidi Krzywacki	University of Helsinki, Finland PME49 Co-Chair
Eeva Haataja	University of Helsinki, Finland
Markus Hähkiöniemi	University of Jyväskylä, Finland
Peter Hästö	University of Helsinki, Finland

PME representatives

Oh Nam Kwon	Seoul National University, South Korea PME president
Nicola Hodkowski	Digital Promise Global, USA
Jenni Ingram	University of Oxford, UK
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Mervi Asikainen	University of Eastern Finland
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Eeva Haataja	University of Helsinki
Markku Hannula	University of Helsinki
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Heidi Krzywacki	University of Helsinki
Anu Laine	University of Helsinki
Laura Niemi	University of Helsinki
Hanna Viitala	University of Helsinki

REVIEW PROCESS OF PME 49

RESEARCH REPORTS (RR)

Research Reports are intended to present empirical or theoretical research results on a topic that relates to the major goals of PME. Research Reports should state what is new in the research, how it builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all papers.

The number of submitted RR proposals was 489, and 207 of them were accepted. Of those not accepted as Research Reports, 124 were invited to be resubmitted as Oral Communications (OC) and 24 as Poster Presentations (PP). After acceptance, 1 Research Reports were withdrawn because of inability to participate in the conference. Altogether, 207 Research Reports are expected to be presented at PME 49.

As in previous years, every RR submission underwent a fully independent double-blind peer review by two to four international experts in the field. The reviewers' reports served as input for the PME 49 International Program Committee to decide acceptance for the conference. Accepted Research Reports are indexed in Scopus: https://www.scopus.com/sourceid/21101194210.

PUBLICATION REPORTS (PR)

Publication Reports are intended to present already submitted, accepted, or recently published empirical or theoretical research results on a topic that relates to the major goals of PME. Publication Reports should state what is new in the research, how it builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all submissions and authors are encouraged to also go beyond what is already published wherever possible – so that the presentation is also of interest to those that are already aware of the published work.

The number of submitted PR proposals was 24, and 19 of them were accepted. All of them are expected to be presented at PME 49. Similarly to RR proposals, every PR submission underwent a fully independent double-blind peer review by two to four international experts in the field. The reviewers' reports served as input for the PME 49 International Program Committee to decide acceptance for the conference. Accepted Publication Reports' abstracts are included in the conference proceedings.

ORAL COMMUNICATIONS (OC)

Oral Communications are intended to present smaller studies and research that is best communicated by means of a shorter oral presentation instead of a full Research Report. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME. Reports should describe on which past research it

builds, give a concise overview of the new results attained, and what the main implications of these results are.

The number of submitted OC proposals was 273, and 157 of them were accepted. Of those not accepted as OC proposals, none were invited to be resubmitted as Poster Presentations (PP). Including resubmissions of Research Reports as Oral Communications, 271 OCs were accepted. After acceptance, 9 Oral Communications were withdrawn because of inability to participate in the conference. Altogether, 262 Oral Communications are expected to be presented at PME 49.

POSTER PRESENTATIONS (PP)

Poster Presentations are intended for information/research that is best communicated in a visual form rather than an oral paper presentation. There is no formal oral presentation associated with Poster Presentations, although there can be a time allotted, after sufficient display time, during which presenters will be available by their posters for discussion with conference participants. Posters should present empirical or theoretical research studies on a topic that relates to the major goals of PME. Reports should describe the research context of the study presented, the study itself, and its implications, with a particular focus on the visual aspects of the presentation.

The number of submitted PP proposals was 109, and 95 of them were accepted. In the end, considering resubmissions of Research Reports as Poster Presentations, 118 PPs were accepted. After acceptance, none of Poster Presentations were withdrawn because of inability to participate in the conference. Altogether, 118 Poster Presentations are expected to be presented at PME 49.

COLLOQUIA (CQ)

The goal of a Colloquium is to provide the opportunity to present a set of three Research Reports that are interrelated in a particular way (e.g. they depart from related or contrasting theoretical stances, use identical instruments or methods, or focus on closely related research questions), and to initiate a discussion with the audience on the interrelated set.

There were no CQ proposals submitted to PME 49.

RESEARCH FORUMS (RF)

A Research Forum is a format of group presentation undertaken by a group of at least five researchers. The goal of a Research Forum is to provide PME members with a comprehensive overview of the state of the art on a topic where substantial research has been undertaken in the last 5-10 years and that is of ongoing interest for the PME members. The researchers are expected to be experts with respect to the research topic

and offer coherently structured activities during 2 slots of 90 minutes each, as well as a comprehensive contribution (30 pages) to the conference proceedings. A Research Forum is not supposed to be a collection of individual research presentations (see Colloquium format), but instead is meant to convey an overview of research, highlighting contemporary academic debates and perspectives in the field. In contrast to the Working Group format for evolving research topics, Research Forums are considered to mark significant advances with respect to established research topics.

The number of submitted RF proposals was 6, and all of them were accepted.

WORKING GROUPS (WG)

A Working Group is a format of group presentation undertaken by a group of at least two researchers. The aim of a Working Group is that PME participants are offered the opportunity to engage in exchange or to collaborate with respect to a common research topic (e.g., start a joint research activity, share research experiences, continue or engage in academic discourse). A Working Group may deal with emerging topics (in the sense of newly developing) as well as topics that are not new but possibly subject to changes. A Working Group needs a clear goal (outcome) and a strategy to reach this goal in collaboration with the participants. It must provide opportunities for contributions of the participants that are aligned with the goal (e.g., share materials, work collaboratively on texts, discuss well-specified questions). A Working Group is not supposed to be a collection of individual research presentations (see Colloquium format), but instead is meant to build a coherent opportunity to work on a common research topic. In contrast to the Research Forum format, meant to present the state of the art of established research topics, Working Groups are considered to involve fields where research topics are evolving.

The number of submitted WG proposals was 29, and 20 of them were accepted. Of those not accepted as WG proposals, 2 were invited to be resubmitted as Seminars (SE). After acceptance, one WG was withdrawn because of inability to participate in the conference.

SEMINARS (SE)

The goal of a Seminar is the professional development of PME participants, especially new researchers and/or first comers, in different topics related to scientific PME activities. This encompasses, for example, aspects like research methods, academic writing or reviewing. A Seminar is not intended to be only a presentation but should involve the participants actively. PME can give a certificate of attendance to participants of the Seminar.

There was no SE proposals submitted to PME 49 originally. Of those not accepted as WG proposals, 2 were invited to be resubmitted as Seminars (SE).

LIST OF REVIEWERS FOR PME 49

In the process of reviewing Research Reports and Publication Reports, 280 researchers from 44 different countries and regions took part. The PME 49 International Program Committee is grateful to all of them for their support. This year review process is exceptionally demanding due to high number of submissions.

A

Abdulhamid, Lawan (South Africa)
Abrahamson, Dor (United States)
Abtahi, Yasmine (Canada)
Adler, Jill (South Africa)
Akkoc, Hatice (Turkey)
Alatorre, Silvia (Mexico)
Albu, Carina (Germany)
Alcock, Lara (United Kingdom)
Alsalim, Lyla Ali (Saudi Arabia)
Anderson, Judy (Australia)
Andrà, Chiara (Italy)
Arcavi, Abraham Remarks (Israel)
Areti, Panaoura (Cyprus)
Armstrong, Alayne Cheryl (Canada)
Asenova, Miglena (Italy)
Ayalon, Michal (Israel)

B

Baccaglini-Frank, Anna E. (Italy)
Bakos, Sandy (Canada)
Ball, Lynda (Australia)
Bansilal, Sarah (South Africa)
Barwell, Richard (Canada)
Baumanns, Lukas (Germany)
Beisiegel, Mary (United States)
Bergqvist, Ewa (Sweden)
Bielinski, Sofia (Germany)
Bikner-Ahsbahs, Angelika (Germany)
Bingolbali, Erhan (Turkey)
Biza, Irene (United Kingdom)
Bobis, Janette (Australia)
Boero, Paolo (Italy)
Bose, Arindam Remarks (India)
Brown, Laurinda (United Kingdom)
Bruder, Regina (Germany)
Bruns, Julia (Germany)

C

Cabral, Tania (Brazil)
Carrapiço, Renata Carvalho (Portugal)
Chan, Man Ching Esther (Australia)
Changsri, Narumon (Thailand)
Charalambous, Charalambos (Cyprus)
Cheng, Ying-Hao (Taiwan)
Chikiwa, Samukeliso (South Africa)
Chimoni, Maria (Cyprus)
Chorney, Sean (Canada)
Choy, Ban Heng (Singapore)
Coles, Alf (United Kingdom)
Cooper, Jason (Israel)
Csíkos, Csaba (Hungary)
Cusi, Annalisa (Italy)

D

Da Ponte, Joao-Pedro (Portugal)
Damrau, Milena (Germany)
Di Martino, Pietro (Italy)
Diez-Palomar, Javier (Spain)
Ding, Liping (Norway)
Ding, Rui (China)
Doerfler, Willi (Austria)
Downton, Ann Patricia (Australia)
Dreyfus, Tommy (Israel)

E

Edmonds-Wathen, Cris (Australia)
Edwards, Laurie (United States)
Ekol, George L. (South Africa)
EL Mouhayar, Rabih Raif (Lebanon)
Erens, Ralf (Germany)
Eriksson, Anna Helena (Sweden)
Evans, Tanya (New Zealand)
Even, Ruhama (Israel)

F

Fellus, Olga (Canada)
Fernandez, Ceneida (Spain)
Ferrara, Francesca (Italy)
Ferrari, Giulia (Italy)
Ferrari, Pier Luigi (Italy)
Font, Vicenç (Spain)
Francisco, John (United States)
Fritzlar, Torsten (Germany)

G

Gal, Hagar (Israel)
Geisler, Sebastian (Germany)
Giberti, Chiara (Italy)
Glanfiield, Florence (Canada)
Gonscherowski, Peter (Germany)
González-Martín, Alejandro S. (Canada)
Graven, Mellony Holm (South Africa)
Gruver, John (United States)
Gutierrez, Angel (Spain)
Göller, Robin (Austria)

H

Hall, Jennifer Elizabeth (Australia)
Halverscheid, Stefan (Germany)
Hanazono, Hayato (Japan)
Hannula, Markku (Finland)
Hattermann, Mathias (Germany)
Heinze, Aiso (Germany)
Helliwell, Tracy (United Kingdom)
Helme, Rachel Elizabeth (United Kingdom)
Hernandez-Martinez, Paul (Australia)
Hershkowitz, Rina Remarks (Israel)
Hewitt, Dave (United Kingdom)
Heyd-Metzuyanim, Einat (Israel)
Hill, Julia L (Australia)
Hino, Keiko (Japan)
Hodkowski, Nicola (United States)
Horne, Marj (Australia)

Hoth, Jessica (Germany)
Huang, Hsin-Mei E. (Taiwan)
Hähkiöniemi, Markus (Finland)

I

Inglis, Matthew (United Kingdom)
Ingram, Jenni (United Kingdom)
Inprasitha, Maitree (Thailand)
Ivars, Pedro (Spain)

J

Jiang, Chunlian (Macau)
Johansson, Helena (Sweden)
Joklitschke, Julia (Germany)

K

Kageyama, Kazuya (Japan)
Kaiser, Gabriele (Germany)
Khan, Steven K. (Canada)
Kieran, Carolyn (Canada)
Kim, Ok-Kyeong (United States)
Kim, Sang Hyun (New Zealand)
Kirsten, Katharina (Germany)
Koichu, Boris (Israel)
Komatsu, Kotaro (Japan)
Kortenkamp, Ulrich (Germany)
Kosiol, Timo (Germany)
Kosko, Karl (United States)
Krause, Christina (Austria)
Krawitz, Janina (Germany)
Krummenauer, Jens Oliver (Germany)
Krzywacki, Heidi (Finland)
Kullberg, Angelika (Sweden)
Kuntze, Sebastian (Germany)
Kuzle, Ana (Germany)
Kwon, Oh Nam (Korea)

L

le Roux, Kate Remarks (South Africa)
Lepellere, Maria Antonietta (Italy)
Lerman, Stephen (United Kingdom)
Levenson, Esther Remarks (Israel)
Lilienthal, Achim J. (Germany)
Lim, Kien (United States)
Lin, Fou-Lai (Taiwan)
Lin, Su-Wei (Taiwan)
Lin, Yung-Chi (Taiwan)
Lindmeier, Anke Maria (Germany)
Lisarelli, Giulia (Italy)
Llinares, Salvador (Spain)
Lüken, Miriam M. (Germany)

M

Makonye, Judah Paul Remarks (South Africa)
Makramalla, Mariam Magdi Maurice (Egypt)
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Maracci, Mirko (Italy)
Markle, Josh (Canada)
Markopoulos, Christos (Australia)
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Menzel, Marielena (Germany)
Mesa, Vilma (United States)
Mesiti, Carmel (Australia)
Metaxas, Nikolaos (Greece)
Meyer, Michael (Germany)
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Mizoguchi, Tatsuya (Japan)
Moreno, MAR (Spain)
Morselli, Francesca (Italy)
Moutsios-Rentzos, Andreas (Greece)
Mulligan, Joanne Remarks (Australia)
Murphy, Carol (Australia)

N

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Nakawa, Nagisa (Japan)
Nannini, Bernardo (Italy)
Nardi, Elena (United Kingdom)
Neria, Dorit (Israel)
Neuhaus-Eckhardt, Silke (Germany)
Nicol, Cynthia (Canada)
Norqvist, Mathias (Sweden)
Nortvedt, Guri (Norway)
Novotna, Jarmila (Czech Republic)
Ntow, Forster (Ghana)

O

Oates, Greg (Australia)
Obersteiner, Andreas (Germany)
Oesterle, Susan Michele (Canada)
Okazaki, Masakazu (Japan)
Okumuş, Samet (Turkey)
Olsher, Shai (Israel)
Orellana, Claudia Andrea (Australia)
Otani, Hiroki (Japan)

P

Palatnik, Alik (Israel)
Palmér, Hanna (Sweden)
Pantziara, Marilena (Cyprus)
Papadopoulos, Ioannis (Greece)
Patsiomitou, Stavroula (Greece)
Pearn, Catherine Anne (Australia)
Pierce, Robyn (Australia)
Pinto, Alon (Israel)
Pinto, Marcia (Brazil)
Pitta, Georgia (Greece)
Pittalis, Marios (Cyprus)
Pitta-Pantazi, Demetra (Cyprus)
Pocalana, Gabriella (Italy)
Potari, Despina (Greece)
Prediger, Susanne (Germany)
Prodromou, Theodosia (Australia)
Proulx, Jerome (United Kingdom)
Psycharis, Giorgos (Greece)
Pustelnik, Kolja (Germany)

R

Rach, Stefanie Remarks (Germany)
Rasslan, Shaker[9] (Israel)

Reid, David A (Norway)
Remillard, Janine (United States)
Rivera, Ferdinand Dychitan (United States)
Roberts, Nicky Remarks (South Africa)
Ronda, Erlina (Philippines)
Rott, Benjamin (Germany)
Rubel, Laurie H (Israel)
Ruwisch, Silke (Germany)

S

Sabena, Cristina (Italy)
Safuanov, Ildar (Russia)
Sakonidis, Haralambos (Greece)
Salle, Alexander (Germany)
Sánchez, Alicia (Spain)
Scheiner, Thorsten (Germany)
Schindler, Maike (Germany)
Schubring, Gert (Germany)
Schukajlow-Wasjutinski, Stanislaw (Germany)
Schwarts, Gil (Israel)
Seah, Wee Tiong (Australia)
Shimizu, Yoshinori (Japan)
Shinno, Yusuke (Japan)
Shvarts, Anna (Netherlands)
Simon, Anna Lisa (Germany)
Soendergaard, Bettina Dahl (Denmark)
Solar, Horacio (Chile)
Solares-Rojas, Armando (Mexico)
Soldano, Carlotta (Italy)
Spagnolo, Camilla (Italy)
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Strohmaier, Anselm R. (Germany)
Sumpter, Lovisa (Sweden)
Superfine, Alison (United States)
Surith, Dhanya (United Arab Emirates)
Sztajn, Paola (United States)

T

Tabach, Michal (Israel)
Tan, Sheena (Canada)
Tatsis, Konstantinos (Greece)
Thanheiser, Eva (United States)
Thomas, Charlott (Germany)
Thomas, Michael O. J. (New Zealand)
Thornton, Stephen John (Australia)
Tortora, Roberto (Italy)
Triantafillou, Chrissavgi (Greece)
Tso, Tai-Yih (Taiwan)

U

Ufer, Stefan (Germany)
Umgelter, Karyna (Germany)

V

Vale, Isabel (Portugal)
Vale, Pamela (New Zealand)
Van Dooren, Wim (Belgium)
Van Hoof, Jo (Belgium)

Van Zoest, Laura (United States)
Vancsó, Ödön (Hungary)
Venkatakrishnan, Hamsa (Ireland)
Viguer Pérez, Mireia (Germany)
Viirman, Olov (Sweden)
Vogel, Rose F. (Germany)
Vos, Pauline (Norway)

W

Wagner, David (Canada)
Wang, Chih-Yeuan (Taiwan)
Wang, Ting-Ying (Taiwan)
Watanabe, Koji (Japan)
Way, Jennifer (Australia)
Weigand, Hans-Georg (Germany)
Widder, Mirela (Israel)
Williams, Gaye Remarks (Australia)
Wolff, Alexander (Germany)
Wu, Lan-Ting (Taiwan)

Y

Yang, Kai-Lin (Taiwan)
Yerushalmy, Michal (Israel)

Z

Zazkis, Rina (Canada)
Zhang, Qiaoping (Hong Kong)
Zhang, Ying (Hong Kong)
Zhang, Ying (United Kingdom)
Zorrilla, Cristina (Spain)

Ö

Öçal, Mehmet Fatih (Turkey)

PME 49
Helsinki, Finland
2026

PLENARY LECTURES

GROUNDING MATHEMATICS IN PERCEPTUALLY GUIDED ACTION: FROM SENSORIMOTOR ENTRAINMENT TO DISCIPLINARY DISCOURSE

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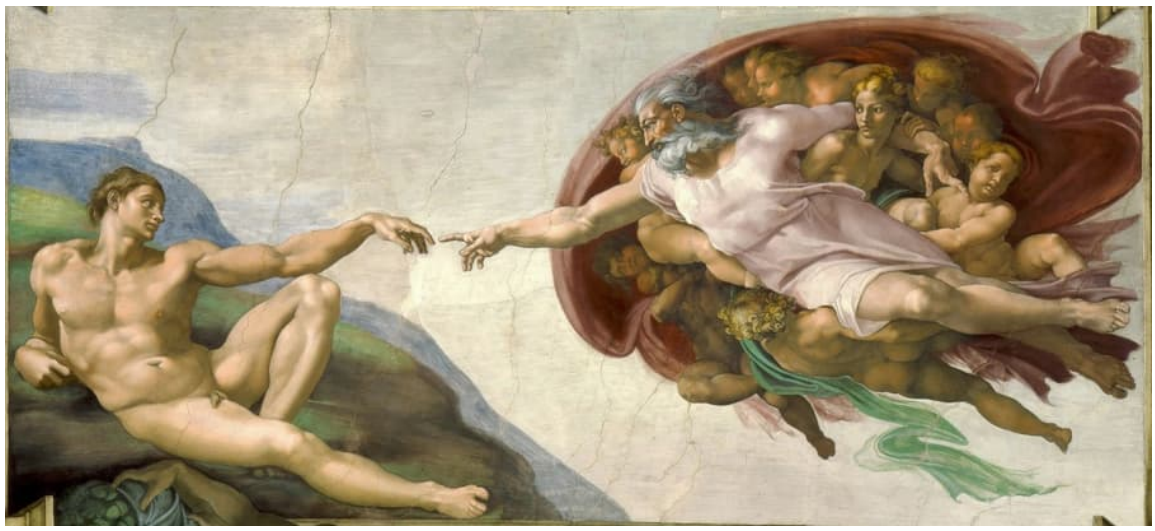
Embodied design is an educational research paradigm pursuing fundamental epistemological and ontological questions through developing and evaluating material and digital resources for mathematics pedagogy. Drawing on humans' evolutionary biological capacities, two design genres came forth in dialogue with cultural–historical techno-scientific practice: perception-based embodied design recruits our ability to perform instantaneous inferences about ecological magnitudes formally analyzed as intensive quantities, such as slope, density, or likelihood; and action-based embodied design elicits our knack for learning to move in new ways to solve motor-control problems, whereby we spontaneously form environmental Gestalts facilitating coordination. Here I show how these two genres cohere theoretically in Enactivism.

EDUCATION AS NEGOTIATION: RENAISSANCE OF A METAPHOR

Figure 1: Learning as guided coordination between evolutionary (student) and cultural (teacher) perceptions of situated phenomena



Note. Michelangelo, *The Creation of Adam* (c. 1508–1512), Sistine Chapel, Rome

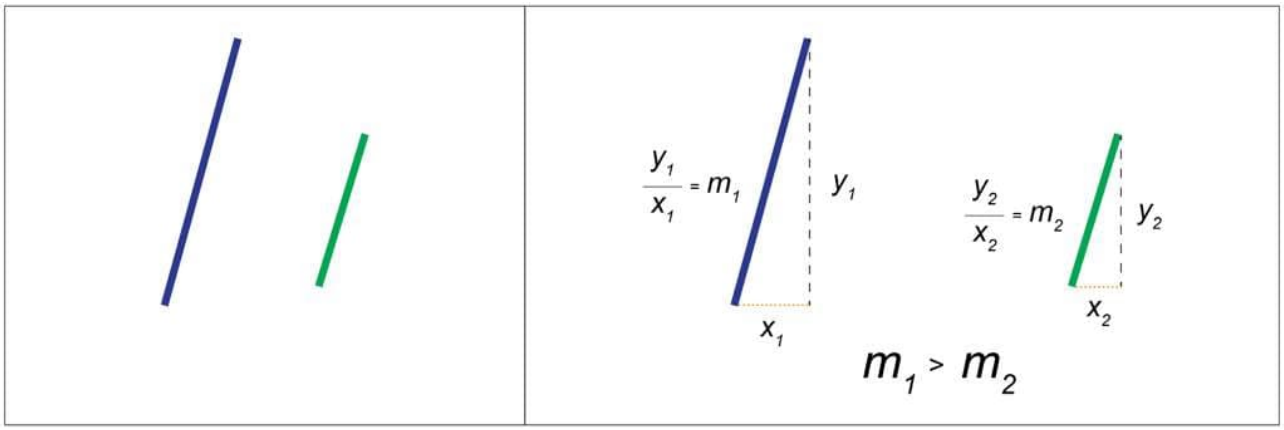
If Michelangelo were still around today, he'd be a very rich guy, because his famous painting is so often used as a metaphor for some idea. Here we go again (see Figure 1). I invite you to consider Adam, on the left, as the "student," and God, on the right, as "culture." Now, as you can see, students arrive into the world wearing very little. Yet, just like all other mobile organisms on Earth, students arrive genetically ready to

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 3-26. PME.

just like all other mobile organisms on Earth, students arrive genetically ready to *perceive* relevant environmental features so as to *act* on them, that is, to develop ecologically adaptive *perception–action loops*. Let me isolate these two components—*perception* and *action*—as distinct resources for mathematics education. By "perception" I will highlight humans' ability to rapidly judge quantitative features of the environment, such as our innate sensitivity to the likelihood of random events (Xu & Garcia, 2008). I call pedagogical activities drawing on this capacity *perception-based embodied design*. By "action" I'll highlight human capacity for learning to move in new ways along diverse terrains in coordination with fellow humans, such as our basic forms of individual or social locomotive motility—walking, running, skipping, dancing (Sheets-Johnstone, 2015). I call pedagogical derivates of this capacity *action-based embodied design*. Ok, so that's about the "student," on the left of Figure 1.

What about culture (on the right of Figure 1)? While kids arrive ready to learn their tribe's language, handle its artifacts, and adopt and adapt its epistemic forms, culture, meanwhile, has sped ahead, developing sophisticated techno-scientific methods that extend the naked mind to the point that these methods don't always conform with naïve ways of being, doing, and thinking. Our wonderful native heuristics, which have served our species extraordinarily well for millennia, now prove inadequate for sending a spaceship around the moon or calculating *pi*, let alone filing our taxes! Consider a student comparing two slanted lines to judge which one is steeper (see Figure 2a). She looks back and forth but cannot decide, because evolution has equipped us to perceive situated phenomena only to the level of precision as has been necessary in our ecological niche (Gigerenzer & Brighton, 2009). When two lines are slanted at a difference of one degree, we cannot select the steeper one with confidence.

Figure 2: Comparing genetic and cultural perception: the case of diagonal lines



a. b.

Culture, on the other hand, perceives this same situated phenomenon quite differently from nature (see Figure 2b): considering each of these lines independently; treating visual features of these lines that nature ignored—the lines' rise (*y* interval) and run (*x* interval); measuring these linear magnitudes (x_1 & y_1; x_2 & y_2); dividing these values to get two quotients (m_1 and m_2); and finally comparing these quotients' magnitudes to

determine which one is steeper ($m_1 > m_2$). Clearly the naked-eye evolutionary gaze and the techno-scientific cultural gaze are different ways of perceiving a line.

Yet these evolutionary and cultural perceptions, I propose, are also different in terms of their epistemic tenor: The *meaning* of slope lives in the naïve gaze, yet the *calculation* of slope, along with its semiotic package of symbolic notation and professional discourse, lives in the cultural gaze (cf. Tall & Vinner, 1981). Wait, but if these two gazes are epistemically different, how might we present mathematical concepts as meaningful (i.e. as "secondary intuition," Fischbein, 1987)? Looking again at Figure 1, what should happen *between* the hands of the student and culture? How might culture hand off its analytic view of phenomena to the natural child? How might the student *ground* the meaning of mathematical symbols (cf. Harnad, 1990)?

This paper is about how evolution and culture can go hand in hand. I will be promoting perception-based and action-based embodied design as pedagogical frameworks for grounding conceptual discourse in evolutionary capacity. The paper explains how these two heuristic frameworks enable students to: (a) elicit their innate sensorimotor inclinations as resources for developing naïve pre-semiotic meanings pertaining to situated phenomena; then (b) dialogue with disciplinary analytic reconfigurations of these same phenomena. It's an ambitious, retrospective paper, in which I further wish to present these two heuristic design frameworks as intellectually cohering on a single theoretical foundation, embodiment, specifically enactivism. And so now we must take a theoretical detour to explain what I mean by embodiment and why, of the various strains of embodiment, our lab favours enactivism as framing both the design and evaluation of educational resources. Then I'll present the designs and compare them.

THE E-TURN IN THE COGNTIVE SCIENCES: FRAMING DESIGN-BASED RESEARCH ON THE SENSORIMOTOR ROOTS OF MATH CONCEPTS

The *E-turn* in the cognitive sciences—E is for embodied, embedded, extended, enactive, etc.—is a big intellectual tent hosting a range of philosophical positions, theoretical speculations, and empirical findings respecting the conjectured modal workings of the mind (Kiverstein & Clark, 2009; Newen et al., 2018). In brief, *embodied* means that our biological constitution informs, shapes, and determines our cognition (e.g., having ten fingers is formative of our quantitative reasoning); *embedded* suggests that cognitive activity is irreducibly situated in natural and socio-cultural ecologies *for* which, and *through* which, it develops—phylogenetically, sociogenetically, and ontogenetically (e.g., naïve perceptual inferences about likelihood were evolutionarily selected for their adaptive advantages); *extended* implies that we engage in the world through our artifactual "prostheses" that participate in our cognitive activity by modifying and augmenting our perception, memory, and reasoning (e.g., a cane, a notebook, an abacus, a navigation app); and *enactive* maintains that all cognition is situated sensorimotor *activity*—there is no other form of

knowing, so even "off-line" cognition is covert sensorimotor activity (e.g., mathematical reasoning is "just another way" of engaging the environment).

As such, the E-turn breaks away from *cognitivism*, the hegemonic 20th century view of the mind as a computational device that processes amodal abstract units of information. The E-turn foregrounds as epistemologically instrumental all the stuff outside of the skull: the various modal, motor, material, social, and cultural constituents of human being. Viewed from the outside as one big happy E-family, inside this home theoretical debates continue, even within each of the E-camps (Gallagher, 2015; McGann et al., 2020; Sheets-Johnstone, 2015). For example, some enactivist scholars adhere to the Humberto Maturana camp, subscribing to his slogan, "Everything that is said is said by an observer," whereas others identify more so as disciples of Francisco Varela, who implicated sensorimotor activity as the mechanics of cognition (Drury & Tudor, 2024). Our laboratory has most closely affiliated with the Varela strain, because the learning environments we build and test are deliberately designed for students to ground mathematical concepts through sensorimotor exploration that builds perception for action. Varela's approach gives us the tools we need to work on our artifacts and data.

Varela et al. (1991) wrote the following:

> In a nutshell, the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided. (pp. 172–173)

The pedagogical derivative of these dense principles, as I have understood them, is that if I want students to develop cognitive structures bearing mathematical meaning, then I should create conditions in which the students learn to act in some new way that instantiates the concept I want them to learn. Initially, the students will struggle to act in that new way, because they don't yet have these cognitive structures, and therefore they cannot yet orient perceptually to the phenomenon to guide task-effective actions; they haven't yet discerned ontologies in the environment that would govern effective interaction (they haven't detected an "affordance," Gibson, 1977; or an "attentional anchor," Hutto & Sánchez-García, 2015). So, the students explore, looking here or there, trying this or that, until, gradually, task-effective sensorimotor patterns are tacitly selected for that consistently elicit the desirable feedback. These sensorimotor patterns iteratively give rise to cognitive structures that enable perception to guide action. It is thus, I believe, that students develop new meanings. Yet for these meanings to become mathematical, students must express the meanings in the semiotic registers of the mathematical discipline, such as in diagrams, tables, and symbolic notation. Notwithstanding, the pedagogically critical phase of learning is in building meaning, not signs (see also Thompson, 2013). As Rousseau put it in 1762,

> What is the use of all these symbols; why not begin by showing him the real thing so that he may at least know what you are talking about? . . . As a general rule—never substitute the symbol for the thing signified, unless it is impossible to show the thing itself; for the child's attention is so taken up with the symbol that he will forget what it signifies. (p. 170)

Citing 18th century Rousseau to explain 21st century enactivist pedagogy suggests that, indeed, the more things change, the more they stay the same. As educational *scholars*, we note that theorizing sensorimotor activity as the driver of conceptual learning resonates with practicing the educational legacy of Friedrich Fröbel, Maria Montessori, Hans Freudenthal, Caleb Gattegno, Zoltan Dienes, Richard Skemp, Seymour Papert, Constance Kamii, Guy Brousseau, James Kaput, and many others who believed children should experience mathematical concepts before they signify these experiences in normative semiotic register. We therefore affiliate with radical-constructivism (Steffe & Kieren, 1994; von Glasersfeld, 1983) and early adopters of enactivism (Pirie & Kieren, 1989). As educational *designers*, enactivism extends constructivist principles from early cognitive developmental psychology—enactivism embraces findings in neurophenomenology and expert ethnography on the ubiquitous function of modal exploration in learning and inventing (e.g., Tao, 2016). Enactivism explains our design practice as creating for students *fields of promoted action* (Reed & Bril, 1996), that is, cultural–historical socio-material settings that pose motor-control problems whose solutions foster the development of enactive skills valorized in the local context. Yet as educational *researchers*, we appreciate enactivism's technical grip on the sensorimotor roots of learning and thinking. In sum, through enactivist lenses we see more clearly what we should design, how we might engage students, what we might measure, analyze, and show, and what theories we should develop.

The objective of this paper is to look back on my thirty years of design-based research to evaluate whether enactivism can account coherently for laboratory findings from implementing two types of educational activities: perception-based and action-based embodied design. The theoretical problem driving this paper is that students' observed behaviors as they engage in perception-based activities appear quite different from their observed behaviors as they engage in action-based activities. Perception-based activities do not require constant manipulation as do action-based activities, so why do I model perception-based activities according to enactivist dogma? Namely, where in perception-based embodied design is the purported sensorimotor activity said to build new perception–action loops grounding conceptual understanding? In my defense, I will challenge researchers' definition of action. To build a new perception–action loop, I offer, we need not act *effectively* on the world—we can, instead, act only *apparently*.

PERCEPTION-BASED AND ACTION-BASED EMBODIED DESIGN

Embodied design is a design-based research program that draws on embodiment theory to create, evaluate, and interpret pedagogical resources for mathematics education. The embodied-design research program evolved through several projects. This section presents the Seeing Chance project (2003–2013) as an exemplar of perception-based design, and the Mathematics Imagery Trainer project (aka Kinemathics, 2008–present) as an exemplar of action-based design. As an action-based embodied design, Kinemathics activities have often been more immediately recognized as implementing

embodied–enactivist principles, because the rationale of this project is for students to learn a new mathematical concept through learning to move in a new way. The movement constitutes a "conceptual choreography," in the sense that it dynamically instantiates the concept itself, even before this new movement form has been modeled as such (just like children learn to stack sand before they think of this as the arithmetical operation of addition). By way of juxtaposition, the Seeing Chance project scaffolds students to *perceive* a static situation in a new way. Whereas the students perform various manual actions to produce the artifacts that lead to this new perception, the actions themselves do not instantiate the concept holistically—the key learning opportunity begins only once the actions have stopped. The student then gazes at the artifacts, struggling to perceive it in a new way. Yet even as the student shows no *overt* movement, I will argue, the student is engaged in *covert* sensorimotor activity. As such, per enactivism, my line of reasoning will be to implicate students' learning across the Seeing Chance and Kinemathics projects as sensorimotor exploration through which cognitive structures emerge that enable task-effective action to be perpetually guided, where action that is either covert or overt to the eyes of the researcher. Accordingly, once I have overviewed each of these two projects, below, the subsequent section will then argue for the two projects' theoretical coherence under an enactivist umbrella.

Perception-based embodied design

Here's a paradox. One the one hand, infants can distinguish between likely and unlikely compound events. Show them a tub containing mixed balls of two colors—perhaps half are green, half are blue. Now pull out a sample of four balls. Infants know that a mixed sample, say two green and two blue, is more likely that a sample of uniform color, say all green (Xu & Garcia, 2008). On the other hand, when evolution meets culture, that is, when intuitive likelihood goes to school, probability becomes *counter*intuitive. What goes wrong? Evolution and culture judge the same situation, arriving at the same inference, yet students cannot bring to bear their innate evolutionary capacity to make sense of the cultural model of the situation. Is students' informal capacity to judge likelihood irrelevant to a formal understanding of likelihood? Or might we figure out how to work *with* children's powerful evolutionary capacity rather than *against* it (Smith et al., 1993)?

The core issue is whether or not students are attending to the order of singleton events in a compound event. For example, when we flip two coins, we might get: two heads (HH); two tails (TT); or a mixed result. These are three different combinations. However, whereas there is only *one* way of getting the 2H combination, and there is only *one* way of getting the 2T combination, there are *two* ways (variations) of getting a mixed-result combination—HT or TH—for a total of *four* unique equiprobable permutations (HH, HT, TH, TT). This means that when we flip two coins multiple times, the cumulative results will increasingly tend toward a distribution where half are a mixed result, a quarter are HH, and a quarter TT. Understanding the logic and

implications of this combinatorial analysis is crucial for making headway in studying probability, going beyond infantile strengths (cf. the case of slope, see Figure 2).

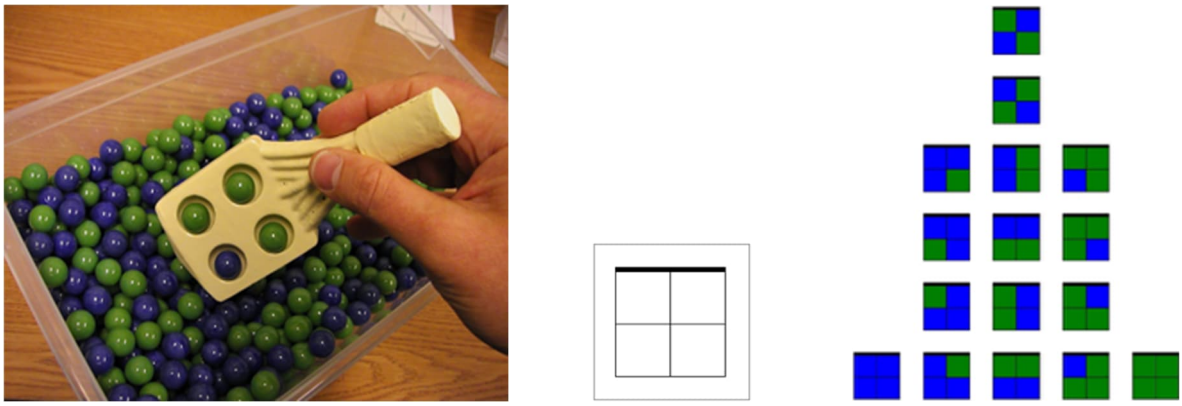
Intriguingly, however, while people intuitively guess that a mixed result of flipping two coins is the most likely event, they will often reject the analytic notion that HT and TH are distinct outcomes, typically claiming, "They're the same! Order doesn't matter!" In fact, some people, once learning that there are three possible compound-outcome combinations (i.e. HH, TT, mixed), will change their initial guess, saying instead that, given the three combinations—and given that order does not matter—we are *equally* likely to receive any of these three events (i.e. HH, TT, and mixed are equiprobable)! These people are caught in an epistemic limbo between mathematically correct solid intuitive statistics pertaining to a situated phenomenon (i.e. that "mixed" is the most likely compound event) and an incomplete analytic modeling pertaining to the same phenomenon (three "flattened" events that, therefore, are equally likely).

Are these our only two options—intuitive naïve heuristics and unintuitive ideal forms? Can educators somehow work with students' mathematically correct intuitive likelihood, along with its belief that "order doesn't matter," even as we attempt to introduce the mathematically correct combinatorial analysis, along with its rule that "order does matter"? How might students develop a "secondary intuition" (Fischbein, 1987)? What pedagogical principle might mobilize such conceptual growth?

The Seeing Chance project (see Figure 3) sought to create opportunities for students to learn combinatorial analysis of compound events by coordinating informal judgment with formal models. Thus, students were to: (a) bring to bear mathematically correct naïve perceptual intuitions pertaining to expected outcome distributions from a random generator; (b) encounter mathematical analysis of the same phenomenon; (c) acknowledge the mathematical model as posing epistemic conflict with their naïve phenomenology; yet (d) reconcile these two epistemic resources as complementary by coming to perceive the mathematical model as congruent with intuitive judgment.

I designed a tub with hundreds of mixed colored marbles, half green and half blue, and a 3D-printed "marbles scooper" engineered to extract a sample of exactly four marbles (see Figure 3a). The idea was for students to guess intuitively that the most likely scoop would be a sample of two green and two blue marbles (hence, 2g2b)—more likely than, say, scooping a 3g1b sample, let alone a 4g sample. While mathematically correct, note how this intuitive guess ignores the *order* of singleton events in the compound event (e.g., they perceive the scoop in Figure 3a as an orderless 3g1b). And yet mathematical theory requires attending to the order of singleton events in the compound event as the analytic means of building the sample space: each combination (e.g., 3g1b), should be spelt out in terms of its possible variations (e.g., GGGB, GGBG, GBGG, BGGG). The pedagogical design challenge was to present this mathematical model, which attends to the order of singleton events in compound events, as a complementary perspective to the students' order-less intuitive perception of the same random generator.

Figure 3: Key non-digital materials from the Seeing Chance design for the binomial. Project page: https://edrl.berkeley.edu/projects/seeing-chance/



a. b. c.

Note. Strictly speaking, the marbles-scooping experiment is hypogeometric, not binomial. Imagine the scooping as a "slow motion" action. Once the first marble has fallen into one of the four scooper concavities, the green-to-blue 1:1 population ratio in the tub changes ever so slightly (e.g., 99:100), thus biasing the chances of the *next* single marble scoop, just as in "no return" experiments. As such, this experiment is unlike the purely binomial case of flipping four coins. However, given the ratio of "4 marbles per scoop" to "hundreds of marbles in the tub (i.e. quasi-infinity)," the experiment stands as a binomial approximation. Not included in this figure are the computer-based frequentist probability experiments that complement these subjectivist and classicist probability tasks.

The design solution was for students to build the sample space themselves. We worked individually with twenty-four children aged 9 to 11 years old. We gave them a stack of cards (see Figure 3b) and crayons and asked them to create a set of iconic images depicting all possible outcomes. We guided the students to include also variations on each combination, even when the students insisted that the order (unique pattern) of the internal colored squares is irrelevant to anticipating the scooping outcomes. Once the sample space was complete, we explored with the students various display configurations that would highlight features of the sample space of possible interest to viewers, eventually steering the students towards shaping the sample space in the form of a 1-4-6-4-1 picture chart (see Figure 3c), which we called the *combinations tower* (hence "CT"). Quite immediately, this form elicited from the students a new reaction. They told us that the CT expresses their earlier judgment that the plurality of outcomes would be of type 2g2b. As such, the students accepted the classicist inclusion of variations on the combinations *only after they had accepted the product of that process.*

The data present intriguing conversations, in which the students go back and forth between repudiating and endorsing the variations, until they appear to achieve cognitive equilibrium. Our analyses took the form of an inquiry into what the students' epistemic grounds might be for finally accepting the full-on sample space. We concluded that the students figured out how to perceive the 1-4-6-4-1 CT sample space as agreeing with their 2g2b > 3g1b > 4g intuitive judgment of expected frequencies,

what we called achieving *inferential parity*. I call this a *semiotic leap*, where an informal felt sense or presymbolic notion (Petitmengin, 2007; Radford, 2003) finds its expression in a formal display, along the way endorsing an analytic differentiation that is conceptually critical (i.e. that order matters). Our participants' semiotic leaps demonstrate how perception-based embodied design enables the guided reinventing, understanding, and grounding of mathematical concepts. Naïve and ideal perceptual constructions of situated phenomena need not be mutually exclusive—they can be cognitively synergistic as conceptually complementary. Thus culture nurtures nature.

In this sense, the teaching~learning interactive process of acculturation might be viewed as a negotiation—perhaps as a mediated conflict resolution—over competing perceptual orientations to situated phenomena (cf. Sfard, 2002). Consider the specific 3g1b card with a blue square in the top-left corner, one of the 16-element CT sample space (see Figure 3c, the bottom card in the 3g1b column). Initially, the student tacitly perceives this card as metonymically standing in for *any* card in its column class. That is, at first the student perceives a particular compound-event outcome as effectively carrying the likelihood of its entire combination class. Yet the mathematical gaze on this card constitutes it as a distinct, differentiated, equiprobable member of its combination column class and, in fact, of its entire 16-element sample space. Notwithstanding, to instinctively behold a permutation as a combination is a powerful evolutionary heuristic response exemplifying our species' adaptive attunement to ecologically relevant instantiations of likelihood (Gigerenzer & Brighton, 2009). It is one of several perceptually privileged intensive quantities that include density, slope, speed, and so on. Rather than discouraging or even attempting to eradicate this innate proclivity to perceive a permutation as a combination, I maintain, educators might instead seek a learning trajectory that endorses this ecological perspective.

As such, I submit, an instructional conversation with students should foreground, celebrate, and explicate their naïve perceptual construction, even as one introduces the ideal mathematical model as a shift in attention (Mason, 1989). Teachers might go about steering a child along a reasoning path of the following general character: (a) I can also see the specific 3g1b card as constituting any 3g1b card; and, as such (b) I can also feel as though this card is more likely to occur than the 4g (all green) card; yet at the same time, (c) we can both note that this card is part of a group of cards sharing the same 3g1b attribute, though it has a unique configuration that is different from the other 3g1b variations in its column; so, perhaps, therefore, (d) our focal card is equiprobable with its column cards; and, as such, perhaps (e) it is equally likely to occur as the 4g.

While our data evidence participants' divergent idiosyncratic conversations with the tutor–researcher, finally they all arrive at the same insight. Students introspect on their naïve perceptual construction, briefly consider the mathematical construction, fall back on their original gaze, and openly acknowledge fluctuating between perceptions of a specific card—as a combination (naïve) or a variation (ideal). Eventually they say that 2g2b will occur more often than other combinations *because* there are more elements

in the 2g2b class than in any other of the four classes (cf. Tirosh & Stavy, 1999). They have found a new way of perceiving the mathematical display as *meaning* their tacit perceptual judgment pertaining to the source phenomenon. They thus endorse the sample space as a semiotic means of objectifying their presymbolic anticipated distribution of experimental outcomes from the random generator (cf. Radford, 2003).

By affording the expression of their naïve inference pertaining to the marble tub, the CT becomes a model of the phenomenon. As such, the CT further comes to serve as a semiotic pivot affording engaged participation in the social enactment of mathematical discourse (Bartolini Bussi & Mariotti, 2008). Students are now equipped with a perception of the mathematical display as constituting the sample space of a random generator, moreover an endorsed model that resonates with their own mathematically correct intuitive judgment. Thus, the students can now practice forms of collaborative argumentation that draw on explicit symbolic artifacts as evidence supporting well-articulated claims. Critically, students are never asked to surrender their initial naïve perceptual construction of the random generator. On the contrary, that perception is valorized as the source of mathematical meaning. Instead, they experience naïve and idea perceptual constructions of a symbolic artifact as contextually valid, cognitively commensurate, and conceptually complementary. How you see is what you think.

We will keep returning to the theme of attentional shifts in perception as pivotal to developing mathematical understanding of situated phenomena. Having demonstrated the perception-based genre of embodied design that taps the evolutionary capacity for perceptual judgment, we now turn to overviewing the action-based genre of embodied design that taps the evolutionary capacity of learning to move in new ways. Action-based activities, too, orchestrate students' attentional shifts, only that they enable students to coordinate task-effective actions manipulating objects in their environment.

Action-based embodied design

The basic logic is simple. Stop a person in the street and say, "Excuse me, Madam, what does it mean to *add*?" If you're lucky and they don't call the police, they will answer you, using both speech and gesture. Watch their hands. You will see gathering, connecting, stacking, filling, extending, and so on, either of imaginary continuous chunks or of discrete quantities, such as their fingers; or you'll see any other schematic variant on dynamic accumulation that shows both the process and product of adding (Lakoff & Núñez, 2000). This is their *meaning* of adding—their enactive *grounding* of the "+" symbol (Harnad, 1990), their *presymbolic notion* (Radford, 2003) or *concept image* (Tall & Vinner, 1981) of the operation. You have elicited an overt enactment of a mathematical conceptual choreography that is always triggered yet usually covert.

Where does such cognitive capacity originate, so that one sunny day, on the street, we can gesture our enactive meaning of a mathematical idea? Silverman (2022) analyzed videography of toddlers playfully extracting sticky clay from a large mound. They gripped the clay in their dominant hand, approached more clay, freed up their dominant hand by stacking its clay content into the nondominant hand, clawed more clay from

the mound, stuck it, too, onto the growing chunk in the other hand, and so on. It is such new coordinated sensorimotor achievement, Silverman speculated, that evidence the birth of new proto-conceptual schemas—enactive know-how that a teacher will elicit some years later in primary school, when these youngsters learn the cultural–historical mathematical practice of adding; the very schemas that will one day be prompted by a friendly stranger they meet on the street, who asks them for the meaning of addition.

So, the basic logic is simple. Mathematically competent individuals behold (*to be* and *to hold*) or comprehend (*to grasp together*) mathematical notions as stabilized forms of egocentric kinesthetic agency, which they first learned to enact by figuring out how to coordinate some new situated dynamic transformation (Gerofsky, 2010; Hadamard, 1945; Tao, 2016). This enactment may be either covert or overt—your brain performs similar neural activity whether you handle actual or imaginary percepts, and your phenomenological experience will by-and-large be the same (Duarte-Mendes et al., 2025; Gallese & Lakoff, 2005; Levy, 2019; Vygotsky, 1926/1997). Fine, you might concede, that's all very well for simple addition, but what about multiplication, rational numbers, algebra, calculus? At what point do students lose a grip on mathematics? At what point do teachers give up teaching for meaning and focus only on teaching for testing (qv. Lockhart, 2009)? What might we do about this? Here's a suggestion.

Back to toddlers. Just like they develop an embodied, egocentric, dynamic, and presymbolic proto-conceptual notion of *adding* through handling clay in a field of promoted action, might students analogously develop notions for multiplication, imaginary numbers, or the Banach–Tarski paradox by manipulating objects in specially designed learning environments? Recall the toddler whose dominant hand was clayful, with no room for more clay, and so, to solve this motor-control impasse, he glued its content upon the other hand's load, thus reinventing the proto-concept "addition." As enactivist design-based researchers, we might devise conditions for students to engage in a task that, likewise, occasions an impasse, whose spontaneous enactive solution transforms students' perceptual orientation to the environment; they would detect an affordance that enables effective action.

Hence the *Mathematics Imagery Trainer*, a pedagogical architecture that fosters proto-conceptual perceptual solutions to motor-control problems. You solve motor-control problems in the Trainer by perceiving new affordances for action, just like that clay-play toddler, thus extending your enactive mind to navigate new immaterial frontiers of cultural–historical mathematical terrains. You learn new mathematical ideas just like you learn to tie your shoes or ride a skateboard (De Jonge-Hoekstra, 2025; Freudenthal, 1971; von Glasersfeld, 1983): it's about developing new perception for action.

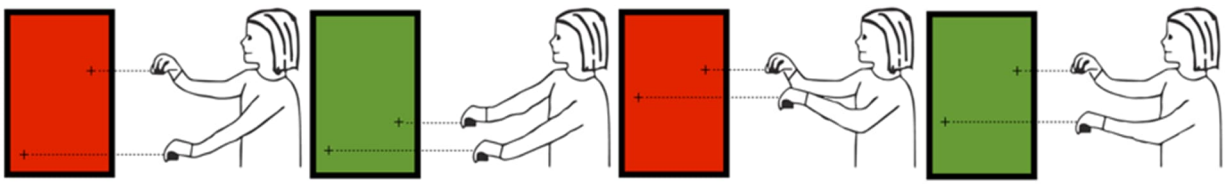
The Mathematics Imagery Trainer design architecture presents a motor-control problem whose enactive solution will ground prospective disciplinary forms. As such, it is a field of promoted action, where the goal action constitutes a dynamic instantiation (a "phenomenalization," Pratt et al., 2006) of the concept in question. Just like toddlers can ground prospective meanings of addition by playing with clay, the

Mathematics Imagery Trainer can, in principle, create analogically playful contexts for students to ground the prospective meaning of any mathematical concept. The basic design, which I first conceived in 2008 to support the learning of proportions, has been implemented in a variety of digital platforms, HCI configurations, and accessibility functionalities, such as multitouch tablets or computer-vision motion detectors with sonification feedback, to support learning a range of concepts (Alberto et al., 2022).

The Trainer's pedagogical procedure is to pose students with a motor-control problem, solicit their available "best shot" enactments, demonstrate that these enactments are contextually inadequate, and create conditions for students to engage in sensorimotor exploration, through which they arrive at new enactive conceptions. In the case of proportional reasoning, young students encounter an interactive device whose additive manipulation regime proves ineffective. They tinker and adjust their movement form until they discover a qualitatively new way of moving, a multiplicative regime. Once students verbally articulate this new strategy, they are offered symbolic artifacts designed to shift their perceptual attention toward improved forms of motor action that, in turn, they express in disciplinary discourse as grounded quantitative conceptions.

Figure 4 is four-panel schematic depiction of an activity progression through which students learn to "move in green," that is, to enact a dynamic conservation. Only later will they develop a conceptual formulation of this enactment form by using available semiotic means. Students are presented with an interface for manipulating two cursors—the left-hand and right-hand cursor, respectively. The screen is initially red, and the students' task is to make the screen green. Typically, they move one cursor up and down, then the other, but the screen is still red. They might then try moving both cursors up and down together, moving them both faster, then slower, but still the screen is red (see Figure 4a). They move the cursors in opposite directions, flailing their hands…, until suddenly the screen turns from red, through color gradations, to green (Figure 4b). They "freeze," their hands, and ask, "Is that it?" We respond, "Well done, but now move both hands keeping the screen green." The child gazes at the screen, wondering what they can keep *the same* while they move to keep the screen green.

Figure 4. The Mathematics Imagery Trainer for Proportion—the Parallels motor-control problem: four key events in learning to enact the conceptual choreography. Project page: https://edrl.berkeley.edu/projects/kinemathics/



a. b. c. d.

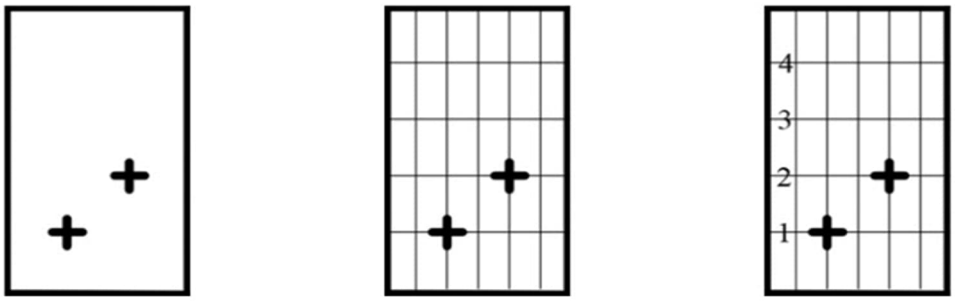
There are no spatial cues hinting how one might enact a dynamic invariance, just a big blank screen. But there are the two cursors that one could perceive as spatially related by attending to the *interval* between them. Perhaps this interval should remain *the same*

even as we lift or lower our hands? The child lifts her hands, keeping the interval constant, but the screen turns red again (see Figure 4c). Why? Because we have programmed the system to turn the screen green whenever the respective heights of the hands above the bottom of the screen relate by a particular ratio, say 1:2. Their hands had chanced on a green combination, say, 10cm and 20cm, respectively, up along the screen. But now they have moved their hands up, say to 25cm and 35cm, thus violating the 1:2. Note that the child is attending to the interval between the cursors (or between her hands). We call this an *attentional anchor*, an environmental aspect or features whose perception facilitates enacting a solution to a motor-control task by collapsing degrees of freedom (cf. "dimensional compression," Riley et al., 2011). More generally, humans learn to perform challenging multilimb coordinations by adapting their perception of the environment (Bernstein, 1996; Mechsner et al., 2001).

Soon enough, the child modifies the spatial interval until the screen is green again (see Figure 4d). We ask her to repeat the task of moving in green. Again, the child moves both hands keeping the interval intact, regresses to red, and fixes the distance. After several trials, the child realizes that her fixes are patterned, for example, as her hands go up, she keeps having to extend the interval between their hands. In response, the child "pre-fixes" the interval by changing it *as* the hands move. She has been attempting to assimilate the multiplicative structure onto her additive scheme, but now she realizes she is dealing with a new type or situation. She says, "I get it—the higher my hands go, the bigger the distance." Thus, she achieves dynamic structural coupling.

Once the child has verbally–gesturally articulated their action pattern, the interview protocol shifts. We illuminate onto the screen a grid of lines that renders the screen's continuous space into a discrete mesh of cells (compare Figure 5a & b). We casually mention to the students that they might, yet need not, use the grid to perform their task of keeping the screen green as they move their hands. As our data reveal, the grid's appearance bears immediate impact on the students' behaviors: (1) their attentional orientation recalibrates from focusing on the spatial interval between their hands to considering instead the grid's lowest horizontal line as their new frame of reference; (2) their bimanual action regime changes from simultaneous to sequential; and (3) their discourse shifts from qualitative ("The higher my hands go, the bigger the distance") to quantitative ("For every 1 that I go up on the left, I go up 2 on the right"). Next, once the numerals appear along the grid's vertical axis (Figure 5c), students discover multiplicative relations ("The right hand is always double the height of the left hand"). Our protocol then guides students through coordinating their pre-grid and grid-based reasoning, such as understanding why the spatial interval keeps growing by 1 unit (in the 1:2 ratio setting) or why raising their hands by 1 and 2 units, respectively, always maintains one hand double as high as the other. These questions have been interesting even for graduate students to consider without recourse to algebra.

Figure 5. Three schematic interface modes of a Mathematics Imagery Trainer for Proportion—the Parallels motor-control problem



a. b. c.

Why do students so readily incorporate the grid, the very moment it appears, as a means of implementing their movement strategy? Our data suggest that the students tacitly recognize in the grid embedded utilities for enhancing three aspects of their naïve strategy: (1) *enacting* the bimanual motor-control with greater precision, because they can now visually identify, maintain, execute, and thus offload precise spatial destinations for their bimanual operations (an instrumental affordance); (2) *evaluating* their own strategy to ensure its efficacy by designating and experimenting with alternative hand location (an epistemic affordance); and (3) *explaining* their strategy to a tutor–researcher or fellow student using verbal–gestural indexical utterances (a discursive affordance). Students judge that these new cultural forms interpolated into their working space offer them *functional parity*—they judge that, working with the grid, at the very least they can perform as well as without it. Moreover, the students figure out how to instrumentalize the grid to achieve a tighter grip on the environment (cf. Rabardel, 1993). The grid's instrumental, epistemic, and discursive affordances suggest its adaptive advantages as extending manual skill. Even though the bimanual form of enactment had earlier changed from simultaneous to sequential, when the grid's potential benefits had still been under scrutiny, some students soon revert to the simultaneous continuous action, even as they maintain the discrete gaze. Thus, students come full circle, incorporating mathematical perception as a cognitive enhancement.

Recall the student's attentional shift from looking at each of her two hands (or the cursors on the screen) to looking at a single object, the spatial gap between the hands. It was that object, a new percept that came forth from the environment, that afforded her a better grip on the situation, literally. Gripping and manipulating this new object led to discovering a transformation pattern constituting a task-effective perception–action loop for moving in green. In turn, she articulated this new grip as an ontology—a new 'thing' she could talk *about*: she "languaged" her perception of the spatial interval as a referent of multimodal utterance, stating, "The higher my hands go, the bigger the distance." She could thus draw the tutor's attention to the new object. Eventually, through adopting the grid and numerals, this new ontology further evolved

from an attentional anchor serving a proto-multiplicative dynamic bimanual coordination to a conceptually robust mathematical assertion.

Such is the quality of learning we are seeing across age groups, media, and concepts, when students work with Mathematics Imagery Trainers built according to principle of action-based embodied design (Alberto et al., 2022). We increasingly believe that this framework, along with its interaction architecture and technological apparatus, is operationalizing tenets of enactivist philosophy as mathematics pedagogy (Pirie & Kieren, 1989). If you want students to learn new mathematical ideas, first create opportunities for them to develop new perceptual orientations to the environment. This is accomplished by creating dedicated fields of promoted action that favor those proto-conceptual perceptual orientations as enabling the enactment of solutions to motor-control problems. Only later, introduce into the working space relevant symbolic artifacts for students to adopt spontaneously as frames of reference for languaging their new perceptual constructions as mathematical discourse. But begin from enactment.

Having overviewed both the perception-based and action-based genres of the embodied design research paradigm, we are about to transition to the thrust of this paper, where I will attempt to defend the claim that both genres can be modeled according to enactivist theory, that is, that they are theoretically coherent.

Perhaps an easier argument to defend first would be that the two design genres are *pedagogically* coherent. Both perception-based and action-based embodied design follow the heuristic motto that students should "experience first, signify later." As such, in both the Seeing Chance and the Kinemathics activities, students initially engage in non-symbolic tasks, drawing on their evolutionary capacity: in Seeing Chance, an exemplar project of perception-based design, students draw on their innate sensitivity to likelihood; and in Kinemathics, an exemplar project of action-based embodied design, students draw on their innate capacity for learning to move in new ways. Furthermore, in both projects, the students then extend their initial situated engagement by endorsing into their task performance available symbolic artifacts that are supplemented into their working space: paper templates for building a sample space in Seeing Chance; a grid and numerals in Kinemathics. Students inspect these new artifacts and recognize their features as bearing utility for performing the task (i.e., the students detect the artifacts' contextual affordances): the sample space in Seeing Chance concretely supports an explicit mathematical argument for the plurality of the 2g2b combination; and the grid articulates explicit movement destinations in Kinemathics. The students are willing to assimilate the available resources into their naïve engagement forms, because they appreciate that these resources enable them to keep performing the task at a level that is, at the least, equally effective in its operational quality to performing the task without the resources: in Seeing Chance, they appreciate that the sample space offers them inferential parity with respect to their naïve likelihood judgment of compound events produced by the marbles-scooping random generator; and in Kinemathics they appreciate that incorporating the grid into

their perceptual orientation to the environments offers them functional parity relative to performing the bimanual movement form in the referential void of an empty screen.

Yet, in both projects, assimilating symbolic artifacts into existing engagement forms causes a "cognitive boomerang effect," changing how students think, because symbolic artifacts, or more generally cultural tools, are not conceptually neutral. Cultural tools

> should be conceptualized instead as embodiments of certain cultural practices, as crystalized templates of action and schematized representations of ways of doing things as discovered in the history of human civilization. Cultural tools are, in this sense, embodiments of the function and meaning of sociocultural practices; they are "objects that can be used for certain purpose" in the human community. As such, they can be appropriated by the child only by acting upon and with them, only in the course of actively reconstructing their meaning and function. (Stetsenko, 2002, p. 129–130)

As such, when students adopt symbolic artifacts to enhance their naïve treatment of a phenomenon, these artifacts "bite back" by implicitly mediating the cultural analytic perception of the same phenomenon, shifting the students into normative disciplinary modeling and discourse. Thus, when students assimilate symbolic artifacts into their naïve forms of engagement, they might accommodate these forms. Still, learning environments can be designed so that students retain their mathematically correct naïve enactive meanings of phenomena while developing sophisticated analytic techniques.

In sum, whereas the two genres of embodied design tap different evolutionary capacity, they demonstrate *pedagogical* coherence in their activity design as well as structural and procedural homology. Perhaps the activities' pedagogical alignment makes sense, given that they were similarly inspired by frameworks such as didactical phenomenology. Yet in the following section I hope to show that the two genres demonstrate also *theoretical* coherence. That is, I will argue that activities exemplifying perception-based embodied design and activities exemplifying action-based embodied design rest on similar theoretical grounds appealing to one and the same type of cognitive activity. Across the two genres of embodied design, I will focus on students' conceptually desirable perceptual shifts. I will implicate these perceptual shifts as emerging to enable action. My argument will hinge on convincing the reader that the following two achievements are cognitively equivalent: (1) perceiving a sample space as expressing a felt meaning of likelihood distribution; and (2) holding an attentional anchor between your moving hands to stay in green. To build this argument, I will draw on radical-constructivism and enactivism to promote a proposed view of action more broadly as *the actuation of intentionality*. Action, I will argue, is about changing your operational relation with the environment, whether this engagement is overt or covert. Bear with me—this should all make sense in a moment.

TRANSFORMING REALITY APPARENTLY OR EFFECTIVELY: HOW THE TWO GENRES OF EMBODIED DESIGN COHERE THEORETICALLY

We begin with the following often-quoted yet enduringly enigmatic passage:

> Knowing does not really imply making a copy of reality but, rather, reacting to it and transforming it (either apparently or effectively) in such a way as to include it functionally in the transformation systems with which these acts are linked. (Piaget, 1971, p. 6)

This peculiar idea of transforming reality *apparently*—what might this mean? Either you *do* or you *don't* transform reality, no? What is the nature of this "as-if" transforming or reality? Piaget appears to be implying that apparently transforming the environment should be theorized as cognitively equivalent to actually and literally transforming the environment! But in what sense could such fanciful "as-if" changing the environment resemble concretely changing its material composition, morphological constitution, concrete properties, location, elevation, rotation, and so on? These as-if transformations seem better suited for discussions of imagination, dreaming, and hallucination, not proper doing. In considering cognition, aren't as-if transformations ontologically distinct from hands-on physical transformations?

Not according to constructivist epistemology. For Piaget as well as for many embodiment thinkers, what matters is the mind, not just matter itself. What *is*, for you, is what you perceive now, whether or not it is physically there, whether or not others see it your way, whether or not it ever existed. Sensemaking is active immersive construction. Wittgenstein (1956) called this *aspect perception*: When John looks at Jastrow's famous duck~rabbit ambiguous figure, we should not say that "John sees the picture as a duck," or that "John sees the picture as a rabbit," because that is not John's experience. Rather, we should just say that "John sees a duck" or that "John sees a rabbit" (my paraphrasing). For John, the image does not "represent" a duck or a rabbit—it *is* a duck or it *is* a rabbit. As John toggles between seeing a duck and seeing a rabbit, John is changing what is there as far as he is concerned—John is transforming reality (apparently). How you see is what you get.

More broadly, ecological psychology (e.g., Turvey, 2019) understands biological organisms' pre-reflective cognitive experience as inherently purposeful, pro-active—things *are* what they afford *us* and *now*. In fact, post-dualist philosophers argue, we don't even see things as things. Rather, we perceive opportunities for action, as bearing what Heidegger (1962) called the ontic quality of being "ready-to-hand." The default phenomenological unit is a perception–action loop, in which the perceiving organism and the perceived object are not separated as "subject" and "object."

Yet if we come to behold a situation in a new way, as enabling us to do, understand, or communicate in ways we could not before, then, even without literally handling the situation, still we now grip the situation in a new way—for all practical purposes, *we have changed the environment*. A new affordance has come forth, the perception–

action loop has been reconstituted, the world has changed: "The environment announces itself afresh" (Heidegger, 1962, p. 105). What we do is what we see.

Implications for design are clear. Educators can set up conditions that occasion opportunities for students to change the situation *apparently*—we scaffold for them a new way of attending to the environment in line with cultural–historical ideal perceptual forms (Shvarts, 2025). For example, students working with the Mathematics Imagery Trainer for Proportion could initially see the spatial interval between their hands as affording conservation only by keeping it the same size as they raise their hands. Yet through exploration, they eventually came to see the interval as affording correlation, as a magnitude they should increase as they raise their hands. Just like Jastrow's picture, it is not the interval itself that changed—it is students' perception of the interval that changed, and this perception enabled a new form of adaptive action.

Still, I am arguing that not only in action-based embodied design but also in perception-based embodied design the cognitive task students face is to figure out how to act on the environment in a new way. To support this argument, I need to broaden our traditional assumption that *action* necessarily implies transforming the world effectively. I need for us to endorse Piaget's invitation to conceptualize action also as transforming the world apparently. I maintain that in both design genres, students engage in sensorimotor exploration for a tighter grip on the world to improve their task performance. They search for new affordances for action, even if they never touch the objects, even if they only see the objects. As such, I adhere to enactivist conceptualization of action as inherently meshed into the phenomenology of perceptual engagement—all perception is necessarily perception for action. Across the design genres, students struggle *to perceive the environment in a new way that would enable them to actualize their intentionality*, whether it is the motor intentionality of moving an object or the representational intentionality of expressing a perceptual judgment (see Merleau-Ponty, 1945/2005). For their enactive cognition and, thus, their conceptual development, it does not matter whether the actualization of intentionality is overtly effective or covertly apparent.

You should push back now, asking me, "Ok, we get the philosophical argument, but what about the empirical evidence? We clearly witness sensorimotor activity in action-based embodied design, but where is the sensorimotor activity in perception-based embodied design?" Fair question. Let's get back to the Seeing Chance data.

At the beginning of the activity, the students were asked to name the most likely random scoop from the marbles tub. They chose 2g2b, judging it as more likely than 4g. Later, when they created the sample space, they initially rejected the variations as unnecessary (i.e. "order doesn't matter"). Eventually, though, they perceived the entire sample space as expressing that 2g2b is more likely than 4g, citing the different number of variations in these combinations' respective columns. Watching the students struggle to perceive the sample space as expressing "2g2b > 4g" suggests that they are undergoing considerable cognitive effort. Yet what is the nature of this effort? What

are students *doing* when they attempt to perceive the display as expressing their felt meaning? In what sense is this semiotic effort a form of action? Other than not haptically handling objects; other than not physically transforming or relocating objects; other than not acting on the objects, in any usual sense of "acting on," the students, I submit, are indeed engaging the objects sensorimotorically. This semiotic effort, short of overt hands-on effective transformation, consists of darting the eyes here and there, gesturing, and talking—these are all sensorimotor activities. The objects themselves are not affected by this engagement, they're passively displayed on the desk. And yet *students' sensorimotor activity is changing the environment*, at least from their subjective perspective. The environment announces itself afresh. It becomes, in a deep sense, something else. They have accomplished a phase shift, reaching new stability, a new attractor, cognitive equilibrium, whereby they have discovered a sensorimotor pattern that lets them perceive the display as affording the expression of their felt meaning. They develop a new cognitive structure enabling perception to guide action, that is, semiotic action. Through dialogue, the students have concretized what had initially appeared to be an abstract mathematical display (Wilensky, 1991).

I'm attempting to promote an idea here, that our notion of what an action *is* might need some revisiting, some accommodating, if we are to assimilate empirical evidence of students' semiotic cognitive activity. This broader theory of what cognitive activity is would challenge the field's traditional binary juxtaposing of thinking and doing, abstract and concrete, offering instead a view of *thinking as doing*. We'd be getting back to the future of radical constructivism (Steffe & Kieren, 1994).

CONCLUSIONS

The embodied design research paradigm explores what mathematics pedagogy might look like if it were epistemologically committed to fundamental tenets of radical constructivism and enactivist philosophy. The research problem motivating this paper was concern over whether two genres of embodied design, each tapping a different evolutionary capacity, coherently instantiate these philosophical tenets. My rhetorical plan for this paper was first to explain this concern by presenting the two design genres as ostensibly incoherent: action-based design promotes learning to move in new ways, whereas perception-based design promotes learning to see in new ways. The idea was to query whether seeing in a new way could be conceptualized as a form of enacting. Yet then I dug deeper into the E-turn in the cognitive sciences to argue that learning to move and learning to see are phenomenologically analogous—they are epistemic variants on discovering new environmental affordances, developing new perception-for-action. What matters for learning is not students' overt interaction with the physical world but their covert sensorimotor activity that gives rise to new affordances. Students learn by transforming the environment, whether effectively or apparently.

Cognitive activity is adaptive sensorimotor coupling with the embedded structures of natural and socio-cultural environments. Cognition is constituted, experienced, and iteratively transformed through purposeful embodied interactions. These ecological

interactions transpire within a set of unanticipated constraints, whereby new affordances emerge, that is, learning occurs. Ecologically adaptive sensorimotor engagement gives rise to cognitive structures that, in turn, stabilize and regulate this engagement by enabling perception to guide increasingly effective action. Perception and action constitute an atomistic duality, whether evolutionarily, neurally, or phenomenologically: we perceive to act. Where this acting falls short of our goals or anticipation, we modify our perception henceforth.

Action is cognitive activity, an actuation of intentionality that transforms the environment, whether effectively or apparently. The sensorimotor ontogenesis of mathematical practice, such as coming to perceive a diagrammatic display as a semiotic reification of one's felt meaning, is semiotic action that changes our environment. Pedagogical interactions create socio-epistemic expectations for students to endorse mathematical discourse. Students partake in the discourse by figuring out how to use disciplinary artifacts, such as a sample space, to articulate their felt meanings respecting problematized phenomena, such as their tacit judgements of likelihood in a random generator. Thus, students achieve a techno-scientific ideal expression of an otherwise naïve sensory impression. But these mathematical artifacts, in turn, highlight within the original phenomena latent quantitative–relational features, such as the order of singleton outcomes in compound events. Thus, the mathematical gaze is adopted.

Just as rock-climbers scan the wall for advantageous levers, so we scan a mathematical display for latent structure that might couple with our semiotic intentionality to objectify our felt meanings. Our cognitive activity, as we attempt and finally achieve this semiotic leap, evolved from groping for better haptic grips on the material environment (cf. Gould & Vrba, 1982). Even as we do not handle the display effectively, we have acted on it apparently, in the sense that its *appearance* has changed adaptively to afford us a new task-effective perception–action loop. As such, our brain is far more agnostic to the sensory quality of our interaction with the environment than we might canonically believe, given our received Eurocentric dualist heritage, with its objectivist ontological binaries of the real and the imagined. In sum, mathematical thinking engages an ancient faculty of the human biological cognitive system. Learning is mental constructing, gradual or abductive adapting, guided cultural re-inventing.

Enactivist philosophy can serve as a powerful blueprint for designing, implementing, facilitating, analyzing, and theorizing mathematical teaching and learning across the ages and the content domains. Central to enactivist educational practice is the design and evaluation of fields of promoted action, be this action effective, apparent, or both, as well as pedagogiccal methodology guiding teachers' child–culture negotiation.

Glenberg (2016) expressed scepticism concerning most technological solutions he was encountering for problems of mathematics education. These interactive artifacts, he worried, inadequately cater to the human mind, which, he emphasized, is a biological cognitive system. Bearing in mind our biological cognitive system, embodied design offers a way for mathematics education—its theory and practice—to move forward.

Author's note

I am grateful for the opportunity to work over decades with the bright embodied enactive minds of numerous study participants, students, collaborators, mentors, and members of the Embodied Underground community. I have refrained from self-citing here, instead referring readers to project webpages. A shorter version of this paper was presented orally in J. Rączaszek-Leonardi (Chair), "Patterns of movement—patterns of thought," the 13th Peripatetic Conference, in Kiry, Poland, October 24–27, 2024. I'm grateful to Muhammed Fatih Doğan, Markku Hannula, Markus Hähkiöniemi, and Nina Peleg for their most constructive comments on an earlier version of this manuscript.

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TEACHING MATHEMATICS USING NEUROSCIENCE – WHAT EVERY TEACHER SHOULD KNOW ABOUT THE MATHEMATICAL BRAIN

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After the development of easy-access measurement and data analysis technologies, neuroscience has entered the field of educational sciences as one of the useful methods to enhance our understanding of teaching and learning. The field of practical didactics is, however, quite far from the neuroscientific research questions. This paper aims to discuss the most important neuroscientific findings on mathematics teaching and learning, and connect them to practical applications in everyday didactics. Some findings are more general, e.g., related to memory, attention, the role of sleep and physical activity, while others are specific to mathematics. Finally, this paper also offers views for a closer collaboration between mathematics teaching and neuroscience.

ON THE ROLE OF NEUROSCIENTIFIC EVIDENCE

Many prominent scholars have considered studies of the brain to be an important part of educational sciences. Rene Descartes was interested in brain physiology and proposed the theory of dualism to explain the relationship between the brain and mental activity in the 17th century. Later in the 1890's, Camillo Golgi and Santiago Ramón y Cajal studied brain structure and proposed the neuron to be the meaningful functional unit of brain activity. Brenda Milner, a British Canadian neuroscientist, is known for her seminal work on the connection of memory and the corresponding brain tissue in hippocampus. She presented data on a patient without hippocampus and without the capability to form new memories, except for motor skills (Scoville & Milner, 1957).

Nowadays, it is considered to be basic knowledge that learning requires very specific neural activity, and that environmental factors affecting the brain are bound to affect learning, too. There is still work to be done in bridging the neuroscientific findings in a helpful manner for teachers and learners, however. The neuroscientific research often works on completely different time scales and environments than schools. Practically, neuroscientific findings may not be generalizable to different age groups, cultures, or topic areas. Often, neuroscience operates at a group level while practical learning needs to be optimised at an individual level. Further, neuroscientific findings don't always reach teachers and educational leaders and decision makers in an easily understandable

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manner. This paper is an attempt to solve some of the issues related to the use of neuroscientific findings in developing teaching and learning of mathematics.

What is memory, actually

Wikipedia defines memory as "the faculty of the mind by which data or information is encoded, stored, and retrieved when needed". This model, starting with encoding and storing information, and ending in the retrieval of the same information, is so well suited with neuroscientific research that most of the brain research on human memory follows this model. But when we compare it to the revised version of Bloom's taxonomy (Forehand, 2010), including steps starting from *Remember*, continuing to *Understand, Apply, Analyze, Evaluate,* and *Create*, it is easy to notice that neuroscience is most capable of studying only the lowest level of education: receiving, remembering, retrieving and reciting information. A true challenge for neuroscience in the future is to climb up the steps of the taxonomy to understand also the higher levels. There are neuroscientific studies on creativity, but they usually don't contain the element of learning – rather, they study creativity based on prior knowledge.

Neuroscientific findings highlight the strong connection between **perception** and memory. First, studies show that in a natural situation, a large part of information is ignored (Panksepp, 1998), and thus impossible to learn. Thus, the lack of conscious perception of information is often the first reason to learning difficulties. Second, studies show that practicing to perceive something, let's say the difference between two numerals, leads to changes at the perceptual level, making the detection of the difference automatic in a step-by-step manner during extensive training (for phoneme learning, see Näätänen et al., 1997). Such learning might require several hours, weeks or even months of practicing, but as a result, the detection of the difference has become automatic already in the perceptual processes. Thus, at the neural level, it is not meaningful to separate perception from memory and learning.

The neural interconnectedness of perception and memory is a good example of the challenges that neuroscientific findings pose on psychological learning models. Psychology would separate perception from learning, but when you study neural activity, the separation is not supported by evidence. Importantly, **working memory**, a key concept in the psychology of learning, is not at all an obvious and unified concept at the neural level (Schröger et al., 2015). The working memory research, studying people's capabilities to hold a list of items in their mind and retrieve it later, are an integral part of the psychology of memory, but from the brain, no specific working memory circuit can be found. In contrast, another neural level concept, **memory consolidation**, has become a key aspect of learning studies (see the section on sleep below) after it was originally found in neuroscientific research of learning (Marshall & Born, 2007). These clear discrepancies between the psychological and the neuroscientific concepts of learning and memory should not discourage researchers and teachers, but rather be an inspiration to widen the perspectives we use when we discuss learning.

NEUROSCIENTIFIC RESEARCH ON LEARNING

This section aims at highlighting such neuroscientific findings that I believe are most relevant for mathematics learning, and can be applied in the practical everyday learning environments.

Attention in the learning situation

Neuroscientific studies highlight human attention as a fragile skill – all of us suffer from attentional problems in demanding environments and everyone's attention can be distracted by strong stimuli. The voluntary directing of attention, i.e., the learners' conscious efforts to keep their attention directed to the learning material, teacher's voice, etc., is easily compromised by external environmental factors like background noise (especially speech), moving objects (also in the peripheral parts of the visual field), and even things like uncomfortable chairs or clothes, smells, temperature, light conditions, visual patterns in the room, etc. Additionally, internal factors like thirst, hunger, the urge to move the body, or recurring thoughts like fearful or otherwise emotional events that have happened before or are expected to happen, or things to remember like unfinished tasks or homework, may distract the attention of the learners. Often, the teacher notices the moments of attention distraction in the learners. Removing the most common distractors from the learning environment would be a good goal, but it is not always in the hands of the teacher or the learner – often decisions on learning spaces are conducted by educational leaders, who should know how human attention works in different ages.

Recently, the percentage of learners with diagnosed attentional problems like ADHD has increased dramatically (Salari et al., 2023). Diagnosed or not, we all fall into a multidimensional continuum of attentional skills with varying abilities in directing our attention, holding it focused, resisting distraction, and reacting to important events outside of our attentive focus. Every child needs to practice their attentional skills in an environment that supports this practicing – math tasks that require problem solving and reasoning require also highly developed attentive skills. It is important to think about scaffolding: First, the learning environment of small children should support the attention skills so much that it is easy for them to keep the attention directed to the task at hand. For young learners aged 9 years and younger, only a short duration of attention control is typically possible, and teachers may notice that the pupils need help in keeping their attention focused for longer than just a few minutes at a time. Only after lots of practicing and experiences of succeeding in directing one's attention in supportive learning environments, the older learners can proceed to more demanding environments that have interruptions, noise, etc. Teachers and education professionals must be included when teaching environments and buildings are designed so that the diverse needs of all types of learners and learners of all ages are taken into account in the planning.

Attention skills are also affected by extracurricular activities, since the human brain is very malleable. If the learners spend several hours per day in tasks where their attention is externally directed quickly from one item to another, like when browsing fast videos or multitasking on social media platforms, their capabilities to voluntarily keep their attention focused according to their own will may be weakened, and this affects their attentive capabilities also at school (Jones et al., 2024). In contrast, if the learners spend a lot of time in tasks where they voluntarily keep their attention focused on one item of their interest, like when reading a book, playing a musical instrument, or being engaged in arts, sports, or play, they simultaneously practice their skills of directing their attention, which may be beneficially visible in the school context, too. The amount and ways of using digital tools in the free time affects attentive capabilities (Moisala et al., 2016).

Specifically, hobbies have been shown to increase attentive capabilities (Roman-Caballero et al., 2022). In hobbies, learners are often engaged with tasks that are of high interest to them. Following directions from the adults, maintaining the focus of one's attention at the task, monitoring one's own performance and learning, and collaborating with others are all elements of hobbies that may help the learners develop their attention skills. Especially musical hobbies like learning to play a musical instrument or singing in a choir seem to be beneficial for practicing attention skills (see chapter on music below).

Curiosity changes brain activity so that learning becomes easier

Kang et al. (2009) studied curiosity in the brain and revealed that when a person is curiously pondering a question, their brain activity in the caudate nucleus and inferior frontal gyrus is increased. These structures are related to anticipation of reward. When the answer is revealed, brain activity in the hippocampus and parahippocampal gyrus increases, which helps them learn and remember better. Enhanced learning has been proposed to be the primary function of curiosity (Kang et al., 2009). These studies can be applied in teaching and learning by trying to connect the learning content into something that evokes curiosity in the learners, i.e., something that is of interest to them outside of school. It would be important that the teacher knows what their students are interested in. Math tasks are easily connected any topic, for example football, celebrities, pets, different cultures, etc. A connection to a topic of interest for the learner makes the learner curious and helps learning.

Gruber et al. (2014) studied how much curiosity towards one item can help learning other items, too. They measured brain activity when the learners were presented with questions evoking varying levels of curiosity. Between the questions, the learners were shown photos of neutral, unknown faces. Later, it was found that the learners remembered those faces better that were shown in between the curiosity-evoking questions compared to those shown in between dull questions. Gruber et al. call this **curiosity state**, during which the midbrain and hippocampus were especially active (Gruber et al., 2014). These results can be applied in teaching by aiming to have at

least some curiosity-evoking elements in all teaching and placing the elements that may evoke less curiosity between those (Kidd & Hayden, 2015).

Active, learning minds are connected with active bodies

According to the recommendations published by the Finnish Ministry of Education and Culture (2021), children and adolescents aged 7 to 17 years are recommended to be physically active in a versatile, brisk and strenuous manner for at least 60 minutes a day, while excessive and extended sedentary activity should be avoided. Unfortunately, only one third of children and adolescents fulfill the criteria in Finland (Kokko & Hämylä, 2025), and the situation is similar in other Western countries, too. This is unfortunate since physical activity and fitness are correlated to mathematics learning (Rasberry et al., 2011) and classroom-based physical activity interventions show a positive effect on attentive behaviour during lessons and improvements in academic achievement (Watson et al., 2017). Unfortunately, it is still typical that most of the math lesson is spent sitting.

From the neuroscientific viewpoint, the strong connection between physical activity and learning is easy to understand. Structurally, the brain tissue is healthier and contains more neurons, more connections between neurons, thicker cortical surface, stronger white matter tracts, and less signs of ageing in individuals who have a physically active lifestyle compared to sedentary individuals, and the increase of physical activity changes the brain structure to be healthier (Erickson et al., 2015). Functionally, brain areas related to learning, memory, attention, and planning of one's own actions are more active in individuals with physically active lifestyles (Erickson et al., 2015). In Finland, schools are not only responsible for teaching physical education as a school subject, but all teachers, including subject teachers of math, are obliged to promote a physically active lifestyle.

Educational systems must react to changes in the surrounding society. When the learners move less and less during their free time, and when the extracurricular physical activity is more polarised than ever (a few move more, while many move less than before), schools need to find effective ways to make the school day and the math lesson physically active. Teachers need support, training, and ready-made programs that can be easily administered (Tancred et al., 2020). This is especially important for subject teachers in mathematics, whose studies often contain less general didactics than those of class teachers. Physically active breaks of 2-15 minutes, physical education, active transportation to school, as well as combining the academic content to physical activity during lessons are all part of a healthy, learning-enhancing school day (Vetter et al., 2020). A physically active math lesson could mean that a normal lesson is interrupted with high-intensity interval of physical activity where the learners move very vigorously for a few minutes, and then return to continue the lesson. Another, very effective solution is to make math learning tangible using objects to manipulate, move and turn in order to solve the math problem. For example, when teaching mathematical

operations with fractions, giving the learners some tangible objects that are easily divisible into fractions makes the learners more active and the task easier to understand.

Sleep and rest are key elements of learning

The connection of sleep and learning has been studied extensively in neuroscience already for decades. Research shows that while we are sleeping, our brains are not resting but rather are actively processing information – everything that we heard, saw, read, or experienced during the day is processed again in the subconscious brain processes during sleep (Born et al., 2006). These **memory consolidation** processes are different from the conscious problem solving and learning processes that we conduct during the day time. Memory consolidation affects long term memory, and it involves also connections to previously learnt material – even connections that we didn't come to think about during learning (Born, 2010). For example, if a learner is presented with the concept of integrating for the first time on a math lesson, they might learn it mechanically but not yet connect it to their daily life during wakefulness. It might happen that while they are sleeping, their brain makes a connection between the new concept and the time when they were trying to estimate if they have enough paint to paint a wall. In the morning, their understanding of integrating is deeper due to this connection. Of course, they might make the same connection while awake, but it may be more likely during sleep.

Learning is compromised without proper nutrition

Our brains use a lot of energy – in a sitting individual, brain activity uses approx. 20% of all energy use of the body, despite the fact that it is only 2% of our body weight (Raichle & Gusnard, 2002). Breakfast and lunch during the school day are vital for learning, since long periods without food make the learners tired and anxious and thus not able to learn. In a longer time scale, a healthy and balanced diet is important for learning since it supports brain health, mental health and learning-related structural changes in the brain. According to WHO (2026), "Today the world faces a double burden of malnutrition that includes both undernutrition and overweight, especially in low- and middle-income countries. There are multiple forms of malnutrition, including undernutrition (wasting or stunting), inadequate vitamins or minerals, overweight, obesity, and resulting diet-related noncommunicable diseases."

In Finland, schools offer a healthy and balanced lunch for free for all learners. Nutrition is not without problems, though. Overweight and obesity are common – from 2-16-year-old girls 20% and boys 34% are overweight or obese (THL, 2023). In addition, many children and adolescents skip breakfast, possibly resulting in low blood sugar level before and high after lunch, compromising learning. Further, not all children and adolescents eat all components of the lunch due to their personal taste, and some children and adolescents find the school lunches otherwise difficult for example due to noise. Eating disorders are becoming more common in Western countries. Lifetime prevalence of eating disorders vary from 0.4 to 4% (Qian et al., 2022). All eating

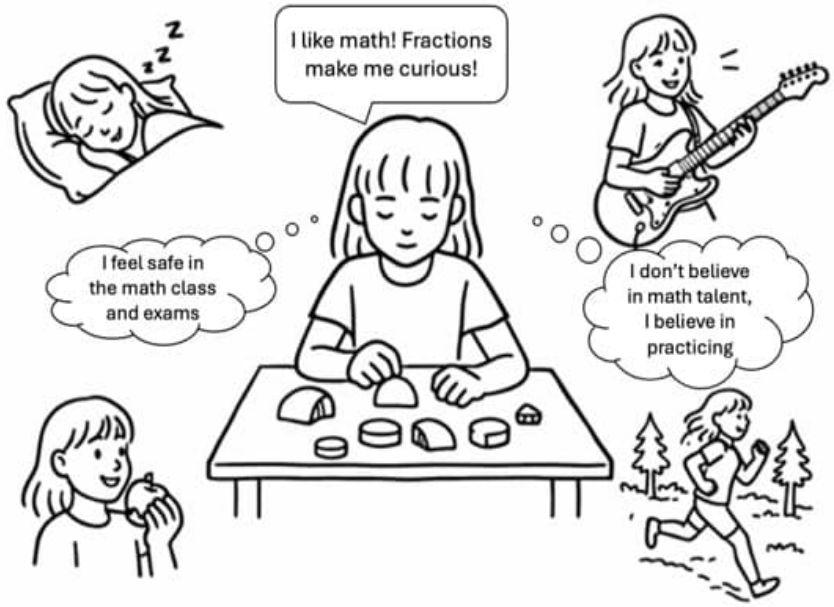
disorders compromise learning since they severely affect brain health and cause neural changes.

Music may boost or hinder learning

Several researchers have studied the effects of musical hobbies on children's intelligence and attention (Schellenberg and Lima, 2022). Correlational studies show a connection between brain development, cognitive capabilities and musical hobbies, but from correlation studies, it is not possible to make causal conclusions. However, several recent intervention studies have found similar effects in children who were randomised into a musical activity group compared to children who were randomised into another group like visual arts. The current conclusion of the field is that musical activities, since they are very demanding for attention, are helpful for advancing attentive skills in children (Roman-Caballero et al., 2022). This means that learners who have music as an extracurricular hobby may find it easier to focus on the learning content, especially in noisy conditions.

Schools can use music in many ways to support learners' attention. Music as a school subject serves several important purposes, not only teaching musical skills and music subject knowledge, but also helping learners practice their attentional skills, become more connected with their emotions, and making musical hobbies more easily accessible to learners after they have gained experiences, interest and confidence in school music lessons. Offering possibilities for extracurricular musical hobbies at school is important especially for those learners who would otherwise not have access to such hobbies. Having a school choir, orchestra, and band rooms to be used by the learners are great investments that the school can make to advance learning. Using music during breaks can make the breaks more active since energetic music playing in the background makes people move more. After such a break, the learners may have it easier to focus during the next lesson (see chapter on physical activity above). Sometimes the use of background music in math lessons may help some learners, especially those with fixed mindset, anxiety, or attention problems. There is some evidence showing that individually selected background music may be very helpful to some learners. The use of musical teaching methods in the teaching of mathematics is a growing field which is increasingly offering new change-making didactic solutions that help some learners significantly.

Figure 1. A summary of factors affecting learning according to neuroscientific research.



The learner's amount and quality of sleep and their nutrition are foundational for all learning. A physically active lifestyle and being allowed and encouraged to move while learning can boost learning success. Learners may benefit from musical activities to support their attentive skills and executive functions. Curiosity, psychological safety and growth mindset are related to higher motivation and lower stress during learning and make learning more successful.

In summary, a learner who feels safe in the math class and exams, who is curious and can show it, who has slept well and leads a physically active healthy lifestyle, has good attentive skills and knows how to use music for the benefit of their own learning, is in a very good position to learn mathematics.

THE BRAIN ON MATHEMATICS

Research in cognitive neuroscience suggests that the human brain is biologically prepared to process numerical information. Simply put, working with numbers is natural for the brain, while working with letters is not. At school, learning mathematics is heavily dependent on language skills due to the way that mathematics is taught, but in expert mathematicians, the neural circuits related to mathematical functions have very little overlap with the circuits related to language processing. At school, mathematics learning can be hindered by dyscalculia, fixed mindset or math anxiety. The neural underpinnings and potential applications of the research related to these concepts are discussed below.

Mathematics is natural for the brain – especially compared to reading

Drawing on the work of prof. Stanislas Dehaene in cognitive neuroscience, basic number sense appears to rely on neural systems that evolved long before formal schooling or symbolic mathematics. In contrast, reading and writing are cultural inventions that require the brain to reallocate new neural circuits that originally evolved for other purposes, a process that Dehaene calls **neuronal recycling** (Dehaene & Cohen, 2007). Reading and writing are new cultural inventions, emerging globally a few thousand years ago and in Finland only a few hundred years ago. The human genome has therefore not had sufficient time to evolve dedicated neural circuits specifically for literacy. Dehaene describes how learning to read repurposes an area in the ventral visual cortex (the visual word form area), which becomes specialized for recognizing letter patterns, and thus reading and writing require extensive learning and explicit instruction. In contrast, early numerical abilities emerge spontaneously during development, even in cultures with limited formal education.

Humans, like many other animals, possess an intuitive ability to perceive and manipulate numerical quantities. This capacity is often referred to as the **approximate number system (ANS)**, a nonverbal neural system that allows us to estimate approximate quantities, compare magnitudes, and perform simple arithmetic operations (Gilmore et al., 2011). Neuroimaging studies show that these abilities engage a network centered in the intraparietal sulcus of the parietal cortex, a region that supports the representation of numerical magnitude. This system develops early in infancy and operates without any formal instruction so that the mere act of looking at a group of objects automatically includes an estimate of the number of objects.

It is of high relevance that numerical cognition is not uniquely human. Research demonstrates that a wide range of animals possess rudimentary number abilities. For example, primates, birds, fish, and even some insects can perform simple numerical comparisons (Nieder, 2021). Studies with rhesus monkeys, chimpanzees, crows and parrots show that these animals can even learn symbolic representations of numbers and perform approximate addition or subtraction. Importantly, these findings suggest that the neural foundations of numerical cognition evolved very early in evolution.

Although we are equipped with a basic numerical intuition, advanced mathematics still requires cultural learning and symbolic systems. Education can build upon the brain's innate quantitative mechanisms, and extend them through language, notation, and formal reasoning.

What is the relevance for math education and for teachers from the research in the approximate number system (ANS) in humans and other animals and the studies comparing innate mathematical skills and unnatural, cultural reading abilities at the neural level? First, I think that it is encouraging for teachers and learners to think about ANS as something natural, something that we share with other animals. Understanding how deeply ANS is rooted in the visual system and in tangible real-life objects, and

not in speech or written language, could be an inspiration to develop teaching of early mathematics towards more visual, more tangible, and less language-rooted direction. Second, such thinking could inspire solutions for individuals with math anxiety: more time spent working on tangible math tasks, practically finding solutions to math problems, focusing more on approximating solutions, and less on discussing math concepts, let alone discussing the importance of math. Third, understanding the neural basis of dyscalculia (see below) and its connection to ANS could help develop early remediation methods for dyscalculia for children aged 3-10 years. Those methods could be rooted in approximations and tangible practicing of basic operations.

Brain areas related to expert mathematics

When we move from very basic math operations and approximations to expert mathematics, the relevance of the ANS decreases and an important part of learning is based on defining and understanding mathematical concepts via language. However, Amalrick and Dehaene explain in their seminal paper (2019) that mathematical thinking relies on a specialized brain network that is largely separate from the classical language system. In the school context, mathematical statements are often presented using language, and the learning of mathematics heavily depends on language capabilities. However, Amalrick and Dehaene show that at the expert level, the brain processes related to mathematical meaning are organised in a distinct cortical network not related to the language network.

Functional neuroimaging studies comparing mathematicians with non-mathematicians have revealed that expert mathematical reasoning consistently activates a set of regions known as the math-responsive network centered in the intraparietal sulcus in both hemispheres of the brain (Amalrick and Dehaene, 2016). The intraparietal sulcus is widely associated with numerical magnitude processing and quantity representation, and it appears to form the core neural substrate for mathematical intuition. In expert mathematicians, this region becomes highly engaged when evaluating the truth of mathematical statements or reasoning about abstract mathematical concepts.

The network includes also regions of the inferior temporal cortex, which are thought to support the recognition of mathematical symbols, formulas, and structured patterns. These areas allow mathematicians to access stored conceptual knowledge about mathematical objects such as functions, geometric structures, and algebraic relationships. Together, these parietal and temporal areas form a distributed system specialized for representing mathematical meaning.

Thus, advanced mathematical thinking does not primarily rely on the classical language areas of the brain, such as Broca’s or Wernicke’s regions. Although mathematicians often need to read mathematical statements written in natural language, the conceptual processing of those statements largely bypasses the language system. Once the linguistic structure of a sentence is understood, the brain shifts processing to the parietal–temporal math network. This observation supports

Dehaene's claim that mathematics builds upon evolutionarily older neural systems for representing quantities and spatial relationships. Even highly abstract mathematical concepts appear seem to rely on the ancient neural circuits originally involved in processing magnitude, spatial structure, and logical relations.

Interestingly, these same studies indicate that becoming a professional mathematician does not dramatically change the basic architecture of the brain, which is the case for example in musicians. Instead, expertise appears to refine and strengthen existing neural circuits, allowing mathematicians to manipulate complex symbolic and conceptual structures more efficiently. In expert mathematicians, therefore, mathematical reasoning emerges from the interaction of several neural systems: the intraparietal sulcus supporting numerical and structural representations, temporal regions encoding symbolic and conceptual knowledge, and frontal areas contributing working memory and cognitive control during complex reasoning. Through years of training and practice, these systems become finely tuned to handle the abstract structures characteristic of advanced mathematics.

Thus, in sum, the brains of expert mathematicians illustrate how human cognition can build highly sophisticated intellectual abilities on ancient neural foundations. Mathematical expertise does not depend primarily on language but instead relies on a specialized parietal–temporal network that supports the representation and manipulation of abstract quantitative and structural concepts.

Dyscalculia and its neural underpinnings

Developmental dyscalculia is a specific learning difficulty characterized by persistent problems in understanding numbers, learning arithmetic facts, and performing mathematical calculations (Butterworth et al., 2011). Unlike general learning difficulties, dyscalculia affects mathematical cognition specifically and occurs in individuals with otherwise typical intelligence and educational opportunities. Research in cognitive neuroscience has shown that dyscalculia is associated with differences in the brain networks that support numerical processing. Individuals with dyscalculia often display reduced activation or atypical structural development in the intraparietal sulcus during numerical tasks. These differences lower the ability to compare quantities, estimate numbers, or grasp numerical relationships. In addition, typically the intraparietal sulcus interacts with other brain areas to form a network for mathematical thinking. In learners with dyscalculia, this network functions less efficiently, leading to difficulties in learning arithmetic and other mathematical skills.

Mathematical reasoning also relies on a broader frontoparietal network that includes regions of the prefrontal cortex, which support working memory, attention, and cognitive control. In individuals with dyscalculia, these regions may be recruited differently during mathematical tasks. Some studies suggest that children with dyscalculia rely more heavily on frontal areas, possibly reflecting compensatory strategies when core numerical representations in parietal regions are less efficient.

Further, in dyscalculia, the mapping between number symbols and quantities may develop more slowly or less accurately, making it difficult to learn arithmetic facts or understand place value. Practically, these differences may make math learning more straining for learners with dyscalculia.

Importantly, dyscalculia does not arise from a single cause or identical neural pattern in all individuals. There is no typical dyscalculic brain structure. Instead, dyscalculia reflects a variety of developmental pathways in which numerical processing networks develop differently. Some individuals may show primary difficulties in magnitude representation, while others may struggle more with symbolic number processing or the coordination of numerical and executive systems.

Understanding the neural basis of dyscalculia has important implications for education. Because the condition involves differences in how numerical information is represented in the brain, effective support often focuses on strengthening number sense through visual, spatial, tacktile, and multisensory approaches. Interventions that emphasize concrete representations of quantity, for example number lines, manipulatives, tangible objects, and visual models, can help reinforce the connection between symbols and magnitudes.

Mindsets matter in mathematics

Mindset is defined as an implicit belief that an individual can hold about basic human abilities, such as intelligence or attention (Dweck, 2006). Mindsets form a continuum from fixed mindset, which refers to believing that specific human attributes are static and unchangeable, to growth mindset, which refers to considering these attributes as malleable and shaped and developed with effort. There is some neuroscientific research on mindsets, and it is especially interesting to follow the research on math-related mindsets. For example, Ita Puusepp et al. (2021) studied 97 Finnish pupils aged 8-9 years. They recorded brain responses and physiological activity during a math task where the learners were constantly receiving feedback. They found that growth mindset was related to the automatic brain responses that were elicited by attention allocation and by the feedback. Practically, this means that the beliefs of the learners affect already the subconscious brain processes and embodied physiological reactions related to learning and feedback – already before the learners made any decisions about their conscious efforts to learn. In addition, when recording the autonomous nervous system activity of the learners in the same situation, Puusepp et al. (2024) found that fixed mindset was related to decreasing arousal and lower performance. This indicates that high arousal in the beginning of the task may not be optimal for task performance.

The neuroscientific findings on mindsets towards mathematic skills and learning highlight the fast and automatic nature of the effects. Already at the perceptive level and in the automatic processes of attention allocation and receiving feedback, the brains of the learners with growth mindset function in different ways that those with fixed mindset. Math teachers could apply this information by discussing mindsets in

the classroom especially in situations where learners use tell-tale phrases of fixed mindset, for example “I’m so bad at math”, “I have no math talent”, “math is always so easy/difficult for you”, etc. A strong fixed mindset prevents the learner from succeeding in math simply because it changes the brain activity related to perceiving the task, allocating attention to it, and towards feedback on performance. A teacher who recognises the success of learning math in their pupils should also make the same success visible to the learners themselves, since sometimes it is not clear for the learners that it is indeed their practicing and effort that made them learn, not some inborn talent.

Mathematics anxiety and its neural underpinnings

Math anxiety refers to feelings of tension or fear that interfere with the manipulation of numbers and the solving of mathematical problems (Luttenberger et al., 2018). Research in cognitive neuroscience and educational psychology shows that math anxiety is not merely a matter of attitude or motivation; it is associated with measurable changes in brain activity and physiological responses at the autonomous nervous system level that affect how mathematical information is perceived and processed.

Neuroimaging studies indicate that individuals with high levels of math anxiety show increased activity in brain regions involved in emotional processing and threat detection (Artemenko et al., 2015). One important structure is the amygdala, which plays a central role in detecting danger and generating emotional responses such as fear or stress. When learners with math anxiety anticipate or encounter mathematical tasks, the amygdala may become more active, signaling a perceived threat. This response can trigger physiological stress reactions that severely interfere with cognitive performance.

Another important region is the insula, which is associated with the awareness of internal bodily states and emotional discomfort. Elevated activity in the insula during mathematical tasks has been observed in individuals with high math anxiety, suggesting that these individuals experience math-related situations as emotionally aversive and also notice how their bodies react to this aversion. For example, they may notice their hands sweating or their shoulders becoming more tense.

Math anxiety affects the brain systems that support reasoning and working memory. Mathematical problem solving relies heavily on the prefrontal cortex for working memory, attention, and cognitive control. When anxiety is high, these cognitive resources are occupied by worry and intrusive thoughts. As a result, fewer mental resources remain available for preceiving the task, performing calculations, or reasoning through mathematical problems.

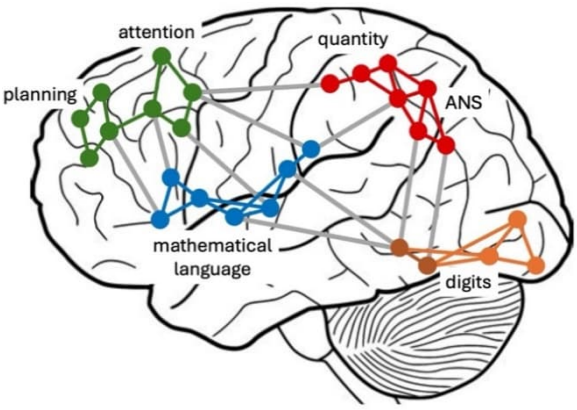
Studies have shown that individuals with math anxiety often display altered activation patterns in the frontoparietal math network. Under conditions of high anxiety, the efficiency of this system is thus reduced, leading to slower or less accurate mathematical performance.

A highly relevant finding from brain imaging studies is that the neural responses associated with math anxiety arise even before the task begins. In some experiments, participants who report high math anxiety show increased activity in pain- and threat-related brain networks already when they anticipate solving a math problem (Lyons and Beilock, 2012), or, in a school context, know that the next lesson is going to be math. This suggests that the expectation of mathematical difficulty may have a key role in triggering stress responses that undermine performance. Some of this stress seems to be reduced after the task begins and the learner notices that they can perform the task. Thus, for a teacher, it is a good strategy not to talk about the future tasks too much but to plan the lectures so that in the very beginning, easy math tasks are performed in order to reduce the anxiety before introducing a bit more demanding tasks step by step.

Math anxiety often develops during school years and can be influenced by negative experiences with mathematics, repeated failure, or social expectations. Over time, these experiences may reinforce a cycle in which anxiety leads to avoidance of mathematics, which in turn limits practice and skill development. From a neural perspective, repeated anxiety responses can strengthen connections between emotional networks and situations involving numbers or calculations. Importantly, the presence of math anxiety does not reflect a lack of mathematical ability. Many individuals with strong cognitive potential may perform below their capabilities when anxiety interferes with the brain systems needed for reasoning and working memory.

Understanding the neural mechanisms of math anxiety highlights the importance of educational environments that reduce fear and support positive engagement with mathematics. Strategies such as supportive teaching practices, gradual exposure to more challenging problems, and activities that build confidence can help reduce anxiety and allow the brain's numerical reasoning systems to function more effectively (Petronzi et al., 2021). There is tentative evidence that listening to relaxing music while performing math tasks may reduce math anxiety. In addition, connecting math tasks to everyday environments such as cooking or sports may reduce anxiety via showing the learners that they have already succesfully performed mathematical operations in their everyday life outside school.

Figure 2. The mathematical network of the human brain (shown from the left)



The math-responsive network in and around the intraparietal sulcus is shown in red. This area is responsible for automatic number approximations and simple mathematical operations already in small children, and in expert mathematicians, it is activated in mathematical problem solving. Visual areas (in orange) in the occipital lobe and word-form area (in brown) are responsible for perceiving and understanding written digit symbols. Broca's area, auditory areas and Wernicke's area (in blue from front to back) in the temporal and frontal lobes are active in tasks related to speaking, listening and understanding math-related language and concepts, respectively. In addition, frontal lobe areas related to executive functions such as planning and sequencing tasks and allocating attention are highly relevant in performing mathematical tasks.

TEACHING MATHEMATICS USING NEUROSCIENCE

Neuroscience is an emerging resesarch field that moves from laboratories into everyday life both in terms of research methods and research questions. It is not self-evident how math teachers can use neuroscientific research results in developing their work. Researchers in the field of neuroscience must make the research results available for teachers, while teachers should follow the latest results of neuroscience of learning.

Neuroscience provides a lot of understanding in brain wellbeing and learning, and this information is relevant also for math learning. Studies show that good diurnal rhythms, good sleep quality and duration are all relevant for learning. In addition, good nutrition is a prerequisite for good learning. Further, research shows that physically active lifestyle as well as physical activity and movement prior to and during learning are beneficial for attention allocation, learning, and memory. In addition, there is evidence that musical activities may enhance learning via strengthening attentive skills and executive functions more generally. For a math teacher, however, it may not be self evident how to affect the sleeping patterns, nutrition, physically active lifestyle or musical hobbies of learners. These are general lifestyle patterns that are more affected by societal factors than individual teachers. In contrast, during the math lessons, the use of physical activity and tangible manipulatives can help the learners comprehend some mathematical concepts more easily.

Neuroscience can offer more understanding on math-specific educational issues. The comparison between the neural cirquits related to mathematics and reading provides an interesting setting for considering the teaching of elementary mathematics and reading. The role of language is important in math teaching, but in some situations it may be problematic, like in the case of learners with language-related problems or lack of language abilities. Importantly, the brain networks related to mathematical thinking seems to be distinct from the language network both in elementary mathematics and in expert-level math problem solving.

Research on mindsets shows differential neural patterns in learners with fixed mindset compared to growth mindset. In addition, the activity of the autonomous nervous

system is related to mindsets. Discussion on the role of "math talent" and practicing is important for all learners, as well as the methods that emphasize the role of effort and practicing in math learning.

For math teachers, the understanding of math anxiety and its neural consequencies is important. Teachers need help in order to battle math anxiety. Planning teaching and learning so that math anxiety is taken into account is a good practice in math teaching when the teacher knows or suspects that some learner suffer from math anxiety.

Developing the connection between neuroscientific research and everyday math teaching and learning requires collaboration between researchers and active teachers and developers. Testing new teaching methods, finding new avenues for research, and reacting to changes and possibilities in the school and in the society should be a common effort of teachers and researchers. This ensures the highest quality of math teaching and learning as well as the most relevant neuroscientific research that can lead the field forward.

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MATHEMATICAL MAKING AND SENSE-MAKING: DIGITAL AND AI LITERACY "IN THE MAKING"

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As artificial intelligence (AI) increasingly performs mathematical reasoning and generates correct solutions instantly, a more fundamental question emerges: what counts as doing mathematics in the AI era? If machines can calculate faster, optimise more efficiently, and even produce elegant proofs, what does meaningful collaboration between humans and AI look like in the mathematics classroom? In this plenary, I argue that what is at stake lies in foregrounding construction over computation. I advance mathematical making as a way of rethinking creativity in the AI age—not as the production of novel answers, but as the practice of building, interpreting, and refining structured worlds. Across 3D modelling, touchscreen interaction, and programming environments with AI-infused tasks, my research has evidenced students' mathematical reasoning and problem-solving as constructing relationships, shaping systems, and making sense of dynamic structures—a process I conceptualise as "mathematical making". From this perspective, AI and digital literacy are not about mastering tools, but about understanding and critically examining the mathematical structures that underpin computing and intelligent systems. If AI can operate within predefined systems, mathematics education must cultivate those who can design, question, and remake those systems.

INTRODUCTION

Artificial intelligence (AI) has begun to unsettle long-standing assumptions about what it means to cognitively engage in doing and learning mathematics. Contemporary AI systems solve equations, generate proofs, and optimise model with a fluency that rivals—and sometimes surpasses—human performance. In 2025, Google's Gemini Deep Think and a new experimental OpenAI model both solved 5 out of 6 International Mathematical Olympiad problems; unlike previous AI models that relied on formal or symbolic techniques, these systems tackled mathematical problems using natural-language reasoning—mirroring how humans think and communicate (New York Times, 2025). In school mathematics, where success has often been measured by procedural fluency and efficient solution generation for pre-defined problems, this development is not merely technological; it is epistemological. If machines can compute, conjecture, and even "explain," what, then, should mathematics education aim to cultivate? What should be valued in mathematics learning and problem solving in the AI era?

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 45-68. PME.

These questions are no longer abstract. In Hong Kong, a large-scale AI-in-education initiative has recently been launched across K–12 contexts with the explicit aim to "empower students for tomorrow: cultivating a future-ready generation for the age of AI" (Hong Kong Jockey Club Charities Trust, 2026). In this project partnered with the Education Bureau and which I am involved in, generative AI tools are developed and integrated during subject (e.g., mathematics) teaching and learning across compulsory education. While such initiatives promise innovation and technical fluency, they also signal a deeper paradigm shift. If AI can assist with or automate many forms of mathematical production, then mathematics education must reconsider not only how AI is used, but what remains fundamentally important in mathematical thinking and learning.

In an age increasingly defined by intelligent machines, Papert's (1980) constructionism offers a timely reorientation for mathematics education—one that foregrounds creativity as the human capacity to design, interpret, and remake mathematical worlds. Mathematical creativity, therefore, requires reconceptualisation. It is not limited to producing novel solutions, but involves building, reconfiguring, and making sense of systems of relations. In this paper, I argue for foregrounding construction or making in mathematics education—in the sense of constructing models, representations, and refining them—rather than merely applying established procedures.

To develop this argument, I introduce the idea of mathematical making and trace its development across various tool-mediated contexts: from embodied material engagement, to multimodal digital interaction, to programming and AI-mediated modelling tasks. These contexts are not simply technological variations. Rather, they represent progressively expanding forms of mathematical world-building, where learners construct artefacts, coordinate representations, generate data, and critically evaluate models. As I will demonstrate across these settings, creativity becomes visible as a constructive, multimodal, and iterative–critical practice. By following this trajectory, digital and AI literacy can be understood as emerging within practices of mathematical making, as learners engage in constructing, analysing, and revising rule-governed systems.

The sections that follow are organized as follows. I review four interrelated strands of literature that together establish the research context, conceptualisation, empirical findings. The first strand, *Mathematical Creativity in the AI Age*, situates the paper within contemporary debates about the nature and value of human creativity under conditions of increasing human–AI collaboration. The second strand, *Multimodality and Mathematical Sense-Making*, examines multimodality as the condition for mathematical sense-making, drawing upon research demonstrating that mathematical thinking as deeply rooted in learners' engagement with multiple representational and embodied resources. Building on these perspectives, the third strand, *Game-based Mathematical Making*, introduces a design-oriented approach that highlights mathematical activities as enacted through making, play, and iterative design, offering

a concrete methodological context for studying creative mathematical practice. Through empirical illustrations drawn from my research, I argue that re-centering construction offers a timely reorientation for mathematics education in an age increasingly call for human-AI collaboration.

MATHEMATICAL CREATIVITY IN THE AI AGE

Creativity is widely recognised as a foundational competence for learners in the twenty-first century and is frequently identified as a central educational goal worldwide (Organisation for Economic Co-operation and Development, 2019). Within mathematics education, early research examined the relationship between mathematical creativity and mathematical ability (Kattou et al., 2013), as well as academic achievement (Bahar & Maker, 2011), often with a particular focus on gifted learners (Leikin et al., 2009). Across this body of work, researchers have typically theorized mathematical creativity through three core dimensions: fluency, flexibility, and originality (Balka, 1974; Silver, 1997; Joklitschke et al., 2022). Fluency is reflected in the range of mathematically correct and meaningful ideas generated by students when solving a problem. Flexibility is demonstrated when learners shift between representations, make use of diverse strategies, or connect different mathematical domains, including overcoming stereotypical approaches (Leikin, 2009). Originality is defined as solutions that are either novel to the student or unconventional within the context of the classroom (Silver, 1997; Leikin, 2009). These dimensions, initially drawn from creativity research more broadly (Torrance, 1966), form the basis for evaluating and supporting creative engagement in mathematics education.

Given such conceptualisations, empirical studies have framed mathematical creativity as learners' production of multiple or novel solutions, shifts in representation, and acts of generalisation (Leikin, 2009; Klein & Leikin, 2020). While such research has yielded important insights, it has also tended to frame mathematical creativity within a predominantly cognitive and task-oriented paradigm. Creativity becomes visible primarily through solution outcomes, often evaluated in relation to correctness, efficiency, or divergence from standard methods. Within this framing, creative performance is measured against established criteria of mathematical validity, leaving comparatively limited room to theorise creativity as a broader human practice of mathematical sense-making.

Recent critiques have noted that prevailing framings of mathematical creativity as individual cognitive production rather than situating it within embodied, social, and material forms of engagement (Levenson, 2022; Assmus & Fritzlar, 2022). In other words, less attention has been paid to the embodied and multi-sensory dimensions of creative activity, and to capturing creativity as a dynamic, exploratory, and materially situated practice. This limitation becomes particularly visible in contemporary learning environments involving digital tools, interactive media, and making-oriented activities (Weng et al., 2022), where mathematical ideas are explored through inquiry, experimentation, and iterative refinement. In such contexts, creativity is not confined

to generating alternative answers or solutions; it unfolds through learners' interactions with artefacts, representations, and technological systems. Yet, the creative mathematical actions emerging from these multimodal engagements remain under-theorised.

The rapid rise of AI intensifies the urgency of this reconsideration. As mathematical activity increasingly entails interacting with, interpreting, and extending AI-generated outputs, the limitations of viewing creativity solely in terms of efficiency or novelty become more apparent. Our central question shifts from whether learners can produce multiple correct solutions to whether they can design, interpret, and reshape the mathematical systems within which solutions are generated. In this context of expanding human–AI collaboration, a renewed conceptualisation of mathematical creativity emerges, one that foregrounds construction, multimodality, and sense-making.

MULTIMODALITY AND MATHEMATICAL SENSE-MAKING

The study of multimodality in mathematical thinking has gained prominence following the embodied turn in educational research (Shapiro & Stolz, 2019). From the perspective of embodied cognition (Wilson, 2002), mathematical thinking is not detached from bodily experience but deeply rooted in perception, action, and interaction (Edwards et al., 2014; Lakoff & Núñez, 2000). This perspective challenges a dualistic view of cognition as purely abstract and symbol-based, instead positioning mathematical reasoning as emerging through engagement with the world.

A growing body of research demonstrates that mathematical meaning emerge through learners' coordinated engagement with multiple semiotic resources (Arzarello, 2006), including speech, gesture, diagrams, and interaction with physical and digital artefacts (Hall & Nemirovsky, 2012). Empirical studies have shown how mathematical ideas are developed and communicated through gestures (Abrahamson, 2014; Ng & Sinclair, 2015; Shvarts & van Helden, 2023), action-oriented language (Nathan et al., 2014), and body-based metaphors (Núñez, 2006). These findings suggest that multimodality is not merely a channel for expressing pre-formed ideas; it constitutes a fundamental condition under which mathematical thinking becomes enacted and transformed. Mathematical meanings are created through coordinated action across modes (Nemirovsky & Ferrara, 2009).

In educational contexts, a multimodal lens broadens our understanding of mathematical competence. It recognises that learners may demonstrate understanding through gesture, movement, visualisation, manipulation, or digital interaction, not solely through written or spoken symbolic forms (Ng, 2016). This perspective is particularly salient in early mathematics learning, where children often rely on sensorimotor, visual, kinesthetic, and auditory experiences as they develop symbolic understanding (Sinclair & Pimm, 2015). Multimodality, therefore, provides a more inclusive account of participation and sense-making.

Research adopting this lens has generated important insights into learners' mathematical competence, including the meaningful participation of marginalised students in mathematical discourse and activity (Moschkovich, 2009). Yet much of this work has examined multimodality at the level of individual interactions or moment-to-moment sense-making. Less attention has been devoted to how multimodally rich environments can be systematically designed, orchestrated, and implemented within mathematical practice in everyday classroom (Danish et al., 2020). At the same time, as digital tools are capable of generating different forms of representations, multimodality becomes more than a lens for analysing mathematical activity; it becomes a design principle for cultivating mathematical creativity in digitally mediated classroom contexts.

GAME-BASED MATHEMATICAL MAKING

A long-standing characteristic of human culture is the desire to play games—to invent rules, explore possibilities, and engage in challenges for enjoyment, curiosity, and social meanings. Across history, games have served as sites for experimentation, creativity (Pellegrini, 2009), and the development of mathematics—in areas such as probability and statistics, mathematical modelling, and game theory (Pan et al., 2022). In contemporary contexts, this human impulse to play has gained renewed relevance in light of recent advances in AI. Notably, *AlphaGo* by Google's DeepMind (Sumner, 2016) illustrates a tradition in AI research of using games as testing grounds for computational and mathematical innovation. While systems like *AlphaGo* play games to maximise predefined objectives, human engagement with games is fundamentally different: it is exploratory, imaginative, and expressive. This contrast underscores the educational significance of the game-based context as uniquely foregrounding creativity, interpretation, and sense-making at a time when routine problem-solving is increasingly automated by AI.

Accordingly, *game-based learning* (GBL) has high potential to transform the current educational landscape; it places complete games—often purposefully designed as educational or serious games—at the centre of learning. Such methodology differs from gamification, or merely overlaying game elements onto otherwise conventional activities (Qian & Clark, 2016). Grounded in Vygotsky's (1978) Social Constructivism, GBL is an active and social process of sense-making that unfolds through problem solving, exploration, and interaction with others in rule-governed environments. A growing body of research across K–12 education (e.g., Karakoç et al., 2022; Tsai & Tsai, 2020) reported generally positive effects of GBL on academic outcomes, numeracy (Tokac et al., 2019), engagement (Jabbar & Felicia, 2015), and social-emotional learning (Alotaibi, 2024). While most research on GBL has focused on the educational effect of pre-designed games, less attention has been given to the educational potential of game creation as a form of learning (see review by Kafai & Burke, 2015).

When learners design games rather than merely play them, the mathematical terrain shifts. Game creation requires us to formulate and refine rule systems, simulate dynamic behaviours, construct probabilistic structures, model data, balance constraints, and iteratively test and revise designs. These processes require students to draw on mathematical creativity—to invent something new and mathematical which did not exist before in the learning community (Riling, 2020). Moreover, game-making is inherently multimodal: students sketch flow charts, manipulate spatially, perform imagined moves physically and digitally, narrate strategies, and refine artefacts based on feedback. In this sense, the design of a game becomes the design of a mathematical system—an activity that integrates creativity, modelling, and representations. In this way, game creation offers a concrete and timely context for examining mathematical creativity as a socially situated and multimodal practice.

Against this backdrop, I introduce mathematical making as a conceptual construct, referring to the process of constructing and iteratively refining mathematical structures, representations, or systems through coordinated multimodal activity, such that meaning emerges through the act of creation itself. *Game-based mathematical making* represents a particular instantiation of this broader process. When learners design games, they construct rule-governed worlds in which mathematical relationships regulate movement, probability, scoring systems, feedback loops, and strategic interaction. The act of balancing constraints, simulating outcomes, and refining mechanics requires learners to engage with mathematics as a dynamic system rather than a static body of knowledge. Through iterative design and play-testing, mathematical structures become objects of reflection and transformation.

EMPIRICAL ILLUSTRATIONS IN THIS SPACE

My empirical work over the past decade has examined mathematical creativity, multimodality, and digital making across a range of technology-rich learning environments. Across touchscreen mathematics applications (Ng & Yeung, 2025; Yeung & Ng, 2023), dynamic geometry environments (Ng & Sinclair, 2015; Ng, 2016), and more advanced making contexts such as 3D printing and 3D modelling (Ng & Ye, 2022), computer-aided design (Ng & Chan, 2019), and screen-based programming (Ng & Cui, 2021), a consistent finding is that mathematical creation is fundamentally multimodal; learners reason not only through symbols, but also through gesture, spatial action, material and digital construction, and multilingual talk. The following subsections offer three empirical illustrations that show how mathematical making and sense-making can be studied in practice: (i) material–digital making through 3D-pen modelling and block-based programming, (ii) multimodality as a lens for tracing computational thinking and inclusive participation, and (iii) emerging forms of AI and digital literacy as learners engage in programming and game-design tasks that foreground the mathematical ideas underpinning computational systems.

In the following, I illustrate my relevant studies, offering three empirical illustrations that show how mathematical making and sense-making become visible in practice.

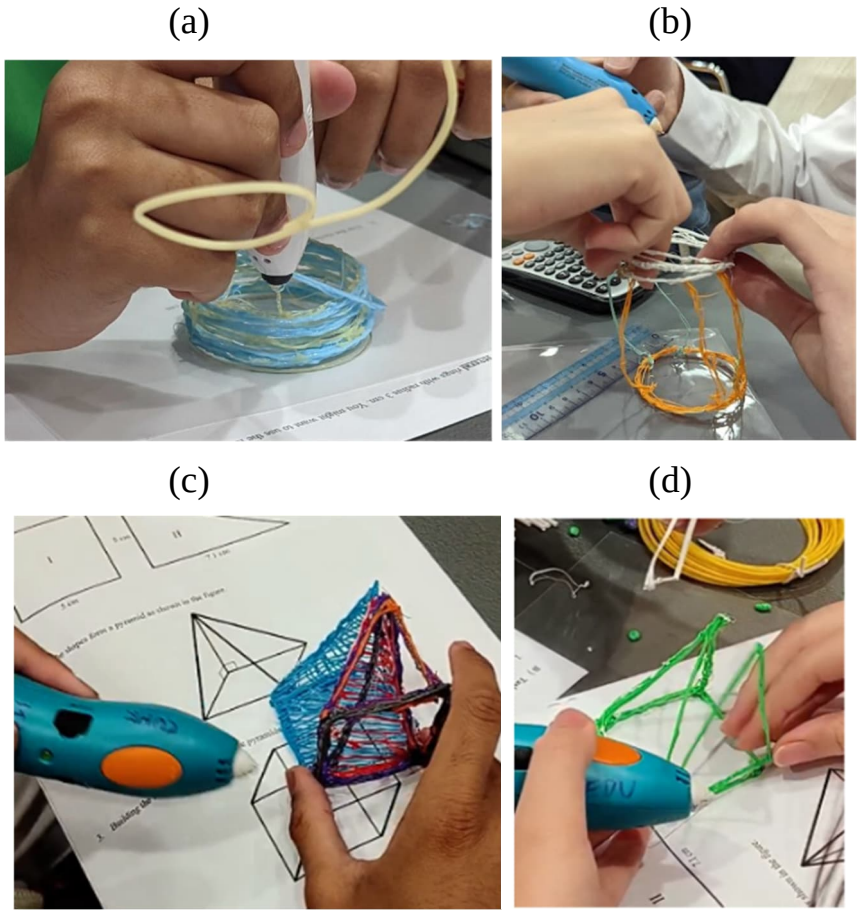
They are organised progressively: from embodied material construction, to multimodal interaction in digital environments, and finally to programming and AI-related game design tasks where learners engage with increasingly complex mathematical systems. This progression foregrounds the central claim of this plenary: that in the AI age, creativity is not reducible to solution production but emerges through iterative acts of construction across material, digital, and computational contexts.

1. Mathematical Making with 3D-Pen

One strand of my work has examined how human–technology interactions can "give rise to new possibilities for doing mathematics" (Ye et al., 2024, p. 695) within a learning community. In studies of students' 3D model construction using handheld 3D printing devices, I investigated what it means to understand learning as a form of making (Ng & Ferrara, 2020). Mathematical activity in these settings was not separable from the students' hands, the tool, and the surrounding material environment; rather, learning and making were entangled processes unfolding through embodied engagement.

To characterise such activity, I proposed the lens of *learning as making* (Ng & Chan, 2019), which captures moments of mathematical inventiveness that exceed conventional distinctions between concrete and abstract, or between body and mind (Ng & Ferrara, 2020). Rooted in embodied cognition and constructionism, this line of research embraces the idea that the act of creating is itself a site of mathematical thinking. When students engage in tactile, tool-mediated construction, such as using 3D Pens to generate geometric models (Figure 1), their work is not limited to aesthetic or mechanical choices. Rather, it often involves creative mathematical actions (Riling, 2020), defined as actions "that transitio[n] a given mathematical context into a new version of mathematics by creating ways of doing or thinking about mathematics that were previously not possible for a particular community of mathematicians" (p. 17). When using 3D Pens to create models, students may engage in such actions as they improvise new structural strategies, explore spatial relationships, and shift between dimensional entities.

Figure 1. Various moments of students' mathematical making via (a) stacking; (b) stretching; (c) sticking; and (d) filling.



In our recent classroom-based interventions (Yeung et al., 2026), we examined how embodied making activities with 3D Pens can serve as a context for fostering mathematical creativity (Figure 1). Students engaged in a range of physical crafting actions—stacking, stretching, sticking, and filling—that not only allowed them to construct geometric solids, but also opened new forms of mathematical sense-making. For example, *stacking* offered an embodied experience of building volume through the accumulation of two-dimensional cross-sections (Figure 1a); *stretching* transformed height into a lived process of extension, linking length to bodily movement and coordination (Figure 1b); *sticking* foregrounded spatial composition, as students composed solids through sequences of anchoring lower-dimensional components (Figure 1c); and *filling* reframed area as an emergent property of accumulated coverage (Figure 1d). Through these actions, students actively constructed mathematical properties, transforming geometric concepts into situated, material experiences (see also, Ng & Ferrara, 2020; Ng & Ye, 2022).

Mathematical making, in this context, is situated, adaptive, and aesthetic decision-making as learners negotiate the affordances and constraints of materials and tools. It unfolds as an iterative process of refinement and adjustment, deeply responsive to

material feedback (Papert, 1980). Each action generates new conditions for thinking, prompting further sensing, responding, and re-making. In this way, mathematics is not imposed from outside the activity but emerges through cycles of embodied construction.

2. Mathematical Making with Touchscreen Applications

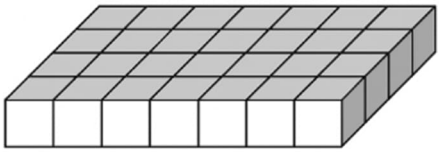
A related strand of my work has focused on multimodality and inclusive mathematical sense-making, particularly in culturally and linguistically diverse classrooms (Ng, 2016; Ng & Yeung, 2025; Zhang et al., 2026). The primary medium in this work is the touchscreen mode of interaction, which enables learners to construct digital artefacts directly with their fingers. Through analyses of gesture, touchscreen interaction with the application TouchCounts (TC), and collaborative dialogue, this research documented students' rich embodied expressions as they developed numerical and computational ideas. As they engaged with and received feedback from touchscreen applications, their gestures became increasingly coordinated and sophisticated.

This development is illustrated in the episode below, in which two young children reason about iterative processes resembling computational loops through gesture while constructing numbers in a problem-solving situation (Yeung et al., 2024). Importantly, the illustration highlights the competence of learners who are often positioned as low-performing, challenging deficit-oriented views of learning. We also demonstrate that multimodal engagement can reveal forms of mathematical making that may remain less visible in classrooms where symbolic fluency is foregrounded.

The episode further shows how early computational thinking (CT) can develop and become manifest without the use of a programming tool, but within a touchscreen application. This suggests more broadly that CT is not confined to coding environments, but may emerge across diverse tool-mediated learning contexts. In the task presented by the adult researcher (myself), two first-graders were invited to devise efficient strategies for determining the total number of unit blocks in a rectangular model without resorting to one-by-one counting while using TC (Figure 2). This contrasted with an earlier moment in which Susan responded, "I don't know," when asked, "How do you know when to stop (with TC)?"

Figure 2. (a) An example of the rectangular model task given to the participants. (b) A participant devising a strategy of finding the number of unit blocks in a random rectangular model without one-by-one ‘counting’ (Yeung & Ng, 2024)

(a) (b)



While solving the task of finding the total number of unit blocks in a 5x7 arrangement, the following interaction unfolded among Susan, myself, and TC (adapted from Yeung & Ng, 2024):

Susan:	One (Putting five fingers on TC)
TC:	Five!
Susan:	Two (Putting five fingers on TC)
TC:	Ten!
Susan:	Three (Putting five fingers on TC)
TC:	Fifteen!
	...
Susan:	(Iterating the process of putting five fingers on TC) Seven
TC:	Thirty-Five
Susan:	(Total number of blocks is) Thirty-Five!
	...
Researcher:	Can you explain what you did?
Susan:	I counted there are one (pointing to the unit of blocks), two (pointing to the unit of blocks), three (pointing to the unit of blocks), four (pointing to the unit of blocks), five (pointing to the unit of blocks), six (pointing to the unit of blocks), seven (pointing to the unit of blocks), so I tapped seven times (on the screen). [Figure 3]
Susan:	I didn’t count (them one by one), but I counted this, this, and this (referring to the three steps of the gesture scheme).

Figure 3. An illustration of child-tablet interaction and the corresponding computational concepts being enacted (Yeung & Ng, 2024).

As shown above, Susan taps her left hand with five fingers against TC while TC chants "Five." Following each tap, she uses her right hand to extend one additional finger, tracking the number of repetitions. Through this coordinated two-handed activity, Susan does more than execute a counting procedure. She constructs a dynamic representation of the 5 × 7 structure: her left hand stabilises the constant "five" as a unit of accumulation, while her right hand materialises the evolving count of iterations. The touchscreen, her hands, and the spoken feedback from TC together form a distributed system in which the mathematical structure of repeated addition is enacted rather than merely stated.

From a computational perspective, one can identify iteration and variable-like behaviour in her actions. Each tap performs an iteration—adding five to the total—while the progressively extended fingers of her right hand track the number of iterations completed. Yet beyond identifying these constructs, what is striking is how Susan *makes* the multiplicative structure visible and manipulable through embodied coordination. The constant (five) and the varying count (seven iterations) are not abstract symbols but are physically instantiated in her two hands. In this sense, she is not simply applying a known strategy; she is constructing a system that models the rectangular array.

As she repeats the process, her body synchronises the rhythm of tapping and extending, stabilising the relationship between the unit size and the number of repetitions. The iterative loop is not programmed in code but enacted through rhythmic bodily action. The touchscreen responds with numerical feedback, which in turn informs her next movement. This cycle of action and feedback exemplifies mathematical making as an iterative process of constructing, testing, and refining a structure until it coheres with the intended configuration.

The episode thus illustrates how mathematical making can foreground both multiplicative reasoning and early CT without relying on formal programming tools. Susan's embodied interaction transforms the rectangular array from a collection of unit

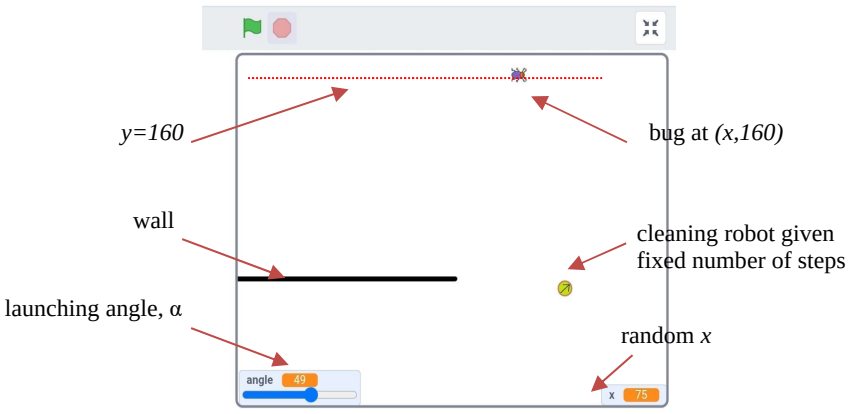
blocks into a structured system governed by repeated accumulation. Iteration and accumulation emerge as properties of the number-making activity. In this way, CT appears not as a separate skill but as one dimension of a broader process of mathematical making, where learners design and enact structures while solving problems with their hands on the touchscreen application.

3. AI and Digital Literacy "in the Making"

A third strand extends digital making into programming and game-based contexts (Ng & Cui, 2021; Ng et al., 2026). In these environments, students design digital artefacts using child-friendly programming tools, engaging in what was conceptualised as *white-boxing*: making visible and manipulable the mathematical logic underlying computational and AI systems. Through iterative design cycles, learners connect mathematical concepts, CT, and emerging AI literacy. Game-based tasks require students to model behaviours, balance constraints, and refine rule systems, foregrounding system design as a creative mathematical practice.

The designed task illustrated below, as detailed in Ng et al. (2026) represents the culmination of this trajectory. Students were introduced to a scenario in which a cleaning robot, constrained to move in straight lines and bounce off walls, must determine an appropriate launch angle to reach rubbish appearing at random positions along a fixed horizon (Figure 4). The problem required students to reason with three interdependent parameters: the launch angle α, the x-position of the rubbish, and the fixed number of steps the robot could take. Rather than treating these as isolated variables, students were invited to construct and refine a dynamic system in which geometric constraints, iteration, and spatial relationships interacted.

Figure 4. A task designed with Scratch (https://scratch.mit.edu/projects/1188791727/)

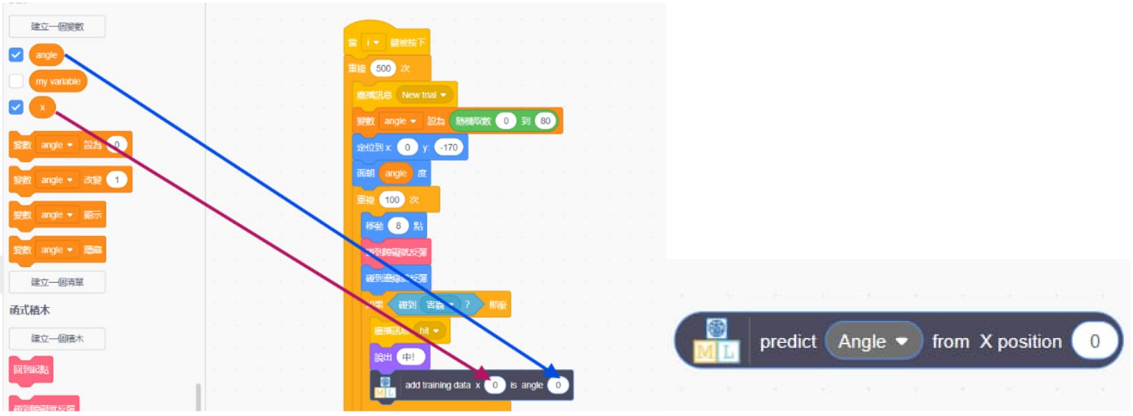


The task unfolded across three phases, each extending the scope of mathematical making. First, in Scratch, students programmed the robot's bouncing behaviour and explored how varying the launch angle affected its trajectory. Here, mathematical making involved constructing a rule-governed environment: students specified constraints, tested outcomes, and iteratively refined code to stabilise desired

behaviours. The robot's motion was not simply observed; it was shaped through deliberate structural decisions.

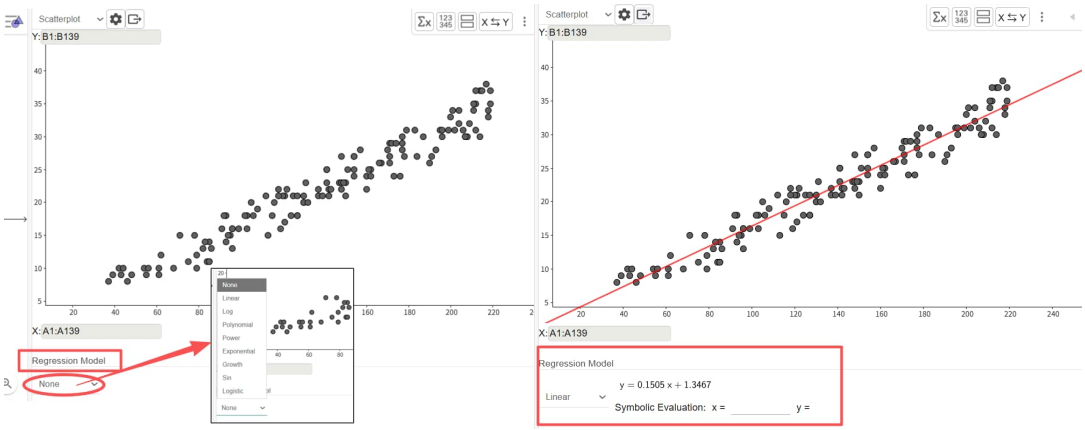
Second, students generated datasets consisting of ordered pairs (rubbish x-position, launch angle) that resulted in successful hits (Figure 5). These data were used to train a simple machine learning model in ML4K; https://machinelearningforkids.co.uk/). At this stage, students moved from making trajectories to making data-driven relationships. Importantly, AI was not presented as an opaque system but as one whose behaviour emerged from the mathematical structure of the dataset they constructed. The training process foregrounded how patterns in data could be translated into predictive rules.

Figure 5(a-b). The second part of the designed task, with ML4K.



The final phase deepened this engagement by importing the training data into GeoGebra to visualise scatter plots and explore lines of best fit (Figure 6). Students examined whether the data followed linear or non-linear patterns and interpreted the regression line as a mathematical model relating input to output. Here, AI literacy became inseparable from mathematical reasoning. When students articulated that "if the x-position increases, we need a larger launch angle," they were not merely describing machine behaviour; they were identifying functional relationships. When they recognised that inaccurate or insufficient data would affect prediction quality, they were reasoning about the limitations of modelling.

Figure 6(a-b). The third part of the designed task, with GeoGebra.



(a) Scatter plot (b) The line of best fit

Table 1 summarises this developmental trajectory, from initial procedural conceptions of AI ("AI is a system that knows everything") to increasingly relational and critical understandings grounded in data patterns and functional modelling (Ng et al., 2026). Such a trajectory developed through model construction, pattern identification, and critical evaluation. As mentioned, students initially used a trial-and-error approach in Scratch to collect input data (launch angle and the target's *x* position) and output results (hit or miss). They then ran these data through *ML4K* to assess the extent to which the AI system could predict the launch angle needed to hit a target at *x*. At this stage, students constructed a data-driven training process, but we consider it to be surface-level white-boxing of "how" AI works, focusing on data collection in the machine learning process (Table 1, middle column). This means they realized AI could use the dataset to predict an optimal launch angle rather than conceiving of "AI is smarter than humans" and "AI is a software system that knows everything" (Table 1, left column).

Table 1. Students' conception of AI before and after engaging with the said task.

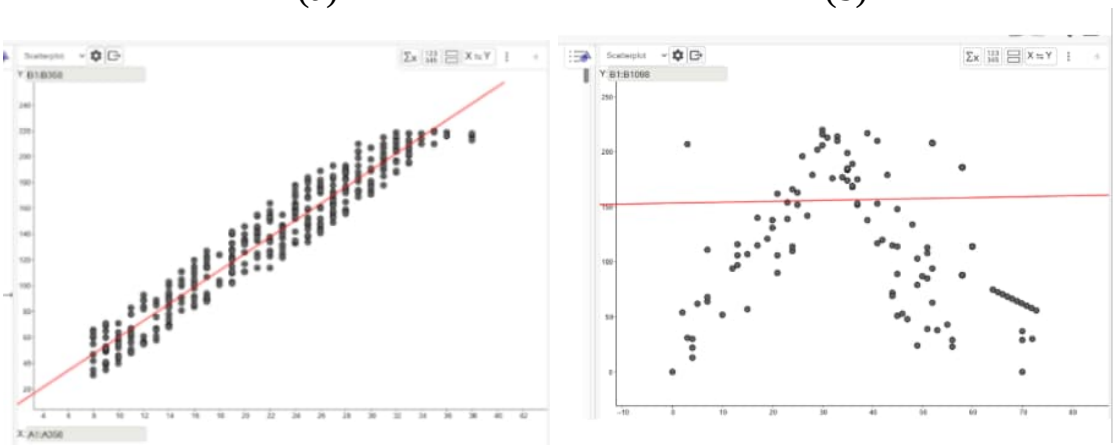
Before the lesson: Inadequate conception of AI	**Surface-level white-boxing: Vaguely describing AI's data collection**	**Critical-level white-boxing: Using mathematics to reason with AI's decision-making**
• AI is a software system that knows everything • Smarter than human; non-human but knowledgeable • Capable of performing complex calculations • It can help me with homework and engage in conversations • Capable of resolving questions beyond our understanding • AI can do something humans do, like walking; some AI knows things on the internet and helps humans • AI is a machine capable of performing diverse tasks but lacking emotional awareness	• I developed and trained new machine learning models, then observed differences generated by additional examples • AI built a model and collected 4,206 data examples • How to use AI and collect 382 examples of data • I learned how to use AI to collect data • I used AI to analyze and target rubbish • AI learns from data gathered by repeated experiments • AI collects data and turns them into learning material	• AI uses existing data or finds patterns from these data to predict the answer • I used the data to calculate the average fit number • AI generates a line from the input data, and I use the line to predict the outcomes • There is a reference line from the data; AI finds the predicted angle from the line • AI finds patterns from data to form rules, then uses the rules to predict new data • AI can be inaccurate since it is summarized from different input data • If there is incorrect data, it will affect the model that AI creates, impacting its prediction accuracy

As students engaged in the third sub-part of the task, they entered a more critical stage of mathematical making. Rather than focusing on how to operate the AI tool, they began to articulate how the system worked, grounding their explanations in mathematical analysis of the dataset. Here, they reasoned explicitly about AI's decision-making (Table 1, right column), observing that the data followed a linear pattern (Figure 7a). They identified the functional relationship between input and output, as reflected in responses such as "they are positively correlated" and "if the x-position of the bug increases, we need a larger launch angle." Those who generated a non-linear pattern (Figure 7b) likewise engaged in mathematical reasoning, offering

explanations such as “I think I haven’t collected enough data” or “maybe some unsuccessful data are mixed in.” Others observed that “AI can be inaccurate since it is summarized from different input data.” These reflections illustrate a further layer of mathematical making: evaluating the adequacy of a model in relation to its data, its assumptions, and its predictive limits.

Figure 7(a-b) a linear pattern and a non-linear pattern generated by students in the designed task

(a) (b)



Across these phases, AI and digital literacy emerged within cycles of constructing geometric rules, constructing datasets, constructing functional models, and interrogating their limitations in a game-design context. In this sense, AI literacy can be understood as mathematical making at the level of systems: learners design, visualise, and refine structured relationships that render prediction possible. The emphasis shifts from “using AI” to understanding and shaping the mathematical structures that enable AI to function. Overall, the 3-part task integrates the affordances of three tools: white-boxing in Scratch, model-making in ML4K, and data visualization in GeoGebra.

FUTURE DIRECTIONS

The empirical illustrations presented above suggest that mathematical making offers a productive lens for rethinking creativity, multimodality, and AI literacy in contemporary mathematics education. From embodied construction with 3D printing technology, to multimodal coordination in touchscreen environments, to system-level modelling in AI-infused tasks, a common thread emerges: learners construct, refine, and critique mathematical structures across material and digital contexts. If AI increasingly performs computation within predefined systems, the educational imperative shifts toward cultivating learners’ capacity to design, interpret, and reshape those systems. This shift raises pressing conceptual, methodological, and pedagogical questions for the field.

1. How is creativity understood and valued across different levels of mathematics education in the context of increasing automation and AI use?
2. What characteristics define mathematical creativity as a multimodal human practice, enacted through engagement with material, digital, symbolic, and interactional resources?
3. How do game design processes (e.g., creating rules, modelling randomness, handling data, testing game mechanics) facilitate learners' creative mathematical practices?
4. How can mathematics education be relevant to diverse sociocultural contexts in the AI age? In low-resources and non-technology-supported classrooms, what forms of AI-literate mathematics learning are possible?

Creativity via Game-based Mathematical Making

A key direction emerging from this work concerns the need to reconceptualise mathematical creativity in light of rapid developments in AI and automation. Much of my earlier research established foundational insights into creative mathematical practice, multimodality, and digital making. However, these studies were conducted prior to the recent acceleration of AI technologies that are reshaping mathematical work across education, academia, and industry. Revisiting this body of work with renewed theoretical intent offers an opportunity to extend and refine existing conceptualisations of creativity, situating them within contemporary AI-mediated contexts.

This direction may begin by examining how creativity is understood and valued across different sectors, including school mathematics education, university research, and STEAM-related industries such as data science, engineering, game design, and interactive media. Creativity in these contexts is not uniformly defined. In some settings, it is associated with innovation and problem-solving efficiency; in others, with experimentation, iterative prototyping, or the design of systems under constraint. By engaging educators, academics, and industry professionals in dialogue, it becomes possible to surface shared concerns as well as tensions in how creative (mathematical) practices are enacted and recognised. Such analysis provides a broader sociocultural grounding for understanding mathematical creativity beyond classroom-based measures of fluency or originality. Through such interdisciplinary synthesis, mathematical creativity can be theorised not merely as an individual cognitive trait but as a multimodal, design-oriented human practice enacted within rule-governed environments.

Exploring this research direction, we may ask not only about how creativity is valued in an era of automation, but also what characteristics define mathematical creativity as a multimodal practice. By positioning game-based mathematical making as a generative context for theory-building, this line of work seeks to articulate a refined

account of creativity appropriate to the AI age—one grounded in construction, design, and the human capacity to shape structured worlds.

Toward a humanities-informed, design-based methodology

A second direction concerns the methodological implications of studying mathematical making in AI-mediated contexts. Design-based research (DBR) has often been framed as a means of iteratively refining instructional designs to address authentic problems (Barab & Squire, 2004). Yet when creativity and AI are at stake, design must be understood not only as problem-solving but as a deeply human practice involving interpretation, judgment, aesthetic decision-making, and ethical reflection. A humanities-informed design-based methodology complements these dimensions.

Iterative cycles of task design, classroom enactment, analysis, and refinement remain central, but iteration is understood as interpretive refinement of human practice rather than mere optimisation. Each cycle offers insight into how mathematical structures are constructed, contested, and stabilised within a learning community. In this sense, DBR becomes a bridge between mathematics education and the humanities: it recognises that designing mathematical experiences involves cultivating imagination, critical reflection, and responsible judgment alongside technical competence. By adopting a humanities-informed stance, research on mathematical making resists reducing creativity to measurable outputs or AI literacy to functional skill acquisition. Instead, it positions classroom design as a site for examining how human practices—collaboration, narrative framing, evaluative reasoning, and ethical questioning—intersect with mathematical system construction in the AI age.

A proposal for AI literate mathematics education with and without technology

A further direction concerns the question of equity. Discussions of AI literacy often presuppose access to advanced technologies, programming platforms, or machine learning tools. Yet in many educational contexts, such resources are limited or unevenly distributed. If AI literacy is understood primarily as the ability to operate technical systems, it risks becoming inaccessible to large segments of the global population. Much of the current discourse in research and practice centres on AI tool use, including teachers' perceptions of AI-assisted instruction and students' engagement with generative technologies outside formal learning environments. Far less attention has been given to how mathematics classrooms can cultivate understanding of the mathematical ideas that underpin AI systems, or how mathematical reasoning can support critical engagement with AI decision-making and ethics.

AI is inherently linked to both computing and mathematics. At its core lie data practices, probabilistic reasoning, modelling, optimisation, and pattern recognition. Mathematics therefore occupies a unique position: it provides the conceptual infrastructure that makes AI possible. Rather than introducing rapidly evolving technologies, mathematics education can embed AI-related contexts within existing

curricular domains, particularly data handling and probability. Topics such as statistical representation, measures of central tendency, variability, correlation, and chance can be reframed through familiar AI-related scenarios. For example, probability problems can be situated in the context of recommendation systems or content filtering, enabling students to reason about how patterns in data influence predictions and decisions.

Drawing on the notion of boundary crossing (Akkerman & Bakker, 2011), AI can be conceptualised as a bridge between disciplines rather than as a standalone technological domain. Within regular mathematics lessons, students can explore how data are collected, how models are constructed, how bias may enter through sampling, and how probabilistic reasoning shapes automated decisions. In doing so, they implicitly engage with AI concepts and ethical considerations without requiring direct access to AI tools.

This perspective broadens participation. It suggests that meaningful engagement with AI-related ideas is possible even in low-resource contexts, provided that mathematics classrooms foreground data reasoning, modelling, and critical inquiry. By repositioning AI literacy within the conceptual core of mathematics, we ensure that preparation for an AI-mediated world does not become the privilege of the technologically advantaged, but a shared educational possibility.

CONCLUSION

In this plenary, I have argued that the accelerating presence of AI in mathematical activity compels us to rethink what we mean by creativity, literacy, and even learning itself. If machines increasingly compute within predefined systems, then the distinctive human contribution lies not in computation but in the capacity to construct, interpret, and reshape those systems.

Mathematical making, as developed in this paper, is *constructive, multimodal,* and *iterative–critical*. It is *constructive* in that learners build objects, relations, models, and rule structures rather than merely operate within them. It is *multimodal* in that meaning emerges through coordinated gesture, speech, symbolic inscription, material interaction, and digital feedback. It is *iterative–critical* in that learners refine, evaluate, and sometimes challenge the adequacy of the structures they create. These dimensions shift attention away from mathematics as a body of abstract forms to be mastered, and toward mathematics as a practice of world-building.

This perspective reframes a long-standing conception in mathematics education that learning is a progression from concrete to abstract. Rather than beginning with abstraction and later "applying" it, learners develop abstraction through construction. Geometric volume becomes stacking and stretching; iteration becomes rhythmic gesture; regression becomes a visible line emerging from data that students themselves have generated. Abstraction is not diminished — it is enacted.

At the same time, the shift is pedagogical. It moves us from learning as passive reception toward making as active participation. AI and digital literacy, understood "in the making," are not competencies delivered through instruction but practices cultivated through design, modelling, and critique. When students recognise that AI predictions depend on patterns in data, that models have assumptions, and that insufficient or flawed data affect outcomes, they are not simply learning about AI. They are participating in mathematical reasoning about systems.

In this light, digital and AI literacy are ontologically unfinished, pedagogically emergent, and socially constructed through mathematical activity. They evolve as learners construct artefacts, coordinate representations, and interrogate models within communities of practice. To prepare students for an AI-mediated world, I argue that mathematics education should therefore foreground the human capacity to build structured worlds, to reason across representations, and to critically evaluate the systems we design. In turn, the future of mathematics education in the AI age lies in cultivating mathematical makers — learners who can design, interpret, and remake the very structures within which computation operates.

Acknowledgements

The work reported in this paper was supported by the following research funding:

- Research Grants Council of Hong Kong, General Research Fund (Ref. 14603720 and Ref. 14603521)
- Research Grants Council of Hong Kong, Early Career Scheme (Ref. 24615919)
- The Chinese University of Hong Kong Knowledge Transfer Project Fund
- The Social Innovation and Entrepreneurship Development Fund

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FROM GAZE TO GUIDANCE: BRIDGING EYE TRACKING AND AI TO UNDERSTAND AND SUPPORT MATHEMATICAL LEARNING

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Understanding students' mathematical learning processes is essential for providing them with adequate support. Eye tracking can offer insights into learners' gaze behavior and can allow for detailed process-analyses; however, its interpretation is complex and difficult to scale for practice. This paper introduces the Gaze-to-Guidance framework, which bridges eye tracking, Artificial Intelligence (AI), and domain-specific knowledge in mathematics education to understand and support mathematical learning. The framework is illustrated through KI-ALF, an AI-supported system that uses webcam-based eye tracking with Grade 5 students. This paper illustrates how AI can extend qualitatively grounded interpretations and enable adaptive support based on learners' processes. This highlights how eye tracking and AI can move from analysis to guidance—in research and also in school practice.

INTRODUCTION

Understanding students' mathematical thinking requires regarding not only outcomes but also the processes through which these outcomes emerge (Kilpatrick et al., 2001; Schoenfeld, 1985). However, such processes are typically difficult to observe in classroom or assessment settings: On the one hand, outcome measures such as test scores or written responses provide only limited insight into how students learn mathematics. Accordingly, gaining access to students' learning processes has become an important concern in mathematics education research (Schindler et al., 2020). On the other hand, observing learning processes by interviewing students has other constraints, for example, with meta-cognitive awareness, and may suffer from accuracy issues (Schindler & Lilienthal, 2018), while conducting individual interviews with many students in school practice is often also too time demanding, especially in times of limited resources in the school world in many countries.

Eye tracking has increasingly been used to address these challenges. Eye tracking, that is, the recording of the focus of persons' gazes, enables the investigation of learners' gaze behavior and to draw conclusions about their *overt visual attention* during task processing (Duchowski, 2017). In mathematics education, it has been employed to study, for example, how learners engage with representations, how strategies differ between novices and experts, and how conceptions and student understanding may be reflected in their gaze behavior (Inglis & Alcock, 2012; Andrá et al., 2015; Obersteiner & Tumpek, 2016; Schindler & Lilienthal, 2019; Thomaneck et al., 2022). A variety of

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studies show the potential of eye tracking to provide fine-grained insights into learners' interaction with mathematical tasks (Schindler et al. 2025a; Strohmaier et al., 2020).

At the same time, interpreting eye-tracking data is often not straightforward. The often-invoked eye-mind hypothesis (or eye-mind assumption) states that the eyes fixate on the referent of what the person is processing (Just & Carpenter, 1980), yet this assumption cannot be taken for granted in mathematics education contexts, as shown by Schindler and Lilienthal (2019) for the domain of geometry. Gaze data can indicate overt attention, but not cognition directly; their interpretation is inherently ambiguous and depends on task characteristics as well as domain-specific knowledge (Hayhoe, 2004; Schindler & Lilienthal, 2019). Consequently, the interpretation of eye-tracking data can be highly demanding for human coders.

In parallel, advances in Artificial Intelligence (AI) have enabled the modeling of complex, temporal, and multimodal data in educational contexts (Schindler, Cevikbas & Lilienthal, 2026). Multiple studies have shown that AI is able to help researchers with meaningful analysis of eye-tracking data in mathematics education, for example, through the use of supervised and unsupervised machine learning approaches (Boels et al., 2023; Schindler & Lilienthal, 2019; Schindler et al., 2022; Simon et al., 2023).

This paper argues that integrating eye tracking, AI, and domain-specific knowledge offers a promising approach for linking the analysis of learners' gaze behavior with its interpretation and the provision of adaptive support. By combining fine-grained gaze data with computational approaches and grounding their interpretation in knowledge about mathematical content and strategies, it becomes possible to move beyond descriptive analyses toward supporting learners' processes through adaptive guidance. To conceptualize this integration, the paper introduces the Gaze-to-Guidance framework, which connects the sensing of gaze behavior with its interpretation and the provision of adaptive support. In doing so, the paper contributes to a perspective that extends the role of eye tracking from observing learning processes toward using them as a basis for supporting mathematical learning, both in research and in practice.

BACKROUND

Mathematics learning as visually mediated process

Mathematical learning is closely tied to the interpretation and coordination of multiple representations, such as symbolic expressions, graphs, or diagrams (Ainsworth, 2006; Duval, 2006). Engaging with these representations requires learners to attend to relevant features and to relate them to underlying mathematical structures. In this sense, arguably most school mathematics activities can be understood as visually mediated, as meaning construction depends on how learners perceive and organize visual information (Duval, 2006; Schindler et al., 2025b).

From this perspective, visual attention plays a central role in mathematical learning processes. Eye-tracking research builds on this by capturing spatio-temporal sequences

of gaze points that can be used to investigate overt visual attention and thus provide access to learners' interaction with representations (Duchowski, 2017; Schindler & Lilienthal, 2019). This offers the possibility to investigate processes that are not easily accessible through verbal reports or written work, and to gain insights into learners' strategies, difficulties, and perception processes (Schindler et al., 2025a, Schindler & Lilienthal, 2018). While gaze data may allow inferences about cognitive and affective processes, such inferences are indirect and require careful interpretation (Just & Carpenter, 1980; Schindler & Lilienthal, 2019; Schindler et al., 2025b). In particular, the meaning and interpretability of gaze patterns depends on the mathematical domain and the structure of the tasks, which calls for domain-specific approaches to interpretation (Schindler & Lilienthal, 2019). Consequently, understanding mathematical learning as a visually mediated process entails not only capturing gaze, conceptualized as, for example, visual attention but also situating it within the domain-specific context and its theoretical background.

Eye tracking in mathematics education

Eye tracking has become an increasingly prominent method in mathematics education research (Strohmaier et al., 2020). While eye-tracking methodologies have been used in psychology and related fields for more than a century, their application in mathematics education is more recent and has grown substantially over the past two decades, supported by advances in non-invasive and increasingly accessible technologies (Schindler et al., 2025a; Strohmaier et al., 2020).

By recording eye movements, eye tracking provides process-oriented data that complement traditional data sources such as written responses or verbal reports. In particular, it enables the analysis of spatio-temporal patterns of gaze, offering insights into how learners attend to different elements of mathematical representations. Empirical studies have used this approach to examine, for example, how learners read mathematical texts and proofs, how they interpret graphical representations, and how their strategies differ depending on expertise or task characteristics (e.g., Andrá et al., 2015; Inglis & Alcock, 2012; Obersteiner & Tumpek, 2016).

Systematic reviews of the field indicate a growing diversity of research foci and methodological approaches. For instance, Schindler et al. (2025a) found that eye-tracking studies in mathematics and statistics education address not only performance differences but also processes such as strategy use, attention allocation, and the coordination of multiple representations. At the same time, Schindler et al.'s review study points to considerable variation in how gaze data are analyzed and interpreted, reflecting the complexity of linking gaze behavior to mathematical thinking.

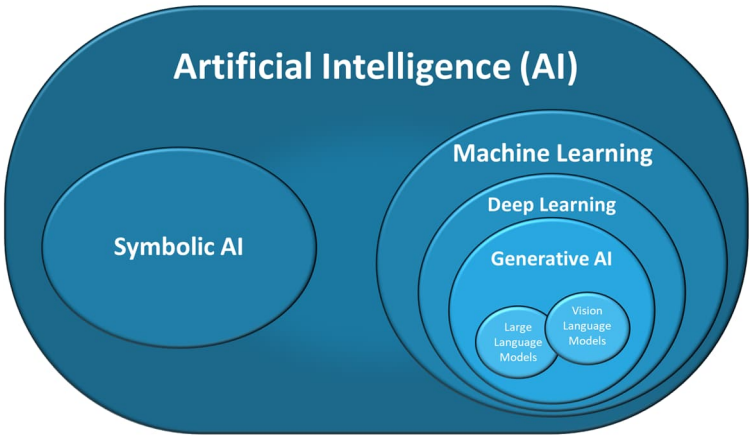
An assumption often underlying eye-tracking research is that eye movements indicate overt visual attention and can thus serve as a proxy for aspects of cognitive processing (see Schindler et al., 2025b). However, as discussed in the literature, this relationship is indirect and context-dependent (Just & Carpenter, 1980; Schindler & Lilienthal,

2019). Consequently, while eye tracking offers valuable insights into learners' work with mathematical representations, its interpretation requires theoretically grounded and domain-specific approaches (see Section 3).

Artificial Intelligence in mathematics education

Artificial intelligence (AI) is increasingly shaping research and practice in mathematics education (Schindler et al., 2026). Broadly, AI approaches can be differentiated into symbolic, rule-based systems and data-driven approaches such as machine learning (ML) (Mitchell, 1997; Russell & Norvig, 2021). Figure 1 provides an overview of major AI approaches relevant to mathematics education and their relationships. Symbolic AI relies on explicitly modeled rules and logical structures, for example to provide step-by-step feedback or analyze solution procedures in intelligent tutoring systems. In contrast, machine learning approaches learn patterns from empirical data rather than relying on predefined rules (Murphy, 2012). A common distinction differentiates supervised and unsupervised learning. In supervised learning, models learn mappings between input data and expert-defined labels and can thereby identify patterns, classify learner behavior, or support adaptive systems. In unsupervised learning, algorithms analyze data without predefined labels in order to identify structures, clusters, or recurring patterns within the data.

Figure 1. Overview of major AI approaches relevant to mathematics education



Within machine learning, deep learning refers to approaches based on multi-layered neural networks that can learn increasingly abstract representations from complex data such as language, images, video, or multimodal interaction traces (Goodfellow et al., 2016). Recent advances in deep learning have enabled the development of generative AI systems that can produce new content such as text, images, or explanations. Particularly prominent examples are large language models (LLMs), which generate and interpret natural language, and vision-language models (VLMs), which combine visual and textual processing. These approaches further extend the possibilities for

conversational interaction, content generation, and adaptive explanations in educational contexts (Håkansson & Phillips-Wren, 2024; Pepin et al., 2025).

These developments have led to a growing range of AI-supported applications in mathematics education. AI systems have been used, for example, to provide automated feedback, identify learner profiles, support adaptive learning environments, and generate mathematical tasks or explanations. At the same time, many current AI-based systems primarily rely on data such as correctness information, response patterns, or interaction logs (Azevedo, 2026). While such data can support adaptive feedback and personalization, they often provide only limited access to learners' underlying mathematical processes and strategies when working on mathematical activities. As a result, important aspects of students' mathematical thinking may remain difficult to capture and interpret.

Moreover, the use of AI in mathematics education raises important challenges. These include questions of reliability and validity, the transparency and interpretability of AI-based classifications, and the meaningful integration of AI into educational contexts (Nanda & Pradhan, 2025; Nguyen & Pham, 2025). In particular, many machine learning models risk functioning as opaque "black boxes," making it difficult to understand how classifications or recommendations are generated. This is especially problematic in educational contexts, where interpretations and support decisions require didactical justification. Consequently, while AI offers substantial opportunities for supporting mathematics learning, its effective use requires theoretically grounded, interpretable, and didactically responsible approaches.

FROM GAZE TO MEANING: THE INTERPRETATION CHALLENGE

While eye tracking can provide information on learners' overt visual attention, deriving meaningful interpretations from eye-tracking data remains a central challenge. Eye movements indicate where learners look, but they do not directly reveal how they think. The key issue is not whether eye tracking can capture attention, but how eye-tracking data can be meaningfully interpreted in relation to mathematical learning and teaching.

A central challenge concerns the ambiguity of gaze data. The same gaze pattern may correspond to different cognitive processes, strategies, or mathematical thoughts. For instance, looking back and forth between two corners in a regular polygon may indicate that the learner is comparing angles, or imagining a line between them, or comparing adjacent areas (Schindler & Lilienthal, 2019). Such ambiguity highlights that gaze data alone are not always sufficient for determining learners' understanding, depending on the domain and task.

Closely related is the domain-specific nature of interpretation. The meaning of learners' gazes and gaze patterns depends on the mathematical content and the structure of the representation. Interpreting gaze patterns therefore requires relating them to domain-specific knowledge about how meaning is constructed in a given task,

for example when coordinating multiple representations. While in some mathematical activities, eye-tracking data and detected gaze patterns can be used on their own to identify learners' processes, in others additional data sources are necessary. And in fact, in empirical studies in mathematics education, eye-tracking data are often enriched by, for example, thinking aloud interviews, stimulated recall interviews, interviews, or observations (Schindler et al., 2025a).

A further challenge concerns the effort and resources required for meaningful interpretation of gaze data. Rich insights into mathematical processes, learners' strategies, and reasoning processes are often obtained through qualitative analyses of eye-tracking data: For example, analyzing eye-tracking data qualitatively allows researchers to see how students proceed on numbers lines (Simon & Schindler, 2022), how they read tree diagrams (Bruckmaier et al., 2019), how they interpret contextual graphs (Thomaneck et al., 2022), how young learners compare quantities and how they proceed in pattern-extension tasks (Pitta-Pantazi et al., 2025), or enumerate quantities through simultaneous enumeration, quasi-simultaneous enumeration, and counting (Schindler et al., 2020). Such approaches allow for nuanced and theoretically grounded interpretations of learners' processes and have contributed substantially to understanding mathematical thinking. However, they are also highly time- and resource-intensive, as they require detailed analysis of individual cases. This limits their scalability, particularly in studies involving larger numbers of learners. Consequently, while qualitative approaches can provide rich insights, there remains a need for methods that can support the analysis of larger datasets without losing sensitivity to process-related aspects of learning.

Taken together, these considerations point to a fundamental gap: While eye tracking can enable detailed observation of overt visual attention, the step from gaze to meaningful interpretation remains theoretically and methodologically demanding. It requires explicit theoretical assumptions about what eye movements represent in mathematical contexts (Schindler et al., 2025b), as well as approaches that can account for the ambiguity, domain-specificity, and resource-intensive nature of interpreting gaze data. In the following, I will argue that and how AI offers new opportunities to address these challenges by supporting the systematic, scalable, and context-sensitive interpretation of learners' gaze behavior.

AI AS THE BRIDGE: MODELING GAZE FOR UNDERSTANDING

The challenges outlined in the previous section—the ambiguity of gaze data, the domain-specific nature of their interpretation, and the effort required for their analysis—point to a central requirement: Approaches are needed that support systematic, context-sensitive, and scalable interpretation of learners' gaze behavior. AI offers promising possibilities to address these challenges by enabling the modelling and integration of complex and heterogeneous data.

A first strength of AI lies in its potential to interpet gaze data. Machine learning methods can be used to identify patterns in gaze behavior across learners and tasks, such as recurring attention patterns or strategy-related differences. In mathematics education, initial studies have demonstrated how such approaches can support the identification of learners' strategies based on gaze data. For example, Boels et al. (2023) applied supervised machine learning methods to classify students' interpretations of statistical representations (histograms and dot plots) based on gaze features, showing that gaze data can be used to identify students' reasoning patterns. Similarly, Schindler et al. (2019) used supervised learning approaches to analyze students' quantity recognition processes, training models to distinguish between learners with and without mathematical difficulties based on eye-tracking data. On the other hand, Schindler et al. (2022) employed a data-driven approach that includes unsupervised learning techniques to identify patterns in eye movements and relate them to students' enumeration strategies. By clustering gaze features and subsequently interpreting these patterns, this study showed how unsupervised approaches can cluster students' enumeration process without relying on predefined categories. Such approaches illustrate how AI can contribute to scalable and systematic interpretations of gaze data, complementing qualitative analyses.

AI also offers possibilities for addressing the effort and scalability challenges associated with qualitative analysis. While detailed qualitative approaches provide rich insights into learners' processes, they are difficult to apply to larger datasets. Recent studies illustrate both the richness of gaze data and the scale at which such analyses can be conducted. For example, Boels et al. (2023) used AI to analyze students' interpretations of statistical representations based on data from 50 students across 25 items, resulting in approximately 1,200 gaze-based process instances. Similarly, Schindler et al. (2019) in the above-mentioned study investigated quantity recognition processes using data from 164 students across 11 tasks, yielding around 1,800 instances through the use of AI. Schindler et al. (2022) extended this line of work by analyzing enumeration processes from 227 students across 36 tasks, resulting in approximately 8'000 process instances. In these contexts, manually identifying strategies for each individual instance would require substantial resources, particularly given the fine-grained nature of gaze data and the need for contextual interpretation. By applying supervised and unsupervised learning approaches, these studies demonstrate how AI can support the analysis of large numbers of cases by automating parts of the pattern detection and classification process, while still building on insights derived from qualitative research. This creates opportunities to extend process-oriented analyses beyond small-scale studies.

At the same time, AI enables the integration of multiple data sources. Depending on the subdomain and tasks used, gaze data alone can be insufficient for interpreting how learners proceed in mathematical activities. AI-based approaches allow combining gaze data with additional information, such as given answers, processing times, or task

characteristics. This multimodal perspective can contribute to robust and context-sensitive interpretations of learners' processes in mathematical activities. For example, Asghari et al. (2026) combined analysis of students' solutions and their eye movements to analyze their strategies in a digital adaptive learning system that supports students with mathematical difficulties in basic mathematical competencies.

However, the use of AI for interpreting gaze data requires careful consideration. AI models may introduce new forms of opacity, and their outputs depend on the quality of the data and the assumptions embedded in the modeling process. In particular, many machine learning approaches—especially complex models such as neural networks—can function as so-called "black boxes," where the reasoning underlying their predictions is not readily interpretable. In educational contexts, this lack of transparency is problematic, as interpretations of learners' processes and resulting support decisions need to be pedagogically meaningful and justifiable. It may otherwise remain unclear why a learner is identified as following a particular strategy or requiring support. Addressing this challenge requires the use and development of interpretable models and explanatory approaches that make algorithmic decision processes more transparent and traceable. Studies such as Boels et al. (2019), Schindler et al. (2019) and Schindler et al. (2022) have shown how this can be realized. Consequently, AI should not be understood as replacing theoretical interpretation, but as a tool that can support and extend it. Rather than delegating interpretation to opaque systems, AI-based analyses need to remain connected to domain-specific theory and educational reasoning. This aligns with recent developments in the field, where AI-based approaches are increasingly explored as a complement to established methods (Schindler et al., 2025a).

Taken together, AI provides a means to address key challenges in moving from gaze to meaning by enabling systematic, domain-sensitive, and scalable analysis of gaze data. At the same time, its use requires approaches that ensure interpretability and transparency of the underlying models, so that resulting interpretations remain meaningful and theoretically grounded. In this way, AI opens the possibility not only of interpreting learners' gaze behavior effectively, but also of doing so in a manner that is both scalable and accountable, thereby enabling interpretations to inform adaptive support. This perspective forms the basis for the Gaze-to-Guidance framework introduced in the following section.

FROM UNDERSTANDING TO GUIDANCE: THE GAZE-TO-GUIDANCE FRAMEWORK

Introduction of the framework

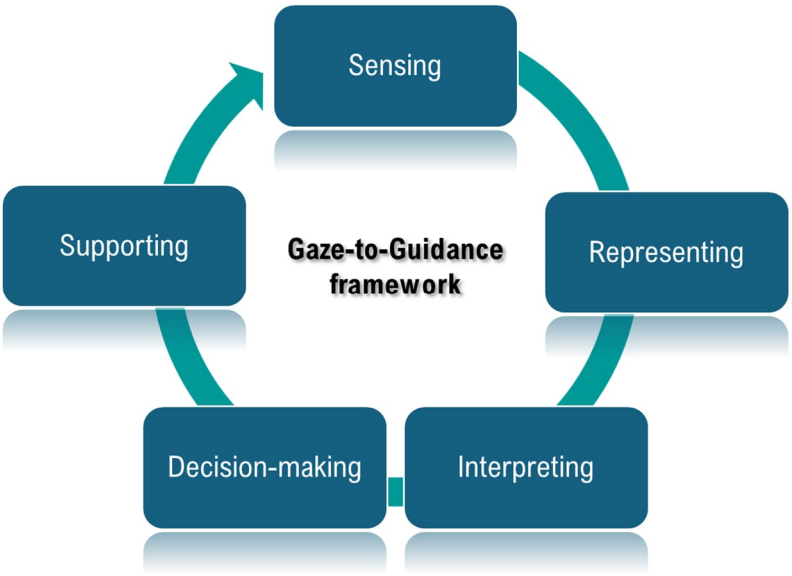
Building on the challenges and opportunities outlined in the previous sections, I propose the *Gaze-to-Guidance framework* as a conceptual model linking learners' gaze behavior to adaptive support in mathematics education. The framework addresses the

gap between capturing gaze data and deriving meaningful, actionable interpretations by integrating insights from eye-tracking research, mathematics education, and AI.

At its core, the framework conceptualizes the process from *gaze to guidance* as a sequence of interconnected components, in which gaze behavior is first recorded, then represented and interpreted, and ultimately used to inform instructional decisions. In doing so, it explicitly accounts for the challenges identified earlier: the ambiguity of gaze data, the domain-specific nature of their interpretation, and the ressources required for their analysis. At the same time, it incorporates the potential of AI to support systematic, scalable, and interpretable analyses.

The framework consists of five main components (Fig. 2): *(1) sensing*, referring to capturing learners' gaze behavior; *(2) representing*, involving the transformation of raw gaze data into meaningful features or representations; *(3) interpreting*, where gaze patterns are related to learners' processes using domain-specific knowledge and data-driven approaches; *(4) decision-making*, in which interpretations are used to determine appropriate forms of support; and *(5) supporting*, referring to the provision of support for the learner. These components are not to be understood as strictly linear, but as part of an iterative process in which interpretations and support can be continuously refined.

Figure 2. Graphical overview of the Gaze-to-Guidance framework



Importantly, the framework does not assume that gaze data are always sufficient for understanding learning processes. In some cases, depending on the mathematical domain and the nature of the tasks, gaze data alone can provide meaningful insights into learners' processes, while in other cases, gaze data alone are not sufficient and need to be complemented by additional sources of information. The Gaze-to-Guidance framework therefore integrates both human expertise and machine intelligence. While AI supports the systematic analysis of gaze data, human expertise is central for

interpreting AI-based results, for designing didactical decisions, and critically reflecting on model assumptions and design choices. By structuring the trajectory from sensing to supporting, the framework provides a basis for both analyzing learning processes and designing systems that can respond to them in real time.

Components of the framework

Within this framework, eye tracking primarily contributes to the sensing of gaze behavior, while AI supports the representation, interpretation, and decision-making processes.

Sensing:The sensing component refers to capturing learners' gaze behavior during their work with mathematical tasks. The sensing component may be realized through different eye-tracking technologies, ranging from dedicated corneal-reflection-based systems to more accessible webcam-based approaches, which differ in their accuracy, robustness, and practical applicability in educational contexts (Schindler et al., 2025b; Asghari et al., 2026). The goal is to obtain data on where and how learners direct their gaze while working on visually presented mathematical tasks. The appropriateness of the designed tasks in relation to the eye-tracking device as well as the technological setup are fundamental to the quality of the gained eye-tracking data. This component is typically realized through eye-tracking technologies, which enable the recording of learners' gaze behavior during their mathematical work.

Representing: The representing component involves transforming raw gaze data into meaningful representations that can be further analyzed. This may include deriving features such as fixations, transitions, using Areas of Interest (AOIs), or structuring gaze data in relation to task elements. Also, gaze heat maps, gaze plots are other representations of gaze scanpaths may serve as representation, on which interpretation and decision-making are conducted. In any case, these representations serve as an intermediate layer between raw data and interpretation, making patterns in gaze behavior accessible for analysis. Choices made at this stage influence which aspects of learners' processes can be captured and how they can be interpreted. In some cases, automated methods can support the extraction and structuring of relevant features, or AI-based preprocessing of eye-tracking data may be beneficial or even needed for the subsequent steps.

Interpreting: The interpreting component focuses on relating patterns in gaze behavior to learners' processes and strategies (e.g., Simon et al., 2023). This requires combining data-driven approaches, such as pattern detection or classification, with domain-specific knowledge about mathematical thinking and task structures. Given the ambiguity of gaze data, interpretations must be grounded in theoretical assumptions and contextual information. Depending on the task and domain, interpretation may rely on gaze data alone or require the integration of additional data

sources. AI-based methods can support this process by identifying patterns in gaze behavior and relating them to potential strategies or difficulties.

Decision-making: The decision-making component concerns the use of interpretations to determine appropriate forms of support. Based on inferred strategies or identified difficulties, decisions are made about whether and how the learner receives support. These decisions may involve selecting feedback, prompts, or tasks that are aligned with the learners' current processes and goals for the further learning process (e.g., Schindler et all, 2025c). Importantly, decision-making should remain transparent and grounded in didactical expertise. AI can assist in this step by suggesting possible support actions based on identified patterns, while final decisions are taken by domain experts.

Supporting: The supporting component refers to the provision of support or guidance to the learner based on the preceding analysis. This may take different forms, such as hints, prompts, visual cues, or adaptive task modifications. The aim is to support learners in developing effective strategies or, for instance, a deeper understanding. Supporting is not a one-time action but part of an ongoing process, where learners' responses can be further observed and inform—possibly iterative—cycles of sensing, interpreting, decision-making, and supporting.

Synthesis of the framework

The Gaze-to-Guidance framework brings together the perspectives developed in the previous sections by structuring the trajectory from observing learners' gaze behavior to providing adaptive support. It addresses the central challenges identified in Section 3: the ambiguity of gaze data, the domain-specific nature of their interpretation, and the effort required for their analysis. In particular, it emphasizes that meaningful interpretation requires combining domain-specific theoretical knowledge with data-driven analysis of gaze patterns.

At the same time, the framework integrates AI inherently. Rather than treating AI as an autonomous decision-making system, it conceptualizes AI as a tool that may support the sensing, representation, interpretation, decision-making, and supporting components. In doing so, it enables the analysis of larger datasets and the identification of patterns that would be difficult to capture manually, while maintaining the need for human expertise in interpreting results and making pedagogical decisions.

Importantly, the framework is not limited to a specific technological setup or mathematical domain. Instead, it provides a flexible structure that can be adapted to different contexts, tasks, and levels of analysis. Depending on the situation, different components may be emphasized, and different forms of data may be integrated. This flexibility reflects the (sub-)domain-specificity of approaches.

Taken together, the Gaze-to-Guidance framework contributes a conceptual foundation for connecting eye tracking and AI together with domain-specific knowledge in mathematics education. By explicitly linking gaze behavior to interpretation and support, it shifts the focus from using gaze data primarily for analysis toward leveraging it for responsive and adaptive learning environments.

GAZE-TO-GUIDANCE IN PRACTICE: KI-ALF

Introduction to KI-ALF

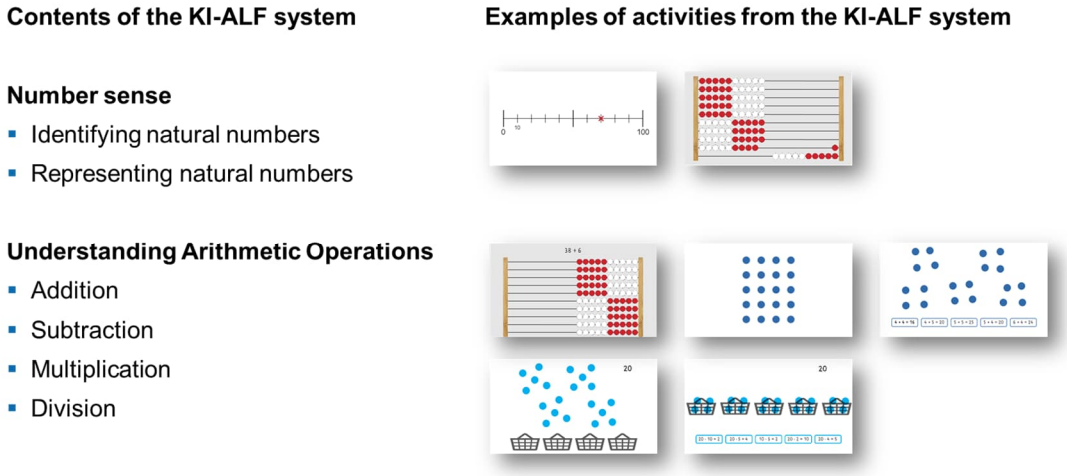
KI-ALF is a research and development project that aims to support students in developing basic mathematical competencies through an adaptive and individualized system. The acronym KI-ALF stands for *"KI-basierte adaptive Lernunterstützung zur Diagnostik und Förderung der mathematischen Basiskompetenzen im inklusiven Kontext,"* which roughly translates to *"AI-Based Adaptive Learning System for Assessing and Supporting Basic Mathematical Competencies in Inclusive Contexts."* A particular focus of KI-ALF lies on school students who experience difficulties in mathematics, for whom timely and tailored support is especially important.

KI-ALF addresses particularly students in Grade 5. Various empirical studies indicate that a substantial proportion of students have not sufficiently developed basic mathematical competencies by the end of primary school (e.g., Ehlert et al., 2013; Moser Opitz, 2013; Schindler et al., 2019) and that students attending comprehensive schools often experience considerable difficulties in this regard (Ehlert et al., 2013). In a study with fifth-grade students in an inclusive comprehensive school, Schindler et al. (2019) found that more than half of the students had not yet adequately acquired these competencies at the beginning of Grade 5. Longitudinal studies across secondary education further show that these difficulties are rarely resolved in school practice. In particular, findings from comprehensive schools indicate that many students continue to struggle with basic concepts. For example, Ehlert et al. (2013) found that around half of the students were still unable to solve division tasks even in Grade 7. As Ehlert et al. (2013, p. 259) conclude, it is doubtful whether these students will be able to understand mathematical concepts that build on such basic knowledge, such as fractions or the place value system. Overall, the findings suggest that in both primary and secondary education it often remains challenging for teachers to effectively support students in developing these highly relevant basic mathematical competencies.

The KI-ALF system focuses on activities that address basic mathematical competencies that are essential for students' further learning in secondary education. In particular, the system engages students in core areas such as number sense, basic operations, and the understanding of representations such as the number line (Fig. 3). Students work on tasks that involve, for example, identifying and representing natural numbers, interpreting and relating different representations (e.g., symbolic and visual), and the understanding of the basic arithmetic operations. The tasks in the KI-ALF system are designed to make students' strategies and potential difficulties visible,

thereby providing a basis for process-oriented analysis and adaptive support within the KI-ALF system through the use of webcam eye-tracking with the help of AI.

Figure 3. Contents and Activity Examples of the KI-ALF system



The central idea of KI-ALF is to combine eye tracking and AI to analyze students' strategies and processes during their work on mathematical tasks and to use this information to provide adaptive support. Dedicated eye trackers currently available on the market are, however, still too costly for practical use in schools, with prices ranging up to 20'000–30'000 € depending on the model and its technical specifications. Additionally, practitioners such as teachers often lack the time, training, and resources to analyze eye-tracking data themselves and to draw meaningful conclusions. To make eye tracking available to schools and democratize its use, we first developed a webcam eye-tracking system as the basis for the KI-ALF learning system. KI-ALF integrates a webcam-based eye-tracking approach with AI-based analysis methods, enabling the capture and interpretation of learners' gaze behavior in real time. In addition to gaze data, further information such as learners' responses and interaction data is considered, allowing for understanding learners' strategies and processes. Based on its analyses, KI-ALF aims to identify learners' strategies and potential difficulties and to provide individual and adaptive support. This support is realized, for example, through explanatory videos or adapted tasks that are aligned with the students' patterns of strategies and responses. In this way, KI-ALF operationalizes the idea of linking process-oriented data with adaptive guidance, making it a suitable example for illustrating the components of the Gaze-to-Guidance framework.

In the following, I will describe how the main components of the framework—sensing, representing, interpreting, decision-making, and supporting—are realized within the KI-ALF system.

Components of the KI-ALF system

Sensing: In KI-ALF, the sensing component is realized through a webcam-based eye-tracking system that captures students' gaze behavior while they work on

mathematical tasks. This approach allows for the collection of gaze data in classroom settings without requiring specialized and costly hardware. AI-based methods are used at this stage to support the estimation and stabilization of gaze positions from webcam input, enabling robust data capture under real-world conditions. The system records where students look on the screen during task processing, thereby providing process-oriented data on their work with different task elements and representations.

To improve the quality of webcam-based eye tracking, KI-ALF for instance incorporates an AI-based automatic correction approach addressing spatial offset, a common issue in webcam eye tracking caused, for example, by participant movement and calibration inaccuracies. Rather than relying on repeated recalibration or additional hardware, the system estimates spatial offset automatically using domain-specific information from the mathematical tasks (Asghari et al., 2023). More specifically, the correction method maximizes the correspondence between gaze data and predefined Areas of Interest (AOIs) within the tasks. Based on this estimated offset, gaze positions are automatically corrected to improve data quality and robustness under realistic classroom conditions, with a substantial improvement of the accuracy (Asghari et al., 2023).

Representing: In KI-ALF, the representing component transforms raw gaze data into structured representations that can be analyzed computationally and interpreted in relation to students' mathematical processes. To achieve this, gaze positions are related to mathematically meaningful AOIs within the tasks, such as numbers, tick marks, or other relevant task elements or structures. Based on these mappings, the system derives gaze-based features including gaze density alongside the AOIs' contextual information, capturing gaze patterns both locally and globally. In this way, gaze data are represented not merely as isolated coordinates, but as process-oriented patterns reflecting students' interactions with mathematical representations and task structures.

KI-ALF considers an augmented gaze-based feature that uses visual task contextual information, alongside gaze density, to identify local and global eye movement patterns. This enables the representation of students' strategies as dynamic processes rather than isolated events. In addition, gaze-based representations are combined with further information such as students' responses and interaction data, allowing for a comprehensive representation of learners' work on the mathematical tasks. For example, in number line tasks, transitions between tick marks and orientation points can be represented to distinguish counting strategies from approaches that use further mathematical structures (Simon et al., 2023).

Interpreting: In KI-ALF, the interpreting component combines domain-specific knowledge about basic mathematical competencies with data-driven analysis methods. AI-based models are used to identify patterns in students' gaze behavior and to relate these patterns to student strategies or difficulties. Importantly, these

interpretations are not derived automatically from gaze data alone but build on extensive prior qualitative research on students' mathematical processes.

For example, earlier eye-tracking studies on structured whole-number representations such as the abacus qualitatively investigated how students proceed when enumerating represented quantities and identified different kinds of strategies associated with varying levels of structure use (e.g., Schindler et al., 2020). These studies showed, for instance, that some students used structured strategies based on grouping and recognizing larger numerical structures, while others relied on counting individual beads or rows. Through such analyses, domain-specific strategy typologies and expertise were developed, which allow for meaningful interpretation of gaze patterns in the respective mathematical (sub-)domain.

Building on these qualitatively developed categories, subsequent work investigated whether AI-based approaches could automatically identify student strategies from eye-tracking data (Asghari et al., 2026, Simon et al., 2023). Here, expert-labelled gaze representations, such as heatmaps and scanpath-based representations, were used as training data for supervised machine learning models. The system then learned mappings between gaze behavior and qualitatively identified strategy categories. Our studies showed that such approaches are promising for automatically identifying students' strategies, although the performance differs between mathematical activities and subdomains (Asghari et al., 2026). While the highest accuracies were achieved for number line activities, lower performances were observed for other tasks such as enumeration in structured representations or visual multiplication tasks, underlining the domain-specific nature of interpreting gaze behavior through AI.

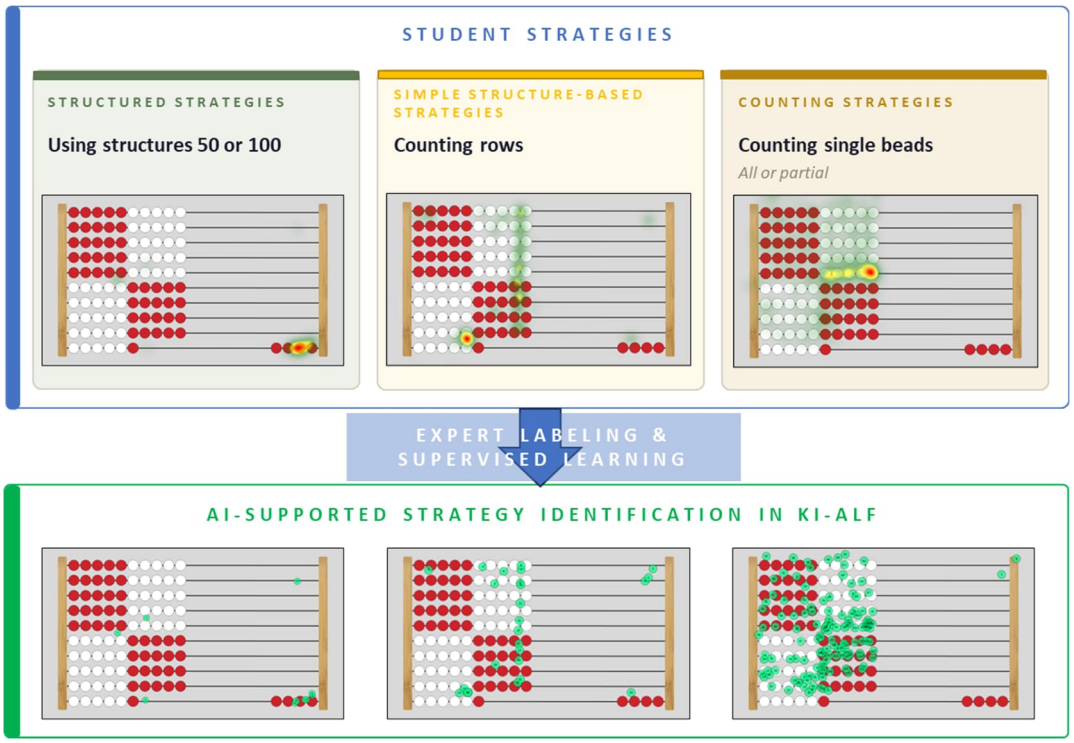
The development from qualitative strategy analyses toward AI-supported strategy identification is illustrated in Figure 4. The upper part shows qualitatively identified kinds of student strategies in enumeration tasks with the abacus, derived from our prior eye-tracking research. Based on expert labeling and supervised learning, the KI-ALF system learned mappings between gaze patterns and strategy categories. The lower part illustrates how corresponding gaze patterns are captured and used for AI-supported strategy identification within the KI-ALF system.

In this way, the KI-ALF system integrates qualitative and AI-based approaches to address key challenges of interpreting eye-tracking data, namely ambiguity, domain-specificity, and the resource-intensive nature of process analysis. Rather than replacing qualitative interpretation, AI-supported analysis builds on domain-specific expertise and extends it to larger numbers of students and process instances.

Importantly, however, the interpretation process in KI-ALF does not rely solely on gaze behavior but also integrates students' responses and error patterns. In addition to identifying how students proceed, the system analyzes the mathematical correctness and characteristics of students' answers. For example, in enumeration tasks with the abacus, the system can distinguish different error types, such as inversions or different

kinds of counting inaccuracies. Combining gaze-based strategy identification with analyses of students' responses allows for a rich interpretation of learners' mathematical processes and supports differentiated forms of adaptive support.

Figure 4. From qualitative strategy analyses to AI-supported interpretation in KI-ALF



At a broader level, the KI-ALF system illustrates how AI can complement rather than replace human interpretation. Qualitative analyses conducted by researchers and educators remain essential for identifying meaningful mathematical strategies, developing theoretically grounded categories, validating AI-based classifications, and critically reflecting on model decisions. These interpretative and didactical tasks require domain expertise and didactical judgment and therefore cannot simply be delegated to automated systems. AI, in turn, is particularly valuable for scaling such analyses by identifying recurring patterns across large numbers of learners and process instances. In this sense, KI-ALF does not treat AI as a "black box," but as a tool whose outputs remain connected to theoretically grounded interpretations and open to human inspection and refinement.

Decision-making: The decision-making component in KI-ALF builds on the interpretation of students' strategies, responses, and difficulties to determine appropriate forms of support. Based on the identified gaze patterns, response behaviors, and kinds of errors, the system assigns kinds of predefined support decisions and corresponding learning trajectories. These decisions are grounded in

domain-specific and didactical considerations about how particular difficulties in basic mathematical competencies can be addressed.

Importantly, the decision-making process does not only distinguish between correct and incorrect answers but also considers how students arrived at their responses. For example, in tasks involving the abacus, the system differentiates between various kinds of errors, such as inversions, counting inaccuracies, or difficulties related to place value understanding. At the same time, it considers the identified strategy patterns, for example whether students rely on counting individual elements or use larger numerical structures. Based on this combination of strategy and error analysis, learners are assigned to different forms of support.

In this way, KI-ALF combines machine learning approaches with symbolic AI. While supervised machine learning is used to identify students' strategies and process patterns from gaze data, symbolic AI is used to connect these identified patterns with explicitly modeled didactically informed rules and support pathways. For example, students showing counting-based strategies may receive support aimed at encouraging the use of structures. The decision-making component therefore functions as a bridge between the interpretation of learners' processes and the provision of adaptive support.

The use of symbolic AI is particularly important at this stage because it ensures that support decisions remain transparent, interpretable, and theoretically grounded. Rather than generating arbitrary recommendations, the system follows explicit didactical decision structures that were developed by domain experts in mathematics education and that can be inspected, critically reflected upon, and refined by researchers and educators. In this sense, decision-making in KI-ALF combines automated pattern recognition with human-designed didactical reasoning.

Supporting: The supporting component refers to the provision of adaptive support and guidance within the KI-ALF system. Based on the previously identified strategies, response patterns, and kinds of errors, the system provides individualized forms of support that are aligned with students' observed mathematical processes and aimed at further developing their basic mathematical competencies. This support is realized, for example, through explanatory videos, hints, visualizations, the use of alternative representations, or adapted practice tasks.

Importantly, the provided support is not only linked to whether an answer is correct or incorrect, but also to how students approached the task. For example, students showing counting-based strategies in number line activities may receive support encouraging the use of orientation points or larger structures instead of counting all tick marks. Similarly, for example, students who count all individual beads on the abacus—even for bigger numbers—instead of using the underlying structures can receive targeted support aimed at recognizing and using larger numerical structures.

AI contributes at this stage by enabling the dynamic selection and delivery of support based on the ongoing analysis of students' gaze behavior, responses, and interactions with the system. At the same time, the support pathways themselves are grounded in didactical considerations and predefined structures. Students' subsequent interactions can then be captured again through webcam-based eye tracking and interaction data, enabling an iterative process in which interpretations and support are continuously refined and adapted over time.

Synthesis of the KI-ALF system

Taken together, the KI-ALF system illustrates how the components of the Gaze-to-Guidance framework can be realized in a concrete educational setting. The example demonstrates how webcam-based eye tracking, AI-based analysis, symbolic AI, and domain-specific didactical knowledge can be integrated to move from capturing students' gaze behavior toward interpreting their strategies and providing adaptive support. In doing so, KI-ALF highlights that meaningful AI-supported interpretation does not emerge from gaze data alone, but builds on extensive prior qualitative research, domain-specific strategy analyses, and human expertise. Rather than functioning as an opaque "black box," the system combines supervised machine learning for identifying gaze patterns with transparent, didactically grounded decision structures that remain open to human inspection and refinement.

At the same time, KI-ALF illustrates both the opportunities and limitations of combining eye tracking and AI in mathematics education. The example shows how AI can support the analysis of large numbers of process instances and thereby extend qualitatively grounded interpretations beyond small-scale studies into school practice at comparatively low cost. However, it also underlines the domain-specific and contextual nature of interpreting gaze behavior, as the quality of interpretation and support depends, among others, on the mathematical content, task design, representations, and technical conditions of data capture. In this sense, KI-ALF exemplifies how the integration of eye tracking and AI can contribute to making process-oriented insights accessible in educational practice while remaining grounded in didactical reasoning and human expertise.

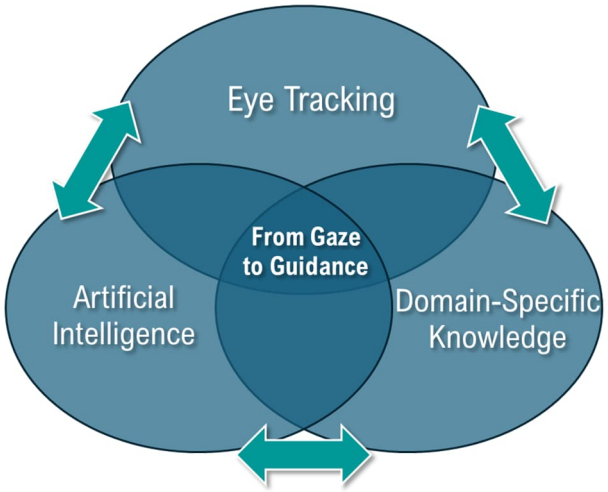
DISCUSSION

Theoretical implications: From gaze to guidance

This paper contributes to the field of mathematics education by extending the use of eye tracking from the analysis of learning processes toward their support. While eye tracking has traditionally been used primarily as an analytical tool to investigate learners' gaze behavior, the Gaze-to-Guidance framework extends this perspective by explicitly linking gaze behavior not only to the analysis of learners' processes, but also to the design of adaptive instructional support. In doing so, it shifts the focus from describing learners' processes to using these processes as a basis for adaptive support.

A central theoretical contribution lies in the integration of eye tracking, AI, and domain-specific perspectives on mathematical learning (Fig. 5). The framework emphasizes that gaze behavior does not carry meaning in itself, but requires interpretation grounded in knowledge about mathematical content, representations, and strategies. At the same time, AI-based methods can enable the systematic identification of patterns in gaze data and can support the extension of such interpretations beyond small-scale qualitative analyses. Meaningful interpretation thus emerges from the interplay between domain-specific theory and data-driven analysis, where AI supports, but does not replace interpretation.

Figure 5. Interplay of eye tracking, AI, and domain-specific knowledge in mathematics education for the Gaze-to-Guidance framework



Furthermore, the Gaze-to-Guidance framework highlights the importance of considering the entire path from sensing to supporting. It conceptualizes data collection, interpretation, and instructional response as interconnected components of an iterative process in which interpretations of learners' strategies and processes continuously inform subsequent instructional support. This perspective aligns with a process-oriented view of the mathematical learning process.

Taken together, the framework contributes to bridging a gap between research on mathematical learning and its application in educational practice.

Methodological implications: Scaling process data responsibly

A central methodological implication of this work concerns the analysis of process data in mathematics education. Gaining meaningful insights into learners' strategies and reasoning processes often requires detailed qualitative analyses, which are time- and resource-intensive and therefore difficult to apply at scale. This paper conveyed a methodological perspective in which AI is used to scale qualitatively grounded interpretations of learners' processes while maintaining their theoretical background. In this way, the integration of AI into the Gaze-to-Guidance framework offers new possibilities for supporting the systematic analysis of large amounts of gaze data.

Importantly, AI does not replace qualitative approaches but builds on them. In the studies underlying the KI-ALF system, domain-specific interpretations of students' strategies were first developed through qualitative analyses. These interpretations then served as the basis for training AI-based models that can identify similar patterns in larger datasets. For example, in the KI-ALF project, AI-based approaches enabled the analysis of thousands of process instances across learners and tasks, which would have been laborious using purely qualitative methods. In this way, AI enables the extension of qualitatively grounded interpretations beyond small-scale studies, allowing for the analysis of large numbers of learners, tasks, and process instances.

At the same time, the use of AI raises important methodological considerations. In particular, the opacity of many machine learning models poses challenges for educational research and practice. As argued in this paper, AI-based approaches should not function as "black boxes," but need to be interpretable and aligned with domain-specific theory. This is particularly important in educational contexts, where educators need to understand why a particular strategy or difficulty has been identified, to respond appropriately. In KI-ALF, this is supported by visualizing the categorizations made by the system, for instance through aggregated representations such as average gaze heatmaps, which make machine-based decisions more transparent and interpretable. At the same time, such visualizations allow researchers to critically reflect on and refine the underlying models, adjusting algorithms and their metrics to ensure that classifications become more closely aligned with didactical considerations.

Furthermore, the framework illustrates the importance of combining multiple data sources. Gaze data alone do not always provide sufficient information for interpreting learners' processes, and their meaning depends on the domain, task, and representation. Integrating gaze data with additional sources, such as students' responses and interaction data, can contribute to more robust and context-sensitive interpretations. Again, decisions about if and how data sources can or even must be combined to draw meaningful conclusions depends on the (sub-)domain.

Taken together, these considerations point toward a methodological perspective in which AI is used to extend, rather than replace, established approaches. By combining qualitative insights, domain-specific theory, and data-driven analysis, the Gaze-to-Guidance framework supports a form of scaling that remains grounded in domain-specific theory and sensitive to the complexity of learners' processes.

Implications for practice: Bringing eye tracking to schools

The integration of eye tracking and AI, as conceptualized in the Gaze-to-Guidance framework and illustrated by the KI-ALF system, opens up new possibilities for supporting mathematical learning in educational practice. Traditionally, eye tracking has been largely confined to research settings due to the cost of equipment, the complexity of data analysis, and the expertise required to interpret gaze data. As a

result, its potential for informing teaching and learning processes has remained largely inaccessible to practitioners.

The developments presented in this paper point toward ways of addressing these limitations. In particular, the use of webcam-based eye tracking, combined with AI-supported analysis, enables the collection and interpretation of process data in classroom settings without requiring specialized hardware, extensive manual analysis, or eye-tracking data interpretation expertise. In systems such as KI-ALF, students' gaze behavior is analyzed during task processing to identify their strategies and potential difficulties, which are then linked to adaptive forms of support, such as explanatory videos or adapted tasks.

Initial experiences from classroom implementations highlight the practical potential of such approaches. Teachers who are using KI-ALF in practice report that KI-ALF can support multiple students simultaneously, enabling individualized support at a scale that would be difficult to achieve through teacher intervention alone, particularly in contexts of limited resources and teacher shortage (see University of Cologne, 2025). Students can work at their own pace and receive support aligned with their individual strategies, which can reduce pressure and support their focused work with mathematical tasks. At the same time, teachers can use the system to identify students who require additional support and direct their attention accordingly.

At the same time, it is important to emphasize that systems such as KI-ALF cannot address all challenges of mathematics learning. However, they can support a substantial number of students in meaningful ways. In addition, the human mathematics teacher can support where additional, more substantial, further, or even more adaptive support is needed.

The use of systems such as KI-ALF in practice requires careful consideration. Teachers need to be able to understand and trust the information provided by the system, which underscores the importance of transparency and interpretability. Rather than presenting opaque outputs, systems should provide insights in forms that are meaningful, for example by making learners' strategies visible through visualizations or categorizations that align with didactical concepts. These insights can support teachers in adapting tasks and providing individual feedback based on students' observed approaches. In this sense, the role of the system is not to replace the teacher, but to support them. At the same time, systems such as KI-ALF can support mathematics teachers in their documentation and reporting of the students' learning process, the support and the progress. They track the whole learning trajectory and can provide teachers with overviews and textual reports that can be used for different purposes.

Yet, the integration of eye tracking into classroom practice also raises questions related to feasibility and implementation. These include issues such as the availability of appropriate technological infrastructure, the integration of such systems into existing teaching practices, as well as constraints related to time, classroom dynamics, and

variability in technical conditions. Although for KI-ALF only a computer, simple webcam and screen are needed, this hardware cannot be assumed to be available and up to date at all schools in different countries. This relates to issues of accessibility. Addressing these challenges will be crucial for ensuring that the potential of such approaches can be realized at scale in practice.

Taken together, the combination of eye tracking and AI offers the potential to move eye tracking from a method primarily used in research toward a tool that can support teaching and learning in everyday classroom contexts. Systems such as KI-ALF contribute to bringing eye tracking from specialized research environments into school practice, where they can support teachers in gaining insight into students' processes and in providing timely and targeted support.

Limitations and future research

This work has several limitations that point to directions for future research. First, realizing the Gaze-to-Guidance framework is always a domain-specific endeavor. The interpretation of gaze behavior depends on the mathematical content, the structure of tasks, and the strategies relevant within a given domain, which requires careful adaptation when transferring the approach to other content areas. Also, methodological challenges remain. Gaze data are inherently ambiguous, and AI-based approaches depend on data quality and modeling assumptions. Ensuring the interpretability of model outputs is therefore essential, particularly in educational contexts.

Future research should extend the approach developed in KI-ALF to further mathematical domains, such as algebra or geometry, in order to examine its generalizability across different mathematical content areas. At present, KI-ALF focuses primarily on Grade 5 students, particularly those experiencing difficulties with basic mathematical competencies. However, the underlying platform offers potential for a broader range of applications, including different mathematical topics, learners with varying ability levels (e.g., also high-achieving students), and different educational stages, such as transitions to vocational or upper secondary education. In addition, the AI-based models and their interpretability used in systems such as KI-ALF can always be developed further and refined. Recent developments in the field of AI open up new possibilities for more nuanced analyses of learners' processes. At the same time, these opportunities require careful investigation to ensure that resulting models remain transparent, theoretically grounded, and educationally meaningful. Furthermore, empirical studies should investigate how such systems can be effectively integrated into classroom practice, including how teachers use system-generated insights and how these affect students' learning processes over time.

Taken together, these directions highlight that the integration of eye tracking and AI is a promising approach that is worth continued development, evaluation, and research.

CONCLUSION

This paper introduced the Gaze-to-Guidance framework as a conceptual approach for linking eye tracking, AI, and domain-specific knowledge in mathematics education. By connecting the analysis of learners' gaze behavior with the design of adaptive support, the framework extends the use of eye tracking from observing learning processes toward using them as a basis for supporting mathematical learning. The example of the KI-ALF system illustrates how this perspective can be realized in practice, showing how gaze-based data and AI-supported analysis can contribute in practice to identifying learners' strategies and providing individual support at scale. At the same time, the paper highlights that such approaches depend on theoretically grounded interpretation, transparency, and the integration of human expertise in their development. Taken together, the integration of eye tracking and AI provides an opportunity for moving from gaze to guidance in mathematics education, enabling process-oriented insights to inform teaching and learning while remaining grounded in domain-specific knowledge in mathematics education.

Funding information

KI-ALF was funded by the German Federal Ministry of Education and Research (BMBF) under grant number 01NV2123. The author is responsible for the content of this publication.

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PLENARY PANEL

ALL MATHEMATICAL SENSE MAKING HAPPENS THROUGH INTERACTION WITH PEOPLE AND ARTEFACTS – NOTHING IS PURELY MENTAL

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The debate motion "All mathematical sense-making happens through interaction with people and artefacts – nothing is purely mental" speaks directly to a longstanding concern in mathematics education: how to understand the relationship between mind, body, discourse, culture, and material mediation in the production of mathematical meaning? This paper presents the debate as structured for the purposes of this publication. Following an introduction by the chair, Phakeng, each panelist's arguments are presented individually. Shvarts and Chan argue in support while Gomez and Maciejewski argue against the motion. The positions advanced here remind us that mathematics education research is always shaped by theoretical choices, and that such choices are never purely technical or methodological – they are also axiological and ethical. Ultimately, we hope that this debate demonstrates how scholars with fundamentally different perspectives can engage one another seriously, critically, and respectfully in the collective pursuit of understanding.

INTRODUCTION (MAMOKGETHI SETATI PHAKENG)

This panel is organised around the Oxford-style motion: "All mathematical sense-making happens through interaction with people and artefacts – nothing is purely mental." The motion speaks directly to a longstanding concern in mathematics education: how to understand the relationship between mind, body, discourse, culture, and material mediation in the production of mathematical meaning. Its significance is heightened by the theme of PME49, *Sensing and Making Sense of Mathematics*, which foregrounds questions about perception, embodiment, representation, and meaning-making in mathematical activity.

The motion is intentionally provocative. It does not merely ask whether interaction with people and artefacts matters for mathematical sense-making. Rather, it asks whether such interaction is universal and whether anything that may legitimately be regarded as mathematical sense-making can ever be purely mental. In doing so, the motion forces us to confront fundamental questions about the nature of mathematics, the nature of cognition, and the role of mediation in learning and knowing.

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These questions have occupied mathematics education researchers for decades. The field has witnessed ongoing debates about the relationship between cognition and participation (Sfard, 1998), between individual reasoning and social interaction (Vygotsky, 1978), and between cognitive, sociocultural, embodied, and discursive accounts of mathematical activity (Lakoff & Núñez, 2000; Radford, 2021; Sfard, 2008). The four contributions assembled in this panel engage these debates from different theoretical traditions and disciplinary perspectives.

The sequencing of the papers is deliberate. Anna Shvarts opens the proposition case by drawing on sociocultural, embodied, and extended theories of cognition to argue that mathematical sense-making is inseparable from interaction with artefacts and culturally organised forms of action. David Gómez follows with a challenge to this position, drawing on research in developmental psychology, comparative psychology, and numerical cognition to argue that some forms of mathematical sense-making can occur independently of interaction with people and artefacts. Esther Chan then advances the proposition through the lens of commognition, arguing that mathematics is fundamentally a form of discourse and that mathematical sense-making is inseparable from participation in socially and historically constituted communicative practices. Finally, Wes Maciejewski presents the opposition position by arguing that mathematics consists of mental abstractions which may be inspired, communicated, and conventionalised socially but nevertheless reside in the human mind.

Readers should note that the papers published here provide the conceptual foundations of the speakers' positions. However, this is not a conventional panel discussion. It is an Oxford-style debate. The live presentations may therefore differ substantially from the written papers and may include passionate rhetorical performances, direct engagement between speakers, counterarguments, rebuttals, and spontaneous exchanges. The written contributions provide depth and context; the debate itself provides the intellectual arena in which these ideas will be tested. We hope that the exchange will keep the audience intellectually engaged and, perhaps, on the edge of their seats.

Theoretical significance of the motion

The motion touches a central fault line in mathematics education research. On one side are perspectives that locate mathematical sense-making in discourse, participation, embodiment, and interaction with cultural artefacts. On the other side are perspectives that maintain that such accounts, while indispensable for explaining learning and communication, do not fully account for the nature of mathematical thinking itself.

Shvarts' contribution is representative of contemporary efforts to move beyond Cartesian separations of mind and body. Drawing on sociocultural theory, embodied cognition, and extended mind perspectives, she argues that mathematical sense emerges through culturally organised interactions with artefacts and representations. Her analysis suggests that even when mathematical reasoning appears internal, it remains shaped by historically developed relationships with symbolic and material

artefacts. In this view, the distinction between internal and external becomes less significant than the dynamic systems of action through which mathematical meaning is constituted (Radford, 2021; Shvarts, 2025).

Gómez challenges this position by broadening the notion of mathematical sense-making beyond formal school mathematics. Drawing on evidence from infancy studies and comparative cognition, he argues that newborn infants and non-human animals demonstrate forms of numerical perception and discrimination before meaningful participation in mathematical discourse or engagement with mathematical artefacts (Feigenson et al., 2004; Izard et al., 2009). His argument raises important questions about whether mathematical sense-making should be understood as emerging solely from cultural participation or whether it also has developmental and evolutionary foundations.

Chan approaches the debate through Sfard's (2008) commognitive framework, which understands mathematics as a specialised form of discourse. From this perspective, learning mathematics involves becoming a participant in a historically developed discourse characterised by particular symbols, routines, narratives, and forms of substantiation (Sfard, 2008, 2015). Mathematical sense-making therefore becomes inseparable from participation in communicative practices. At the same time, Chan extends the discussion beyond the boundaries of commognition itself, arguing that even non-symbolic and pre-linguistic forms of mathematical activity remain dependent on interaction with the environment.

Maciejewski presents a philosophical challenge to these positions. He argues that mathematics consists of mental abstractions. Although social interaction and embodied experience may support the development, communication, and application of mathematical knowledge, they should not be conflated with mathematics itself. His position resonates with longstanding philosophical traditions that distinguish mathematical objects from the social and material conditions through which people encounter them.

Taken together, the four papers demonstrate that the debate is not reducible to a simple opposition between "social" and "cognitive" perspectives. Rather, it reopens enduring questions about the ontology of mathematical objects, the epistemological conditions through which they become known, and the methodological frameworks through which mathematics education researchers interpret human activity (Radford, 2008; Sfard, 1998, 2008).

Sense-making in multilingual mathematics classrooms

The motion has particular significance when viewed through the lens of multilingual mathematics education. Research conducted in multilingual classrooms has consistently demonstrated that mathematical meaning is not secured through formal symbolism alone, nor transmitted through a single language treated as transparent (Adler, 2001; Moschkovich, 2002, 2010; Setati, 2005).

Mathematics classrooms are communicative spaces in which learners and teachers move across languages, gestures, diagrams, inscriptions, examples, and everyday experiences as they engage in mathematical activity. Meaning emerges through explanation, translation, revoicing, code-switching, questioning, and the strategic use of multiple semiotic resources (Barwell, 2009; Moschkovich, 2010). From this perspective, language is not simply a vehicle for expressing mathematical ideas. It is one of the resources through which mathematical ideas are constructed, negotiated, and refined.

Research in South African multilingual classrooms has repeatedly shown that access to mathematics is deeply intertwined with access to language and discourse (Setati, 2005; Setati & Adler, 2000, 2001). Learners do not merely learn mathematical concepts; they learn ways of participating in mathematical practices. Such participation is shaped by histories of language, identity, power, and access. Consequently, mathematical sense-making involves navigating multiple forms of mediation while simultaneously negotiating questions of legitimacy, authority, and belonging.

The multilingual classroom therefore provides a powerful empirical illustration of the mediated nature of mathematical activity. It makes visible what often remains hidden in monolingual accounts of learning: that mathematical participation is always organised through social, cultural, linguistic, and material resources.

At the same time, multilingual classrooms caution against reducing mathematical understanding to observable interaction alone. A learner may be silent and yet mathematically active. Another may participate fluently without demonstrating conceptual coherence. Internal reasoning cannot simply be inferred from visible participation, just as visible participation cannot be dismissed as secondary to cognition. What multilingual classrooms reveal particularly clearly is the need to theorise the relationship between mediation and cognition rather than reducing one to the other.

It is precisely this tension that gives the present debate its significance for mathematics education.

Debate as scholarly practice

The value of this panel lies not only in the topic under discussion but also in the form through which disagreement is organised.

The Oxford-style debate provides a disciplined structure for engaging competing ideas. Participants are required not only to articulate positions but also to respond directly to alternative perspectives. Such exchanges expose assumptions that often remain hidden within particular theoretical traditions and create opportunities for genuine intellectual engagement.

This is especially important in contemporary society. Across many parts of the world, public discourse has become increasingly polarised. Disagreement is frequently treated

as a threat rather than as a resource for learning. Nuance is sacrificed to certainty, and intellectual engagement is too often replaced by ideological positioning.

Academic communities have a responsibility to model a different way of engaging. Universities exist not merely to generate knowledge but also to cultivate forms of inquiry capable of sustaining democratic and plural societies. Scholarship advances not through unanimity but through rigorous engagement with competing ideas.

For international scholarly communities such as PME, this responsibility is particularly significant. Mathematics education is characterised by multiple theoretical traditions, languages, cultures, histories, and experiences. Scholars often arrive at similar questions through very different intellectual pathways. Progress depends not on eliminating these differences but on engaging them productively.

Such engagement requires more than tolerance. Tolerance implies merely enduring difference. Acceptance recognises that others may see, know, argue, and theorise differently because of who they are, where they grew up, how they were socialised, and the traditions within which they work. Acceptance acknowledges that these differences are not obstacles to scholarship but resources for collective inquiry.

The distinction is important. A scholarly community committed only to tolerance risks reproducing hierarchies in which dominant perspectives are merely willing to "put up with" alternatives. A community committed to acceptance recognises plurality as constitutive of intellectual life itself. Such a community creates conditions in which disagreement can occur without hostility, critique can occur without contempt, and difference can exist without exclusion.

In this sense, the Oxford-style debate serves a purpose beyond the immediate motion under discussion. It models forms of scholarly engagement that are urgently needed in a world increasingly characterised by polarisation and intolerance.

Relevance to the PME49 theme

The theme of PME49, *Sensing and Making Sense of Mathematics*, places perception, embodiment, representation, and meaning at the centre of inquiry. The present debate contributes directly to this agenda by asking whether mathematical sense-making is always mediated through bodies, artefacts, discourse, and social interaction, or whether there remain forms of mathematical activity that can properly be regarded as purely mental.

The contributions by Shvarts, Gómez, Chan, and Maciejewski suggest that no simple answer will suffice. Instead, they reveal the layered and multifaceted character of mathematical activity. They remind us that terms such as *mind*, *body*, *artefact*, *discourse*, *representation*, and *sense-making* are not self-evident categories but theoretical constructs whose meanings depend on broader commitments about learning and knowing.

The panel also extends the conference theme in another important way. It treats sense-making not only as an object of research but also as a mode of scholarly life. To make sense together requires more than technical expertise. It requires forms of listening, engagement, and coexistence capable of sustaining productive disagreement. In that respect, this debate does not merely discuss mathematical sense-making; it enacts the challenge of making sense across difference.

Conclusion

The motion before us is intentionally provocative. Yet its significance lies less in determining a winner than in creating an opportunity for the field to revisit foundational questions about mathematical meaning, cognition, mediation, learning, and participation. The contributions assembled here demonstrate both the richness and complexity of these questions. They remind us that debates about mathematics are simultaneously debates about human beings: how we know, how we learn, how we communicate, and how we make sense of the world.

By placing proposition and opposition into direct conversation, this panel invites PME49 participants to engage enduring tensions within mathematics education through the lens of the conference theme. It also offers a reminder that the work of sense-making is not confined to learners and classrooms. It is also the work of scholars who must learn to think, argue, and coexist together in a world that urgently needs more generous forms of intellectual engagement.

ARTIFACTS IN MATHEMATICAL SENSE-MAKING (ANNA SHVARTS)

As all panelists pay attention, the question of whether mathematical sense-making can occur as a purely mental process, independent of interaction with people and artifacts, requires zooming into possible theoretical views on the core terminology: mathematics, sense-making cognition, and artifacts. Historically, philosophical perspectives on the nature of mathematics have ranged from Platonic idealism, which posits mathematical entities as existing independently of human cognition, to various constructivist, sociocultural, and materialist theories that ground mathematical activity in embodied human practice. Each perspective offers valuable insights, and rather than seeking a single "true" philosophical position, I explore which grand theories are illuminating for revealing the relation between mathematical sense-making and interaction with artifacts and other humans. After briefly analyzing the main approaches, I dive into empirical evidence from a synthesized sociocultural and embodied/extended perspective. I showcase the subtle forms of mathematical interaction with ready-made artifacts and their making. Further, I argue for the value-based theoretical choice to theorize artifacts as still present in mathematical sense-making even if they are not physically available in the external environment and sense-making seems “mental”.

Diversity of theoretical approaches: Mathematics, Sensemaking, Artifacts

The theoretical landscape of mathematics education research offers several broad traditions, each offering distinct answers to the question of where mathematical ideas reside. We consider each of them briefly, outlining how they interpret mathematics, mathematical sense-making, and artifacts.

Platonism is perhaps historically the most widespread view that is accepted by the majority of working mathematicians and the general public: mathematical objects exist in an abstract, mind-independent realm, unrelated to artifacts or other people. Its widespread version holds that concepts are defined formally and sense-making proceeds via logical propositions, which are transparent to human experience. Another version of Platonism, that survives the critique of formalism by Gödel's incompleteness theorems, assumes that mathematical truth cannot be accessed through formalism, but requires transcendental phenomenological intuition (Tieszen, 2011). Neither of those theoretical views assigns any significance to cultural artifacts or other people in mathematical sense-making. As our Motion, as well as the Opposition and Mathematics Education Research in general, assumes that interaction with artifacts and other people is essential for developing mathematical understanding (otherwise, how would education be at all possible?), the Platonistic view on mathematics is not interesting for further consideration.

The next broad perspective is **cognitivism**, which considers mathematics as a system of abstract mental concepts. It tries to offer a conceptualization of mathematical sense-making in line with the traditional cognitive science view of the mind as an information-processing device with computational and representational powers. Various representations and mental models (e.g., Duval, 2006, but also many others) are at the core of mathematical sense-making from a cognitivist perspective. However, an idea of mentally representing mathematical concepts is problematic (see Thompson & Sfard, 1994): simplifying, representations in cognitive science are meant to represent universal objective physical entities through processing information about them. Yet, the representation of, say, multiplication as a mathematical concept cannot be theorized in the same way: multiplication is, by definition, abstract and not present in reality to be represented mentally. Moreover, classical cognitivist views do not consider artifacts and other people as partaking in thinking and making sense, as those are treated as mental processes. So, again, the cognitivist view is not interesting for dismissing or corroborating the Motion.

A more interesting perspective for our Motion comes from **constructivism**, as developed by Piaget (1970), Dubinsky (2002), Vergnaud (1998), and many others. Here, the actions of a subject with the outside world, including material objects and artifacts, lie at the core of constructing mental models that foster those experiences. While initial mathematics learning (sensory-motor intelligence in Piaget, action in APOS theory, etc.) is grounded in such interactions, later, representations of the previous sensory-motor actions come forth. In the later stages of mathematical sense-

making, the constructivist perspective states, that actions with objects and artifacts are internalized and form mental schemes that can be manipulated without referring back to the artifacts and actions with them. This view seems to be a perfect candidate for the Opposition to the Motion: even though it does not assign a specific theoretical role to artifacts and other people, it centers the idea of initial interaction and further processing of mental constructions, without such interactions.

However, overcoming the solitude of cognizing subjects and making mathematics a shared knowledge poses conceptual difficulties for constructivism (Lerman, 1989; Radford, 2008). The constructivist perspective either needs to jump back to Platonic ideas that personal experience intuitively or a priori grasps transcendental universal truth, or to socio-cultural relativism, settling mathematical knowledge in language and cultural practices (Lerman, 1989). This moves us to the next set of theories, which also mark a more contemporary debate on the ontology of mathematics and the role of the artifacts and other people.

Here, I combine a broad set of theories that work towards overcoming Cartesian dualism in understanding mathematical sense-making. Those are **socio-cultural** approaches (e.g., Sfard, 2008; Radford, 2021, and many others), **new materialism and agential realism** (de Freitas & Sinclair, 2014), and a wide range of theorizing that can be agglomerated under the umbrella of **radical e-views: embodied, extended, enactivist perspectives** (e.g., Reid & Mgombelo, 2015; Abrahamson, 2021; Shvarts, 2025). All of those rather diverse perspectives center on the role of material environments, artifacts and humans in dynamic interactions (or intra-action) in the process of becoming. Mathematics appears as trans-individual knowledge exactly because it is seen as a system of artifacts within bio-socio-cultural-historical practice, which reaches beyond individual agency in its unfolding materiality. So, as both Motion and Opposition suggest, the interaction with artifacts and other people is in focus on those perspectives. Yet: is it required for any mathematical sense-making?

Interacting with artifacts: uncovering mathematical sense

Let us now dive deeply into tracing the subtle ways in which specific interactions with mathematical artifacts uncover mathematical sense. The examples come from eye-tracking studies and studies of blindness to account for various types of interactions. With those examples, I show that mathematical sense is uncovered through specific forms of actions, which allows a subject to perceive mathematical affordances of the artifacts and thus make sense of them mathematically.

Let me start with the canonical example of the perception of coordinates on a Cartesian plane (Krichevets et al., 2014) where attending to the points' coordinates requires a specific cultural form of action: looking along the axis and counting. An analysis of the children's perception reveals that this cultural way of action needs to be established, and only then does mathematical perception of the points develop (Shvarts, 2018). As multiple studies show, such vertical/horizontal interaction is essential for various mathematical situations related to the Cartesian plane, such as searching for a mean

value on a histogram (Boels et al., 2025) or operating with vector fields (Hahn & Klein, 2025). Another example comes from proving in geometry (Marco & Shvarts, 2025): a student argues that two segments are equal. His verbal expression is limited to "Ah, yes", while careful fixation-per-fixation analysis of his eye movements reveals that the equality statement is confirmed by focusing on the corresponding sides of the equal right triangles.

Okay, the aforementioned examples cover "visual" areas of mathematics. What about algebraic manipulations? Recording eye movements in reading expressions with brackets (Schneider et al., 2012) reveals that patterns of eye movements repeat the structure of brackets, focusing at first on the operations that need to be performed earlier. The shape of the brackets as "embracing" an expression appears essential for comprehension, facilitating the target interaction (Goldstone, 2010). Similarly, Braille readers need to be taught efficient strategies in reading algebraic expressions through touch to make sense of them (Leendert et al., 2023). So, specific *cultural forms of interaction* with the artifacts are critical for mathematical sense-making.

All those results exemplify that mathematical sense-making requires developing an ability to interact with artifacts in a particular way, so that subjects can perceive mathematics. Artifacts become incorporated in the dynamic system of students' perception and action (Nemirovsky et al., 2013, Ferrari & Ferrara, 2020; Shvarts et al., 2021) or in their cultural practice (Radford, 2021). This incorporation of the artifacts (rather than interiorization of procedures, as constructivist approaches suggest) reveals their cultural affordances and enables higher-order thinking (Shvarts, 2025).

(Re-)inventing artifacts: reifying/crystallizing mathematical sense

Another form of artifacts' involvement in mathematical sense-making consists of their gradual development by students. The transition between a process and an object is considered at the heart of advanced mathematical thinking. While constructivist researchers assume such transition happens through encapsulation of process into mental objects, socio-cultural researchers highlight the transformation in language and material culture and language that supports such reification of actions (Sfard, 2008) and crystallization of new artifacts (Leontiev, 1987; Radford, 2021; Shvarts et al, 2024). Such reification happens both in the development of culture and later in the development of a particular student who comes to re-invent this culture as they progressively signify their interaction with others and the environment by introducing new artifacts.

Mathematical sense-making without artifacts?

Socio-cultural ideas often assume that artifacts can interiorize and become mental, yet preserving semiotic mediation, as the popular accounts of Vygotsky would assume (Vygotsky, 1987; Bartolini Bussi & Mariotti, 2015). In this case, if artifacts are interiorized, they are not visible yet present in a mathematical sense-making. Let us

zoom in on the more subtle interpretations of those Vygotskian ideas, thus clarifying the notion of artifacts.

Imagine a cup and a small ant sitting on its edge. The way an ant encounters a cup has nothing to do with making sense of a cup as a cultural artifact: it is rather a mountain for it than a cup. An educated human and an ant interact with a cup in completely different ways, revealing its different affordances. A cup, as a cultural artifact, is an example of how material and ideal are merged together (see Shvarts et al., 2024): an action of drinking (rather than standing on a cup or throwing it) is determined by its physical affordances available for a biological body and its cultural affordances available for a cultural body (sometimes, those ideal/cultural properties are referred as supersensible, Roth, 2020; Leontiev, 1978). Similarly, an artifact becomes mathematical only when it is acted upon in a particular cultural way – mathematically (see above the example of Cartesian coordinates).

Consequently, artifacts and their affordances cannot be defined without processes of creating or interacting with them. *Artifacts-in-interaction* become instruments of action as they come to be a part of perception-action system of an acting person a person's system of perception-action loops (Nemirovsky et al., 2013, Ferrari & Ferrara, 2020, Shvarts et al., 2021). Eyes and hands are ready to act with an artefact in a particular way: the body anticipates the expression within brackets as first in processing, the right direction on a number line as presenting growing positive magnitude, and so on. In each action with an artifact, the imaginary anticipation of how to act and how the artifact will respond to an action is blurred with the actual response from the artifact's physicality.

Now, let us remove the physical, tangible component of an artifact from this system of perception-action loops. Try to write your name without a pen. Try to add 17+86 with closed eyes. Imaginary interaction still preserves the artifacts' properties well enough to continue with sense-making even when the tangible component of an artifact is absent. Anticipation of an artifact developed for regulating an interaction with it serves now as imagination for continuing sense-making through interaction with an *imaginary artifact*. Such a theoretical view allows reconsidering cognitive science results about specific types of mental representations as evidence of the behavioral dynamics shaped by imaginary artifacts. This way, mental calculations based on an imaginary abacus require planning of finger movement and often their execution through gestures (Brooks et al., 2017); planning auxiliary lines in geometry involves eye fixations on their future positions (Epelboim & Suppes, 2001). When artifacts are not directly visible, their involvement in the sense-making can be traced through experimental techniques: For example, a reaction time to a target number depends on its position on an imaginary number line and how close this position is to the left or right hand that gives the answer (e.g., Fias, 1996). Such an imaginary number line can even be shared in collective imaginary space (Atmaca et al., 2008). The dynamics of seemingly mental

cognitive operations are determined by the affordances of the artifacts that enabled their development in mathematics learning.

In those experiments, residual motor processes help in tracing how artifacts are present in mathematical sense-making even when their physical component might not be directly tangible. Further, external components of action (movements, gestures and eye movements) might completely *contract* (Radford, 2021; Chumachenko et al., 2025). The sense-making might seem to be "mental" – without any observable component. Yet the regulative component of actions with artifacts is preserved, and the affordances of imaginary artifacts enable mathematical reasoning.

How should we conceptualize the situation when we observe neither the physical component of interaction nor an artifact in a mathematical sense-making? The material-historical development of the artifacts and actions with them leaves a trace on cultural forms of sense-making that individuals appropriate and develop. This challenges the constructivist ideas of action internalization into mental schemes. Would it be valuable for research in mathematics education to consider mathematical sense-making as floating in transcendental ideas of an ideal world? Or would it be more productive and actionable to consider mathematical sense-making to be a natural achievement of phylogenesis, anthropogenesis, and ontogenesis? Purely mental is a meaningless category from a non-dualist embodied-material bio-socio-historical perspective on mathematics. Mathematical sense-making is happening in the body-mind, in interaction with artifacts—either physically available or imaginary—and through *intra-action* or *reification and crystallization* that leads to the development of mathematics itself as a dynamic system of artifacts in action.

MATHEMATICAL SENSE MAKING IS BROAD, AND ITS EDGES DO NOT REQUIRE INTERACTION WITH PEOPLE AND TOOLS (DAVID MAXIMILIANO GÓMEZ)

In order to discuss the motion, it is crucial to agree on what do we understand by mathematical sense making. While used with some frequency, definitions of this concept are not abundant. One such definition was provided by Schoenfeld (1992/2016), by posing mathematical sense making as a part of learning to think mathematically: "developing competence with the tools of the [mathematical] trade, and using those tools in the service of the goal of understanding structure" (p. 1). This idea of sense making focuses on understanding structure of the world using mathematical tools. Schoenfeld (1991) also used the expression "suspension of sense making" to refer to situations where students answered apparently non-sensical things when solving a mathematical problem, such as when they ignore realistic considerations in problem solving. There are two aspects to ponder about this conception of sense making. First, that it is limited to formal mathematics, leaving out the possibility of sense making in informal mathematics, that is to say the mathematical ideas present in children before schooling. This interpretation is not well aligned with

recent developments and findings from Developmental Psychology demonstrating early mathematical abilities. Second, the notion of sense making is closely linked connecting mathematics with real phenomena. Altogether, mathematical sense making can be conceived more broadly as finding (or attributing) mathematical meaning to a phenomenon through one's cognition, by means of, for example, an idea, an intuition, a perception, or a pattern. From this perspective, I will present two arguments, one more empirically oriented and another one more theoretically oriented, demonstrating contexts in which mathematical sense making may occur purely mentally. That is to say, I will not argue that mathematical sense making occurs only mentally in every situation, but rather that there are important situations in which the mind processes mathematical meaning without support from people or artefacts/tools.

Before moving forward, I must point out the introduction of the notion of imaginary artefact by the team in favor of the proposition. This concept makes it really problematic to discuss the proposition, as many mental processes might be subsumed under the umbrella of "interacting with a mental or imaginary artefact". The very process of reading a symbolic mathematical statement could be considered interaction with a cultural artefact (the symbols), rendering the proposition almost tautological and uninteresting. Therefore, for a productive discussion this notion cannot be taken without adequate qualification.

To discover contexts in which mathematical sense making is not constrained by experience with people and artefacts, we must start acknowledging that mathematical thinking (and, therefore, sense making) is not limited to the extent of formal education, but instead it emerges and develops through children's early years and goes beyond the end of school into adulthood. While I will not question the necessity of interactions with people and tools in the early stages of mathematics learning, the thinking processes engaged by a person who is already mathematically competent also constitute mathematical sense making. I will thus draw from two human learning and development theories to argue that the reasoning of a mathematically competent person is another context in which mathematical sense can be made without the kind of interactions that the motion poses as inescapable.

But mathematically competent people is not the only context in which ascribing a role to interactions with people and tools is problematic. I will draw from research and theories from Mathematical Cognition and Comparative/Developmental Psychology, focusing on a critical question relating to the origins of mathematical thinking. Not limiting ourselves to formal mathematical thinking leads us to also consider precursors of this thinking and ask at what point in development we can find evidence of such precursors in children. Connecting this with the motion under debate, do these precursors need experience interacting with people and/or artefacts in order to emerge? I will show empirical evidence of very early mathematical cognition, specifically mathematical perceptions that can be found in very young infants and newborns in their first days of life. Some of these perceptions are even shared by non-human animals,

showing that some mathematical thinking precursors are linked to phylogenetically developed innate brain mechanisms, devoid of the need of interactive experiences.

Mathematically competent thinking

Theories about learning and development may focus on different aspects of these processes. Some theories, like Ausubel's (1968; see also Bryce & Blown, 2024), emphasize the transformations that a learner's cognitive structure undergoes throughout the learning process while others, like Vygotsky's (1981; see also Mahn & John-Steiner, 2013), focus on the role of social interactions for promoting learning and, in turn, supporting the learner's development. Interestingly, Vygotsky's work directly connects with the ideas of interactions with people and artefacts/tools present in the motion under debate, by understanding interactions with tools as a social interaction with the culture providing that tool (Taber, 2025).

Some key elements of Vygotsky's theory include mediation and internalization (Moll, 2009). A teacher or a knowledgeable peer can mediate between a novel piece of knowledge or skill and a learner, scaffolding the learner's road in a problem-solving environment toward that knowledge. Hence, learning starts in an intersubjective interaction, where the learner's access to knowledge is mediated by others, and by engaging in these interactions the learner progressively internalizes that knowledge (Taber, 2025). As a result, the initially intersubjective interactions progressively become intrasubjective processes allowing the learner to use the novel piece of knowledge or perform the novel skill in the future without the need of external mediation (Moll, 2009). In contrast, Ausubel's (1968) theory focuses on the transformations exerted on the learner's internal cognitive structure by the acquisition of new knowledge, not on the processes leading to these transformations. Nonetheless, both theories consider a final stage where learning has occurred, represented by internalization into an intrasubjective process or by transformation of the internal cognitive structure. The key point in relation to our debate is that, although people and artefacts may have been crucial throughout the learning process, they are no longer necessary once learning has taken place, as by that time the learner has internalized the relevant interactions. That is to say, a person who has mastered a mathematical concept can then use it for sense making without further interactions with others or with artefacts. This does not exclude the possibility that such knowledge or skill becomes the object of further learning through interactions. However, the motion under discussion makes a claim about mathematical sense making, not mathematical learning.

Nonetheless, the opposing team has suggested that artefacts remain present in advanced stages of mathematical competence, by resorting to imaginary artefacts, whose usage would be detectable through experimental techniques. However, presenting examples of tasks where one can detect the presence of such imaginary artefacts is not proof that imaginary artefacts/tools underlie every mathematical cognitive process. Indeed, one can think mathematically in terms of mental objects,

such as visualizing sets of objects and comparing them by number or size or perform some arithmetic operation with them. Such non-symbolic tasks certainly require mental/imaginary objects, but these objects are not tools themselves. Indeed, not every object is a tool, and even neurophysiological data discriminates between tools and graspable objects (Mruczek et al., 2013). Altogether, while imaginary artefacts may play a role in the case of mathematically competent adults, the next context provides an even stronger case.

Early mathematical skills

Decades of research in Comparative and Developmental Psychology have contributed to understanding some foundational skills believed to act as precursors of mathematical thinking. In the interface between Mathematics Education and these areas, the field of numerical and mathematical cognition has identified skills providing support for performance in simple numerical/mathematical tasks and, therefore, for the development of complex mathematical thinking. These skills include the ability to estimate, in a quick and approximate manner, how many elements are in a large set of objects, or the ability to assess, quickly and accurately, the number of elements of a set of up to four elements (Feigenson et al., 2004). Quick and approximate estimations are attributed to an Approximate Number System (e.g., Odic & Starr, 2018), which allows people to build a rough estimate of the cardinality of a set of objects presented by means of vision, audition, or touch (e.g., Barth et al., 2003; Hauser et al., 2003; Ziegler & Drewing, 2022). These estimates are believed to sustain abilities such as comparing two sets' cardinalities in order to decide which of them has more elements (Odic & Starr, 2018), both within the same perceptual modality and across modalities (Barth et al., 2003). Neuroscience research with numerate adults has linked this ability to activity in a region of the cerebral cortex called the *intraparietal sulcus* (e.g., Piazza et al., 2004). Crucially, this ability has also been documented in native speakers of languages with reduced sets of number words (e.g., Mundurukú: Pica et al., 2004; Awetý: da Silva Sinha et al., 2026). Moreover, performance in tasks requiring the Approximate Number System is comparable with that of native speakers of Western languages such as French and Brazilian Portuguese (ibid.).

Other lines of research have investigated human infants' numerical skills. Wynn (1992) showed that infants of about five months of age, after being presented with a situation in which two puppets are located one at a time behind a screen (1+1), look at the scene for longer times when the outcome presented is unexpected (1+1=1) than when it is expected (1+1=2). An analog finding emerges when infants are shown two puppets and, after being hidden behind a screen, one is extracted (2-1): longer looking times to an unexpected outcome (2-1=2) than to an expected one (2-1=1). Infants' expectations in this kind of task have been attributed to an Object-Tracking System (Feigenson et al., 2004) that can follow up to 3-4 objects simultaneously. Moreover, evidence for an Approximate Number System has been documented for six-month-old infants, who discriminate sets of objects whose cardinalities differ in a 1:2 ratio but not sets whose

cardinalities differ in a 2:3 ratio (Xu & Spelke, 2000). Ascribing these abilities to interactions with people or tools is problematic. Infants' interactions with people rarely focus on this kind of numerical knowledge at such young age, and infants' intentional tool usage seems to emerge during the second year of life (Rat-Fischer et al., 2012). Still, it could be argued that infants' reaching behaviors, attested by sixteen weeks of age (Berthier & Keen, 2006), could play an analog role to early tool usage enabling these early demonstrations of mathematical skill precursors. But this argument loses strength when considering that numerical precursors show up much earlier in development: Izard et al. (2009) tested looking behaviors in newborn infants during their first days of life (average age of 49 hours, range 7-100 hours), showing that they spontaneously looked for longer times at a screen with a small (resp. large) number of elements when listening to a sequence of a few (resp. many) syllables. That is to say, newborn infants are able to make sense of the world in a manner that is congruent with their seminal mathematical abilities.

Investigations with non-human animals have demonstrated that many other species exhibit behaviors consistent with the Approximate Number System and/or the Object Tracking System studied in humans: rhesus monkeys (*Macaca mulatta*; Hauser et al., 2000), New Zealand robins (*Petroica australis*; Hunt et al., 2008), ruin lizards (*Podarcis sicula*; Miletto Petrazzini et al., 2017), mosquitofish (*Gambusia holbrooki*; Dadda et al., 2009), and guppies (*Poecilia reticulata*; Agrillo et al., 2012). Even insects, such as the honeybee (*Apis mellifera*), exhibit basic numerical capacities (Pahl et al., 2013). Altogether, this points toward a phylogenetically shared system (or set of systems), common to many species including humans, that allow basic numerical sense making despite vast differences in their bodies, brains, and genetics. The broad extent to which these abilities are shared (from mammals to insects) further contributes to questioning the necessity of tools for numerical abilities to emerge.

Going back to the motion and the claims of the other team, the context of early development poses a case in which it would be highly problematic to ascribe a role to tools. Such tools should be available to be used by 2-day-old infants and a vast array of non-human species, and claims for their existence should be accompanied by strong evidence, which the opposing team has not provided.

Back to mathematical sense making

In summary, three crucial elements for our debate are: (1) mathematical sense making is not limited to formal mathematics; (2) learning and sense making are distinct processes; and (3) mental/imaginary manipulation is not the same as artefact/tool usage. The number-related precursor abilities of human infants and non-human animals allow them to interact with the world in a mathematically-oriented manner, thus constituting mathematical sense making in the absence of formal mathematics instruction. Moreover, a subset of these abilities is available even to newborn infants before their fifth day of life, making unlikely that these abilities are acquired through interactions with people or artefacts. On the other hand, the conflation between

learning and sense making may be expected in Mathematics Education, as this field is quintessentially focused on learning. Nonetheless, while some learning and development theories attribute to people and artefacts an essential role in learning, once internalization has occurred the learner may not require these interactions anymore to make use of the newly acquired knowledge/skill to find mathematical meanings in the world. And even if these internalized skills were considered imaginary tools/artefacts, not all mental work with objects implies tool usage. Therefore, while mathematical sense making may often stem from interactions with people and artefacts, it is not bound to these interactions.

Acknowledgement

David M. Gómez is grateful for the support of the Chilean National Agency of Research and Development through the grant ANID CIAE CIA250005.

MATHEMATICAL SENSE MAKING FROM WITHIN AND OUTSIDE OF THE COMMOGNITIVE PERSPECTIVE (MAN CHING ESTHER CHAN)

The debate topic can be seen as having two propositions: one is that all mathematical sense making happens through interaction with people and artefacts; the other part is that mathematical sense making cannot happen purely mentally. In order to discuss the propositions, there is a need to clarify what we mean by "mathematical sense making". I will use the framework of commognition (Sfard, 2008) to frame my argument because it offers a clear definition of mathematical sense making and explains the connection between an individual's thinking with the social world.

I will explore two overlapping boundaries that the propositions evoke and reveal. The first boundary concerns two possible interpretations of mathematical sense making, and the second boundary concerns the definition of mathematics based on the framework of commognition (Sfard, 2008). I will argue that mathematical sense making is a collective human endeavour that requires interaction with people and artefacts. It is not purely mental because mental processing of mathematics only constitutes part of mathematical sense making.

Boundary 1: Sense making as understanding *vs* awareness

The term sense making is widely used within mathematics education research but lacks a clear consistent definition (Scheiner & Pinto, 2022). According to Urquhart et al. (2025), sense making can be defined as "the processes through which people interpret and give meaning to their experiences" (p. 3). The term originally focused on the five senses but its meaning has expanded to cover human sense making of their internal and external worlds. The term can be used synonymously with "understanding", "comprehension", or "meaning making" (see Scheiner & Pinto, 2022; Schoenfeld, 1991). However, sense making can also be interpreted as referring to more basic sensory functions in terms of "awareness" and "perception" (sensing). These two different ways of interpreting sense making have different implications for what we

mean by "mathematical sense making". This leads me to the next boundary in terms of how mathematical sense making is defined according to commognition (Sfard, 2008, 2018), a non-dualistic theory of learning that is intended to bridge the divide between social and cognitive theories in mathematics learning. Within this framework, mathematical sense making is understood as a more specific process of understanding and meaning making, compared to the broader process of mathematical learning, which involves modifying and extending one's discourse.

Boundary 2: Mathematical sense making according to the commognitive framework

The commognitive framework was developed to understand and investigate mathematical learning as a social process. It assumes that "thinking is a form of communication and that learning mathematics is tantamount to modifying and extending one's discourse" (Sfard, 2007, p. 567). Communication is seen as a collective endeavour which follows specific social and linguistic rules. People engage in communication, verbally or non-verbally through other symbolic systems, to influence others' thoughts or actions. Discourse is defined as any specific instance of communicating, whether with oneself or with other people (Sfard, 2008; Sfard & Kieran, 2001).

Within this framework, the thinking—communication divide is resolved by equating thinking with self-communication ("to think means to communicate with oneself"; Sfard, 2015, p. 132). The framework adopts a strong participationist position (Sfard, 1998) focusing on learners' participation within a social context. It pays specific attention to situatedness, social interaction, social relationships, and history and culture. According to the framework, mathematics is viewed as a form of discourse following particular rules for thinking and interpersonal communication at both the individual and the societal level. These "rules" include use of symbolic artefacts, such as numerals, algebraic expressions and graphs, to represent mathematical objects; mathematical routines that involve prescribing algorithmic ways to manipulate mathematical objects; and heuristic procedures of proving, defining, and substantiating (Sfard, 2015, p. 132). Individuals participate in the discourse of mathematics which is socially, culturally, and historically established and continually evolving. Based on the framework, mathematical sense making involves organising separate fragments of our experiences into a coherent whole within the broader social discourse of mathematics (Sfard, 2025).

Historical analysis suggests that mathematical sense making is a social process of enculturation, not a purely mental activity. Sfard (1991), for example, described the Greco-Roman historical broadening of the concept of "numbers", from natural whole numbers tied to counting, to include positive rational numbers, positive irrationals, integers and fractions, negative numbers, and complex numbers. She summarised that the initial proposal to extend the familiar mathematical process (e.g., use of whole positive numbers) to include uncommon operations (e.g., describing the length of the

diagonal by integers and their ratio) was often met with strong objections and philosophical debates. Over time, as the new operation—once forbidden—became accepted as useful, it was integrated into the mathematical discourse as a legitimate mathematical object. Her analysis highlights that mathematical sense making—whether by a young learner or an established mathematician—is not purely mental but a social process that involves enculturating a person into the acceptable mathematical discourse and building on that discourse.

Further evidence of the enculturation process comes from studies in the development and use of mathematical knowledge in different cultures (Barton, 1996; Bishop, 1991; D'Ambrosio, 1985), which demonstrate that people from different cultures apply mathematical or numerical concepts even when those concepts are not formally called "mathematics" within the culture. For instance, some Indigenous communities in Australia have complex kinship systems that can be interpreted using group theory methods. These kinship rules can be seen to embody the operations and permutations in algebra (Boyd, 1969; Xu & Ball, 2024). Weaving, an activity present in many cultures (e.g., African, South American, and Asian), requires no symbolic representation, yet the geometric patterns displayed can be seen as mathematical (Barton, 1996; Gerdes, 2000). If we consider engagement with these cultural practices as mathematical sense making, the social dimension is central. At the societal level, the notion of mathematical enculturation (Bishop, 1991) further illustrates how society and environment shape what kinds of mathematical knowledge, practices, and skills are valued.

Inner speech, imaginary conversations, thought experiments, dreams, and "ah-ha" moments

Perhaps the more contentious areas surrounding the debate propositions regarding mathematical sense making involve internal processes such as inner speech, imaginary conversations, thought experiments, subconscious processes (e.g., dreams), and "ah-ha" moments. Mathematicians, for example, have described how they acquired mathematical insights in their sleep. The book *Ramanujan, the Man and the Mathematician* (Ranganathan, 1967) gave the following account:

> Ramanujan and his family were ardent devotees of God Narasimha (the lion-faced incarnation (*avasara*) of God), the sign of whose grace consisted in drops of blood seen during dreams. Ramanujan stated that after seeing such drops [in his dreams], scrolls containing the most complicated mathematics used to unfold before him and that after waking, he could set down on paper only a fraction of what was shown to him (p. 87).

British mathematician Richard Guy also described that he "does mathematics" in his sleep:

> The first indication that I was interested in combinatorics—I didn't know I had the interest, and I didn't even know there was such a subject as combinatorics—was that I used to compose chess endgames. I would sit up late into the night trying to analyze a position. Eventually I would sink into slumber and wake up in the morning to realize that if I had

> only moved the pawns over one file the whole thing would have gone through clearly. My brain must have been checking over this finite but moderately large number of possibilities during the night. I think a lot of mathematicians must work that way. (Albers & Alexanderson, 2011, p. 191)

These accounts demonstrate that mathematical sense making involves mental processes, but they are only part of the sense making process. Hadamard's (1945) study of discovery and invention processes of mathematicians showed that mathematical discovery could happen at an unconscious level. However, sense making is not complete until it returns to the conscious level for verification and formalisation, through language or writing. Ramanujan's notebooks, for example, are known to be an important tool and record for his mathematical discovery (Berndt, 1978). Guy asserted that "for most mathematical problems, immediate thought and pencil and paper—the usual things one associates with solving mathematical problems—are just totally inadequate. You need to understand the problem, make a few symbols on paper and look at them." (Albers & Alexanderson, 2011, p. 190). These examples suggest that while the individual mind is involved in mathematical sense making, it is part of a larger process that also involves interacting with other people and artefacts.

Out of bounds: Biological prerequisites of mathematical sense making

As alluded to earlier, sense making could also refer to more basic sensory functions like "awareness" and "perception" (sensing). As far as mathematical sense making is concerned, these functions could include non-symbolic or non-verbal phenomena such as number sense, pattern recognition, and visual-spatial relationship. However, based on the commognitive definition of mathematics as a form of discourse, these biologically based basic sensory functions are not considered mathematical, and therefore do not constitute mathematical sense making. These functions at best can be considered as prerequisites for mathematical sense making.

Even if these functions are considered as within the realm of mathematical sense making, they are evolutionarily grounded in the social nature of humans (Sfard, 2008). Although non-symbolic or non-verbal mathematical sense making may appear innate or intuitive (Dehaene, 2001), environmental feedback shapes their development and maintenance. Infants are born with a range of perceptual, sensory, and cognitive functions (Cohen & Cashon, 2003). The activities they engage in and environmental feedback allow them to refine and strengthen these functions, enabling further skills development. For example, infants appear to possess universal ability to learn any human languages at birth. They have been found to be able to distinguish between 800+ sounds across all human languages. As they grow, they adapt to their environment by narrowing their ability to distinguish between different sounds in their native language while losing sensitivity to foreign sounds (Kuhl, 2004). This suggests that perceptual-related mathematical sense making is not purely mental but develops and is sustained through environmental feedback.

Conclusion

One could define mathematical sense making narrowly to support the debate propositions. Examining symbolic, non-symbolic, and non-verbal forms of mathematical sense making suggests that they are all part of a broader collective human endeavour, affirming the debate propositions. The propositions highlight the importance of humans and the environment in shaping how mathematics is defined and made sense of. Our minds do not exist in isolation. Recognising this interdependence underscores the power to shape mathematical sense making and learning through intentional social interactions and carefully designed environments.

Acknowledgements

I am grateful for the critical feedback of David Collis, Doug Clarke, James Russo, and Anna Sfard for earlier drafts of my section of this paper. Nevertheless, I accept sole responsibility for any errors or shortcomings present in my argument.

MATH: IT'S ALL IN YOUR HEAD (WES MACIEJEWSKI)

I oppose this motion, and take the opposite view that mathematics is purely a mental activity, often performed with mediating interactions with people and artefacts.

To begin, we must take a stand on what "mathematics" is. The epistemological requires ontological underpinnings. I take mathematics to be the abstraction away from the particular to the universal: mathematics may be rooted in and inspired by our physical reality, including interactions with social beings, but the mathematical objects and operations themselves are mental abstractions, which do become conventionalized and enter our collective mathematical understanding, but nevertheless exist in the minds of those who think about them. So, enumerating one stone is a type of mathematical activity—that of identifying an instance of the mathematical idea of "one"—but mathematics goes beyond that in recognizing there is a concept of "one" detached from stones, or sheep, or stars, or any other physical object. And this one-ness takes on an individual nature: we each have an idiosyncratic understanding of any mathematical object. These understandings might be more, or less, nuanced, under/developed, or deeply insightful, or outside of what we might find comfort in; as Thurston (2006) identifies, "one person's clear mental image is another person's intimidation."

There are many views on mathematics similar to the above and it is worth acknowledging and positioning my argument relative to them. For example, there has been a debate for some time regarding whether mathematics exists in the mind or in the material world, as a property of the universe. This is often framed in terms of Platonic ideals, versus constructed realities—whether mathematics is a discovered property of the universe or a human creation. I've always found this to be an unhelpful dichotomy. Our collective mathematics is deeply rooted in our physical experience and we should not be awestruck that it works so well in describing the universe—if

anything, the effectiveness of mathematics is a testament to the strength of the collective human exercise of mathematics.

Another related philosophical idea is the notion of mind-body duality, that the mind is of a separate nature than the body. Taking this stance, as Descartes (2008) had, creates a tautology in that since we can think of mathematics, without moving or talking and with our eyes closed, then clearly mathematics can be strictly mental. However, I think very few education researchers today believe in such a duality. Indeed, we have a much more advanced and nuanced understanding of the role of the body and communication in mental processes now than ever before. We see a rejection of Cartesian dualism in mathematics education in the form of the embodied cognition (Lakoff and Núñez, 2000) or commognitive (Sfard, 2008) frameworks, which both foreground the body, in the first as positioned in and experiencing the world, and in the second as embedded in a communicative society with shared understandings mediating thought. I argue you can take these theoretical positions and reject Cartesian dualism, yet still view mathematics as a system of objects and ways of manipulating these objects that exists as a mental system not requiring physicality. Both embodied cognition and commognition are completely coherent with the idea that mathematics is rooted in and inspired by physical, and social, reality; it is just that when we learn and express mathematical ideas, we do so via shared understandings, often appealing to physical metaphors or particular instantiations.

In short, embodied and social theories of mathematics, in which I'll include commognition, help explain where mathematics comes from, its socially-conventialized nature, how it is applied, learned, and communicated. However, these theories are not theories of mathematics itself.

Mathematics as a mental activity

To argue against the motion as phrased is to simply find examples refuting its assertion, examples of strictly mental mathematical sensemaking. These are *everywhere*. One of my favourites comes from Henri Poincaré, of the famous resolved conjecture fame:

> At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with those of non-Euclidian geometry. I did not verify the idea; I should not have had time, as, upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the result at my leisure. (Poincaré, 1910).

Phrased in contemporary terms, Poincaré had an "ah-ha!" moment. One could argue that that moment was rooted in him as a commognitive being—I do not question that the objects and framework for his thoughts originated from embodied, social experience gained from working in a conventionalized mathematical system, but I also do not question that Poincaré, in that moment, experienced something entirely within his mind.

I would take this further and suggest all mathematics is, to varying degrees, strictly mental activity. This activity is inspired by, rooted in, interpreted and communicated within, a socially-agreed upon system intended to understand our physical world through abstraction, but nevertheless is mental activity. We struggle to see it as such because of the type of mathematics and the context we observe it in: most school mathematics, from counting to calculus, is not far detached from physical, social reality. If we go further out into mathematics, we observe more frequent, more dramatic examples of mental mathematical sensemaking. To that end, I give a set of examples which not only refute the motion, but also reveal connections to the above theoretical framing: black holes, imagined mathematical objects which became real.

Black Holes

Albert Einstein formulated his general theory of relativity (1920), a theory of gravity, as an application of a revolutionary abstract mathematical idea: there are geometries different from the one/s we experience with our bodies over time; that "geometry" can be abstracted from its original intention of measuring the Earth. This idea stemmed from a deep, centuries-long analysis of Euclid's *Elements*, itself a work inspired by and abstracting our physical realities: the work begins with the creation of a new reality, "[A] point is that of which there is no part." Up for dispute was the necessity of his fifth postulate, that if the sum of the angles made by two lines intersecting a third line, on the same side of that third line, sum to less than 180 degrees, then those lines intersect on that side of the line. Mathematicians tried fruitlessly for centuries to prove this fifth postulate was a result of the other four. It was not until the 19th century that a mathematician, Lobachevsky, simply turned this postulate on its head—by stating that the two lines do not intersect, regardless of the sub-180 sum of their interior angles (1866). Suddenly, what was possible in geometry—in terms of new shapes, but also in terms of new ways to measure those new shapes—exploded and new geometries proliferated.

Einstein developed his ideas off those working in new geometries. In particular, Bernhard Riemann, who was studying ways of measuring on surfaces. Einstein thought of what shape our physical reality might be, not statically, like the shape of the page these words are on, but over time. He asked, what might be a path in this space-time universe? His answer was his field equations (1915). Put differently, Einstein abstracted our physical experience over time into a mathematical entity; a strictly mental construct, at least for Einstein then at that time. But note further that his theory is indeed grounded in reality! Although we struggle understanding flows in space-time, we are each experiencing their effects right now. Gravity is an embodied experience.

The equations describing gravity are mathematical and can be made sense of strictly in the mind, which is what Schwartzchild did when he solved the field equations in an extreme case which suggested the existence of a point—there's Euclid again—in space with a mass of infinite density (1916). A singularity of the equation, revealing a singularity in space-time? This did not correspond well to the existing models of reality

constructed through centuries of collective scientific model building. These singularities would wait 50 years to receive their name, black holes, indicating the strength of their gravity disallowing the emanation of light from their mathematical hearts.

Even when humans in the 1970s observed that black holes actually exist, they did this not by seeing a black hole, but its effects on the surrounding space, bending light and contorting our view of the universe. However, our inability to observe black holes, or experience them in any way, did not stop humans from *thinking* about black holes. One such human was Stephen Hawking, a British mathematician, almost completely paralyzed from the ravaging of amyotrophic lateral sclerosis (ALS), only communicating in a monotonic, emotionless computer voice, programmed by the twitching of a muscle under his eye, one of the last muscles he had control of. (Where is Stephen Hawking in embodied and social theories of mathematics?) In his popular work, A Brief History of Time (1988), Hawking recounts his discovery of the now-named *Hawking radiation* during his lengthy bedtime routine:

> Before 1970, my research on general relativity had concentrated mainly on the question of whether or not there had been a big bang singularity. However, one evening in November that year, shortly after the birth of my daughter, Lucy, I started to think about black holes as I was getting into bed. My disability makes this rather a slow process, so I had plenty of time.

Hawking eventually realizes, again, through pure mental mathematical sense making, that energy could in fact leave the influence of a black hole in the form of thermal radiation from quantum interactions, an effect not yet observed (like at one time its black hole parents), which means black holes can lose mass, shrink and disappear. So, there is theoretical hope that our universe is not destined to be entirely swallowed up in one big singularity—as it stands, our eventual salvation remains a mental exercise.

One last point about black holes: the first "photograph" of a black hole was produced in 2017 by a team lead by Katie Bouman (Akiyama et al., 2019). This reveals an important aspect of the motion guiding this debate: these abstract mathematical theories of our reality are informed by and rooted in physical experience. That's why we knew black holes existed and that we would eventually find them. Our theories are so good at predicting because they are such accurate descriptions, abstractions, of reality.

Back to Earth

Relating this back to mathematics education. At every step in the story, from Lobachevsky to Bouman, mathematicians worked outside of the existing constraints present in the field, creating never-before experienced mathematical chimeras in their minds, later seemingly willed into existence. Our children do that too, we're just quick to ensure their individual idiosyncratic mathematical sensemaking is directed toward our commognitive good.

The reason why we might think mathematical sense making takes place only with at least some interaction with people or artefacts is because the mathematical ideas we discuss tend to be from our direct experience. Seldom do we teach Hawking radiation, and when we do, the audience is very small. Mathematics education concentrates on our common mathematical experiences, like counting or motion, for example, and therefore we should not be surprised that theories for our mathematical sensemaking which centre our collective mathematics building from our embodied experiences also have difficulties in detaching themselves from the system they seek to theorize. Informed by and rooted in our common physical experience, such a theory of mathematical sense making would not be able to frame itself as being anything other than necessary in its own theory. Just as relativity, being born out of the universe, predicts its own presence in the universe. People engage in mathematical sense making with all the physical and social tools at their disposal. Oftentimes, the only tool necessary is mind.

CONCLUSION

In addressing both the motion and the opposition, the papers in this panel provide rich empirical evidence, historical analyses, and diverse theoretical interpretations. Collectively, they raise a series of profound questions that extend well beyond the specific wording of the motion:

- Do infants engage in mathematics before exposure to mathematical discourse and culture, or is it adults who retrospectively interpret their actions as mathematical?
- Do sleeping brains make mathematical sense, or do they merely prepare intuitions that are later articulated through discourse?
- Is insight a purely mental phenomenon, or does it emerge through self-organising interactions with mathematical artefacts?
- Are calculations that leave no visible trace examples of purely mental activity, or do they involve interaction with imagined artefacts and representations?
- Do equations consist of abstractions that reside in the mind, or are they themselves artefacts through which mathematical meaning is constituted?

These questions lead us to even more fundamental ones. What is mathematics? What is sense-making? Is mathematical sense-making primarily individual or collective? What counts as an artefact, and where do interactions with others begin and end? How would developmental psychology, commognition, sociocultural theory, embodied cognition, and related perspectives respond to these questions?

The contributions in this panel demonstrate that answers depend not only on empirical evidence but also on the philosophical and theoretical commitments that researchers bring to their work. Different definitions of mathematics, sense-making, cognition, discourse, embodiment, and artefact lead scholars to interpret the same empirical

phenomena in markedly different ways. Indeed, one of the most striking features of the debate is that the same evidence can be marshalled in support of apparently opposing positions.

For this reason, the tension between the motion and the opposition cannot be resolved through empirical evidence alone. There is no single, theory-neutral "truth" waiting to be uncovered behind the debate. Rather, the positions advanced here remind us that mathematics education research is always shaped by theoretical choices, and that such choices are never purely technical or methodological. They are also axiological and ethical.

The question, therefore, is not only which theory provides the most convincing explanation, but also which theoretical commitments open possibilities for more equitable, humane, and intellectually productive educational practices. The challenge before us is not necessarily to eliminate theoretical plurality, but to understand more clearly the values that underpin our theoretical choices and the consequences that follow from them.

As scholars, we may find it easier to agree on the values we wish to advance than on the interpretations of particular empirical findings. In a field characterised by diverse traditions, perspectives, and histories, our task is not to force consensus but to engage difference with intellectual rigour and mutual respect. Such engagement requires more than tolerance. It requires acceptance: a recognition that others may see, know, and theorise differently, and that these differences are not obstacles to inquiry but essential resources for it.

Perhaps, then, the most important contribution of this debate lies not in determining whether mathematical sense-making is purely mental or always mediated. Rather, it lies in demonstrating how scholars with fundamentally different perspectives can engage one another seriously, critically, and respectfully in the collective pursuit of understanding. In a world increasingly marked by polarisation and intolerance, that may be one of the most important lessons that our scholarly community has to offer.

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PME 49
Helsinki, Finland
2026

RESEARCH FORUMS

CROSS-SYNERGY BETWEEN DIGITAL TOOLS AND THEORETICAL PERSPECTIVES FOR COVARIATIONAL REASONING

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This Research Forum brings together multiple theoretical and design traditions in the study of covariational reasoning and examines how these perspectives can be put into productive dialogue through digital tools. Building on work that frames covariational reasoning through different theoretical lenses, we propose cross-synergy between digital tools as both a design and a cross-perspective unifying construct. In this contribution, we present and analyze three synergistic redesigns that combine digital tools and conceptualizations of covariation from prior research. The Research Forum culminates in a discussion of what these redesigns reveal about covariational reasoning and how cross-synergy can guide future design and research on digital tools in mathematics education.

INTRODUCTION

Covariational reasoning is a form of reasoning central in mathematics education. Across different research traditions, there is broad agreement on at least two aspects: covariation is a form of reasoning supporting a dynamic approach, as opposed to static approaches, in functional thinking (Carlson et al., 2002; Paoletti et al., 2023), and it can support students in constructing reciprocal relationships between quantitative structures and physical phenomena (Panorkou & Provost, 2025; Saldanha & Thompson, 1998). An extensive body of literature exists showing how covariational reasoning can support the understanding of mathematical concepts such as variable, rate of change, proportion, exponential growth, functions of one or two variables (e.g., Thompson et al., 2017), and physical, dynamic, and contextualized situations, such as the law of gravity or the greenhouse effect. Moreover, covariational reasoning, given its dynamic nature, can offer a powerful bridge between abstract mathematical concepts and concrete, physical, embodied, and aesthetic experiences, often supported by the affordances of technology.

Precisely because covariational reasoning spans diverse contexts and experiences, the breadth of contexts and school levels in which covariational reasoning is studied, not only as a cognitive activity but also as a construct guiding educational design processes

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 131-159. PME.

or as a possibility to reconceptualize dynamically mathematical concepts, raises a challenge: what we see as covariational reasoning, and how we design to elicit it, depends on the theoretical lens and representational means through which learners engage with changing quantities. This diversity becomes especially consequential when covariational reasoning is investigated in embodied and digital contexts, where theoretical lenses shape what becomes visible and what becomes designable. While the rapid expansion of digital tools can open new pathways for covariational reasoning, they can also fragment attention across representations and interaction styles if these tools are not carefully coordinated.

A growing body of research examines how digital tools can support covariational reasoning, but what counts as a productive relationship between tools and meanings is often theorised differently across perspectives. We address this tension by foregrounding *cross-synergy* as a unifying theoretical construct for connecting tools, meanings, and perspectives in the study and design of learning environments for covariational reasoning. Following on existing literature addressing synergy between artifacts (e.g., Mariotti & Montone, 2020), we elaborate on cross-synergy in our collaborative work as both (1) a design construct for coordinating aspects of tools' designs and representations so that meanings emerging through one can be related to, and transformed by, meanings emerging through another; and (2) a cross-perspective construct, where theoretical commitments and design conjectures become resources for revisiting what counts as covariational reasoning in the activity and for fostering the emergence of new or complementary conceptual meanings.

This Research Forum grew out of the *Workshop on Covariation 2025* (Brixen, Italy), where a shared insight became visible: covariation is theorized through multiple, sometimes only partially connected, perspectives. The workshop did more than highlight diversity; it catalyzed collaboration by creating productive contact between distinct theoretical commitments, methodological traditions, and design approaches. This Research Forum continues that work. We use cross-synergy to organize our collaborative design efforts and to examine how different theoretical lenses shape what we conceive as covariational reasoning, and how digital tools can be synergistically designed to support such reasoning. Hence, the contributions of this Research Forum can be summarized as follows: theoretically, we elaborate on cross-synergy as a construct capable of integrating various theoretical accounts on covariational reasoning; design-wise, we develop and present novel approaches to coordinating digital tools, representations, and interactions so that key conceptual links become structurally necessary; and methodologically, we document and reflect on a creative and collaborative way of working—across cultural and theoretical traditions—to design synergistically with covariational reasoning as educational goal. Finally, the implications of our Research Forum go beyond the specific mathematical topic of covariation. By examining how cross-synergy between digital tools and their underlying theoretical perspectives can be implemented and studied, we aim to

contribute to a design- and research-oriented way of working across tools and across theoretical perspectives in mathematics education.

STATE OF THE ART ON COVARIATIONAL REASONING

While the notion of covariational reasoning has roots in mathematics itself, its early theoretical formulations in mathematics education were largely developed within a *constructivist* tradition. Confrey and Smith (1994, 1995) describe the covariation approach to functions as coordinating sequences of values for two variables so that changes in one are related to changes in the other. They contrast the covariation approach with the correspondence approach to functions, often grounding students' covariational reasoning in sense-making with and about tables of numeric values.

Thompson (1994) focused on students' sense-making about non-numeric magnitudes changing together within a situation. From this perspective, a student is reasoning quantitatively about a situation when she conceives of objects in the situation as having attributes that are measurable. When those measurable attributes are dynamic, the student can envision the measures to have magnitudes that vary. Covariational reasoning is the coordination of those varying magnitudes. Theories of covariational reasoning that build on this notion describe ways of reasoning about relationships between varying non-numeric magnitudes within an individual's image of a situation (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017).

Research on covariational reasoning has then developed most systematically in the U.S. context. The contributions of Saldanha and Thompson (1998), Carlson et al. (2002), and Thompson and Carlson (2017) form a sustained program of inquiry whose findings have been refined over decades and remain active today. This work is strongly shaped by a *cognitivist* and *constructivist* perspective and rooted in quantitative reasoning: covariational reasoning is framed as "someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (Saldanha & Thompson, 1998, p. 299), and Carlson et al. (2002) declared their interest in deepening "the cognitive activities involved in coordinating two quantities while attending to the ways in which they change in relation to each other" (p. 354). Building on these ideas, Thompson and Carlson (2017) proposed a taxonomy of cognitive (and purportedly developmental) levels, grounded in previously identified mental actions, to describe how learners envision variables varying together.

Studies of covariational reasoning suggest an interesting tension: although many undergraduate mathematics students struggle to reason about how quantities change together (Carlson et al., 2002), younger students—sometimes as young as 13—can successfully do so when learning environments are well designed (Ellis et al., 2015; Paoletti et al., 2023). This research illustrates that covariational reasoning as a way of thinking can be developed through repeated opportunities to engage with intentionally designed tasks. A common feature of such tasks is the use of digital tools to create opportunities for students to reason with and about covariation across multiple

representations, often dynamically linked (e.g., Ellis et al., 2015; Johnson et al., 2017; Paoletti et al., 2023).

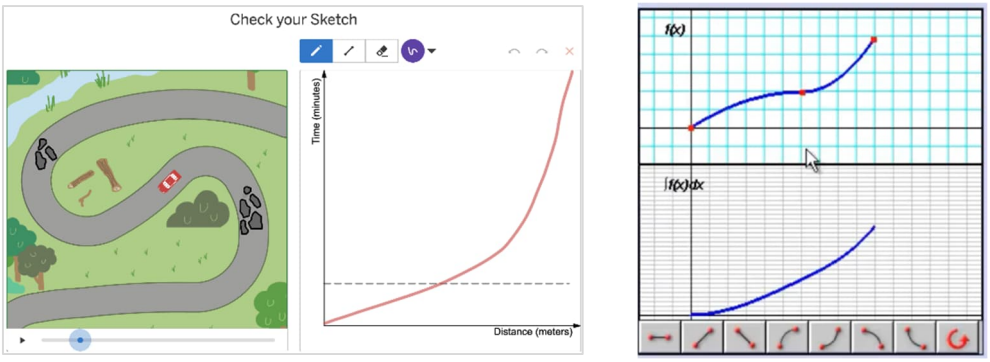
A common use of digital tools is to present students with representations of dynamic phenomena, sometimes dynamically linked to formal mathematical representations. For example, Johnson et al. (2017) designed a digital tool on GeoGebra which dynamically links a simplified model of a Ferris Wheel to a graph. Students can press a button to start the Ferris Wheel's motion or drag a point around the wheel to power the animation. As the Ferris Wheel rotates, dynamic segments in the situation support students' attention to the intended quantities. As the situational quantities covary, a point in a graph which represents the paired magnitudes of those quantities moves, leaving behind a trace through the coordinate plane. In this example, the digital tool provides opportunities for action (dragging a point around the Ferris Wheel to power the animation of motion) on intentionally designed mathematical representations (a graph dynamically linked to the animation of motion) which have been carefully designed to provide a theoretically meaningful response to those actions (motion of the point which leaves behind a trace in the coordinate plane).

As new digital mathematics environments are created, new forms of digital interaction to support covariational reasoning are possible. Margolis (2025) investigated differences in students' covariational reasoning across a sequence of sketch-to-animation (StA) tasks implemented on the Desmos Classroom platform (now the Amplify Classroom platform). Like the digital tools described above, StA tasks afford students the opportunity to sketch a graph representing the relationship between two quantities constructed from a dynamic situation. However, StA tasks leverage a novel form of feedback by allowing students to view the interpretation of their sketch in the context of an animation of the dynamic situation. For example, in the Race Task (Margolis, 2025), students are prompted to sketch a graph representing one car that starts fast and ends slow and another that moves backwards at least once (Screens 7-8 in https://classroom.amplify.com/activity/6530fdddf9b2d8c074313cbb). On the next screen, students can view an animation of the cars moving along a road based on their sketch (Figure 1, left). Like the digital tool from Johnson et al. (2017), Margolis' (2025) StA tasks provide opportunities for students to engage in mathematically meaningful actions on or with mathematical representations and then view automatic feedback from the digital tool. However, sketching is a form of activity that is significantly less constrained than actions like dragging points in the animated situation. Additionally, the representational link is reversed in StA tasks—students act within the graphical representation and then see the interpretation of their action in the animated situation. Margolis (2025) illustrates that these aspects of the digital task design can support or constrain the forms of covariational reasoning that students use as they engage with a task.

European research on covariation draws on a wider range of theoretical perspectives, which has led to different conceptualizations of covariational reasoning. In some of

these conceptualizations, covariational reasoning is detached from its connections to quantitative reasoning and instead privileges a dynamic conceptualization of functional relationships.

Figure 1: On the left, a static image from the StA feedback animation on The Race Task (Margolis, 2025); on the right, the main interface of the Calculus Integral Sketcher (Swidan et al., 2020)



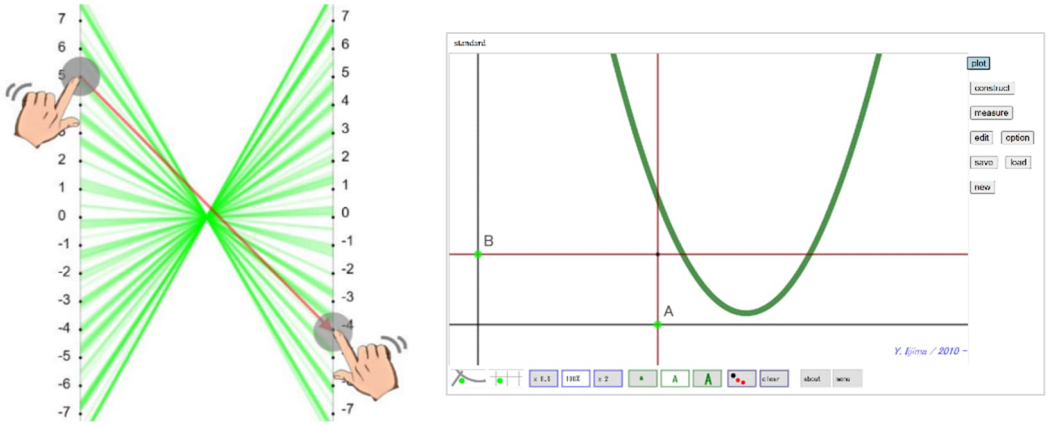
One prominent line of work takes a *sociocultural* view in which knowledge is co-constructed through interaction with peers and experts and mediated by tools, so requiring carefully designed teaching sequences promoting the evolution of personal meanings into culturally shared mathematical ones (e.g., Antonini & Lisarelli, 2021). Adopting the semiotic bundle framework, Arzarello (2019) showed how a dynamic simulation representing a ball rolling down an inclined plane, together with a parameter that changes the plane's inclination, can support students in making sense of relationships between a distance–time function and a parameter (the change of the plane angle)—what he termed second-order covariation. Swidan et al. (2020) showed how the Calculus Integral Sketcher (CIS) can support students in making sense of relationships between a function and its antiderivatives—what they termed covariation of covariation. CIS was designed to support students' construction of increasingly sophisticated meanings for the relationship between a function and its antiderivative. The environment simultaneously displays two Cartesian systems: the upper system presents the graph of a function, while the lower system presents the graph of its antiderivative. Crucially, these representations are not static. Students actively construct the function graph by assembling segments from a set of icons (e.g., constant, linear, curved), and the antiderivative graph is generated dynamically in response to these constructions. This design positions covariation as a relationship whose meaning emerges as students interpret coordinated change through interaction with representations and digital tools, through which the culturally developed meanings embedded in the tool become visible in classroom discourse. The CIS is not accessible anymore, but an image of its interface is shown in Figure 1 on the right.

Bagossi (2026) puts those previously mentioned conceptualizations in conversation with earlier notions of covariation by naming three "orders" of covariational reasoning. First-order covariational reasoning refers to covariational reasoning as known in the literature. Second-order covariational reasoning is defined as envisioning and conceptualizing families of invariant relations among varying quantities and their parameters as they vary. Third-order covariation—aligned with covariation of covariation—is about how changes in one covarying relation are linked to changes in another related relation. Reflective of the socio-cultural theoretical foundation, these definitions suggest that covariational reasoning involves not only mental images, but all multimodal resources involved in conceptualization processes (gestures, speech, representations, interaction with tools and others).

A second theoretical line of research is an *embodied-cognition* view, which emphasizes that learning covariation involves not only mental processes but also bodily interactions with the environment. For example, Wei et al. (2024) showed how bimanual movements with Nomograms in the Numworx environment can support students' covariational reasoning. Nomograms, also known as Parallel Axes Representations (PAR, Nachmias & Arcavi, 1990), are a function representation consisting of two parallel number lines for input and output values, connected by arrows that map x to $f(x)$. In a digital learning environment such as Numworx, Nomograms can be reimagined as digital-embodied tools (Wei et al., 2025). Each axis contains a draggable point, and learners manipulate input and output values, often bimanually, to generate or complete a family of arrows that constitute the target function (Figure 2, left). Importantly, the Numworx digital-embodied interface provides real-time feedback (e.g., an arrow turns green when the two points form a valid input-output pair, and the green trace remains visible). This combination of (a) two linked, manipulable variables, (b) immediate perceptual feedback, and (c) accumulating traces makes the digital-embodied Nomogram a unique tool to "see" how a function coordinates changes across variables. In Nomogram-based designs, covariation is treated as learners' coordination of two variables while focusing on how they change in relation to each other. Nomogram tasks operationalize this idea by making the two variables simultaneously controllable and perceptually comparable. Learners coordinate their hand movements to discover or maintain a functional constraint (e.g., same speed in opposite directions for $f(x) = -x$). From an embodied instrumentation perspective (Shvarts et al., 2021), the digital-embodied Nomogram is not only a representation of a function but an artifact that students learn to use through action. In this view, learning is shaped by the coupling between the learner, the artifact, and the task. Covariational relationships emerge as relationship-in-action within a body-artifact functional dynamic system: through repeated interaction, students' coordinated movements can stabilize into more fluent ways of acting, which then support later reflection on the underlying mathematical meaning.

Adopting the same theoretical perspective, Bagossi et al. (2026) observed how pairs working with the Digital Tracer and engaging in bodily coordination can develop a covariational perspective on functional relationships. The Digital Tracer, implemented in the GC/html5 environment and experienced on multitouch devices, consists of a quadrant of the Cartesian plane with a point that can be dragged along the *x*-axis and another along the *y*-axis of the plane (Figure 2, right; https://yiijima.sakura.ne.jp/GCes/e-01/gchtml/gc_00016-Tracciatore_Parabola-02.htm).

Figure 2: On the left, a digital-embodied Nomogram for the function $f(x) = -x$; on the right, the Digital Tracer

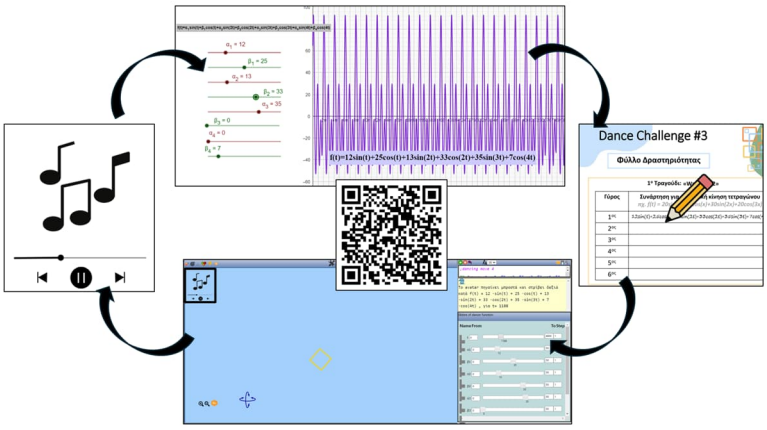


By dragging the points, users can control the intersection point determined by the two perpendicular bars passing through each point. Such an intersection point leaves a colored trace, providing immediate visual feedback on the coordination between the actions performed. A trace button allows for easy activation and deactivation of the trace of the intersecting point. Participants are usually requested to engage in pairs and attune to each other's movements in order to create a trace along a given curve. What is formed is called an intercorporeal functional dynamic system (Shvarts & Abrahamson, 2020): two individuals engaged in a coupled coordination of motor actions with a digital tool. The underlying embodied design fosters stabilized perception–action loops that support the conceptualization of covariation as smooth and continuous. Indeed, while the movements along the two axes can be associated with the variations of *x* and *y*, the coordination of such movements allows one to bodily experience the covariation of the two.

Another theoretical line of research draws on *new materialist and aesthetic* perspectives, treating covariational reasoning as a materially enacted, distributed aesthetic practice that can unfold in artistic contexts (such as music and dance) through assemblages of bodies, concepts, and tools. This framing resonates with inclusive materialism in mathematics education (De Freitas & Sinclair, 2014), in which mathematical concepts participate in the material field of action, and new mathematical

relations are articulated as actualisations of the virtual through gesture-diagram-tool interplay. Within such framing, aesthetic forces (e.g., relational coherence, felt fit/misfit, resonance/dissonance) operate as orienting constraints that reorganise what counts as a covariational problem and how it is pursued. Karavakou and Kynigos (2025) examined this line in an artistic context framed as "Choreographing with periodic functions", in which students became choreographers. They iteratively designed song-matched, periodic function-driven animations by coordinating two digital tools, GeoGebra's graphic calculator with MaLT2's programmable dynamic modelling environment (Figure 3).

Figure 3: Combination of GeoGebra and MaLT2 for exploring aesthetically driven covariational reasoning (Karavakou & Kynigos, 2025)



GeoGebra (https://www.geogebra.org/calculator/j6ghx8d8) functioned as a rapid "choreo-graphing" workspace for exploring periodic functions through sliders controlling parameter values and inspecting graph features (e.g., period, range, direction, and local intensity). MaLT2 (http://etl.ppp.uoa.gr/malt2/) enabled students to embed function expressions directly in logo-movement commands and to temporalize functional behaviour by varying a time parameter to generate oscillatory motion that could be replayed alongside the music. Students' covariational reasoning was continually reshaped through this aesthetic interplay of graph, sound, and animated movement, foregrounding constructs such as aesthetically driven covariational reasoning, supported by the duo of GeoGebra and MaLT2.

THEORETICAL FRAMEWORK: CROSS-SYNERGY

In mathematics education research, the increasing diversification of theoretical perspectives and digital tools has generated rich but often fragmented accounts of learning processes. This is particularly evident in research on covariational reasoning, which has been theorized through cognitive, sociocultural, embodied, and aesthetic lenses, each foregrounding different aspects of learners' activity. While each of these perspectives offers valuable insights, they often remain theoretically compartmentalized. In the following sections, we review literature on the concept of

synergy across artifacts, clarify terminology relevant to our work, and then introduce the construct of *cross-synergy* as a unifying and promising construct for conceptualizing how meanings, tools, and theoretical perspectives can be productively coordinated. Broadly speaking, cross-synergy refers to situations in which the combined functioning of elements or representations gives rise to forms of meaning-making that would not emerge from their independent interactions. As such, cross-synergy shifts attention from individual components to the relations established among them.

Synergy between artifacts in the literature

The notion of "synergy between artifacts" has been theorized in slightly different ways by different authors. Whilst artifact is a term consistently used to identify any device created by human beings for any objective.

Maschietto and Soury-Lavergne (2013) introduced the term duo of artifact to describe the combination of a concrete and a digital artifact designed to address the same mathematical content. They analyzed how students used the manipulatives and carefully considered signs and interactions to design the digital counterpart. Designing the interactions between the manipulatives and digital components (specifically, the concrete pascaline and its digital counterpart, the e-pascaline) enables learners to explore the same mathematical concepts through complementary representations, allowing them to make connections between manipulative actions and abstract reasoning and thereby deepening their conceptual understanding.

Building on the framework of semiotic mediation, Faggiano, Montone, and Mariotti (2018) introduced the notion of synergy between artifacts, typically involving pairs of artifacts. Their work on axial symmetry illustrates how alternating between a manipulative and a Dynamic Geometry Environment can foster a synergic process in which meanings developed through one artifact are reactivated, reinterpreted, and refined through engagement with the other. Crucially, synergy was not attributed to redundancy across artifacts, but to their complementary affordances and the relationships established between the meanings they support. Subsequent work (Mariotti & Montone, 2020) further emphasized that synergy is not an automatic outcome of multi-artifact environments. Rather it is a process often triggered in moments of interpretative difficulty or epistemic tension. Thus, synergy is a phenomenon in which references across artifacts establish relationships between meanings emerging from their use. In this sense, synergy is intrinsically semiotic: it concerns how meanings generated through one artifact resonate with, extend, or transform meanings generated through another. When learners encounter challenges in making sense of a phenomenon within one artifact, references to another artifact can function as pivot signs, enabling the transfer and reorganization of meanings. In this way, synergy supports the evolution from situated, artifact-bound meanings toward more generalized and mathematical meanings.

Recent work by Panorkou and Provost (2025) extends the notion of synergy beyond pairs of artifacts to consider representations as digital simulations, tables, graphs, and questioning. In their design framework for eliciting covariational reasoning in integrated STEM contexts, synergy operates both as synergy of artifacts and synergy of artifacts and questioning. Synergy of artifacts refers to the intentional sequencing and coordination of artifacts that embody the same underlying quantitative structure while offering distinct affordances. For example, simulations support dynamic exploration and the perception of covariation, tables foreground discrete correspondences between values, and graphs make global trends visible. When these artifacts are orchestrated synergistically by researchers and designers, learners can progressively weave a semiotic web that supports increasingly sophisticated forms of covariational reasoning. Synergy between artifacts and questioning highlights the role of discourse and prompts in activating and sustaining these connections. Carefully designed questions can direct learners' attention to relationships across artifacts, fostering transitions from static thinking to a more covariational one. In this sense, synergy is not located solely in artifacts themselves, but in the broader design ecology that coordinates tools, tasks, and interactions. This expanded view resonates with earlier notions of synergy, while also broadening its scope to include instructional moves and epistemic practices. Importantly, it shows that synergy can operate across disciplinary boundaries, supporting the visibility of mathematical structures within scientific contexts.

Cross-synergy: a design and cross-perspective construct

The variety of theoretical and design perspectives from which our research originates has led to the necessity of identifying common terminology to make our collaborative work possible and fruitful. While in the previous section we intentionally used the specific terminology adopted in our individual research and in the literature, we now make explicit the meanings of some key terms we adopt in our joint work and coherently use in what follows. We use the term *tool* to denote the ensemble of the material component, the educational goal underlying a specific design, and all those design and instructional choices influenced by the theoretical perspectives from which those applications stem. In our work, we will focus specifically on digital tools designed to support covariational reasoning. For the definition of *variable*, essential to talk about covariation, we follow the approach of Jones (2022), who uses it to denote any potentially varying value, including values of real-world quantities and values of mathematical function inputs and outputs.

Building on existing literature, we suggest *cross-synergy* as a construct unifying theoretical traditions. Cross-synergy can be conceptualized as both a design and a cross-perspective construct. As a design construct, cross-synergy guides the intentional coordination of tools, representations, and interactions so that each element enhances the learning potential of the others. As a cross-perspective construct, it allows us to disentangle how meanings are connected across tools, representations, and theoretical

framings, so offering learners opportunities to grasp such connections, particularly in moments of sense-making, conflict, or generalization.

Within research on covariational reasoning, cross-synergy offers a unifying construct for bridging theoretical perspectives that emphasize different dimensions of learning. Cognitivist accounts foreground mental images of quantities varying together, sociocultural approaches emphasize mediated action and discourse, embodied perspectives attend to bodily coordination and perception–action loops, and new materialist perspectives highlight materially enacted, affective, and aesthetic engagement. Rather than privileging one lens over others, cross-synergy provides a way to conceptualize how these perspectives can inform one another through design and analysis. By focusing on how meanings might be constructed across representations and interactions, cross-synergy shifts attention to the processes through which learners might coordinate multiple sources of meaning. This makes cross-synergy particularly suited as a cross-perspective construct: it allows researchers to examine how different theoretical commitments foreground distinct aspects of activity, and how their combination can reveal dimensions of covariational reasoning that might remain invisible within a single framework.

In this contribution, cross-synergy is therefore positioned as a central construct for both the design and study of learning environments for covariational reasoning. It frames the investigation of how digital tools and theoretical perspectives can be brought into productive dialogue, not by collapsing their differences, but by leveraging them as resources for deepening mathematical meaning.

METHODOLOGY

In this section, we describe how our group collaboratively developed three new digital tools that convey synergistic redesigns for covariational reasoning, thereby applying cross-synergy. Each digital tool builds on our prior tool-based research but culminates in a single redesigned environment that coordinates different design elements to foreground complementary covariational meanings. Building on the theoretical framing introduced in the previous section, we treat cross-synergy both: (a) as a design construct for coordinating aspects of the original tool designs and representations so that meanings emerging through one can be related to, and transformed by, meanings emerging through another; and (b) as a cross-perspective construct for working productively across different theoretical commitments about what counts as covariational reasoning in the activity.

The redesigns were produced through a creative, iterative workflow, consistent with approaches to collaborative prototyping in heterogeneous design teams (e.g., Soldano et al., 2025), combining whole-group and subgroup work. Whole-group sessions established common criteria for what counts as cross-synergy and a common methodological language. Subgroups then developed each redesign, producing drafts of task sequences, prompts, and cross-representation links. These were subsequently

reviewed in whole-group rounds to ensure the redesigned activity remained coherent across perspectives and that cross-synergy was not reduced to a simple juxtaposition of disparate representations within a single tool. The design corpus included: (a) concise analytic demonstrations of each original tool (its core interactions, representational system, and the covariational meanings it was designed to make salient), (b) successive redesign descriptions (task sequences, prompts, interface, linking decisions), and (c) meeting notes and decision logs capturing design rationales and revisions over time.

Step 1: Building a shared design problem and selecting tool pairings

We began by specifying a shared problem of practice: while digital tools can open powerful pathways for engaging with covariational reasoning, they can also fragment learners' attention across representations and interaction styles. We therefore sought tool pairings in which cross-synergy was plausible, because they invite distinct interaction styles and their underlying conceptualizations of covariation are complementary rather than redundant. This selection process yielded three redesign targets:

1. *It takes two hands* synergizing Digital Tracer & Nomogram: to support students' first-order covariational reasoning through coordinated manual movement, allowing for the anticipation of the trace of a dynamic point and the coordination of numerical values on parallel axes;
2. *Sketch to dance* synergizing Sketch-to-animation & Choreographing with periodic functions: to support students' second-order covariational reasoning by connecting sketch-to-animation interaction with aesthetic refinement of covariational patterns in graphing-as-creation;
3. *Sketch, Check, and Revise for Integral (SCR4I)* synergizing Sketch-to-animation & Calculus Integral Sketcher: to engage students in third-order covariational reasoning through a dynamic approach by connecting sketch-to-animation feedback with calculus-oriented meaning making on functions and families of antiderivatives.

To guide tool selection and redesign work, we used a shared tool-meaning mapping table to document three aspects of students' interaction with each digital tool that are particularly relevant to covariational reasoning: (1) the set of *actions* available to the learner; (2) the set of mathematical *representations* on which those actions operate; (3) the tool's automatic *feedback* in response to the learner's actions. Using this table, we articulated how each tool's action-representation-feedback structure is implemented to foreground particular covariational meanings, and we used these comparisons to specify where complementary structures could be coordinated. Making these aspects explicit contributed to our operationalization of cross-synergy as a design construct.

Step 2: Making tool-meaning commitments explicit through theoretical bricolage

Step 2 was a collaborative process of negotiating what each theoretical lens makes salient about covariational reasoning and what it tends to background, so that these differences could be productively leveraged as redesign resources and to structure the activity with it. Operatively, we began by assembling a *theoretical bricolage* (Gravemeijer & Cobb, 2006), treating our different theoretical lenses (at different levels of generality) as resources to tinker with when formulating shared design principles for cross-synergy.

To translate this bricolage into actionable design work, we used the prementioned shared tool-meaning mapping table to make explicit what each original tool tends to make visible, doable, and discussable about covariation (i.e., which covariational meanings are commonly produced through its representational and interactional structure) and what tends to remain tacit when each tool is used alone. Comparing tools through the table helped us articulate tool-specific covariational meanings and potential complementarities. Methodologically, this step operationalized cross-synergy as a cross-perspective construct. Differences in theoretical commitments and design conjectures were treated as productive resources for identifying systematic privileging of meanings, and the 'gaps' that could potentially be addressed through complementary representations intentionally linked.

Step 3: Designing task sequences

Each subgroup then designed task sequences organised around core ideas of cross-synergy. This step was operationalized by: outlining when the task flow made references across representations structurally consequential for progress; outlining when progress depended on coordinating meanings emerging through interactions with different representations; defining methodological choices such as task formulation, peer or expert interactions, and meta-reflections revealing the underlying theoretical assumptions.

Step 4: Iterative refinement through cross-group reviews

Finally, redesigns were iteratively refined through cycles of subgroup testing and whole-group reviews. Subgroups identified strengths and weaknesses, where learners could plausibly proceed while attending to only one representation or conceptual meaning (i.e., where cross-synergy would likely fail), and revised the prompts and the task sequence. Whole-group review rounds then served as cross-perspective checks, examining whether cross-synergy remained faithful to the meanings each initial tool was designed to elicit, while also opening new opportunities to link and transform those meanings across the new tools. The outcome of this methodology is a set of three synergistic redesigns, each comprising: (i) a rationale for the pairing in terms of complementary covariational meanings emerging from the initial tools coupled; (ii) the intended conceptual meaning reorganization in covariational reasoning that the

synergistic tools are designed to foreground; and (iii) a task sequence offering users opportunities to activate and sustain conceptual connections inferred by the underlying theoretical assumptions.

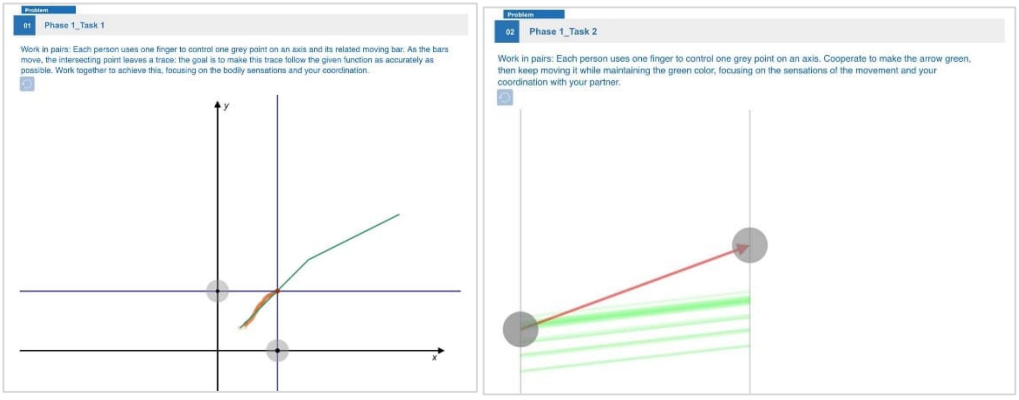
PRESENTATION OF THE SYNERGISTIC REDESIGNS

In this section, we present in detail the three synergistic redesigns that we obtained by combining pairs of digital tools and linking the conceptual meanings of covariational reasoning that emerge in one design with those that emerge in another. After an overview of each digital tool, we then outline how cross-synergy manifests in each tool both at the design and cross-perspective level.

It takes two hands

It takes two hands arises from the synergistic redesign of the Digital Tracer and the Nomogram. The digital tool is conceived as a three-phase sequence: beginning with separate, non-numeric experiences in each representation, moving to explicitly numeric experiences, and culminating in a synergistic phase where the two representations are dynamically connected while keeping their distinct interaction structures. In this way, the design gradually supports students' first-order covariational reasoning. Students first feel two variables through coordinated pairs of finger movements, then name and describe these changes numerically, and finally generalize them as an invariant functional relationship across representations. The redesign is fully implemented within the Numworx environment and can be accessed at: https://bit.ly/4djmGaF

Figure 4: The two tasks of phase 1 of *It takes two hands*

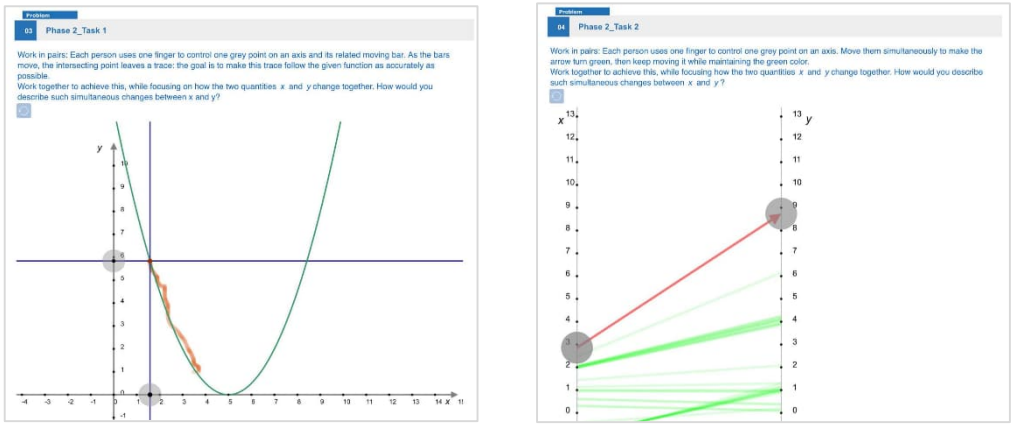


Phase 1 promotes an exploratory and non-numerical orientation in which learners focus on coordinated movement and experience controlling two variables simultaneously. Participants work in pairs, and each partner is supposed to control one grey dot on an axis in both tasks. In task 1.1 displaying a Digital Tracer, learners see a green broken trace (two segments with different slopes). Each partner drags a point on one of the perpendicular axes so that they steer the intersection point along this trace (Figure 4,

left). The goal is to follow the given trace as accurately as possible. In task 1.2 there is a Nomogram associated with the function $y=x+1$ (Figure 4, right). Each partner controls a point on one of the two parallel axes. Their joint movements generate arrows that link the two points and leave correct traces that map input values to output values. The goal is to maintain the arrow green, that is, to maintain a valid input-output pair. In phase 1, learners are requested to focus on their bodily sensations and coordination with the partner.

In phase 2, a more numerical approach is promoted (Figure 5). The main design difference compared to the previous phase is the presence of ticks and numerical values on the axes in both tasks 2.1 and 2.2. While the goal of tasks has not changed, learners are now asked to focus on how the variables involved, *x* and *y*, are changing together. The functions underlying the two tasks are quadratic. The tasks invited learners to discuss their observations in pairs and then to formulate a written description. While phase 1 centers on qualitative, bodily coordination, phase 2 overlays this activity with a numeric structure, preparing students to express covariational relationships more formally.

Figure 5: The two tasks of phase 2 of *It takes two hands*

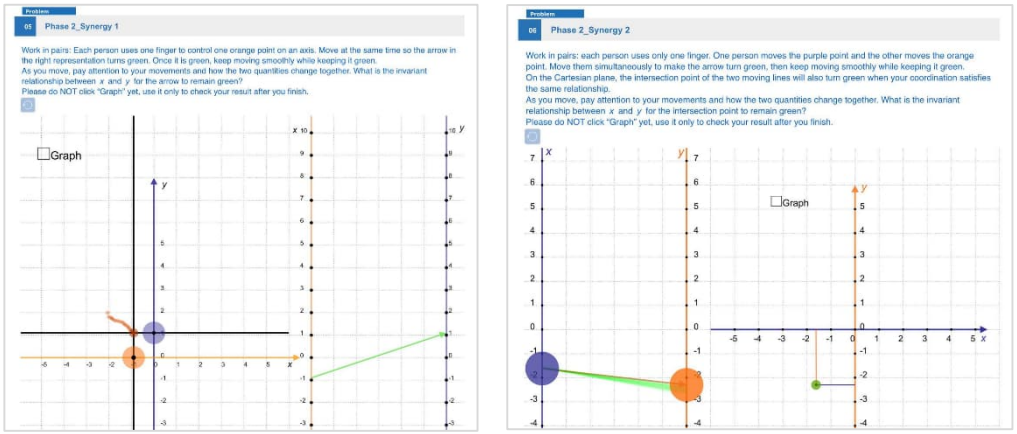


Phases 1 and 2 are supposed to gradually familiarize learners with each representation individually. Phase 3 shifts the focus from the single representation to a more abstract conceptualization of the functional relationship, as representation-independent invariant. In both tasks of phase 3, the Digital Tracer and the Nomogram appear as dynamically linked (Figure 6).

In task 3.1 (Figure 6, left), users working in pairs are supposed to attune to controlling the movement of the constrained points in the Digital Tracer. Such movements cause a simultaneous response in the Nomogram. If the intersection point matches one of the points of the hidden function, the corresponding arrow in the Nomogram becomes green. The underlying function is an absolute value function. In task 3.2 (Figure 6,

right), users are supposed instead to attune to controlling the movable points in the Nomogram. Such movements produce a simultaneous effect in the Digital Tracer.

Figure 6: The two tasks of phase 3 of *It takes two hands*



When the input-output pair in the nomogram matches a point of the hidden function, the intersection point in the Tracer becomes green. In this task, the underlying function is a hyperbola. In both tasks, the request is to find the invariant relationship between *x* and *y* that keeps the arrow/intersecting point green and to express this relationship in mathematical terms. Learners have the possibility to check their conjectures eventually by clicking on the "Graph" button, which shows the hidden function. Graphically, phase 3 maintains the numerical elements introduced in phase 2 and adds a common background grid in both tasks. A specific color is associated with each axis and its corresponding movable point (orange for the *x*-axis and blue for the *y*-axis) in both the representations (Digital Tracer and Nomogram). The synergistic redesign keeps the representational structures and action possibilities of the two original digital tools, but they now participate in a single, dynamically connected covariation problem.

Cross-synergy on the design level. The design of *It takes two hands* builds on a three-stage progression that has characterized research on action-based design (Abrahamson & Lindgren, 2014) and empirical studies with the Nomogram (Wei et al., 2024; 2025): non-numerical, numerical, and abstract. Phase 1 is the non-numerical one, and this is reflected in the design choice of using non-numbered axes to focus on the movements of the two variables. This choice is supported by the tasks' request asking to focus "on the bodily sensations and your coordination" and then to describe their movements. Learners gain experience with what it feels like to coordinate two movements so that a goal is achieved (following a trace, keeping an arrow green), yet without verbalizing rates or values.

Phase 2 is the numerical one, offering numbered axes and corresponding ticks. The tasks' request prompts to focus "on the simultaneous changes between *x* and *y*" but in terms of amount of change as prompted by the design choices. This phase supports a

transition from felt coordination to quantified descriptions of covariation. In phase 1 and 2, the Digital Tracer and the Nomogram appear as standalone representations. They are synergistically and explicitly connected only in the third phase. In phase 3, the numerical design elements introduced in phase 2 are maintained jointly with a background grid. This third phase configures as the most abstract one, not just because the given trace usually provided in the Digital Tracer representation is absent. Instead, learners are invited to infer and express the underlying invariant relationship that is coherently visible in both representations with mathematical terms.

The actions from the two original tools, the Digital Tracer and the Nomogram, are still available in the redesign. Users can drag movable points along the perpendicular bars of the Digital Tracer and the parallel axes of the Nomogram. In tasks 1.1 and 2.1, users can act only on the Digital Tracer, while in tasks 1.2 and 2.2 only on the Nomogram. In the synergistic phase, after experiencing the two representations separately, only one of these actions drives the underlying functional relationship per task. The original Digital Tracer also offered a trace button at the bottom of the screen, allowing users to deactivate the trace option and, for instance, reposition on the screen. Such action has been reproduced in the Numworx interface with a reset button on the top left of the screen which provides a clear interface. The trace functionality is acknowledged as important to represent both the trajectory of the intersecting point and to visualize the dynamic and global behavior of the function (Antonini & Lisarelli, 2021).

The visual feedback provided in the initial tools remains unchanged in phases 1 and 2: the arrow in the Nomogram becomes green or red, providing corrective feedback and leaving a green trace for each correct input-output value pair; the Digital Tracer provides only informative feedback, making visible through the trace functionality the result of learners' coordinated movements. In the third phase, instead, feedback becomes a synergy mechanism. In task 3.1, dragging the points in the Digital Tracer triggers green/red arrow in the nomogram. In task 3.2, it is the intersecting point in the Digital Tracer that becomes green if the arrow in the Nomogram is positioned correctly. In both tasks, a "Graph" button has been introduced, allowing learners to check their conjecture with the correct invariant relationship.

Cross-synergy on the cross-perspective level. As previously introduced, both the Nomogram and the Digital Tracer were designed in line with the theoretical perspective of embodied instrumentation (Shvarts et al., 2021). Hence, for us authors-designers, this shared grounding offered a natural common language and terminology to discuss how to synergize our tools. The redesign in all three phases reflects the importance of valuing perceptual and sensorimotor aspects in the learning process in order to develop stabilized actions that then can lead to meaningful mathematical conceptualization. In *It takes two hands* redesign, and in particular in phase 3, the intentionality guiding learners was imported from the Nomogram, that is to maintain the arrow/point green. What is formed in the interaction with the redesign in all phases is instead an intercorporeal dynamic functional system, as it was in the original design

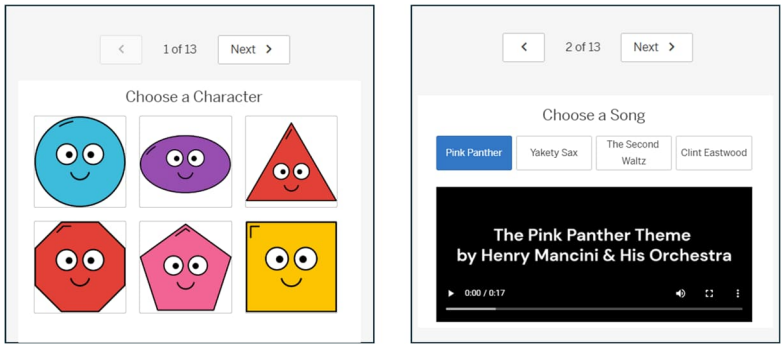
of the Digital Tracer, given that learners are asked to work in pairs and attune to achieve the same goal. While all the tasks individually promote a dynamic approach to the function concept, the display of two different kinds of representations, Nomogram and Tracer, in sequence in the first two phases and dynamically connected in the last one, strengthens the idea that what makes the relationship invariant is the relation between the simultaneous changes between the variables involved, independently from those representational elements that provide a frame to suitably represent such a relationship. In this sense, the redesign enacts cross-synergy as theorized in our framework: meanings that are first developed within the representations individually become resources for interpreting and reorganizing meanings about invariant relationships in the synergistic phase. The juxtaposition of a Cartesian-plane-based tracer and a parallel-axes nomogram shows that what remains invariant is not the diagram but the relationship between simultaneous changes in variables. While the Digital Tracer reproduces a global trajectory in the Cartesian plane, the nomogram foregrounds local input-output pairings. Together, they support learners in recognizing covariation as a relationship-in-action that can be traced, quantified, and finally expressed symbolically.

Sketch-to-Dance

Sketch-to-Dance (SktD) brings together Sketch-to-Animation (StA) task design principles, where learners sketch a graph and interpret its meaning through feedback animation that enacts the sketch, and Choreographing with periodic functions, an aesthetically oriented choreographic context, where learners create song-matched movement by iteratively shaping motion, guided by intensities of fit/misfit that emerge in the music–periodic function–movement coupling. In this sense, SktD can be characterised as an StA-inspired redesign in an aesthetically driven activity system, designed to support second-order covariational reasoning by enabling students not only to coordinate time and motion, but also to manipulate, use, and refine the motion-time relationship itself.

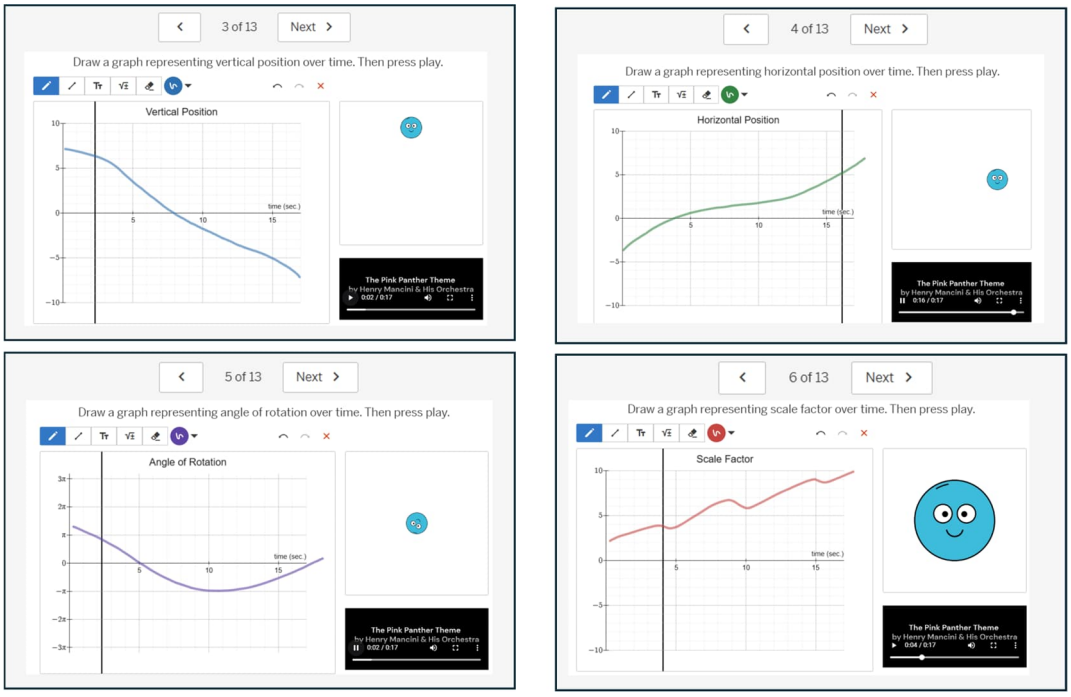
The synergistic redesign keeps the artistic goal from the choreographic context (to make the avatar move to the music), but changes how learners can act mathematically: instead of having to anticipate motion by a closed-form trigonometric expression (edited through parameters and then tested-animated via steady-slider-movement in another tool), they freehand sketch the relationship directly as a graph-over-time and immediately watch and hear how the functional relationship corresponding to the graph becomes animated motion with the song. The SktD redesign is implemented as a staged sequence in a single Amplify classroom environment (https://classroom.amplify.com/activity/686fc96f59bd55fdbf0a1955). A 13-screen sequence was designed to support a three-phase choreographic activity. This redesign operationalizes variation and refinement as the main design engine.

Figure 7: Screenshots from SktD; Screens 1-2



In phase 1, after choosing an avatar and selecting the initial soundtrack (Song 1: Pink Panther) (Screens 1–2; Figure 7), learners work through a single-motion sketching cycle across four types of motion: vertical position, horizontal position, angle of rotation, and scale factor (Screens 3–6; Figure 8). On each screen, they iterate draw–play–revise as a time cursor sweeps across the time-graph during playback, enacting the sketched relation as avatar motion. The same cycle is then repeated with three additional soundtracks (Yakety Sax, The Second Waltz, Clint Eastwood), so the song becomes a designed source of variation: learners revisit the same type of motion under changing musical structures (rhythm-tempo, melodic contour, style, etc.), revising features such as rate of change, smoothness, range, periodicity, and monotonicity to achieve a locally judged fit.

Figure 8: Screenshots from phase 1 of SktD; Screens 3-6



In phase 2, learners explore two different types of function combination (Screens 7-8; Figure 9). With Song 4 pre-selected, they move to Screen 7 and select at least two types of motion; the tool overlays their previously sketched graphs so they can compose them into a more complex motion. They may return to Screens 3-6 to revise the sketches and test alternative combinations against the song. Learners then go back to Screen 1, choose a favorite song, and proceed to Screen 8 to create a two-avatar choreography by drawing at least one graph for each avatar, distributing relations across characters within a shared musical time base.

In phase 3 (Screens 9-13; Figure 10), learners shift from freehand sketching to symbolic expression: they type equations for one or more motion (supported by a multi-curve plot and a motion-equation table), and they test the resulting motion against the corresponding soundtrack. This phase positions equation-writing as a bridge from enacted, sketch-based covariation to more formal function families, as operated in Choreographing with periodic functions.

Figure 9: Screenshots from phase 2 of SktD; Screens 7-8

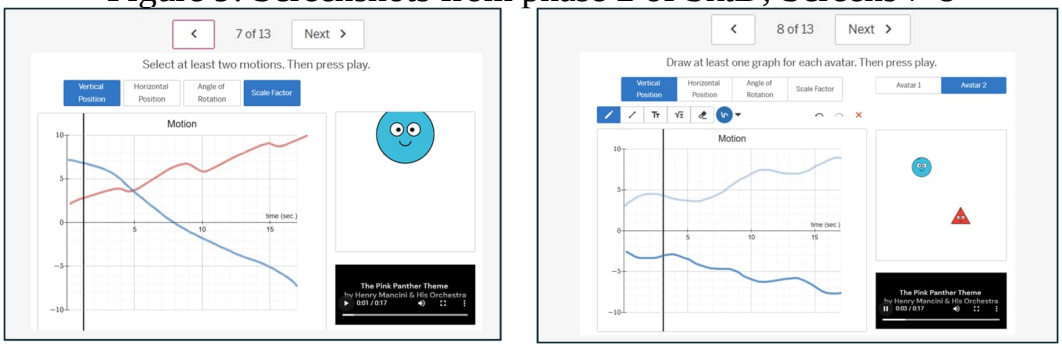
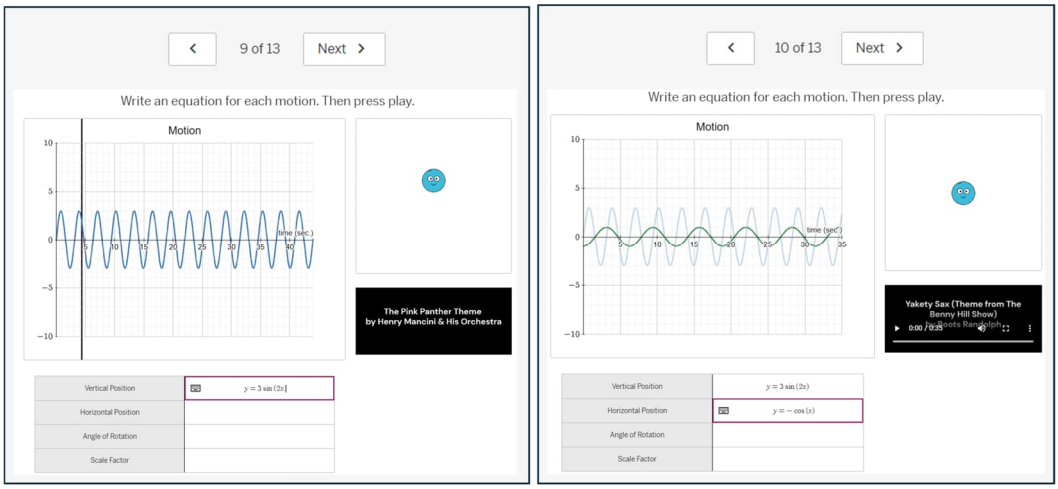


Figure 10: Some screenshots from phase 3 of SktD; Screens 9-10



Cross-synergy on the design level. The design of SktD builds on a three-phase progression that moves from direct, sketch-based enactment to composition of relations, and finally to symbolic generalisation. In phase 1, learners iteratively

choreograph by sketching time-graphs for single motion dimensions, repeatedly replaying and revising across four contrasting songs. In phase 2, they combine motions (Screen 7) and then distribute relations across two avatars (Screen 8), shifting the work from "one varying variable over time" to coordinated covariation across multiple variables. In phase 3, learners type equations for one or more motion dimensions, reconnecting the activity to the symbolic function families foregrounded in choreographing periodic motion. This cross-synergy on the design level is operationalised through the environment's action–representation–feedback structure, as captured in the tool-meaning mapping work. The core mathematical representation is a pre-labeled Cartesian coordinate plane (quadrants I and IV) used as a motion-time graph, where time progresses along the horizontal axis, and the sketched curve is interpreted as a changing motion variable. This representation is maintained across screens so that variation is carried primarily by (a) the selected motion (vertical position, horizontal position, rotation, scale), and (b) the song (re-fit across soundtracks), enabling learners to revisit "the same" representational form under systematically changing musical conditions. In phase 3, the representational field expands to include a multi-curve plot and a motion–equation table, making the relation between symbolic expressions and enacted motion explicitly inspectable.

The redesign intentionally shifts the primary mathematical action from tuning parameters of a closed-form equation to sketching graphs as the main way to propose and revise a covariational relationship. Learners also act by playing/pausing/replaying song excerpts, using repeated testing as part of refinement (an action that becomes mathematically consequential because it controls the time-based enactment of the sketch). Phase 2 adds compositional actions: selecting and combining two or more motions (Screen 7) and distributing graphs across two avatars (Screen 8), so that learners coordinate multiple time-dependent variables as choreographic components rather than working with a single curve in isolation.

SktD adopts an StA-style inscription–feedback loop but aligns its interpretive function with the aesthetic feedback logic foregrounded in Choreographing with periodic functions. As the song plays, a vertical time cursor sweeps across the time axis and continuously reads the sketched curve, updating the avatar's selected motion attribute (position, rotation, or scale) in real time; this creates rapid draw–play–revise cycles in which replay makes differences in timing, smoothness, intensity, and shape perceptible. Thus, feedback is not treated as verification of a "right" graph but as the production of felt fit/misfit that redirects action—aligning with the MaLT2–GeoGebra choreographing approach, where representational feedback (graph/parameters) and experiential feedback (motion-with-music) are interpreted through aesthetic criteria (e.g., too fast/slow, abrupt/smooth) rather than correctness. In phase 3, entering equations adds comparative visual feedback (e.g., a highlighted curve and motion–equation table), supporting inspection of how symbolic choices map onto enacted motion.

Cross-synergy on the cross-perspective level. SktD was designed as a productive cross-perspective integration between two meaning systems of covariation that are often studied separately. From a constructivist perspective, as embodied through StA tasks, the graph functions as a means of making covariational reasoning visible. Graphs can be understood as representations of covarying variables in an animated situation, and sketching them entails learners' construction and coordination of such variables through goal-directed action and feedback within the environment. From an aesthetic–materialist stance, mathematics is embedded in an artistic context, and functions are approached dynamically, with feedback interpreted through aesthetic sensibilities (of fit or misfit) rather than objective correctness. The redesign leverages these differences as a cross-synergy mechanism: it retains StA's sketch-enact-revise structure while reorienting what counts as "success" toward attuning covariational features to musical events. In this sense, the redesign also creates conditions for second-order covariational reasoning, since students are invited to iteratively reshape the covariational relationship itself as an object of reflection and design. For example, across phases 1 and 2, students are likely to determine that the frequency of a periodic function can be manipulated (via horizontal dilation) to better match the tempo of the song. And then in phase 3, students can change the parameters of a trigonometric function and immediately see the resulting graph. Taken together with their activity in phases 1 and 2, the students can develop connections between their symbolic choices and the resulting graph and enacted motion.

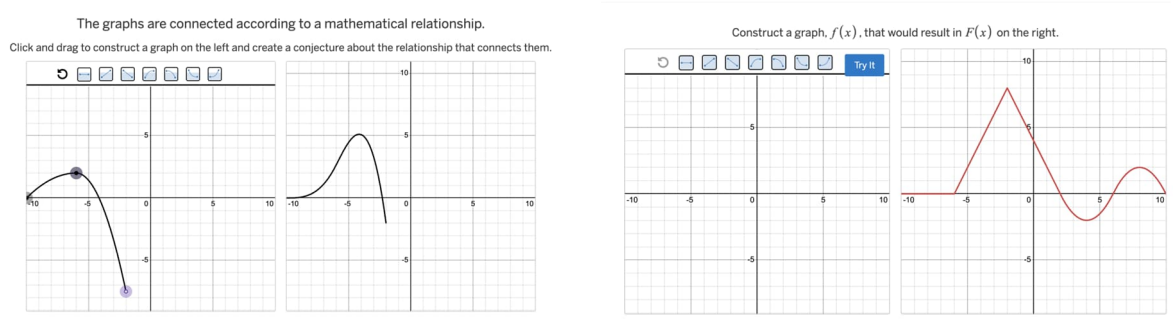
Sketch, Check, and Revise for Integral

The *Sketch, Check, and Revise for Integral* (SCR4I) redesign arises from combining the core representations of CIS, two dynamically linked Cartesian systems for f and F (Figure 1, right), with the StA sketch–check–revise design logic (Figure 1, left). The synergistic redesign is conceived to support learners' third-order covariation by making the relationship between a function and its antiderivative the object of interpretation and prediction, and by structuring learners' activity in staged cycles of anticipating, constructing, and comparing linked graphs. SCR4I maintains the mathematical goal and available actions from CIS while adopting slight variations in mathematical representations and feedback structure from StA. As in CIS, in the left side Cartesian system, the graph of f is displayed and can be constructed by selecting different graph-shapes to create a piecewise function and then dragging to translate or dilate those graph-shape segments (Figure 11, left); in the right-side system, the graph of F is displayed as the dynamically generated antiderivative of f. The redesign is supposed to stimulate a dynamic conceptualization of "covariation of covariation," in which learners reason about how changes in the covarying relation represented by f correspond to changes in the related covarying relation represented by F.

The redesign unfolds in three phases. In phase 1, an interpretive and qualitative approach is promoted: learners are asked to interpret the relationship between the two graphs as they appear in the two coordinated Cartesian systems, attending to how

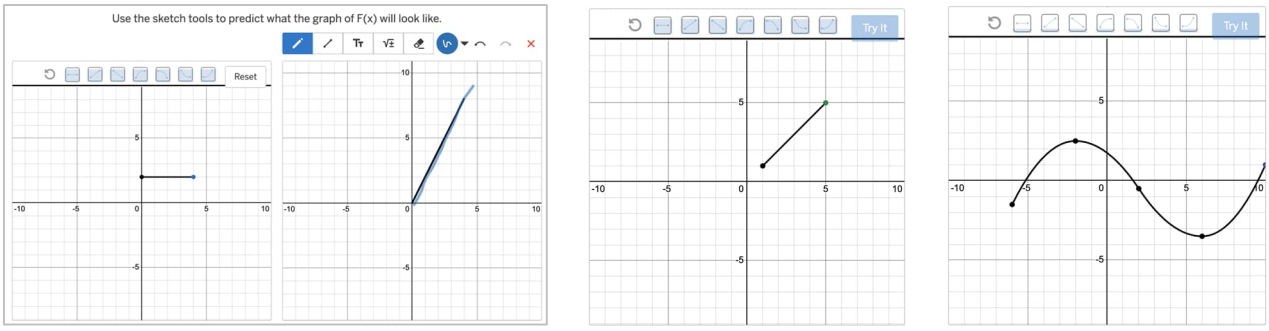
features of *f* correspond to global and local features of *F*. As the learner constructs the graph of *f* on the left, the graph of *F* on the right immediately updates, providing real-time feedback. Phase 1 is intended to have the same action-representation-feedback structure as CIS. In phase 2, learners are requested to predict how the graph of *f* should look given the anti-derivative graph, and to check the correctness of the graph they constructed (Figure 11, right). This prompt shifts learners from engaging in interpreting the link between *f* and *F* to predicting, constructing, and checking while maintaining the same set of available actions and representational links between the two Cartesian systems. Unlike the linked graphs in phase 1, the feedback in phase 2 has a slight delay, so that the antiderivative of the student's constructed graph only appears once they have completed their graphing activity and pressed "Try It". This form of feedback is similar to the StA tasks in which learners construct a graph via sketching and then view the interpretation of their sketch in the animated situation.

Figure 11: On the left, phase 1 of SCR4I; On the right, phase 2 of SCR4I



In phase 3, learners are requested to predict, construct, and check what the graph of *F* will look like for a particular instantiation of *f* (Figure 12). Students sketch their prediction for *F* and then press "Try It" to see the graph of *F* revealed. This design supports students in making comparisons between their predictions and the generated graph of *F*, and in revising their predictions accordingly.

Figure 12: Some available graph shapes from phase 3 of SCR4I



Cross-synergy on the design level. On the design level, cross-synergy is achieved by coordinating CIS's two-graph dynamic linkage with a prediction-driven cycle of StA

activity. CIS provides the dual-graph action-representation-feedback structure that makes the function–antiderivative relation directly inspectable and manipulable. The sketch–check-revise logic contributes a structured sequence of tasks in which learners must first form an interpretation, then commit to a prediction about a target graph, and then test this prediction through the app's dynamic generation of the linked graph. This sketch-check-revise activity is supported by slight shifts in the mathematical goal (e.g., construct f such that F looks a certain way) and feedback (e.g., construct f prior to viewing the related F). Thus, the cross-synergy is not a simple addition of features; it is a re-organization of activity so that interpretation of the f - F relationship becomes progressively more explicit, anticipatory, and revisable. The redesign leverages dynamic and visual feedback specifically to support comparison between "what should happen" (prediction) and "what happens" (generated graph), making learners' covariational meanings available for inspection and refinement.

Cross-synergy on the cross-perspective level. On the cross-perspective level, the cross-synergy is grounded in the socio-cultural and constructivist perspectives. From this standpoint, the redesign is conceived not merely as a mediating tool for students' personal meaning-making, but as a cultural one that embodies historically constituted mathematical knowledge—namely, the relationship between a function and its antiderivative. The function–antiderivative relationship is not treated as something to be constructed anew by learners, but as an existing mathematical object that becomes progressively explored and conceptualized through interaction with the tool. Within this view, learning is understood as becoming aware of mathematical meanings that are already embedded in the cultural tool. The staged prediction cycle, which comes from the constructivist StA design, plays a critical role in this process. By first interpreting the relationship between the two graphs, then predicting how the graph of f should look, and subsequently predicting the form of F across different function shapes, learners are repeatedly confronted with discrepancies between their personal anticipations and the mathematically grounded behaviour instantiated by the SCR4I redesign. Through this process of comparison and reflection, students can give meaning to the relationship between two covarying relations—how variations in the covariational relationship of f generate variations in the covariational relationship of F. In this way, third-order covariation is framed as an awareness that emerges through mediated engagement with a cultural tool that materializes the mathematics of accumulation. The cross-synergy thus lies in aligning prediction-driven activity, adopted from the constructivist StA design, with a socio-cultural view of learning, in which third-order covariation becomes accessible as a culturally grounded mathematical relationship.

CONCLUDING DISCUSSION

This Research Forum discusses how cross-synergy can expand the conceptual space of covariational reasoning when digital tools originating from different theoretical traditions are intentionally combined. In the sense used here, cross-synergy does not

refer to the simple coexistence of digital tools or to occasional references between them. It refers to a purposeful combination of such tools in a synergistic redesign so that meanings emerging from both the original tools become necessary for interpreting different steps of the activity. When design aspects and theoretical instances enter into such cross-synergy, learners engage with mathematical relationships through interactions with the digital tool in an intentionally designed educational activity. Conceptual meaning, therefore, develops through the interplay of representations, actions, and feedback offered by the digital tool.

Each digital tool discussed in the Research Forum was originally designed to foreground a particular way of engaging with covariational reasoning. Some emphasize the emergence of global patterns through movement, others highlight the maintenance of local constraints between variables, and others make visible relationships between mathematical relations. When digital tools function independently, these perspectives remain largely separate. In the synergistic redesigns, however, learners encounter situations where meanings developed through one representation must be interpreted in relation to another. Through this process, covariation becomes visible not simply as the behaviour observed within one tool, but as an invariant relationship that persists across different tool-based forms of activity. The examples here discussed —including the Digital Tracer, Nomogram, Sketch-to-Animation environments, Choreographing with periodic functions, and the Calculus Integral Sketcher—illustrate how cross-synergy between digital tools can open conceptual spaces that are difficult to access when each tool is used alone. Across the redesigns, cross-synergy appears when the activity requires learners to actively relate meanings generated through different representations. Rather than focusing attention on a single representation or tool, learners move between representation-specific perspectives and attempt to reconcile them. Through this process, mathematical relationships should become more explicit and more stable, because they remain coherent across different representational contexts. In this sense, cross-synergy should enable learners to experience mathematical ideas as relationships that hold across forms of interaction.

By reflecting on our experience of implementing cross-synergy as a design construct, three design considerations can be synthesized from the redesigns for creating productive cross-synergy between digital tools. The first design consideration is *reciprocal prediction.* Productive cross-synergy at a design level might emerge when learners form expectations within one representation and examine how those expectations unfold in another. Predictions therefore might function as bridges between representations: an anticipation developed in one representation becomes a hypothesis that can be explored through another activity step. The second consideration is *cross-representation* feedback. In synergistic redesigns, feedback generated within one representation provides information that must be interpreted through another one. Feedback, therefore, becomes a mechanism that exposes relationships between representations rather than merely signalling correctness. Interpreting such feedback

requires learners to integrate meanings emerging from multiple representations. The third principle is *enforced comparison*. Cross-synergy does not arise automatically when tools or representations are placed side by side. Tasks must create situations in which progress depends on relating meanings produced through different representations. Alternating which representational aspects drive the activity, structuring work around prediction and verification cycles, or asking learners to articulate relationships that remain stable across representations are examples of ways to support this process. These considerations highlight that cross-synergy between digital tools is not simply a matter of combining tools. It depends on designing activity structures in which the relationship between representations becomes mathematically meaningful. In such redesigns, representations function as complementary perspectives that illuminate different aspects of the same mathematical relationship.

Even though our implementation of cross-synergy has led to the creation of three synergistic redesigns, we believe that this is not the only way to implement it. A redesign of an educational activity where two tools co-exist is definitely another way of implementing cross-synergy, and even though we have focused on digital tools in line with our research interests, cross-synergy could also involve non-digital tools. While our work has mainly focused on the design and theoretical level, our redesigns should be tried out with different learners to have the chance to appreciate if redesign conjectures and considerations are actually effective. In this sense, we conceive the synergy construct, as presented in the literature, as complementary to cross-synergy in addressing the engagement of learners with such synergistic tools.

Our work also suggests that cross-synergy is not produced through a single recipe, but through collaborative processes that bring different expertise, theoretical, and design priorities into dialogue, thereby opening a field of design possibilities. Cross-synergy can thus be understood as a flexible collaborative methodology for creating mathematically meaningful relations among tools, tasks, and perspectives.

FUTURE DIRECTIONS OF RESEARCH

Beyond covariational reasoning, the notion of cross-synergy offers broader implications for mathematics education. Many mathematical ideas become meaningful through the interaction of different forms of activity, tools, and perspectives. Designing activities that employ a cross-synergy approach can enable learners to encounter mathematical concepts as coherent structures that remain stable across multiple forms of engagement. Cross-synergy between tools also provides a productive framework for cross-perspective and design-based research. The digital tools discussed in this Forum originate from theoretical traditions that emphasize different dimensions of mathematical activity, including cognitive coordination of quantities, embodied interaction with tools, socio-cultural mediation, and aesthetic engagement. When tools grounded in these perspectives are synergistically and intentionally combined, their cross-synergy should promote aspects of reasoning that may remain hidden within a

single theoretical lens. In this sense, cross-synergy offers a way to bring theoretical perspectives into dialogue through design. Future research may therefore investigate how learners move within such synergistic redesigns and how invariant mathematical relationships are articulated across digital contexts. Attention to these processes may contribute to a deeper understanding of how mathematical meanings develop through complex digital tools.

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STUDYING CHILDREN'S LEARNING OF NUMBER STRUCTURE – FINDINGS AND CHALLENGES

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The Research Forum provides a comprehensive overview of contemporary research on young children's learning of number structure in the teaching of foundational arithmetic. Number structure (rather than counting strategies) is, in this research, seen as the foundation for conceptually solid knowledge of numbers and operations. This topic has been undergoing substantial research in the last 10 years, represented in the RF by six invited research groups. By conceptualizing the generality and specificity of this research regarding theory, methods, and outcomes, the RF is in line with the PME49 conference theme, "Sensing and making sense of mathematics."

TOPIC AND FOCUS

The goal of the Research Forum is to provide PME members with a comprehensive overview of contemporary research on young children's learning of number structure in the teaching of foundational arithmetic. This topic has been undergoing substantial research in the last 10 years, with six invited research groups at the forefront (see presentations below). They represent a global interest in the learning of number structure but differ in research approaches and theoretical groundings. Conceptualizing the generality and specificity of this research regarding theory, methods, and outcomes at the Research Forum will contribute to advancing this research branch. The focus will be on the research outcomes: What are the theoretical underpinnings guiding the research, how is the research conducted, and what can the respective invited research group's studies make claims of, to contribute to the field of knowledge?

BACKGROUND

In 2018, a Research Forum was organized at PME with the theme "Early arithmetic learning" (Björklund et al., 2018). It highlighted the teaching of arithmetic skills in the early years through the lens of different theoretical and methodological approaches that helped us understand how socially and culturally influenced education may promote children's development. Taking this as a starting point, followed up in following PME conferences, and other international conferences on mathematics education, we observed a rising interest in early arithmetic research and particularly studies on how to teach for sustainable mathematical development. We wish to highlight a particular research branch that complements the dominant theories and research of recent

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 160-189. PME.

decades, by focusing on number structure (rather than counting strategies) as the foundation for conceptually solid knowledge of numbers and operations.

Recent research in this branch has highlighted that young students' awareness of mathematical structure is key to mathematical development. Students who experience number structure are oriented toward mathematical relationships (e.g., part-whole relationships), which makes it possible for them to solve arithmetic tasks using number relations and to reason about non-numerical structural aspects—for example, by using algebraic notations.

Research has found significant effects of teaching approaches that facilitate the development of structural awareness of numbers as a fundamental aspect of arithmetic skills (e.g., English & Mulligan, 2013; Davydov, 2008; see RF contributions below). Such findings have been made by research groups in several countries, with many of these led by the invited researchers. What conjoins these research efforts is a theoretical interest in developing arithmetic skills based on structural ways of perceiving numbers and quantities, with empirical findings of learning through this approach showing positive gains. A key literary work is Venkat et al.'s (2019) theoretical paper, which chisels out what mathematical structure entails regarding young learners. This also aligns with Davydov's (1982) work, which describes the core of arithmetic learning as discerning the internal logic of a situation involving quantities.

A structural approach contrasts with the more traditional, widespread use of counting-based approaches to early arithmetical learning, which Coles (2017) described as "a counting world." Across contexts, there is evidence of substantial proportions of children failing to transcend counting-based approaches, which leaves them ill-prepared for making progress beyond very low whole number ranges. What has been shown to impact learning outcomes, and to avoid the learning of limiting ways of understanding and using numbers, are approaches that emphasize the experienced meaning of numbers and relationships within and between numbers or quantities. One example of an attempt to theorize what it means to learn the fundamental meanings of numbers in this way can be found in Björklund et al. (2021), with educational implications that seem to cross countries, contexts, and cultures.

This Research Forum aims to highlight different ways to gain knowledge about students' awareness of number structures by discussing the methods and theories used for studying this, and teaching interventions' results for students' learning outcomes and progress in mathematical knowledge and skills. This is important since we need to learn more about the different approaches' consequences for findings as well as the theoretical and methodological challenges with using the approaches. This Research Forum focuses on both general and specific features of studying students' learning of number structure. We intend that the gathering of researchers with substantial research experience in this field adds to knowledge about research on early arithmetic learning as well as showing the strengths and limitations of the methods and theories that have been developed based on ideas of teaching and learning with a structural approach.

The invited research groups contribute presentations of their respective strands of research studies in which the theoretical frameworks and methods they used are highlighted in relation to observed learning outcomes. Specifically, the focus will be on the kind of knowledge the research adds to the existing field of arithmetic teaching and learning. The Research Forum will add value to the mathematics education community by discussing and concluding on theoretical and methodological advancements.

In each of two sessions, three research groups will present their research strands. By bringing forth different perspectives and highlighting what questions and inquiry focus the theoretical or methodological perspectives induce as guiding questions, we anticipate that constructive and creative reflections will emerge that contribute to ideas and insights, moving the position of the structural approach forward. The first session focuses on theoretical foundations, study designs, and methodologies, while the second focuses on the knowledge contribution the conducted studies provide to the field of early mathematics education.

IMPLEMENTING VARIATION THEORY IN EARLY ARITHMETIC TEACHING – CHANGING YOUNG LEARNERS' WAYS OF EXPERIENCING NUMBERS

Angelika Kullberg, Camilla Björklund, Ulla Runesson Kempe, Anna-Lena Ekdahl, Jessica Elofsson & Maria Alkhede

BACKGROUND

This presentation gives an overview of a series of studies conducted for more than 10 years in which we have been implementing and trialing the variation theory of learning in early arithmetic education using a structural approach. Variation theory (Marton, 2015) is based on decades of empirical classroom observations and interviews with learners in different knowledge areas, including mathematics. The theory is continuously being tested and tuned in iterative teaching projects characterized by collaboration between teachers and researchers. It provides a model for describing learning in terms of discerned and undiscerned aspects of what is intended to be learned. The learners' perspective is key, and the studies focus in particular on how learners experience numbers and how their ways of experiencing numbers and number relations can be developed by providing opportunities to discern necessary aspects of numbers in designed teaching situations.

The studies implement theoretical principles of number learning from a structural perspective (e.g., Schmittau, 2004) and, in particular, include the use of finger patterns to aid young learners' discernment of number (Neuman, 1987). A structural approach to teaching further means developing children's ability to coordinate numbers' cardinality and ordinality meanings, their awareness of units, and their ability to reason

about and use part-whole relationships to solve arithmetic problems. Conditions for successful learning and the use of finger patterns to structure numbers have been observed, tried out, and confirmed, for example, in Swedish preschool studies (with 5-year-olds, Björklund et al., 2021) and Grade 1 (Kullberg et al., 2024). In particular, the teaching interventions resulted in significant changes in the learners' early arithmetic skills, measured in both quantitative and qualitative terms. This provides evidence for the success that a structural approach conducted through variation theory principles has shown in both the short and long terms.

THEORETICAL FOUNDATION

The starting point and theoretical foundation are grounded in 1) the variation theory of learning (Marton, 2015) and 2) the theoretical principles of number learning from a structural perspective (e.g., Schmittau, 2004). The research draws on Dagmar Neuman's (1987) work on school beginners' number knowledge, emphasizing children's ways of experiencing numbers and how this shapes their understanding of what is possible to do with numbers. Although Neuman made efforts to theorize children's understandings of numbers based on the different ways in which they experience numbers, she did not fully come up with a theoretically driven conclusion regarding how children come to change their ways of experiencing numbers—knowledge that would be significant for developing teaching practices aimed at facilitating children's development of arithmetic skills.

Variation Theory, developed later than Neuman's studies, adds a model for describing learning in terms of discerned and undiscerned aspects and how teaching can facilitate the discernment of necessary aspects of numbers. A central feature in our projects is the directedness towards a specific mathematical object of learning and how this and its aspects are handled in teaching. The theory's usefulness is in its explanatory value – explaining why learners do what they do in arithmetic problem-solving – and in its direct applicability in teaching design.

Based on empirical research in early mathematics education, how certain aspects of numbers are presented to learners plays a significant role in their learning of numbers. In particular, research showing the constraints of counting-based strategies compared to a structural approach to arithmetic problem-solving (e.g., Neuman, 1987; Ellemor-Collins & Wright, 2009) has been guiding the projects. In contrast, we design teaching that strives to make it possible for learners to experience aspects and relationships of numbers with representations or artifacts that visualize units constituting composite sets in a part-whole relationship.

DEVELOPING EARLY ARITHMETIC TEACHING

Characteristic of the projects we have conducted is the close collaboration between teachers and researchers, and the iterative process of planning and evaluating intended and enacted teaching in relation to experienced learning among the children. The empirical grounding of the theoretical ideas is significant, and critical for the success

(in terms of better and sustained learning outcomes among the learners) is the teachers' participation because the enactment of theory-driven teaching needs to be responsive to the specific learners in the specific situation and time.

In line with variation theory, learning cannot be understood or teaching developed separately from what is intended to be learned (a learning object) – how and what to learn are inseparable. The strong foundation in theory for how learning is made possible (in terms of changes in ways of experiencing numbers) is embedded in the teaching, both in designed teaching activities and in the teachers' interactions with the learners. From a variation theory perspective, variations in ways of experiencing numbers derive from what aspects of numbers in a mathematical task (such as cardinality, ordinality, part-whole relations, and representations; see Björklund et al., 2021) are discerned by a learner. A learner who only discerns numbers' ordinality, but not cardinality, will inevitably handle numbers differently from how a learner who has discerned numbers' cardinality makes use of numbers in problem-solving. Furthermore, a learner who has not discerned the numbers' part-whole relationship will not be able to reason about and solve, for example, missing addend problems.

In all our intervention projects based on variation theory principles and a structural approach, children were interviewed before and after being involved in a school-year-long teaching intervention. The intention is to establish a baseline for the participating learners' knowledge of numbers from which teaching should commence and to assess learning outcomes in terms of discerned aspects of numbers. Our interviews focus on qualitative features of number knowledge that encompass discerned (and undiscerned) aspects of numbers. In accordance with variation theory conjectures regarding what it means to know something and learn about something, the undiscerned aspects are explicitly what teaching should attend to. The interviews provide crucial information for teaching design. Principles for how to make these aspects discernible for learners are embedded in teaching activities and teachers' actions. When following up in post-interviews, the discerned aspects observed in the ways the learners handle arithmetic tasks are also what will be considered learning outcomes. In this way, our contribution to the field of arithmetic teaching and learning provides insights into what it means to have number knowledge and skills, and how teaching can be designed to facilitate this kind of learning.

A theoretically intriguing and significant question for educational practice is how to make an aspect possible to discern, or, in other words, how to advance, a learner's way of experiencing the meaning of numbers. From the series of research studies in which we follow both learners' ways of experiencing numbers and what is offered to them to discern in teaching activities, this is one example:

In one of our early studies, we found in the pre-interview that preschool children had a wide variety of number knowledge; in particular, some children seemed to lack cardinal knowledge of numbers (Björklund et al., 2022). We then designed teaching interventions using artifacts to facilitate the discernment of numbers as composed sets

(making cardinality the primary object of learning). The artifact "five-snake" constituted a bead string with five red beads that could be moved along the string to make groups. In teaching, this five-snake was used in combination with finger patterns on one hand (five fingers to fold or unfold to represent the number of beads grouped on the string). By keeping the whole (five) invariant and varying how the beads were grouped on the string (4 and 1, 3, and 2), it was possible to discern the part-whole relationship within the number five.

The same structured artifact, but with five white beads added, making a "ten-snake" in a semi-decimal structure, was used to broaden the part-whole relationship within the number ten. The semi-decimal structure was used to emphasize units that did not need to be counted. The learner can perceive five beads of the same color in an instance, connecting the number of beads to his or her hand (five fingers), and make use of this five-unit to identify, for example, 7 as "all of the red beads/the whole hand and two more". This allowed discernment of units within units, building on the basic aspect of cardinality (earlier emphasized in the five-snake as a baseline) but now broadened to foreground part-whole relations. The additive structure of numbers that was made discernible could then be used to solve simple arithmetic tasks without counting in ones but instead using the structure on the bead string and finger patterns to see how parts relate to one another and to the whole simultaneously. Simultaneity was found to be key in this learning, which means that the units (grouped beads or fingers) and the whole (total number) are explored and experienced as a coherent relationship—if you know the total and can see one unit, you know the missing or hidden unit. In this way, the learners learned to solve arithmetic tasks, not as counting strategies to apply but as part-whole relations.

Furthermore, in studies with primary school learners, the snake artifact was extended with five more units, allowing the learners to use the same structural approach to reason about tasks exceeding ten, thus emphasizing ten as a benchmark in arithmetic problem-solving and making use of units and decomposition of units. When extending the number range, more advanced arithmetic tasks could be introduced that needed new aspects to be discerned, such as the relationship not only within numbers but also between numbers. For example, after the intervention, students in the first grade were able to use part-whole relations of number to solve tasks like 15–7=_. One student explained, "And I pretend I have one more hand, and then I take it away [the extra hand] and two more [folds two fingers on the right hand], then it is 8," showing that she had discerned the 5 in the 7 and could use this knowledge to solve the task by bridging 10.

Post-interviews showed that by taking part in the interventions described above, the learners learned to experience numbers as structural phenomena, which means they solved arithmetic tasks according to the numbers' part-whole relationships rather than applying certain counting procedures. This result was found in both the younger and older learners who participated in our studies. Variation theory allows us to study what

has been afforded to students in teaching and what students have discerned, described by the offered, and experienced critical aspects. This is beneficial in that it also works as a means to assess whether teaching has been successful – did the learners develop their ability to discern necessary aspects to solve arithmetic problems as intended?

The successful results for the learners and the theoretical insights from the projects add to the field of mathematics education research a complementary view on how to teach numbers and how young students learn about numbers and arithmetic. Insights from earlier projects have been implemented and tested in new projects with other teachers and classes, broadening both the ages of the learners and the variety of learning contexts. Teaching is characterized as variations on the same theme, keeping the principle constant but varying learners and activities in which theoretical concepts are embedded—encompassing a larger iterative cycle of implementation, evaluation, and development—contributing to a process of constant development on a broader scale and providing evidence of successful learning and teaching.

CONCLUSION

In our research studies, teachers are part of implementing and trialing knowledge from earlier projects in new education contexts and thereby contribute to the development of arithmetic learning theory, ensuring ecological validity. The foundation is cumulative knowledge building on early arithmetic learning and teaching in several studies, each building on earlier innovations and insights (iterations of lessons and activities), which means that knowledge and experiences from earlier projects are made use of, trialed, and further developed in subsequent projects. The greatest contribution, however, is that learning and teaching arithmetic can be explained in commensurable terms; we can identify what aspects the teaching has afforded and what students have discerned concerning the target capability. We suggest that this is a way to ensure validity and sustainability in knowledge building as an important cornerstone of the field of arithmetic education research and practice. We dare to make this suggestion because of the empirical studies' learning outcomes, which indicate strong effects on children's sustained and developable knowledge of numbers.

SUPPORTING EDUCATORS' FOCUS ON EARLY NUMBER STRUCTURE IN AN IRISH PRE-SCHOOL INTERVENTION

Hamsa Venkat, Brenda Lattimore, Cora Gillic, Nicola O'Reilly & Geraldine French

BACKGROUND

Attention in the mathematics education to early number structure has largely been located in two camps. In one camp, number structure is linked to the idea of generalised arithmetic and early algebraic approaches to number teaching and learning (Blanton et al, 2011). This approach is contrasted with an 'arithmetical' focus on calculation. In the second camp, structural approaches are contrasted with counting-based approaches.

In the latter, sequential counting in ones is the primary approach used to both enumerate and to answer numerical additive questions. As noted in the introduction to this Research Forum, there is growing international evidence of counting-based approaches leaving problematic proportions of children locked into this way of working, unable to move beyond the early number ranges in which such approaches can work in effective and efficient ways.

Our work is in the latter group, and has drawn, for over a decade, on research teams across South Africa, Sweden and Ireland with an interest in structural approaches to early number learning. This choice, in South Africa and Sweden, was linked to prior evidence of children failing to move beyond the use of counting in ones for their calculations (e.g. see Schollar, 2008, in South Africa and Neuman, 1987, in Sweden). More recent evidence suggests that this problem is more widely recognised internationally (see Hopkins et al's, 2022, findings from Australia).

'Structural approaches', for us, refer to approaches that focus, from the start, on number relations (Venkat et al, 2019). These rest–in turn–on seeing numbers as part of a network of properties within the number system. In counting-based approaches, sequential counting in ones is the primary approach used to enumerate and to answer numerical additive questions. Given our shared interests in the pre-school and early primary phases in which children are simply staring to get familiar with numbers in the number system, structural approaches–for us–take in early subitizing, as one way to overcome the recourse to counting, and early attention to number relations.

In early number work, number relations have tended to focus on cardinal relations with quantitative enumeration of discrete activities feeding into the emphasis on counting (Coles, 2017). However, work across South Africa (Venkat et al, 2021) and Sweden (Björklund et al., in review) has noted the need for a growing bank of ordinal counting skills (forward and backward counting from any number and knowing the number words that come before/after any number) to be in place for moves forward with structural approaches. These approaches incorporate cardinal, ordinal and part-whole relations.

In this presentation, we focus on an Irish pre-school intervention that has drawn on this earlier work to build an intervention that takes subitizing and structural approaches that focus on cardinal, ordinal and part-whole relations. The intervention is located in an urban setting identified as located in a community dealing with multiple social and economic challenges. Our focus, in this presentation, is on outlining the Irish pre-school policy and curriculum context. This allows us to locate our approaches to supporting educators' attention to number structure, via tasks and tools focused on subitizing and number relations, in a curricular context in which child agency, play-based learning and integrated working are seen as critical.

THE IRISH PRE-SCHOOL POLICY AND CURRICULUM CONTEXT

Irish Early Childhood Education practice is guided by the Aistear Curriculum Framework (NCCA, 2024). The framework emphasises child and educator agency, and advocates for integrated play-based learning that is led by children's motivations and interests. There is limited specification of learning outcomes beyond a recognition that young children should broaden their literacy and numeracy through sense-making of their experiences. While prior research has noted that supporting emergent numeracy learning requires strong mathematical content knowledge and knowledge of children's mathematical development (Gasteiger & Benz, 2018), Irish survey research points to limited access to professional learning opportunities and practical resources for supporting early mathematical learning among early childhood educators (O'Neill, Gillic and Kingston, 2022). Recent Inspectorate findings (DE, 2024) have pointed to the need for more intentional pedagogies to move beyond superficial number activities, in ways that maintain children's agency within learning.

One concern, seen in mathematical outcomes in the Irish primary schooling system is that mathematical outcomes in TIMSS Grade 4 from 2023 indicate statistically significant lower outcomes for children in schools serving areas of relative socioeconomic disadvantage in comparison to wealthier populations (McHugh et al, 2024). This points to the need for early interventions focused on mathematics.

It was these gaps–in professional learning opportunities and socioeconomic divides–that we sought to fill with an intervention focused on structural approaches to early number learning in an urban inner-city pre-school. Our intervention model was linked to the pedagogies and priorities that are predominant in the Irish curriculum landscape.

SUPPORTING ATTENTION TO NUMBER STRUCTURE IN AN IRISH PRE-SCHOOL INTERVENTION

The strong commitments to play-based and integrated working and children's interests-led working led us into a collaborative and iteratively developing model of professional learning focused on number relations. This model has involved monthly professional learning interactions that involve some input from the university team, with tasks and activities relating to aspects of number relations, and time for educators to share their reflections on their own work with the tasks, and children's activity in the context of their use.

Our focus on cardinal relations began with a focus on quantitative enumeration linked with the language of more/less, taller/shorter, bigger/smaller as a way of emphasising a focus on relationships between quantities. Attention to ordinal relations initially attended simply to supporting children's learning of the ordinal sequence, linked with the language of 'What comes next/what comes after?' and 'What number is before _?' In part-whole tasks, we are connecting subitizing of finger pattern numbers with questions about the number of fingers open and number of fingers closed, breaking towers (or other sets) into two parts, or working with a known whole and a visible and

a hidden part. Across all aspects of number relations, we are encouraging initial working in the 1-5 range before proceeding into relations in the 1-10 range.

Our work began with baseline interviews with the 18 3- and 4-year-olds in the setting. These interviews showed disparities of awareness of number relations between children and disparities of awareness across the different aspects of number relations that we were interested in. At one extreme, the group included a child who was able to recite the counting sequence up to 30 relatively fluently, and knew the number after given numbers in the 1-20 range. This child was also able to arrange a mixed-up set of 1-10 digit cards in order. However, her cardinal awareness was much less secure, with rush to say the number sequence leading her to enumerate with errors. Her subitizing range was accurate for 2 and 3 beads, and for 4 fingers, but on asked how many fingers were on one hand, and then two hands, she responded by counting these quantities. She was unable to state the number of closed fingers on one hand after counting the open fingers, indicating limited proficiency with part-whole relations in the 1-5 range.

At the other extreme, some children were unable to verbalise the ordinal number sequence from 1-3 either as rote counts or in cardinal count tasks. Also of interest, in a setting with many children with limited spoken English, were relative strengths in reading and ordering symbolic numbers on the digit cards in relation to verbal cardinal or ordinal counting. More broadly, and important for thinking about our intervention approach, was evidence, simply, of a lack of interest in answering our cardinal count-related questions in the baseline activities.

CONCLUDING COMMENTS

Our baseline findings support the view that ordinal counting does need to be supported if we are to make progress on structural approaches to number learning at pre-school level. To this end, we have developed overview record sheets for educators to do some mapping of any progress they see on children's ordinal, cardinal and part-whole understandings, with progressions in each category linked to both attention to number relations rather than counting, and to extensions in the number range.

THE ROLE OF STRUCTURING MATHEMATICS IN POSTGRADUATE STUDIES AND RESEARCH IN SOUTH AFRICA

Corin Mathews

The Research Forum *Studying children's learning of number structure – findings and challenges* highlights a shared international concern: early arithmetic instruction remains dominated by counting-based activity that privileges procedural focus over relational understanding. As a result, learners often develop fragile numerical knowledge that is difficult to generalise beyond immediate tasks. The work of Mathews and Prinsloo contributes to this collective endeavour by positioning number structure

not merely as a pedagogical emphasis, but as an epistemic reorientation in how number is constituted as an object of mathematical knowledge. Across a coordinated programme of postgraduate research in South Africa, structuring is investigated as a way of knowing number that foregrounds relations, compositions of units, and invariant numerical properties across representations.

This body of work shows that structural awareness becomes visible when learners engage with number relationally rather than sequentially, as instruction foregrounds part–whole coordination, composite units, and base-ten organisation. Structural reasoning is therefore treated as foundational to arithmetic proficiency, conceptualised as recognising relational compositions of units rather than merely accumulating counts. This perspective aligns with developmental theories of mathematical concept formation, which emphasise learning through engagement with relational systems rather than empirical aggregation of discrete units (Davydov, 2008). Within this orientation, number is understood as a structured system of quantitative relations, and learning involves discerning how units are composed and coordinated. Arithmetic activity thus shifts from procedures on quantities to operations on relational structures, explaining why counting-focused instruction often produces procedural success without relational understanding. Although grounded in postgraduate research and teacher education in South Africa, this programme addresses key Research Forum concerns regarding how number structure is constituted, investigated across representations, and used to interpret learners' mathematical knowing.

STRUCTURING NUMBER AS AN EPISTEMIC DISTINCTION

A central theoretical contribution of the Mathews–Prinsloo research programme is the claim that counting and structuring represent qualitatively different epistemic orientations toward number. Counting treats number as a sequence of discrete accumulations enacted through temporal action. Structuring, in contrast, conceptualises number relationally through part–whole coordination, unitising, and base-ten organisation. These orientations are not alternative strategies within the same conceptual space but reflect fundamentally different ways of experiencing number. This distinction resonates strongly with developmental instruction theory, where conceptual growth occurs through learners' engagement with the internal relations that constitute mathematical systems rather than through repetition of procedural routines (Davydov, 2008). From this perspective, the unitising of ten is not a procedural convenience but a conceptual reorganisation that restructures the number system itself. Ten becomes a composite unit that reorganises numerical activity, enabling generalisation, flexible calculation, and structural transformation. Empirical evidence from South African classrooms supports this distinction. Learners demonstrating structural competence coordinate composite units flexibly and reason with part–whole relations across representations:

> Learners who can count efficiently, partition ones and tens, work with groups of ten and understand number relationships when solving addition problems, operate with high levels of base-ten thinking. (Mathews, 2025)

Within a structural orientation, arithmetic activity becomes relational rather than count-based. Learners operate on composed units, anticipate numerical transformations, and coordinate quantities across representational systems. In contrast, counting-based activity constrains learners to linear progression and restricts their capacity to generalise numerical relations. Extending this perspective to teacher education reveals that structural awareness cannot be assumed even among pre-service teachers. Research shows that prospective teachers frequently revert to procedural enactments when working with representations designed to foreground structure:

> While PSTs recognised the value of ENLs in fostering conceptual development, the findings emphasise the need for more structured pedagogical scaffolding within teacher education programmes. (Mathews & Prinsloo, 2025)

These findings underscore that structural reasoning is not an automatic consequence of exposure to representations but requires deliberate pedagogical mediation. By making learners' structural awareness visible, the structural approach provides teachers with principled ways to design tasks that foreground part–whole relations, emphasise base-ten coordination, and diagnose whether learners are reasoning relationally or procedurally. This epistemic distinction is further clarified through variation theory, which explains learning as a process of discerning critical aspects of an object of learning through systematic variation (Marton, 2015). From this perspective, structural awareness becomes possible when instruction varies numerical features while preserving invariant relational properties. Counting persists when variation obscures relational invariants, whereas structuring emerges when learners experience coordinated variation in part–whole configurations, composite units, and base-ten decompositions. Variation thus provides a pedagogical mechanism that enables learners to experience numbers as a structured system.

OPERATIONALISING NUMBER STRUCTURE ACROSS REPRESENTATIONS

Responding to the Research Forum's methodological emphasis, the Mathews–Prinsloo programme treats structuring as an empirically investigable phenomenon rather than an abstract construct. Structural awareness is operationalised through task design and analytic criteria that make relational reasoning observable, including learners' coordination of composite units, base-ten partitioning, relational explanations, and responsiveness to designed variation. These indicators act as traces of discernment, signalling awareness of critical structural aspects of number (Marton, 2015). A central methodological focus on empty number lines shows how learners' jump strategies reveal whether number is experienced through sequential counting or coordinated composite units, with explanations further indicating relational or procedural reasoning. Representations are thus treated not as neutral tools but as epistemic

mediators that open specific dimensions of variation for structural discernment. Multimodal artefacts such as student-created mathematics picture books provide rich sites for examining how structural awareness is represented visually and narratively:

> The study further reinforces the importance of intentional mathematical structuring in picture books, showing how variations in number representations, sequencing, and patterns influence children's ability to discern mathematical relationships. (Prinsloo et al., 2025)

By analysing number lines, assessment tasks, and learner-generated texts within a coherent theoretical framework, this research demonstrates that structuring is not representation-specific but a general feature of mathematical knowing. The methodological contribution lies in showing how diverse data sources can be analysed through a common structural lens.

KNOWLEDGE CONTRIBUTIONS TO EARLY MATHEMATICS EDUCATION

The Mathews–Prinsloo research programme contributes to early mathematics education in four key ways. First, it reframes learner difficulties as epistemological rather than procedural, showing that struggles stem from fragile structural conceptions rather than computational weakness. Second, it advances methods for studying structural awareness through observable indicators and variation-theoretic analysis. Third, it shows that representations shape learners' access to relational structure, underscoring the importance of intentional task design. Fourth, it extends structural research into teacher education, demonstrating that pre-service teachers need explicit support to develop structural reasoning. Collectively, the studies provide evidence that persistent arithmetic difficulties arise from weak structural conceptions while offering methodological tools for researching number structure across contexts.

STRUCTURING MATHEMATICS WITHIN POSTGRADUATE RESEARCH

A distinctive contribution of this work is its grounding in postgraduate mathematics education research. Embedding structural approaches within master's and doctoral studies shows how theory-informed inquiry can produce contextually grounded insights while informing international debates. Postgraduate research provides a productive space for integrating theory, methodology, and pedagogical experimentation, particularly within South Africa's resource-constrained contexts. This integration ensures that theoretical advances inform practice, positioning structural approaches as guiding principles for curriculum design, task construction, and assessment.

CONCLUSION

This contribution has positioned the Mathews–Prinsloo research programme as advancing the Research Forum's collective effort to understand young children's learning of number structure. By conceptualising structuring as a distinct epistemic orientation and operationalising it across multiple representations, this body of work contributes theoretical clarity, methodological depth, and empirical insight. The

findings show that moving beyond counting-dominated arithmetic requires more than new teaching techniques; it requires reorienting how number is conceptualised, how learning is studied, and how instruction is designed. Structural awareness emerges when learners experience number as a relational system, when tasks foreground invariant properties, and when pedagogy deliberately supports discernment. Situated within postgraduate research and teacher education in South Africa, this programme demonstrates how structural approaches can be studied, supported, and sustained across contexts, contributing to the broader international effort to reframe early mathematics learning.

WAYS OF EXPERIENCING NUMBERS – THE POSITIONAL SYSTEM IN A LEARNING ACTIVITY PERSPECTIVE

Helena Eriksson, Inger Eriksson, Sanna Wettergren, Marie Björk & Gunilla Pettersson Berggren

INTRODUCTION

This paper presents a tentative result from a collaborative design research project funded by the Swedish Institute for Educational Research (2022–2024). The overarching aim of the project was to create an online professional support – a teaching guide – consisting of tasks, tools and theoretical principles based on the El'konin Davydov Curriculum (EDC) and learning activity theory (LA) (Davydov, 2008).

The EDC is based on the idea that students first need to explore the structural aspects of knowledge content and then use it in various concrete operations. This principle is described as "ascending from the abstract to the concrete" (Davydov, 2008, p. 106). According to Davydov (2008), exploring a general structure that is not visible requires tool-mediated work. In LA, so-called learning models can be developed and help students in their exploration of the 'abstract.'

Grounded in the EDC and LA, the project emphasises working with different base numbers to promote students' exploration of the general structure of the positional system since the familiarity of the base-ten system often obscures its underlying structure, which different bases help to expose (Slovin & Dougherty, 2004; Vygotsky, 1986). The general structure of the positional system, e.g., the relationship between the different digits in a multi-digit number, is both multiplicative and additive (Slovin & Dougherty, 2004). To understand the general structure, the EDC suggests that students need to work in different bases in order to uncover the structure (Davydov et al., 2012; Slovin & Dougherty, 2004; Vygotsky, 1986). In our project, the students used learning models, such as line segments and tables, to explore the general structure of the positional system and to use this structure to note numbers in different bases.

Research on the positional number system has largely focused on what students need to discern in order to grasp its general structure, for example, the ability to group

number units into successively larger units regardless of base. However, there is still limited knowledge of how students actually experience and make sense of the structure of the positional system, particularly following instruction designed to foreground its general features. This lack of knowledge constitutes a problem for research as well as for teaching, as it restricts our understanding of how particular instructional designs may support students' ways of experiencing numbers. The aim of this paper is to describe and discuss primary school students' ways of experiencing numbers after participating in lessons planned with tasks designed based on the EDC and LA (Davydov, 2008; Davydov et al., 2012).

METHODOLOGY

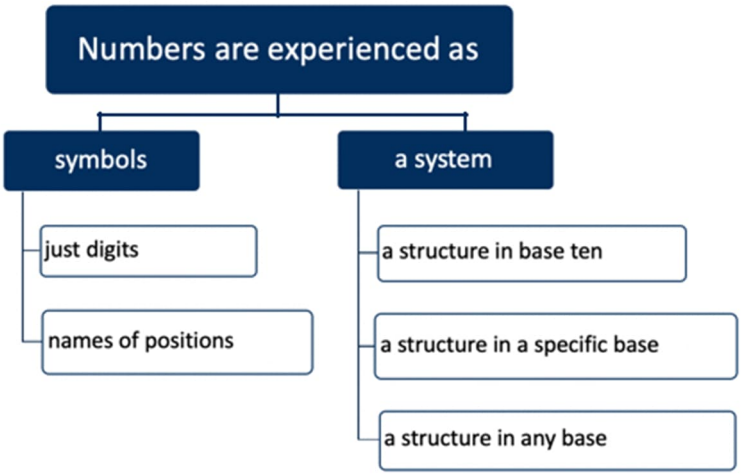
To explore primary school students' understanding of the positional system, the research project conducted more than 20 video-recorded research lessons with Grade 2 and 3 students (7–9 years old) in two socio-economically and culturally diverse schools. The lessons were designed in line with LA, focusing on problem situations, allowing students to identify a problem to be solved jointly with peers and the teacher. All the problems focused on the positional system in different bases. After participating in research lessons, 30 students were interviewed in pairs to explore their qualitatively different ways of experiencing numbers. The interviews were video recorded with the camera directed towards their hands, writing, and drawings. In each interview, the students were asked to reflect on tasks presented by the interviewer. As an example, during the first question, the interviewer drew two lines representing the addition of 9+1: a long line representing the value 9 and a short line representing the value 1. The interviewer said: If I have nine and then add only one, it becomes ten. Why does nine have only one digit when ten has two digits?

The transcribed interviews were analysed phenomenographically (cf. Kullberg et al., 2024, see also Eriksson, 1999). Such analysis focuses on how a phenomenon can be experienced in qualitatively different ways within a group of people. The analysis can be described as a constant comparative reading of the transcriptions, and in our case, we repeatedly returned to the video recordings to validate it. Further, the analysis goes beyond what is said verbally, as underlying meanings, gestures and intonations can provide clues to meaning. The result is presented with a set of categories that form a so-called outcome space.

FINDINGS

As a result, the outcome space consists of two qualitatively distinct ways of experiencing numbers: as *symbols* and as a *system*. The category 'symbols' divides into two subcategories: *as just digits* and *as names of positions*. 'A system' divides into three subcategories: *as a structure in base ten, as a structure in a specific base,* and *as a structure for any base* (see Figure 1).

Figure 1: The outcome space



In this presentation, we have chosen to illustrate only the category that shows students' experiences of numbers as a system, with its three subcategories.

Experiencing numbers as a structure in base ten

In this subcategory, students experience numbers as a structure of the base ten system. Their reasoning seems to be guided by an established familiarity with number units, such as ones and tens. For example, the digit *one* in the tens position is perceived as representing a unit of ten ones. However, this understanding remains embedded in the base ten system and is motivated by reference to how numbers are usually structured, rather than by an understanding of the positional system as such.

Experiencing numbers as a structure in a specific base

In this subcategory, students experience numbers as a structure that is dependent on a specific base. When working in another base, such as base three, students identify that the number unit to the left of the ones position must be constituted according to that base and therefore cannot be named or interpreted using base-ten terminology. Students reason about the need for a new number unit that emerges when the ones are exhausted in the given base. This is reflected in the students' attempts to name the unit to the left of the ones position in ways that align with the structure of the specific base, for example, by referring to the digits that constitute the number rather than to established base-ten units such as tens. In this way, number units are experienced as base-dependent and internally consistent within each base.

Experiencing numbers as a structure for any base

In this subcategory, students experience numbers as a positional structure that remains the same across bases. The base is understood as a defining parameter that determines when a new number unit is formed, while the underlying structure of the positional system remains the same regardless of the base. Students reason about the emergence of a new number unit in relation to reaching the base, expressing that the transition to a new position occurs "when you get to your base." In this way, the formation of a new

unit is no longer tied to a specific base but is understood as a general principle of the positional system.

Overall, the outcome space indicates that the fundamental qualitative difference in students' ways of experiencing numbers concerns whether numbers are experienced as symbols or as a system. Within experiences of numbers as a system, qualitative variation can be further related to differences in which aspects of the system are discerned. In the first subcategory, familiar base-ten number units are discerned, through which the positional organisation of numbers is made meaningful. In the second subcategory, the role of the base is discerned, and number units are experienced as constituted in relation to the base in use. In the third subcategory, students discern both the base and the structure of the positional system simultaneously, enabling a generalised experience of the positional system as applicable to any base.

CONCLUSION

Previous research suggests that students' familiarity with the base-ten system can obscure the underlying structure of the positional system, which can also be successful in work with decimal numbers. Based on the phenomenographic analysis, we argue that students' engagement in measuring, grouping, and exchanging in different bases supported them in moving beyond taken-for-granted base-ten reasoning. In line with the principles of the EDC, this enables students to reflect on numbers in a more general and abstract way, contributing to an emerging theoretical understanding of the positional system. Consequently, we highlight the importance of carefully designed tasks, tools, and theoretical principles that make the underlying structure of the positional system visible. Thus, it seems powerful to work with the structure of the base system, as described in this study, in the first school year and even before the base ten is presented.

Acknowledgements

We express our sincere gratitude to the teachers at Sjöstadsskolan and Tjärnaskolan who participated in planning and testing of tasks in the research lessons. We would also like to thank all the participating students.

RELATIONAL QUANTITATIVE THINKING IN EARLY GRADES: STUDIES FROM QUEBEC

Elena Polotskaia, Annie Savard & Nathalie Silvia Anwandter Cuellar

INTRODUCTION

Research on word problem-solving in primary school mathematics raises the question of the structures of word problems and students' recognition and use of these structures (e.g., Vergnaud, 1982; Steffe & Johnson, 1971). Our studies in this domain draw on the notion of quantitative relational thinking proposed by many (e.g., Smith III &

Thompson, 2008; Davydov, 1982). More specifically, Davydov (1982) suggests that the idea of quantitative equality/equivalence, as well as the idea of quantitative relationships, is fundamental for the development of mathematical thinking in students. He stated that some of these quantitative relationships are prerequisites to the study of numbers in primary school. In our studies, we further develop Davydov's ideas to propose five basic quantitative relationships as cognitive tools promoting a structural view of quantities. We suggest that this type of relational thinking may facilitate problem-solving and is foundational for algebraic thinking. Following Davydov's idea, we suggested introducing quantitative relationships (Polotskaia et al., 2019, 2023; Polotskaia & Savard, 2018, 2020) into the primary school curriculum to complement and reinforce students' mathematical thinking development. The following presents a short overview of our studies in the subject for the past 15 years, aiming at understanding how young learners may develop their thinking about quantities and quantitative relationships to solve mathematical word problems, as well as at the development of teaching tools and practices allowing students to go beyond procedural knowledge of arithmetic operations and memorized number facts.

QUANTITATIVE RELATIONSHIPS

Our first research (Polotskaia, 2014) helped to suggest several hypotheses about the way the word problem-solving unfolds in primary school:

- the students pay more attention to numbers than to the logic of the story;
- the teachers do not have tools to attract students' attention to the story's logic and structure, represent and discuss this structure (and not the numbers involved);
- the curriculum (in Quebec) is not explicit about the study of structures in arithmetic.

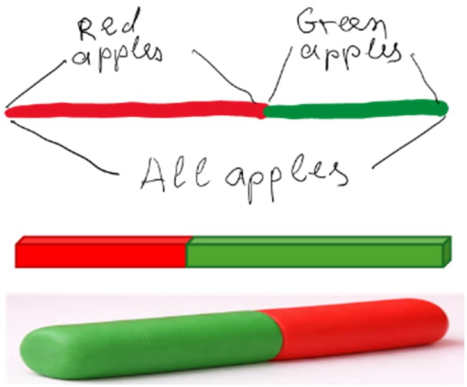
To address these issues, drawing on Davydov's theory of Developmental Instruction (2008), we hypothesized that introducing *quantitative relationships* and their graphical and material representations while solving word problems may enhance students' mathematical thinking and success. According to Davydov (1982), a relationship is defined as the reciprocal roles of three quantities. For example, a set of apples is composed of two sets: green apples and red apples. Thus, *Total* is the quantity of all apples, *Part1* is the quantity of green apples, and *Part2* is the quantity of red apples. *Total*, *Part1* and *Part2* are roles of quantities involved in this relationship. The three quantities obey the relationship: Total should be equal to the sum of Part1 and Part2, the difference between Total and Part2 should be equal to Part1, etc. For example, the values 20 apples, 5 apples and 7 apples are not in this relationship; they do not obey it, they are not valid to describe our situation with apples.

A quantitative relationship expresses the internal logic of a situation involving quantities. Yet a relationship is not an arithmetic operation. The relationship helps to determine which operation allows one to find an unknown quantity from two known

quantities (Davydov, 1982). For example, to find the number of green apples, one can use the total of apples and the quantity of red apples: *Total Apples – Red Apples=Green Apples*. The arithmetic operation, in this case, will be subtraction. To find the *Total of Apples*, one needs an addition operation: *Red Apples + Green Apples*.

To represent a quantitative relationship visually or materially, multiple tools can be used: schematic drawing, modelling clay, Cuisenaire rods, paper strips, etc. Figure 1 shows three of the possible presentations of the apples' situation.

Figure 1. The apples' situation relationship is represented schematically, with Cuisenaire rods and with modelling clay.



DEVELOPING TEACHING TOOLS AND PRACTICES

Employing research-design methodology, we conducted two studies: one about additive relationships and their experimental integration into the curriculum of Grade 2 of primary school (Polotskaia et al., 2019); and one about multiplicative relationships and their experimental integration into the grades 3 to 6 curriculum (Polotskaia et al., 2023). Together with school board consultants and teachers, we developed and tested multiple learning scenarios that directed learners' attention to the quantitative relationships more than to numerical values or calculations. The teachers and the students learned how to describe relationships orally, represent them visually or materially and how to use the representations to find a solution strategy.

Our data (Polotskaia & Savard, 2018) show that relational thinking positively affects young learners' approach to solving arithmetic word problems, making previously difficult problems more manageable. Yet, in all our studies, we observed a period of confusion and difficulty in students and teachers at the beginning of the introduction of relational thinking. This confusion, in our view, is due to the strong attitude young learners develop towards numerical information as the most important for school mathematics. The learner's attention is attracted to numbers and possible calculations, leaving behind the relationships and the logic of the situation itself, its mathematical structure. The transformation of this attitude, from numerical to relational and structural, is the key issue in our studies.

Our latest studies (Anwandter et al., 2024) were focused on possible algebraic thinking that can be fostered and observed in preschool learners (4-5 years old). We developed a set of game-like activities to let children play and experiment with weights, discuss and solve related problems. For example, in a game, a child needs to solve an equation formulated as an equality of weights of several blocks-animals (mouse 10g; cat 20g; dog 30g and pig 40g). The children do not know the weight in grams, but they can discover some relationships between weights (for example, 2 mice = 1 cat). Figure 2 shows that the weight of a mouse together with a cat is equal to the weight of an unknown animal. The child needs to figure out which of the four animals available in the game is hidden on the right side of the balance.

Figure 2. The equation tells that a mouse and a cat weigh the same as a hidden animal

The games were formulated so that the numbers did not play a major role. The maximum number of identical blocks used in a game was 4. Thus, even the children without a strong knowledge of numbers under 10 could participate. Our observations confirmed the young learners' use of equality structure and of elements of algebra-like reasoning.

CONCLUSION

We continue the research on relational and structural thinking in primary school. We try to understand the long-term effects of this learning on students' development of algebraic thinking and on their success in word problem-solving. We also try to evaluate the need for teacher training and curricular change. However, it is increasingly clear that learners are more likely to grasp a relational way of thinking in contexts where numbers are not explicitly involved. Thinking relationally and structurally is proven to be beneficial for learners. Our observations also suggest a hypothesis that young students may experience a high cognitive challenge where they need to operate numerical and structural information simultaneously. We also believe that an important teacher training is needed to support the integration of relational and structural thinking in primary school classrooms. Thus, more research is needed to develop teaching strategies adapted to young learners' needs and capacities.

EFFECTS OF INTERVENTIONS FOR LEARNING TO PERCEIVE AND USE STRUCTURES IN SETS

Priska Sprenger, Christiane Benz & Frank Reinhold

INTRODUCTION

Number structure can be represented by structures in sets of objects. These structures, or structured sets of objects form a foundational concept for part-whole relationships, and are crucial for arithmetic learning (e.g., Kullberg et al., 2024; Mulligan & Mitchelmore, 2013). The aim of this research is to investigate whether a play-based intervention focusing on structural use of discrete quantities up to ten in the final kindergarten year can enhance the ability to perceive the structure of a set and the use of this structure when determining cardinality. Often children remain in a "counting world" (Coles, 2017), relying primarily on counting-based strategies to determine cardinality. In contrast, a structural approach emphasizes children's awareness of part–whole relations and their ability to "see" numbers as composed of sub-sets. Our study adopts this structural perspective and investigates how explicitly structured visual arrangements up to ten can support young children in developing such structural awareness. However, the act of perceiving structures is not directly observable. By analysing the eye movements of children in a 10-dot array (egg-carton format), we aimed at getting insights about their perception of structures and their use of these structures to determine cardinality. By making structural perception processes visible, we seek to derive implications for how play-based activities in kindergarten can be designed to foster number structure.

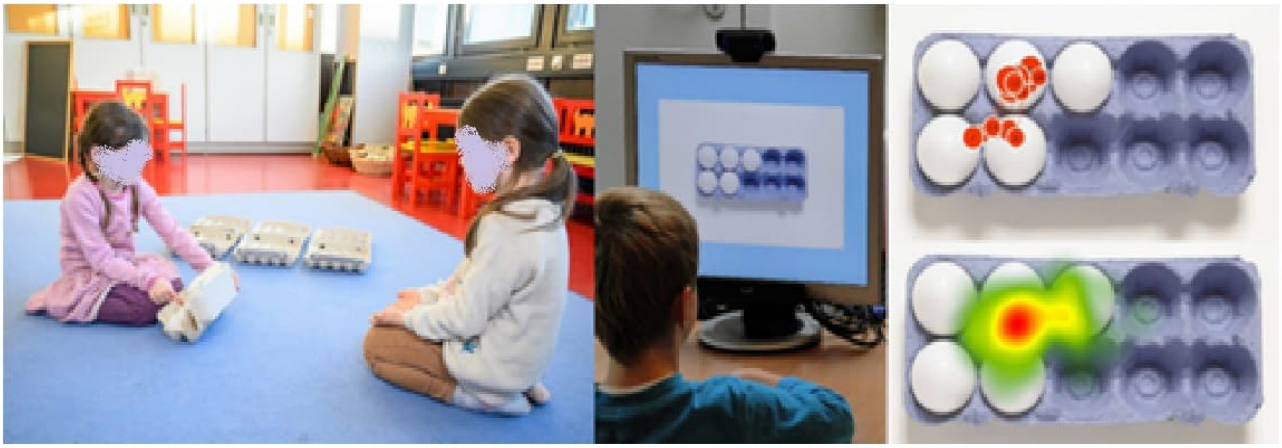
METHODOLOGY

To address this aim, a longitudinal study was conducted over the last year of kindergarten. 95 children (5 years old)—divided into treatment condition (n = 55) and control condition (n = 40) were interviewed three times (T0: pre-test, beginning of the year; T1: post-test, after a play-based intervention of 4 months; T2: follow-up test, end of the year). The parents' and children's consent were obtained. In individual qualitative interviews, children viewed photos of egg cartons on a screen and stated the number of eggs when they knew it (Figure 1). They were then asked to explain their reasoning. Six items containing either five, seven, or nine eggs were evaluated, and all interviews were videotaped. In order to describe the covert structuring processes and thus the children's approach as objectively as possible, three data sources were used: Video data (1) and eye-tracking data (2) until the child said the number (Figure 1) and video data (3) during explaining how the child came to the result. These data sources build the basis for a three-step-evaluation-method: A triangulation procedure leads to interpretative hypotheses about perception and determination processes. The three data sources are complementary and interdependent (Sprenger & Benz, 2020).

To operationalise children's structural perception and structural use in determining cardinality, we identified eye-movement patterns indicating structural perception, such

as pendulum movements between two subsets (eye-tracking data (2)), as well as typical gestures or verbal explanations. These were coded into categories such as "(no) structural perception" or "(no) structural use" (video data (1) and (3)). By triangulating these coded gaze patterns and (verbal) strategies, we were able to infer children's underlying structural processing. The intervention aimed to create a play and learning environment for children's active mathematical exploration (Figure 1). Kindergartens in the treatment group received materials to facilitate playful exploration and support structural perception and its utilization.

Figure 1: Part of the play and learning environment (left), child sitting in front of the screen during the interview (middle), examples of eye-tracking data (right)



To describe children's learning of structural perception and use of structure for determination we used Linear Mixed Models (LMM) to estimate the effect of the intervention. Task type, Time Point, and Intervention Condition, and the respective interaction terms were used as fixed effects in the full model. Random item intercept and random child intercept, as well as child random slopes for task type and time point were considered. For the overall intervention effect, we compared a model without the predictor intervention group to the full model. Tukey contrasts were utilized to test longitudinal effects.

RESULTS

We asked whether we can stimulate kindergarteners' perception of structures and determination with a structural use of discrete quantities up to ten through a play-based intervention. We expected a positive development for children in the treatment condition from pretest to posttest to follow-up, and no notable development in the control condition. Table 1 and Figure 2 show that this is true; a goodness of fit test showed a significant increase in model fit from the baseline model (which does not contain a parameter regarding the intervention condition) to the full model, $X^2(6) = 16.61$, $p = 0.011$. Since we neither found a significant interaction effect of task type and intervention group, nor a three-way-interaction of task type, time point, and intervention group, we considered the differences in perception and determination a

significant but global difficulty generating factor, *EMM* = 0.56, 95% CI [0.42, 0.69], not affecting the intervention nor the longitudinal nature of the data. Tukey contrasts are therefore averaged over both levels of task type. This analysis showed that the positive longitudinal development in the treatment condition (Figure 2) was significant, both from T0 to T1, *EMM* = 0.304, *SE* = 0.076, $p < 0.001$, and from T1 to T2, *EMM* = 0.234, *SE* = 0.056, $p < 0.001$, while in the control condition, students decreased non-significantly from T0 to T1, *EMM* = -0.038, *SE* = 0.089, $p = 0.998$, and increased non-significantly from T1 to T2, *EMM* = 0.157, *SE* = 0.065, $p = 0.150$.

Figure 2: Estimated Marginal Means describing the longitudinal effect in the development of perception and determination for both intervention conditions separately.

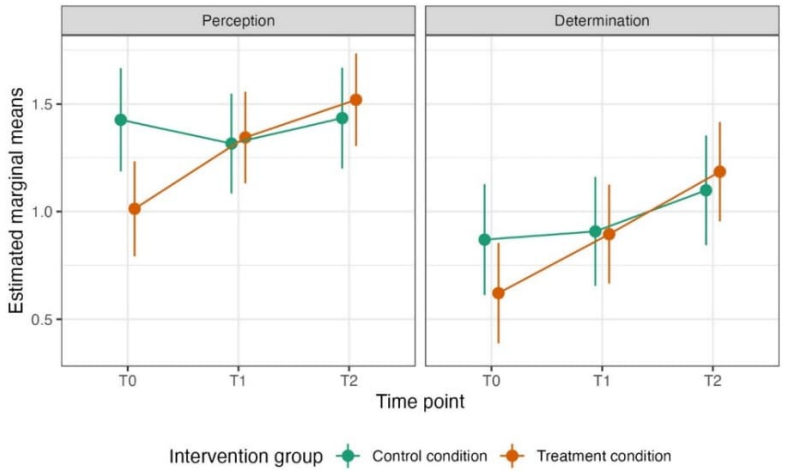


Table 1: Estimates from the Linear Mixed Model

Fixed effects	*EMM*	*SE*	*p*
Intercept	0.870	0.131	<.001
Type [Det. → Perc.]	0.557	0.067	<.001
Time [T0 → T1]	0.038	0.097	0.696
Time [T0 → T2]	0.229	0.098	0.020
Condition [Control → Treatment]	-0.249	0.141	0.077
Two-way interactions			
Type [Perc.] × Time [T1]	-0.148	0.089	0.098
Type [Perc.] × Time [T2]	-0.221	0.088	0.012
Type [Perc.] × Condition [Treatment]	-0.165	0.089	0.065
Time [T1] × Condition [Treatment]	0.236	0.128	0.065
Time [T2] × Condition [Treatment]	0.336	0.129	0.009
Three-way interactions			

Type [Perc.] × Time [T1] × Condition [Treatment]	0.206	0.118	0.082
Type [Perc.] × Time [T2] × Condition [Treatment]	0.163	0.116	0.161

Random effects	*Var*	*SD*	*PCV*
Children (*N* = 95)	0.385	0.621	54.7%
Type [Perc.]	0.019	0.139	90.0%
Time [T1]	0.235	0.484	18.2%
Time [T2]	0.243	0.493	40.9%
Items (*k* = 6)	0.035	0.187	0.0%

Note. Task type (factor, 0 = perception, 1 = determination), Time Point (factor, T0 = pretest, T1 = posttest, T2 = follow-up test), and Intervention Condition (factor, 0 = control condition, 1 = treatment condition)

DISCUSSION

The significant improvements in the treatment group, contrasted with the stable performance in the control group, indicate that the intervention effectively fostered children's perception and use of structures in sets. To demonstrate this intervention effect and to gain insight into children's structural perception, the use of eye-tracking proved indispensable (Sprenger & Benz, 2020). However, eye-tracking data are not self-explanatory, since eye movements only indicate where and for how long a person looks at a particular object. Therefore, the data must first be interpreted qualitatively. In this context, the three-step evaluation method, with eye-tracking data as one source within a data triangulation framework, proved to be an appropriate approach. At the same time, the high complexity of data collection and analysis implies that this method is not well suited for everyday educational practice when studying structural use.

For early mathematics education, our findings contribute three main insights. First, they provide longitudinal evidence that play-based activities explicitly designed around structured visual representations up to ten can significantly strengthen children's structural perception and use of structure. Second, they show that such structural competencies can be observed already in the final year of kindergarten, suggesting that basic support for number structure should not be postponed to primary school. Third, by visualizing the structural perception of five-year-old children using eye tracking in the context of data triangulation, the study offers design principles for teaching activities that encourage children to engage with part-whole relationships in everyday objects such as egg cartons.

The study has some limitations. Inferences about children's perception are based on interpreted eye-tracking data that, although supported by video and verbal data, may introduce subjectivity into the identified gaze patterns and statistical analyses. In addition, the sample consists of 95 five-year-old children from German kindergartens working with a specific egg-carton task format, which restricts the generalizability of

the findings to other age groups, cultural contexts, and representations of numerical structures. By articulating both the potential and the limitations of this triangulated eye-tracking approach, the study adds to the broader discussion in early arithmetic research on how number structure can be made visible and teachable in early years settings.

CONCLUDING REMARKS

Through all the presentations, we can conclude that the object of interest is the same – a structural awareness of numbers as a foundation for teaching and learning – but there are differences in what and how it is studied. The presentations can broadly be seen in two main contributing fields, which do not exclude the other but are emphasized in the presentations above: focusing on methods and results and on theories of learning.

FOCUSING THEORIES OF LEARNING

One common feature of the research presented above is the attention to mathematical content and how this content is handled in teaching and learning situations. Several of the studies adopt variation theory of learning as a theoretical framework (see, e.g., Kullberg et al., Venkat et al. and Mathews above), which lends well to this kind of research as this theory conjectures that learning is always the learning of 'something' and the learner's way of experiencing a (mathematical) phenomenon is both the outset for teaching and the outcome, because learning is described as changes in one's way of experiencing that phenomenon. What to teach then becomes a function of what aspect students need to discern; that is, learning (and teaching) is always connected to what is intended to learn. A similar connection between content and how learning commences is found in the Davydov curriculum adopted by, for example, Eriksson et al. and Polotskaia et al. (see above), where, in particular, teaching is directed at the internal logic of a situation involving quantities. The theoretical conjectures direct the designs of the teaching, aiming to make the learner discern connections and principles that advance the learners' understanding and skills in solving arithmetic problems.

Theory embedded in intervention studies, often conducted in close collaboration between teachers and researchers in both design and conduct, ensures ecological validity and, thus, theories of learning that are relevant to teaching practices. Without a sound theoretical basis, our understanding of how particular instructional designs may support students' ways of experiencing numbers and quantitative relationships will be restricted.

FOCUSING METHODS AND RESULTS

The methods used in studies within the field of the structural approach to early number learning are significant because of the need to justify what theories conjecture to be suitable ways to learn and teach. A theory that cannot be implemented in teaching practice or that does not lead to the expected learning outcomes bears little significance in this strand of research, which is very closely connected to and aims to contribute to

the development of educational practice. A good example is the artifacts produced and utilized within research-informed intervention studies, where, for example, picture books (see Mathews above), egg cartons (see Sprenger et al. above), bead-strings (see Kullberg et al. above), or iconic representations (see Polotskaia et al. above) embed theoretical principles as facilitating means for learners to discern the properties of numbers or quantitative relationships.

A common feature of the research methodologies is the starting point from the learner's perspective, aiming to capture how numbers and their properties appear to them. The methods chosen for capturing this are delicate because they involve young children, and data are often generated in authentic teaching situations where the border between research and education becomes blurred. We can see in the research studies that qualitative methods are commonly used, such as interviews and observations. Eriksson et al. (see above) took a phenomenographic approach to studying young learners' experiences of numbers. Such findings show distinct categories that reveal differences in how learners perceive numbers. An interesting methodology that complements this is eye-tracking (see Sprenger et al., above), which adds the possibility of capturing intuitive and nonverbal perceptions. When learners are very young, as in Sprenger et al.'s study, this methodology contributes to the observations of their actions and utterances and reveals their directed attention interpreted as structural awareness.

From the different methods applied in the presented studies, all of which aimed to contribute to developing mathematics teaching, it must be pointed out that not all methods apply to teachers' educational work but are rather complementary. While eye-tracking is impractical for teachers' daily work, interviews can help teachers understand children's ways of experiencing numbers in tasks, whether it is based on counting or structural awareness. Observations also reveal information about young students' knowledge of number structure and skills to make use of, for example, quantitative relationships, which are significant to capture both for research inquiries and teaching design. An observer's theoretical perspective will, however, inform what conclusions are drawn from a learner's encounter with numerical tasks, which is why research and teaching need to make explicit the grounding for their interpretations. Mathews (see above) pointed out this in studies with preservice teachers, where it was shown that knowledge of structural awareness and the relation to arithmetic development among children cannot be taken for granted. This is a motive for highlighting different theories and related methodologies that inform studies of arithmetic teaching and learning as a contribution to early mathematics education, teacher training, and the research community.

One thing that stands out in the group of researchers included in the Research Forum is that the results from the studies of early arithmetic learning and teaching highlight learners' perspectives and qualitatively different ways of understanding and making use of mathematical relationships. A consequence that we see is a close connection to

teaching practice and how teaching can be designed to support a learner in advancing his or her understanding of numbers as a structured system of quantitative relations.

IN SUM

Altogether, this compilation of papers included in the Research Forum "Studying children's learning of number structure – findings and challenges" contributes to a broadened understanding of research on the teaching and learning of number structures. The frameworks represented by the invited research groups are considered complementary and, therefore, necessary to discuss in a forum where the particularity and the generality may be brought to the fore and, thus, contributing to further developments and initiatives.

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WHAT IS MATHEMATICS-SPECIFIC WHEN WE BUILD KNOWLEDGE IN MATHEMATICS EDUCATION RESEARCH?

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Besides their role in practical problem solving, research fields aspire to advance human understanding by building knowledge. Mathematics education research often asserts its special contribution to human understanding on the grounds of its focus on mathematics in various education practices. This mathematical sensibility in the mathematics education research gaze can enable unique contributions to the larger field of educational research and the interdisciplinary table of educational policy-making. What exactly are these mathematical specificities in our contributions to knowledge building? This research forum explores various ways in which mathematics shapes the mathematics education research gaze, contributing to the clarification and development of knowledge building in our field.

A NEEDED REFLECTION ON THE NATURE OF OUR GAZE

Patricio Herbst, Felix Lensing, and Nathan Lombard

Our field's production of knowledge is aided by a mathematics education research (MER) gaze, with which we interrogate the practices of mathematics education. The term *gaze* stems from social research, where scholars have used it to refer to the specialized ways in which disciplines construct and interact with their objects of study (e.g., Foucault, 2000). Indeed, different disciplines may focus on the same happenings in the world but do so through different gazes and as a result aspire to produce different kinds of transposable knowledge. By transposable knowledge we mean knowledge claims about concepts or constructs which are instantiated by the observations about facts associated with the sample or locale involved in the research. This research forum proposes a reflection on the nature of our field's gaze and urges a commitment to research that further develops this gaze. We suggest that such reflection and commitment are essential for establishing the *autonomy* of our research field through the distinctness of our transposable knowledge, and for warranting a seat at the table in discussions of educational policy and research funding: Our field does not merely consist of the aggregate of scholars whose empirical work takes them to places where people teach or learn mathematics; our field aspires to look at practice in those places with a gaze that allows it to distinguish its claims to knowledge from those that other

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fields can make. With this research forum, we aspire to get a conversation started about the nature of our knowledge-building enterprise and about the need for research that engages in knowledge-building (in addition to research that engages in problem-solving).

An aspect that has been historically constitutive of our field's gaze is our attention to mathematics in the practices of mathematics education. And yet, in recent years some have decried that this attention is not present in some of the research our community has done in the last couple of decades (e.g., Heid, 2010). When the core of our field's research was on students' cognition as they solved particular problems (e.g., Carpenter & Moser, 1982/2020), it might have been obvious how mathematics shaped our gaze and was present in the transposable knowledge our field produced (e.g., the research on those students' solving those problems informed about different uses of addition and subtraction). Rather than assuming that mathematics is vanishing from our gaze or that the way for mathematics to be present in our gaze requires naming topics recognizable to mathematicians, we offer reflections on the question: *How does mathematics shape the MER gaze and how does mathematics show up in the transposable knowledge our field produces?*

WHY DISCIPLINARY SELF-REFLECTION MATTERS TO MATHEMATICS EDUCATION RESEARCH

Those questions continue a reflective tradition on the nature of research in our field (e.g., Biehler et al., 1994; Freudenthal, 1981; Sierpinska & Kilpatrick, 1998; Wittman, 1995). This questioning is still needed for at least two reasons. First, the historicity of both our objects of study and MER itself makes it necessary to repeatedly pose and revisit questions of disciplinary identity as the field continues to evolve. Second, throughout its history, the autonomy of MER—its recognized capacity to identify its objects of study, using its own theories and addressing them with its own methods—has been challenged on both epistemological and sociopolitical grounds. Like any other social science, our autonomy is constantly challenged by societal developments and public opinion, both of which impose upon us many of our research objects and questions (e.g., AI in the classroom?). The meaning of mathematics implied in a research practice affected by such endogenous and exogenous change vectors must be examined.

Two historical characteristics, however, have to some extent reduced the pressure to engage in such reflection. First, in its early history, MER emerged in connection with curriculum development and the study of students' thinking (Suppes, 1969). This led to a tendency to define our relationship to mathematics by drawing on concepts from the discipline and how mathematics is institutionalized in schools. And yet, identifying MER as focused on everything that happens in spaces institutionally designated for mathematics education or on practices that deal with topics sanctioned by the discipline of mathematics leaves outside our field studies of mathematics in cultures, the

workplace, or the playground which have been important in our field, even to better understand the mathematics done in classrooms (e.g., Carraher et al., 1987). Depending too much on the presence of recognizable mathematical topics or on the institutional designation of spaces as mathematical may not be of sufficient assistance for us to recognize mathematical activity in the teaching or learning of other subjects, for example. These latter foci for research suggest the need for critical reflection on how mathematics shapes our research gaze beyond institutional labels or disciplinary references.

Now, if the development of a more reflective discourse of our relationship with mathematics is meant to support our autonomy as a research field, and if this autonomy involves taking stances about what practices we study, what theories we build and use, what benchmarks we use for the application of designs and methods, and what constitutes research results in our field, we ought to consider explicitly what goes into developing such stances. *Ontological* and *epistemological* considerations are surely needed, as we need to reflect on the nature of our objects of study and on how researchers interact with them. But because the possibility of studying those objects is realized by scholars in particular social and institutional positions which afford them differential access to those objects of study, onto-epistemological considerations can at best represent too ideal a type. For a research community populated by scholars coming from many prior disciplinary experiences and locations across the world, *sociopolitical* considerations are also needed. The reflection also needs to take stock of where the community presently is, how its scientific practice enacts or produces ways of knowing within its ways of being, as our scientific practices are bounded by political and economic conditions that include access to resources with which research is carried out—including grants, institutional positions, libraries, equipment, regulations, access to data, educational policies (e.g., access, curriculum, testing), and the very role and status of social research in each culture.

Our examination of how mathematics is present in research in our field thus needs to combine the ambition of improvement with an appreciation of what we are able to do with what we have. We need to thrust our "onto-epistemological foot" ahead, directed on a reasonable vision for progress that may afford us greater autonomy, while keeping a "sociopolitical foot" behind, grounded in the real conditions for the inquiry that we have to sustain and build upon.

SEARCHING FOR THE MATHEMATICAL SPECIFICITY OF THE MATHEMATICS EDUCATION RESEARCH GAZE

How are we to inquire into the mathematical specificity of the mathematics education research gaze? If the question is being raised in earnest, we ought to have no holds barred. We cannot, for example, presume that the practice of mathematicians or the disciplinary knowledge they organize necessarily provides the primary meaning to mathematics in our search for specificity. That may be one possibility—but it is not a

necessity. Likewise, the expectation that the question "What is mathematics?" must be settled before we can investigate the mathematical specificity of mathematics education research should be recognized as a possibility rather than a necessity.

What does seem ineluctable, however, is that the world in which we live includes mathematics in multiple forms: mathematics as a community of knowledge producers and a body of scholarly knowledge; mathematics as a library science classification; mathematics as the name of a school subject; mathematics as an acknowledged tool of professional practices of different sorts; mathematics as a component of public literacy; mathematics as a set of developmental skills; and perhaps more. The practices of teachers, students, teacher educators, curriculum developers, and policymakers often concern one or more of those real-world phenomena. And yet, how we ought to think of these realities also offers possibilities rather than necessities.

We could, for example, follow a structuralist impulse and interpret that set of practices as nestings around a core that provides a primary sense of what's mathematical. Thus, if the status of mathematics in library science is taken as evidence of mathematics as an organized body of knowledge, a first level of nestedness might be mapped onto the practices of humans engaged with that body of knowledge, as in writing mathematics textbooks. And the question could be posed as to how research on textbook writing (or on the textbooks written) maintains attention to mathematics in the claims it lays on transposable knowledge. And, of course, nestings of higher level could be recognized; for example, students studying those textbooks, teachers teaching from those textbooks, or teacher educators teaching prospective teachers to use those textbooks; in all of those nestings, the question could be asked of how research on such activities maintains attention to mathematics in the claims it lays on transposable knowledge. Here, each level of nestedness becomes a potential site for examining how mathematics remains present—and how it is transformed—within the MER gaze.

But we could also resist that structuralist impulse and instead concentrate on the mathematics as it is experienced by students, teachers, or teacher educators in institutions, and investigate how they arrive at meaningful accounts of what they experience as mathematics. We could find in such investigations some grounds for the legitimacy of associations that may be less available in the former approach–associations of mathematics with pleasure or pain, with social sortings or opportunities, for example. We should remain open to constructing a sense of mathematical specificity of our gaze that attends to how people's experiences of mathematics differ from other experiences in regard to what those experiences make people feel and become. And we should wonder whether a synthesis exists that could accommodate different possibilities to inquire into the mathematical specificity of our research under the common purpose of maintaining and deepening the autonomy of our field.

This is particularly important as we think of our field's capacity to sharpen its gaze: How do we put an onto-epistemological foot forward while holding strong to the

sociopolitical foot back? Even the most basic orienting principles—what counts as an ontology, or the meanings and methods of epistemology—are heavily contested among scholars. The resulting diversity can produce a cacophony that might appeal to certain postmodern theorists, but that offers little guidance for flesh and blood researchers who must navigate real-world challenges, arguing for resources, positions, and the value of their work before both the public and other academic fields with greater stature. If we are to act constructively, we must ask whether we can move beyond entrenched fiefs and inherited traditions toward a vision of the field that is both open to multiple approaches and in control of the mechanisms necessary to safeguard its autonomy.

To start to tackle this self-reflective task of unfolding how mathematics can be present in diverse ways within the MER gaze, we now develop analyses of a series of examples from various sectors of MER.

MATHEMATICS AND EDUCATION GAZES MEET IN MATHEMATICS EDUCATION RESEARCH

Andrew Izsák

This forum on mathematics education research (MER) focuses on gaze, or the specialized ways in which disciplines construct and interact with the objects they study. Whether MER has a characteristic gaze raises as a question whether it is a distinct discipline. What is and what is not MER can be contested when researchers view the importance of mathematics and education in their work differently. More fundamentally, what is and what is not MER can be contested when researchers differ in what they recognize as mathematics or as education. How we think about mathematics, education, and basic research can inform how we think about a gaze that is characteristic of an MER discipline that is distinct yet, at the same time, in conversation with mathematics and education.

The adjective *basic* has multiple, related meanings. One meaning connotes something that is introductory, an entry point, and possibly perceived as straightforward. Examples would be a basic course in algebra or a basic course in theories of learning. A second meaning connotes research that does not have immediate practical applications. As an example, Schoenfeld et al. (1993) reported a case of one student's learning that was characterized by complex, multi-level coordination of knowledge and, therefore, revealed limitations of learning theories that described more straightforward accumulation of new knowledge on top of prior knowledge. A third meaning connotes research that (re)examines foundations upon which a field is built. An example from mathematics is work on the structures and logic that underpin the discipline. In what follows, I discuss two examples of "basic" MER in this third sense. In both examples, the objects of study, and therefore the gaze, make contributions that are both mathematical and educational.

The first example comes from Ball and colleagues' work on mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008). These researchers have pursued a complex research program that includes elements directed at teacher preparation policies as well as ones that have uncovered mathematics with which mathematicians are less familiar. In particular, they have refined Shulman's (1986) knowledge categories in order to better understand the mathematical demands of teaching. One part of this multi-faceted research program has involved positing a category of mathematical knowledge employed in teaching but not in other professions, a category they have termed *specialized content knowledge* (SCK). One example of SCK is knowledge of different meanings for division that can be used to design tasks intended to support students' understandings of division with fractions. A related example is knowledge of ways that whole-number factor-product combinations can be leveraged to partition quantities when solving problems that can be modeled with fraction arithmetic (e.g., Izsák et al., 2018). Research that characterizes and maps this specialized mathematical knowledge has been a form of basic research uncovering mathematics with which mathematicians are less familiar, but that is critical for reform-oriented teaching.

The second example comes from research that Sybilla Beckmann and I have conducted to construct a coherent approach to multiplication topics across elementary, secondary, and tertiary grades (e.g., Izsák & Beckmann, 2019). Decades of research exists on topics related to multiplication, including multiplication and division with whole numbers and with fractions, ratios and proportions, rates of change, and so on. At the same time, we observed that oftentimes the field has seemed satisfied with multiple meanings for multiplication, each used in particular circumstances. For example, the oft criticized repeated addition meaning works with whole numbers, while the part-of-a-part meaning works with fractions. At the same time, the field has given less attention to ways in which these different meanings might be connected and seen as different manifestations of the same thing. From our point of view, this omission has consequences for how we might design opportunities for students to use prior knowledge of multiplication when studying new topics across the grades.

In order to develop a more coherent approach to a central swath of school mathematics, we have developed a meaning for multiplication based on coordinated measurement with two different units (e.g., Izsák & Beckmann, 2019). As an example, one can think of a journey measured simultaneously in kilometers and in hours. In the multiplication statement, $N \bullet M = P$, P is the measure of the journey in kilometers, M is the measure of the journey in hours, and N relates measures in the two units, kilometers and hours. On the face of it, this observation might seem innocuous, but it turns out that applying this perspective consistently across situations has led us to critically reexamine widely accepted MER results and to engage in new, at least for us, mathematical thinking.

One example is Vergnaud's (1983) well-known distinction between isomorphism-of-measures and product-of-measures. In his analysis, *speed* • *time* = *distance* is an

example of the former and *length • width = rectangular area* is an example of the later. The coordinated measurement point of view supports a common analysis of the two categories, thereby dissolving the distinction, but this requires reinterpreting *length • width* as *number of square units in 1 column • the number of columns*.

A second example is the recognition that the two different meanings for division—partitive or sharing division versus quotitive or measurement division—leads to two different perspectives on proportional relations. Beckmann and Izsák (2015) referred to these as the multiple-batches and the variable-parts perspectives. Although the multiple-batches perspective has dominated past research on proportional relationships and rates of change, articulating the two perspectives and comparing them has helped us understand important limitations of developing rates of change using multiple batches as the foundational perspective on proportional relationships, especially when laying the ground for thinking about instantaneous rates of change and derivatives in calculus (e.g., Izsák & Beckmann, 2026).

A third example is the development of new ways to connect core calculus results to topics from secondary grades. As an example, if one uses coordinated measurement to span Vergnaud's (1983) categories of isomorphism-of-measures and product-of-measures, then one has an explanation for why rectangular areas can be used to approximate accumulation of a wide range of quantities (e.g., distance, mass, and work). Moreover, Izsák (2025) reported how integration-by-substitution can be justified by thinking about inversely proportional relationships.

These three examples illustrate ways in which we have simultaneously adopted a critical stance towards established MER results and engaged in what, for us, has been new mathematical discovery. Throughout, our work has been shaped by values and practices of the mathematics discipline that include reasoning from definitions, seeking common structure across situations, and reasoning logically.

TRANSPOSABLE KNOWLEDGE FOR EMBODIED GEOMETRY EDUCATION

Dor Abrahamson and Mitchell J. Nathan

We are mathematics-education researchers whose field of inquiry surfaces the other-than-symbolic modalities of mathematical practice, such as gesture, imagination, prosody, material construction, and intersubjectivity. This multimodal gaze on mathematical practice, which characterizes our work, resonates with a recent philosophical turn in the cognitive sciences loosely referred to as *embodiment*, or sometimes as *4E* (i.e., the notion that the mind is embodied, embedded, extended, and enactive; Newen et al., 2018). We believe that implicating the embodied and enactive nature of mathematical reasoning, while constituting a theoretical lens motivating empirical studies, bears fundamental implications for practice, too. For if these theories

hold water—and expert mathematician testimonies support these claims theories (e.g., Hadamard, 1945; Tao, 2016)—then we might wonder how we could create conditions for students to partake in multimodal mathematical practices. This embodied mathematical gaze could inform what tasks educators engage students in, what guidance we offer them, what epistemic climate (Feucht, 2010) we nurture in the classrooms, and how we assess learning.

Didactic Transposition theory (Chevallard, 1989) recognizes that disciplinary knowledge is organized primarily to be put to use and must undergo considerable transformation to be in a form meant to be taught and learned, as it is for classroom lessons and educational technology. In the following, we briefly overview two independent lines of inquiry focused on the mathematical branch of geometry—one pertaining to two-column proof, one pertaining to model construction—that each foreground the function of other-than-textual sensory modalities, semiotic registers, and social behavior that, we claim, are inherent to and constitutive of expert mathematical practice which is embodied and enactive. Our aims, through these project case studies, are to: (1) embellish the mathematicians' gaze as more than visual and, as such, to offer authentic empirical mirrors to that multimodal gaze; and (2) describe unique insights the embodied–enactivist frame offers into the generation of transposable knowledge.

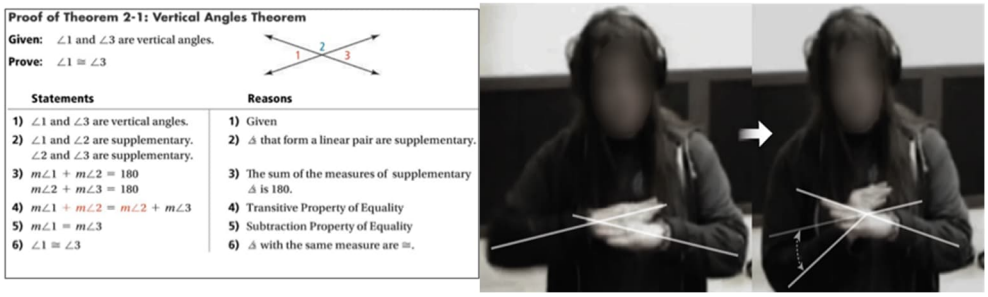
CASE1: GEOMETRY PROOF

Proof is a central disciplinary practice among mathematicians (Lakatos, 2015). When designing educational experiences for geometry proof practices, transposable knowledge starts with the mathematical gaze as it focuses on the nature of deductive proof (Figure 1). Geometry proof education is often out of reach for high school students and their teachers (e.g., Chazan, 1993; Herbst, 2002). Nathan and Walkington (2017) proposed the grounded and embodied mathematical cognition framework as an alternative embodied educators' gaze directed at students' verbal and nonverbal behaviors when producing transformational proofs. This includes surfacing students' embodied simulations of operations performed on imagined objects. This case of the embodied and multimodal nature of a high school student's transformational proof (Figure 1) demonstrates how gestures and operational speech enacting transformations on imagined geometric objects perform embodied simulation in support of insightful geometric reasoning. The student reasons:

> "The statement is true. Because the lines must always [Flat hands cross to make intersecting lines] be lines like they can. [Now crossed arms make lines] ((long pause)) Assuming the lines are straight, and not-, and we're operating on the plane, they the lines are [Now hands make lines that are uncrossed, then crossed] … To draw a straight line from any point to any point. [Switch hands] If one angle, one of the opposite angles was different than the other opposite angle, then the lines would not be straight, [Top (right) hand is now bent at palm] they would be, there would be an angle at the intersecting point in one of the lines."

The resulting embodied pedagogy has been used successfully in multiple laboratory and classroom interventions that lead to enhanced geometry proof performance (Nathan et al., 2022). Such findings provide a basis for didactic transposition from formal mathematics to teachable, learnable embodied geometry.

Figure 1: (Left) Mathematician's gaze of geometry proof based on axioms, ungrounded symbols, highly regulated diagrams, and a logical chain of deduction made through declarative steps of static states that hides processes and applies independently of any physical drawing or object. (*Right)* Educator's gaze of embodied simulations via body movements and operational speech.



Proof of Theorem 2-1: Vertical Angles Theorem

Given: $\angle 1$ and $\angle 3$ are vertical angles.

Prove: $\angle 1 \cong \angle 3$

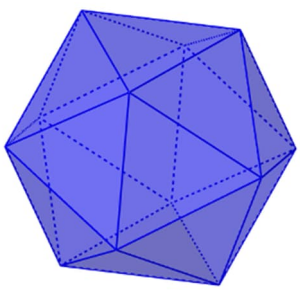
Statements	Reasons
1) $\angle 1$ and $\angle 3$ are vertical angles.	1) Given
2) $\angle 1$ and $\angle 2$ are supplementary. $\angle 2$ and $\angle 3$ are supplementary.	2) $\triangle$ that form a linear pair are supplementary.
3) $m\angle 1 + m\angle 2 = 180$ $m\angle 2 + m\angle 3 = 180$	3) The sum of the measures of supplementary $\triangle$ is 180.
4) $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$	4) Transitive Property of Equality
5) $m\angle 1 = m\angle 3$	5) Subtraction Property of Equality
6) $\angle 1 \cong \angle 3$	6) $\triangle$ with the same measure are $\cong$.

When researchers and teachers neglect the role of gestures and operational speech, they risk presenting only the symbolic, ritualized form of the proof (compatible with the mathematician's gaze), without teaching the dynamic, embodied meaning (the core transposable insight) that makes the proof logical, operational, and generalizable (Harel & Sowder, 2007) as well as convincing to the student. In this research, the mathematical specificity is found in the way bodily action is the transformed mathematical object. The specific challenge of transposing geometry proof is not about making it concrete, but about making the abstract transformations — which are the true mathematical specificities of the content—experiential through the body.

CASE2: OBJECT CONSTRUCTION

Geometry is traditionally taught through textbooks, in which standard shapes and forms are depicted along with formal definitions of their features. 3D geometric forms, in particular, such as a pyramid, present enduring challenges for students (Fujita et al., 2020). For example, students find it challenging to understand how three right pyramids compose a cube, a geometric fact supporting the formula that specifies the volume of a pyramid as one third of its containing cube. In a sense, students cannot perceive the voluminous forms as depicted in flat images let alone scrutinize angular relations between their constituent linear elements or animate complex transformations of these forms.

Figure 2: (Left) Mathematician's gaze of a geometric structure, the Platonic Solid icosahedron (= "twenty-faced"). In geometry, a notation of form {p, q}, named the Schläfli symbol, signifies and provides a uniform method to classify a regular polyhedron with p-gonal faces and q edges at each vertex (Coxeter, 1973). In particular, a regular icosahedron (20 planes, 12 vertices, and 30 edges) features five triangles around each vertex, thus, its Schläfli symbol is {3, 5}. (Right) Educator's gaze on a collaborative construction of a body-scale icosahedron. Participants receive a stack of dowels (edges), a pile of rubber connectors (vertices), and minimal information about the target structure. We monitor the emergence of understanding as situated discourse.



Educational responses to students' sense-making in geometry have varied with scholars' epistemological commitments and historical media. Catering to young students, Froebel (1895) introduced small wood models of 3D-geometric shapes as part of the "gifts" comprising his pedagogical regimen. For adult experts, Monge and Klein similarly sought to render 3D shapes more accessible through engaging with concrete models (Mueller, 1998). Currently, various interactive digital media enable students to inspect 3D shapes by constructing and/or transforming them (e.g., 3D GeoGebra). Mechatronic solutions for blind and visually impaired students offer accessible haptic-tactile inquiry matched with spoken explanation (e.g., the Quad; Lambert et al., 2022). Yet another approach involves the collaborative construction of body-scale Platonic solids, such as an icosahedron (see Figure 2).

Palatnik and Abrahamson (in press) combined theoretical constructs from ecological dynamics and multimodal interaction to interpret the process by which four participants built a body scale icosahedron. The authors thus looked to model individual development of geometric concepts as grounded in the enactive know-how they construct piecemeal through iteratively figuring out a common discourse for planning, executing, and evaluating the coordinated joint actions required for constructing the material structure. If you will, what you build together is what you know. Once the study participants had completed the giant icosahedron, a follow up standard quiz—such as, How many vertices does this shape have? How many edges? How many faces? How many parallel edges? and so on—revealed that the participants drew on their collective enactive know-how to articulate a disciplinary know-what. That is, the phenomenology of mathematical knowing is not about objects—it is itself a form of

re-engaging the objects properly. The study thus offered a pedagogical gaze twin to the mathematical gaze. Future work will examine the durability and generativity of students' geometric know-how and know-that. Perhaps non-trivially, a decade of research on learning through building body-scale geometric models has persistently evidenced students and teachers' joy (Rosenbaum et al., 2024).

CONCLUSIONS

Embodied-enactive perspectives invite a broadening of the mathematical gaze by acknowledging the mathematical contributions of nonverbal behaviors. Students' embodied simulations of operations as exhibited by their dynamic depictive gestures and operative speech (Case 1) and enactive–discursive know-how developed through multimodally coordinating co-operative action (Case 2) reveal the invariant and variant properties of mathematical objects to support claims about universal properties of shape and space. We thus contribute a view of the mathematical gaze, along with its distinctive semiotic registers, as developmentally grounded in sensorimotor engagement with real and imaginary objects.

TRACING THE MATHEMATICAL GAZE IN TEXTBOOK STUDIES IN MATHEMATICS EDUCATION

Vilma Mesa and David Kollosche

Textbook studies are an interesting field to study the impact of the mathematical perspective: On the one hand, textbooks are mostly subject-specific, although there might be some subject-crossing textbooks on specific topics, for specific agendas (e.g., integrated STEM education for first-graders). A database search of "analysis of school textbooks" lists almost exclusively subject-specific studies. Indeed, the analysis of school textbooks might require subject-specific tools to analyze subject-specific knowledge structures and learning trajectories. On the other hand, the overall structure of textbooks is to some extent comparable across subjects, which suggests that at least some theoretical framings and methodological approaches are transferable between subjects and might constitute a part of a subject-independent field of textbook analysis.

We will show that textbook studies in mathematics education have resulted in knowledge building that is subject-independent as well as in knowledge building that is subject-specific, presenting first two studies that investigate the overall makeup of mathematics textbooks, and next a study that zooms into the analysis of specific aspects of mathematics textbooks using a conceptualization specific to mathematics education.

REZAT'S (2009) STUDY OF STUDENTS' TEXTBOOK USE

In his doctoral dissertation, parts of which have been published in English as well (e.g., Rezat, 2006), Rezat (2009) approached the question "how students use the mathematics textbook to learn mathematics" (p. 12, translation by DK). Rezat decided to use

grounded theory to approach this question but found that its coding paradigm was too general for the question at hand. Therefore, he exchanged it by approaches that build on Vygotsky's concept of instrumental acts, which help to analyze how individuals interact with artefacts such as textbooks. In order to describe the affordances of mathematics textbooks, Rezat further distinguished the following structural levels: delimitation of textbooks by school year, mathematical field or otherwise; structure of textbook chapters; and structure of small coherent sections. For each of these structural levels, Rezat (2009) conducted a qualitative content analysis (Mayring, 2021), which resulted in the identification of types of structural element such as introductory exercises, introductions, explanations, and sample exercises within the structure of small coherent sections. These findings allowed him to compare the structures of different schoolbooks and to associate student use with textbook structure.

Thus prepared, for two to three weeks, Rezat (2009) had secondary school students mark everything they looked at in their textbook and take notes on what they did with the text in every instance. Additionally, he observed the mathematics lessons of the students participating in the study and interviewed the students about their textbook use. He found, among other things, that students use their textbook to do their homework, to exercise further, to learn contents out of sheer curiosity.

Tracing the mathematical gaze in Rezat's (2009) work, it can be argued that the theoretical framework employed, the research methods used, and even the concepts identified such as *types of structural elements* and *student use cases* are not specific to mathematics but could be applied to many other school subjects. Still, Rezat's insights are valuable for our understanding of the structure and use of mathematics textbooks. It is noteworthy that Rezat's references concerning textbook analysis nearly all originate in mathematics education research and that Rezat has not published his findings in a publication medium beyond mathematics education.

DOWLING'S (1998) SOCIOLOGICAL PERSPECTIVE ON TEXTBOOKS

Dowling's (1998) textbook analysis pursues knowledge-building that is specific to mathematics. Dowling aims at developing a sociological "language of description […] designed to enable the analysis of empirical data" (p. 1, without original emphasis). His socio-linguistic perspective supplies him with concepts, which he develops further with a mathematical gaze: For example, alienating common assumptions as socially shared myths, Dowling begins his book by discussing mathematical myths such as the myth that "mathematics justifies its existence on the school curriculum by virtue of its utility in optimizing the mundane activities of its students" (p. 9). To give another example, he distinguishes texts by the degree in which their expressions and contents are specialized (p. 135). This is especially important for academic mathematics, where both contents and expression are highly specialized, while learning arrangements might resort to using preliminary ideas about concepts and everyday language for their description. His theory building allows Dowling to show how English middle school

and grammar school textbooks reinforce different myths about mathematics and provide text that allow for different degrees of understanding of academic mathematics.

Dowling's (1998) work is an interesting example for a reflection on how a mathematical gaze can influence the development of mathematics-specific theory and research methods: Although the theoretical and methodological tools are taken from subject-independent disciplines, Dowling's careful interrogation of the discourses in and about mathematics lead his choice of tools and allow him to develop mathematics-specific methods of analysis (such as the analysis of the specialization of mathematical texts) or to arrive at mathematics-specific insights that add to theory building in mathematics education (such as the myths he identified).

ANALYSIS OF MATHEMATICS TEXTBOOKS TO IDENTIFY POTENTIAL CONCEPTIONS

The following studies pertain to the application of a conceptualization with clear roots in mathematics education, Balacheff's cK¢ model of conceptions (Balacheff & Gaudin, 2010), to understand how mathematical notions as presented in textbooks can lead to the multiple conceptions of those notions. Balacheff's cK¢ model of conceptions attends to *knowings* (from the French *connaissance*, or, a learner's personal constructs of mathematical notions) and differentiates it from *knowledge* (from the French *savoir*, or the intellectual constructs recognized by the community). In an instructional situation, the subject's cognitive dimension interacts with aspects of the task that bring into play the knowledge at stake, in action-feedback loops with a *milieu* that end once the task has been completed. Balacheff models conception as a quadruplet that includes a problem (in a class of problems or "disequilibria" that call for mathematical notions needed to reach equilibrium, p. 190), the semiotic representations, or the "linguistic, graphical or symbolic means which support the interaction between the subject and the *milieu*" (p. 190), the operations or tools for "actions on the *milieu*" needed to simulate what the learner does to tackle the problems and "to transform and manipulate linguistic, symbolic or graphical representations" (p. 190), and the control structures, the "components supporting the monitoring of the equilibrium of the (student-*milieu*) system" (p. 190) or the strategies (e.g., metacognitive processes, use of definitions, checking existing answers) that allow learners to decide whether they had solved the problem and whether they had done so correctly. The cK¢ model gives a process for characterizing conceptions grounded in the analysis of students' productions (what they write, what they say) and not on assumptions of what they are thinking.

This approach has been applied to textbook analyses to identify conceptions about functions, angles, span, linear independence in a wide range of textbooks (see Gerami et al., 2024; Mesa & Goldstein, 2016). The analysis uses the textbook content to identify the problem at stake, noting the expectations that the author has imposed on

the content to identify the operations, the semiotic registers, and the control structures that are explicitly mentioned in the textbook. These constitute possible conceptions that could be assumed would emerge if students had read the textbook. Analysis of student responses has allowed for an expansion of the conception framework, to identify the criteria of application of the control structures and whether the inferences from the criteria are appropriate or not. This expanded system allows for an analysis of students' work that pinpoints the places in which a control structure use was warranted.

These fine-grained analyses of content in textbooks and student productions give better models of students' conceptions that do not depend on the elaboration of internal mental models (e.g., as in schemas), but that are directly tied to the mathematical knowledge at stake. In addition, it provides specificity to the control structures, as tied to mathematical notions, and, with the possibility of systematizing these analyses, it is possible to anticipate all the conceptions that textbooks can implicitly promote.

Concerning the subject-specificity of research approaches, we understand that Balacheff's model answers to analytical challenges that are paramount in mathematics education but not necessarily so in research on other school subjects, especially the concern for semiotic representations and operations—as these are objects specific to mathematics. Most noteworthy, however, is how the application of Balacheff's model to specific content areas of the mathematics curricula gives rise to local and content-specific theories as to what it means to understand and learn a certain content on levels that allow both for the analysis of written learning materials and of student productions.

DIVERSITY OF MATHEMATICAL GAZES IN RESEARCH ON MATHEMATICS CLASSROOM INTERACTION

Susanne Prediger

Research on mathematics classroom interaction is another interesting case in which knowledge building used two mechanisms, importing theories from other disciplines and establishing home-grown theories that refer more to mathematics-specific aspects. Interestingly, the kinds of mathematical gazes are highly diverse, as will be discussed.

Research on mathematics classroom interaction started in the 1970s with a focus on surface structures such as teachers' and students' talk time. Bauersfeld (1979) was one of the first who investigated deeper structures of classroom interactions by importing ethnomethodological and microsociological perspectives into mathematics. By his interactionist research approach, he and his students could identify subtle mechanisms for how typical interaction patterns between teacher and students shape the learning opportunities in mathematics classrooms, elaborating on well-known constructs such as funnel pattern and sociomathematical norms (Voigt, 1994). Subsequently, the interactionist research approach has influenced other research groups, such as, Yackel

and Cobb (1996), who integrated it into their formerly constructivist approach. Other similar qualitative research approaches such as Ethnomethodology, Discourse Analysis, or Conversation Analysis were also imported from sociology to mathematics education research to study classroom interaction. Yet, they were often criticized for their missing mathematical gaze: When the main research focus was on interactive mechanisms on *how* norms or mathematical themes are established co-constructively in classrooms, this often neglected *what* was established mathematically (see Ingram, 2018, for a brief historical overview). In a survey on Discourse Analysis research in mathematics education research journals, Ryve (2011) counted that the majority of the 108 analyzed papers focused generic questions, while only a minority studied mathematical aspects of their discourses in view. To further grow the academic field with strong examples of mathematics-specific research, it is thereby interesting to make explicit what forms of mathematical gaze can be adopted: In which different ways can qualitative classroom interaction research be specific to mathematics?

Even research that does not address mathematical contents can still be specific to the role of mathematics in society, for example, the role of mathematics as an instrument reproducing inequity, being considered too hard for marginalized students. Jungwirth (1991), for example, studied the mechanisms by which teacher-girl interaction contributed to interactively "constructing girls' mathematical failures," while teacher-boy-interaction prioritized a pattern of "concealing the failure." Similarly, Gellert et al. (2018) determined patterns of interactive stratification for students with low socioeconomic and/or immigrant status. Based on these reconstructive findings, classroom design research started to identify productive practices which contribute to overcome inequities by strengthening marginalized students' agency and identities (DIME, 2007). It seems that the role of mathematics can raise particular challenges which are less virulent in other subjects and require subject-specific reflection. The authors themselves characterize their perspective as sociological (Gellert et al., 2018) without claiming to provide a unique contribution which could not be provided by sociologists. However, their work was still very important for theorizing on mathematics education, even beyond the practical purposes of strengthening equity.

For capturing learning opportunities specific to mathematical learning contents, Cobb et al. (2001) promoted the construct of mathematical practices and studied which practices of argumentation can and should be established. Selling (2016) took this further to identifying explicit teaching practices that make the expectations around argumentation explicit, once students have inquired possible argumentations individually. This research on mathematical practices might have contributed to establish them as important curricular content in educational standards of various countries.

Other content-specific forms of the mathematical gaze related to particular kinds of mathematical knowledge, such as, conceptual understanding of concepts: Case studies revealed that, although teachers often work with multiple representations, important

learning opportunities are missed as long as surface translations (focusing only the numbers) are accepted and teacher-student-interaction does not make collectively explicit how underlying mathematical structures occur in the representations (Post & Prediger, 2024). Such kind of research contributes to content-specific theorizing on enhancing understanding, by identifying important conditions of productive teaching, which are of relevance across various mathematical topics.

Topic-specific forms of mathematical gaze can further unpack the research for topic-specific theorizing, for example, on important learning steps in a trajectory towards well-justified and understood volume calculation (Ademmer & Prediger, 2025). Although these steps (most importantly, explicating the multiplicative unit structures) were already specified in epistemological papers and design research papers focusing student learning, their constant neglect in classroom interaction is worth overcoming by topic-related reconstructions of classroom interaction. One major outcome is the typical pattern of productive topic-specific support and topic-specific pathways of classroom discourse, showing for example that even if one student quickly offers a formula, it is worth to relate it back to the meanings of counting in rows and layers. While the ideal learning trajectory is well known from design research studies, it is that topic-specific research on classroom interaction that can contribute to theorizing on productive teachers' steering pathways up and down the levels of the learning trajectory. Topic-specific classroom research is still rare and should be strengthened in the future in order to establish topic-specific theoretical accounts for each mathematical topic and their connections across topics.

Table 1: Different forms of mathematical gaze in classroom interaction research

	Descriptive research for understanding typical patterns	Research for identifying productive teaching practices
Specific to the role of **mathematics in society** as an instrument of inequity	Mechanisms for constructing girls' failure (Jungwirth, 1991) Stratification mechanisms (Gellert et al., 2018)	Equitable teaching practices strengthening (marginalized) students (DIME, 2007)
Content-specific to particular mathematical practices	What is established as a good argumentation? (Cobb et al., 2001)	Encouraging (Cobb et al., 2001) and explicit teaching practices on argumentation (Selling, 2016)

Content-specific to particular forms of mathematical knowledge	What opportunities for conceptual understanding are created/ missed? (Post & Prediger, 2024)	Teaching practices for explicitly connecting multiple representations (Post & Prediger, 2024)
Topic-specific to particular mathematical topics (e.g., area of prisms)	What opportunities for justifying volume calculation are provided along a learning trajectory? (Ademmer & Prediger, 2025)	Productive teaching practices for enhancing the justification of volume calculation (Ademmer & Prediger, 2025)

CONCLUSION

Summing up, the mathematical gaze can relate to different levels of classroom interaction research, even if for the moment, only a minority of studies hold a content-related or topic-specific form of mathematical gaze (Ryve, 2011). While a content-specific focus on mathematical practices is well established, expanding the topic-specific research could strengthen the mathematical gazes on classroom interaction and more consequently connect it to topic-specific epistemological and design research studies. This would allow, in the long run, to establish also theorizing activities on transposable knowledge across different topics of similar epistemological natures, such as in, classroom interaction for developing understanding for (different kinds of) formulas and the role of making explicit the underlying mathematical structures.

While there is no single criterion for judging the best way for mathematics-specific classroom interaction research, the maturing academic research field should elaborate on different forms of mathematical gazes in Table 1 and establish systematic connections to other research areas within mathematics education research to strengthen the knowledge building outcomes.

THE MATHEMATICAL GAZE, AND ITS NECESSITY, IN MATHEMATICS TEACHER EDUCATION

Gil Schwarts and Irene Biza

In this paper, we argue that research on mathematics teacher education and teaching practice should incorporate a mathematical gaze as an essential component. We briefly discuss the importance of this perspective, how it may be supported, and illustrate it through three research cases. In line with Prediger's (2024) call to foreground the content-related dimensions of research on mathematics teachers' collaboration, we seek to draw attention to the following issue: Even when situated within mathematics

education, research on professional learning often focuses on teachers' experiences and reflections or on the design features of learning environments, while giving comparatively limited theoretical attention to how mathematics itself shapes the phenomena under investigation. We suggest that this issue is structural: In professional learning settings, unlike in classrooms, mathematical content is nested within discussions about teaching. In these contexts, attention often shifts toward generic teaching practices—such as discussion moves or classroom management—while the mathematical dimension receives limited attention. And while mathematics does not disappear in professional learning settings, its status changes: It becomes a layer within a complex configuration. When research on mathematics teaching phenomena attends only incidentally, or not at all, to the mathematics, the interpretation of those phenomena is compromised. In our view, mathematical and pedagogical discourses are entangled to form mathematics teaching discourses, and this entanglement should be reflected in the object of inquiry in mathematics teacher education. How, then, can research capture this entanglement? Herbst and Chazan (2012), who specify the didactical contract as subject-specific through constructs such as instructional situations that have specific norms, is an example of research capturing such entanglement. In another example from research, Weingarden and Heyd-Metzuyanim (2025) employ the diagrammatic visual of the Realization Tree, which represents how mathematical objects can be discussed in classrooms, to foreground an analysis of teaching discourse closely tied to the mathematical aspects of teaching. Prediger (2024) demonstrates how a general theoretical framework "gains its explanative power for content-related purposes when filled in content-related ways" (p. 281). We now briefly present three cases with more detail to illustrate the necessity (both conceptual and operational) of the mathematical gaze on how we inquire and analyze mathematics teaching and teacher learning.

MATHEMATICAL GAZE AND RESEARCH ON MATHEMATICS TEACHERS AND TEACHING: THREE ILLUSTRATIVE CASES

In the MathTASK program on research and development of mathematics teachers' pedagogical and mathematical discourses, Nardi, Biza and colleagues design fictional classroom situations that are informed by MER and teaching practice, and invite teachers to identify issues related to teaching and learning in those situations and to reflect on how they would respond. In Nardi et al. (2012) secondary mathematics teachers were invited to respond to one such situation in which two students debate whether a given line is tangent to the graph of a function, using algebraic and graphical arguments, respectively. Analysis of teachers' responses identified the complexity of the positions teachers take when they interpret students' responses and plan their actions (Nardi et al., 2012). To unpack such complexity, researchers analyzed the warrants (and their origins) in teachers' arguments. Initially, they drew on Toulmin's (1958) field-dependence account and Freeman's (2005) classification of warrants, but they found that such a classification was not enough to capture the mathematical

specificity in teachers' multiple professional and institutional roles. To surface such specificity, they adapted Freeman's (2005) classification of warrants with attention to teachers' role as mathematicians and as teachers of mathematics. For example, they introduced the institutional warrant as "a justification of a pedagogical choice on the grounds of it being recommended or required in a textbook (institutional–curricular) or on the grounds that it reflects the standard practices of the mathematics community (institutional–epistemological)" (p. 161). A teacher may say a visual argument would not be acceptable as a full argument because within the mathematics community, visual arguments are not precise (institutional–epistemological) or because visual arguments are not acceptable in mathematics high-stake exams (institutional–curricular). This observation signals that interpretation, address, and uptake of teachers' argument need attention to the mathematical content as an epistemological object as well as a learning object.

In another study, Biza, Nardi, and colleagues unpacked the interrelationship between mathematical and pedagogical discourses and their critical role in teachers' competency in interpreting students' responses and responding to them (Biza et al. 2018). The study took place in the context of a post-graduate mathematics education program that incorporates MER with mathematical content. They used a similar classroom situation about a line tangent to a function graph, this time with teacher–student exchanges, to identify how MER and mathematical content are reified in participants' responses. Here, the term reification is defined as "the gradual turning of processes into objects" (Sfard, 2008, p. 118). Reification of pedagogical discourse addresses how MER theories and findings are reified into participants' responses "in order to describe the pedagogical and didactical issues of the classroom situations and the intended practice" (Biza et al. 2018, p. 64), while reification of mathematical discourse addresses how mathematical theories and practices are reified into participants' responses "in relation to the identification of the underpinning mathematical content of the classroom situations and the transformation of this mathematical content into the intended practice" (p. 64). The mathematical and pedagogical discourses are nested and shape what the mathematics teaching discourse is with attention to both mathematical and pedagogical routines and narratives.

Our third example draws on Schwarts' work on novice mathematics facilitators transitioning from classroom teaching to a teacher educator or facilitator role (Schwarts, 2025). In classrooms, mathematical authority is often anchored in the epistemology of the mathematics discipline, which provides relatively clear criteria for what counts as true or valid. When mathematics teachers move into facilitation, however, their practices as discussion leaders undergo substantial changes. From a generic gaze, these changes can be attributed to shifts in roles, norms, or authority relations. A mathematical gaze, however, reveals a more subtle shift: not only in authority, but also an epistemological shift. It concerns how mathematics teachers, as facilitators, draw on, suspend, or reconfigure mathematical criteria of correctness and

justification as they navigate pedagogical dilemmas. In other words, the phenomenon is not only a change in social positioning but also a reorganization of the role of mathematics and its epistemology within professional discourse.

Through three cases, we illustrated the necessity of the mathematical gaze into research on mathematics teacher education and teaching practice towards three directions: (a) to increase the analytical power of existing theoretical instruments (e.g., the adaptation of Freeman's classification), (b) to highlight the nested role of mathematical and pedagogical discourses in teacher education and teaching (e.g., by addressing and evaluating both mathematical and pedagogical discourses), and (c) to situate social phenomena (e.g. shifts in authority or professional roles) into the disciplinary characteristics of mathematics teaching, thus supporting a deeper and subject-specific analysis of this phenomena.

HOW DOES MATHEMATICS SHAPE THE MER GAZE AND WHY MIGHT THIS MATTER?

Daniel Chazan

This Research Forum has provided examples of the gaze of researchers on mathematics education when it focuses on specific aspects of mathematics teaching and learning. As the examples bear out, the mathematical in MER's gaze is always more than just the conjectures, theorems, and axioms of the discipline. For example, in Abrahamson and Nathan's contribution, doing mathematics is conceptualized as an embodied, embedded, extended, and enactive activity, which allows them to see new aspects of student learning and to imagine other ways in which to teach mathematics. More generally, our gaze often combines attention to mathematics and educational matters.

Even when the forum shifts attention to textbook studies, where subject specificity seems prominent, the mathematics in the MER gaze combines ways of conceptualizing school mathematics, such as, topics, as well as more disciplinary perspectives that inform curricular decision making. The particular combination may change over time. Modern or New Math can be thought of as bringing early 20th century advances in the content of the discipline into the school curriculum across the world to provide students with new learning opportunities. By contrast, the NCTM Standards and the Common Core Standards in the US concentrated on ways in which the practice of doing of mathematics in the discipline might shape how students learn mathematics in school, and thus, again, their opportunities to learn. The careful studies of textbooks discussed by Mesa and Kollosche illustrate other ways in which the mathematical is present in MER, including a sensitivity to ways in which mathematical conceptions that originate in one curricular context may reappear in another to which they are less well suited.

Moving into the areas of MER where the mathematics in the activity is in some sense mediated and the objects of study do not necessarily carry mathematics as clearly with

them, the mathematical in the MER gaze remains a combination of yet different elements. For example, as Prediger suggests, the mathematical in the MER gaze may be imbued with understandings of the societal roles that mathematics plays as gatekeeper from, and gateway to, opportunities for individual advancement. And, as outlined in the rows of Table 1 in Prediger's contribution, there are other ways in which the mathematical in the MER gaze can differ when classroom instruction is being studied. Similarly, Schwarts and Biza's contribution suggests that the mathematical in the MER gaze in studies of teacher development can include many elements of mathematics, from analyses of mathematical practices relevant to teachers' own mathematical activity, to understandings of the particular mathematics that is at the heart of the teaching teachers will do, to capturing the mathematical components of teachers' gaze as they work as teachers responsible for the education of children, not only instructors of particular mathematical content.

In the proposal for this Research Forum, we argued that Mathematics Education Research often asserts that it has the potential to make a special contribution to human understanding by focusing on the mathematical in education practices. What is it about a mathematical sensibility in the mathematics education research gaze that offers the potential to make unique contributions to the larger field of educational research and to being deserving of a seat at the interdisciplinary table of educational policy-making?

Based on the above summary of the initial offerings in the Research Forum, I would suggest that our field's claim is not based in a singular mathematical sensibility that researchers on MER hold in their gaze, but rather in the variety of mathematical sensibilities that MER makes available to the researcher of mathematics teaching and learning. This warrant fits with the contributions to the forum in emphasizing the importance of having a claim to a mathematical sensibility in our gaze, while at the same time challenging MER to grow and develop even more instantiations of mathematical sensibilities which will continue to enrich the insights that our field might develop. Rather than using one particular definition of the mathematical to police the boundaries of our field, we could continually consider the redefinition of the mathematical in ways that make our field relevant to the variety of ways in which mathematics is present in modern societies. If this line of argument is sensible, perhaps we should also be pushing ourselves to not only attend to the mathematical in how we examine matters of teaching and learning, and also consider ways in which we might be able to argue that the very claims generated by our research themselves might legitimately be described as mathematical as well. Of course, to do so would require us to explicate in what way such claims are mathematical, and especially how our descriptions of the mathematical may depart both from the discipline and from the school subject.

In closing, separate from attempting to answer this question about in what ways our mathematical sensibilities might warrant the importance of the insights of our field, I would like to underscore the importance of asking, and re-asking, this question. The

potential for unique contributions to educational research and educational policy-making are what we—when we act as stewards (Herbst, 2023) of research in mathematics education—use to argue for investment in the field. For example, we as a field may need to argue periodically for funding for research, as well as support for journals.

Similarly, we as a field may want to argue for the importance of subject specificity in the training of doctoral students and in the focus of their dissertations, rather than having mathematics educators solely educated in generalist education courses or graduate level courses in mathematics. Instead, we might desire a doctoral education with attention to the history, philosophy, and sociology of mathematics, as well as to the results and methods of MER. Without a broad education of the mathematical sensibilities of future mathematics educators in the gaze they bring to educational matters, our field may be less likely to continue to respond to the mathematical throughout society.

CONCLUDING COMMENTS: ON GAZE AND BASIC RESEARCH

Patricio Herbst

How is mathematics present in the pluralistic research done in our field? Clearly, school mathematics is present in the practices we study—as the tasks, topics, or courses of study our research participants are occupied with. But some of the details of those particulars are to be forgotten when we make knowledge claims about those practices, while other particulars are likely to be taken as instances of abstractions that participate in the knowledge claims we seek to make. How do our knowledge claims about practice maintain a relationship with mathematics even if they rise above the particular mathematics of the practice being studied? Some studies make very apparent the mathematical entanglements of their knowledge claims, while other studies do not. But, in general, few studies rely on an explicit articulation of how they navigate the transition between the mathematical nature of the actual practices they study and the mathematical aspects of the knowledge of practice the researchers produce. In other words, mathematics education research studies vary widely in how they construct the relatedness of their object of study with mathematics. But our field values knowing more about this (see Yan, Mamolo, & Kontorovich, 2025) and our research forum has sought to promote that.

Because our field is pluralistic in what our research studies focus on, we cannot expect every research study to relate to mathematics in the same way. Furthermore, as the papers above demonstrate, the organization of knowledge in the discipline of mathematics does not provide sufficient resources for researchers in our field to identify how their knowledge claims are related to mathematics. We contend that it is important for our research field to elaborate on the various ways in which our objects

of study relate to mathematics and to produce research that enhances the relatedness to mathematics of our knowledge claims. Because we often claim that our field provides knowledge which is distinct from that produced by other fields of education research, and we often rely on the special place of mathematics in our research to make that claim, it is important for our field to engage in work that clarifies the various ways in which our engagements with mathematics education practice result in knowledge claims that are related to mathematics. The unique contribution our research domain makes to scholarship, policymaking, and practice depends not only in our capacity for solving problems of mathematics education practice but in our capacity to provide fundamental understandings of mathematics education practice that distinguish our field from other fields that also sometimes draw their empirical data from places where mathematics is taught and learned (or that includes those).

Basic research has been defined as research which pursues fundamental understanding (see Herbst et al., 2022). Basic research is not antithetical to research that is use-inspired: Indeed, Stokes (1997) drew inspiration from the work of Louis Pasteur to characterize a kind of research that was both motivated by use and pursuant of fundamental understanding. The practices of mathematics teaching and learning are full of problems to solve to which many different fields are attracted to apply their gazes. Our special opportunity, similar to that of Pasteur's when he developed microbiology, is to demonstrate that our gaze affords us distinct understandings of such practices, fundamental understandings which are not mere applications of ideas from other disciplines. Research that hones our gaze, seeking to clarify how our knowledge claims are related to mathematics, can help us achieve such fundamental understanding and support our field's autonomy.

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HOW THEORIES WORK: FOUNDATION, FORM, AND FUNCTION

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Communicating about theories is an enduring academic practice in Mathematics Education Research (MER). This practice has been enriched by the use of various theories in research, through dialogues between theories or the networking of theories. Our Research Forum extends this ongoing debate by offering the dimensions of Foundation, Form, and Function (3F) as a framework to communicate with, within, and across theories. Variation Theory serves as a meta-theoretical perspective to guide the use of the 3F Framework. Six contemporary theoretical advancements in MER are discussed: (Re)Conceptualising Resources, The Documentational Approach to Didactics Across Cultures, A Dialogic Theory of Learning, Theorising a Role of Time in Covariation, Inclusive Materialism, and Socio-ecological Theorising.

INTRODUCTION

The international community of mathematics education researchers can be characterised as a field of diverse theory cultures (Bikner-Ahsbahs et al., 2023). Even within a single domain, such as the study of students' affect, various theoretical approaches are used, representing different theory cultures. When scholars cooperate across these cultures or reviewers evaluate submissions, they must be "open to multiple perspectives and tolerant to different paradigms of research" (Hannula, 2006, p. 209). Early approaches to navigate this diversity used triangulation of theoretical perspectives (Maasz & Schlöglmann, 2006), which, at best, resulted in comparing and contrasting theories. Mok and Clarke (2015) advanced this approach, using triangulation for complementary research accounts with the aim to illustrate the "multi-faceted portrayal of a complex social situation (abstract)" (p. 96) across educational cultures. These approaches contributed to deepening the understanding of diverse

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theoretical traditions and their foundations rather than maintaining and improving the coherence of the field.

Recent theoretical developments speak to the need for tools and frameworks to guide communication about theories. Bosch et al. (2025) offered the With-Within-Across analytic framework, positing three stances on theoretical working. The stances distinguished researchers' orientations to theories, illuminating different purposes that theoretical working can serve in research programs. In their review study, Scheiner and Bosch (2025) showed that dialogues between theories employ converging or diverging forces in research. While they found converging forces to be more prevalent, they noted the value of diverging forces, to distinguish the strengths of different theories, including core assumptions. To avoid fragmentation, Maier (2006) proposed to overcome competition between theories by regarding one's own position "more relatively as a particular contribution to a comprehensive concept of learning mathematics" (p. 96). Nevertheless, a comprehensive concept of learning mathematics has to be conceptualised if it is to be part of a scientific discourse. Therefore, the stance Maier proposed is a methodological view on diversity, but it does not avoid the growth of fragmentation in the field.

A shift of view on theoretical diversity was proposed through the introduction of the networking of theories. This approach extended the emphasis of tolerance to the theoretical diversity by embracing theoretical diversity as richness (Bikner-Ahsbahs & Prediger, 2014), in the sense that mathematics education researchers from different theory cultures can learn from each other. Radford (2008a) conceptualised dialogue/networking between theories by building relations between theory cultures on the level of principles, methodologies, or research questions by which he defined theories. To foster theoretical dialogue, Bikner-Ahsbahs and Prediger (2014) proposed four pairs of networking strategies, positioned on a continuum of increasing integration: understanding other theories and making one's own theories understandable, comparing and contrasting theories, combining and coordinating theories, and locally integrating and synthesising theories.

Despite their value, the use of networking strategies is not an easy process. To date, there are no comprehensive frameworks or analytic tools available to guide the use of networking strategies in research. In addition, theories are difficult to understand and use because they are built on assumptions which are rarely articulated, and the ways theories can be used to solve problems are not straightforward. This hurdle is a particular obstacle for early career researchers, and it can bring decisive consequences for scientific communication on theories as well as research results in the field, be it during conferences, in publications, or review processes. Thus, participating in a discourse on theories or reading an article that uses various theories can be a challenge for emerging as well as experienced researchers, particularly when the kind of theorising leads to advancements within theory cultures. These problems point to a two-fold challenge: to participate in a discourse on theories (1) scholars must be

familiar with the theories in question, their cultures and traditions, their logic as well as their functions in research and development, and (2) analytic tools or frameworks for discussing theories from outside those theory cultures are lacking.

A FRAMEWORK FOR UNDERSTANDING THEORIES

Our Research Forum addresses this two-fold challenge by using the dimensions of *Foundation*, *Form*, and *Function* (3F) as a framework to communicate with, within, and across theories (Bikner-Ahsbahs & Johnson, 2026). The 3F Framework emerged from reflection on mathematics education researchers' conceptualisations of theories. The dimensions of foundation and function build on Radford's (2008a) proposal that theories are systems of principles, methodologies, and questions. The dimension of form points to differences in underlying logical structures across mathematics education researchers' conceptualisations of theories. For instance, Niss (2007) proposed theories to be a different kind of system than Radford (2008a). To guide the use of the 3F Framework, Variation Theory (Marton, 2015) serves as a meta-theoretical perspective.

The foundation of a theory tells us about the theory's roots and its main assumptions, including ontological, epistemological, and axiological commitments (Radford, 2008b). For example, Knowledge in Pieces (KiP) theory (diSessa, 1993) has its foundations in science education in the United States, positing a granular perspective on the development of knowledge.

The form of a theory refers to the structure of how the theory is built up. This could be as a web of knowledge elements, as demonstrated in KiP theory, or as a hierarchical building of concepts and propositions, as suggested by Niss (2007).

The function of a theory addresses what the theory can achieve (e.g., to describe, explain, or prescribe). For example, KiP theory provides an asset-oriented approach to students' knowledge development; students develop new knowledge by growing their interconnected networks of knowledge elements, not by replacing flawed elements.

In summary, the foundation of a theory gives meaning to its principles. Its form tells something about how the concepts and propositions are related and constructed. The functions of a theory shape the relevant background for the methodology and the research questions, as they tell us what the specific theory can achieve.

Variation Theory (Marton, 2015) is a general educational theory of learning, which we combine with the 3F Framework to leverage in-depth understanding of theories in discussions with, within, and across theories. From a Variation Theory perspective, to bring about necessary conditions of learning, learners experience variation in critical aspects of an object against an invariant background. Critical aspects are features that learners can discern, or separate, from objects. There are three patterns of variation: contrast, generalisation, and fusion, which are to happen in that order. Applying Variation Theory to the learning of theories in mathematics education research, the critical aspects to be discerned are three dimensions (or features) of theories (i.e.,

foundation, form, and function), and more specifically theory cultures. Within a single theory culture, the dimensions provide contrast. After contrast, there is generalisation, which demonstrates that foundation, form, and function are not just dimensions of a single theory culture; these dimensions can apply to different theory cultures as well. Fusion occurs when simultaneous variation in theory cultures and dimensions of those theory cultures is experienced, which should happen after contrast and generalisation (see Bikner-Ahsbahs & Johnson, 2026).

AIM, FOCUS, AND LAYOUT OF THE RESEARCH FORUM

We present and discuss different kinds of theories and different forms of theorising. The aim is therefore:

> To introduce six theoretical advancements together with the 3F framework, guided by Variation Theory as a meta-theoretical perspective. Our aim is two-fold: (a) to sensitise the audience to the foundation, form, and function of theories, and (b) to use the framework to practise communicating with, within, and across theories, including new ones as well as theoretical approaches with which audience members are familiar.

To address this aim we begin with an invitation to the problem. We then share recent review results on the dialogues between theories (Scheiner & Bosch, 2025) and the With-Within-Across analytical framework distinguishing stances on theoretical working (Bosch et al., 2025) in the following presentation:

- Developments in theoretical dialogue and theoretical working in mathematics education (Thorsten Scheiner and Marianna Bosch)

Next, we present the 3F Framework together with our meta-theoretical Variation Theory perspective (Bikner-Ahsbahs & Johnson, 2026). Subsequent presentations discuss six contemporary theoretical advancements that we will examine; these are:

- (Re)Conceptualising Resources in Mathematics Teacher Education (Jill Adler and Nicola Bretscher)
- The Documentational Approach to Didactics across Cultures (Yusuke Shinno, Takeshi Miyakawa, and Chongyang Wang)
- The Dialogic Theory of Learning (Ana Carolina Faustino and Denner Dias Barros, on behalf of Ole Skovsmose)
- Theorising a Role of Time in Covariational Reasoning (Kevin Moore)
- Inclusive Materialism (Ayush Mukherjee and Nathalie Sinclair)
- Socio-ecological Theorising (Alf Coles, Kate le Roux, and Armando Solares-Rojas)

The extended abstracts of these seven contributions follow, with a final section looking back and ahead. The first contribution (Scheiner and Bosch) provides an invitation to theoretical dialogue and theoretical working. Tenets of Variation Theory ground the organisation of the six remaining contributions. To provide contrast, each elaborates briefly on foundation, form, and function specific to their theoretical approach. Then,

to provide generalisation, Bikner-Ahsbahs and Johnson look back on the foundation, form, and function of different pairs of theories. We conclude with a look ahead to consequences and future direction for this work.

RECENT DEVELOPMENTS IN THEORETICAL DIALOGUE AND THEORETICAL WORKING IN MATHEMATICS EDUCATION

Thorsten Scheiner[1] and Marianna Bosch[2]

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Theoretical diversity is a defining feature of mathematics education research, yet how researchers engage productively with that diversity remains underproblematised. Drawing on a systematic review of theoretical dialogue and a framework distinguishing stances on theoretical working, this contribution examines what these two works together reveal about theoretical practice in the field. Three modes of dialogue and three epistemic stances towards theories illuminate complementary dimensions of the same underlying challenge: how theories work, how they relate to one another, and how researchers orient themselves towards them. This integrated analysis reveals a field inclined towards convergence, yet constrained by a lack of explicit tools for communicating across traditions.

THE DOUBLE QUESTION OF THEORETICAL PRACTICE

Mathematics education research is characterised by a remarkable proliferation of theoretical approaches. This plurality presents a persistent tension: diverse perspectives illuminate aspects of educational phenomena that no single framework could apprehend alone, yet fragmentation risks impeding cumulative knowledge development and obscuring common intellectual ground. Responses to this challenge have evolved considerably. Early moves towards triangulation and pragmatic tolerance gave way to the 'networking of theories' programme (Bikner-Ahsbahs & Prediger, 2006; Prediger et al., 2008), which reframed diversity as richness and proposed strategies for productive engagement across theoretical traditions. Yet the field continued to face two distinct, if related, challenges. The first is structural: how do theoretical approaches actually interact in research practice, with what purposes, and through what techniques? The second is agentive: how do researchers orient themselves towards theories, and how does that orientation shape the knowledge they produce?

Our systematic review of theoretical dialogue (Scheiner & Bosch, 2025) and our framework distinguishing stances on theoretical working (Bosch et al., 2025) address these two challenges in a mutually illuminating way. Taken together, they reconstitute the question of theoretical practice: not merely how to connect theories, but how the relationship between theory and researcher shapes what theoretical engagement can achieve. The present paper synthesises our joint contribution as conceptual preparation

for the 3F Framework introduced in this Research Forum (Bikner-Ahsbahs & Johnson, 2026).

TWO DIMENSIONS OF THEORETICAL PRACTICE

At the heart of both contributions lies a shared concern: the gap between the potential of theoretical diversity and what the field actually realises from it. Together, they reveal that what limits the productivity of theoretical engagement is not a lack of sophisticated strategies for connecting theories, but a more fundamental opacity about how theories work and what orientation towards them a given research purpose demands.

In our systematic review, we proposed three modes of theoretical dialogue—the intentional engagement between theoretical approaches: *inter-theoretical* dialogue, which examines relationships between distinct approaches; *multi-theoretical* dialogue, which uses multiple approaches to address a shared phenomenon; and *meta-theoretical* dialogue, which transcends individual approaches by developing frameworks capable of encompassing or adjudicating among them. In Bosch et al. (2025), we offered an analogous but differently oriented triad: working *with* theories (theories serving as analytical tools for pre-identified problems), working *within* theories (operating inside a theory's constitutive logic to refine and extend it), and working *across* theories (a meta-theoretical orientation in which theories themselves become objects of investigation). The resonance between these two tripartite distinctions is not coincidental: both distinguish engagement at the level of individual frameworks from engagement across frameworks, and both identify a reflexive, meta-theoretical mode as demanding yet consequential for field-level development.

THE DYNAMICS OF CONVERGENCE AND DIVERGENCE

A striking finding from our systematic review of 78 peer-reviewed articles is the predominance of convergent tendencies. The review found that 74.3% of papers used predominantly unifying strategies—seeking to integrate or synthesise theoretical approaches—while only 9% used diverging strategies aimed at highlighting distinctions and preserving theoretical identities. This asymmetry is significant not because convergence is inherently problematic, but because of what it obscures.

Read alongside our companion framework, the dominance of unifying forces acquires a fuller explanation. Working *with* theories—arguably the most prevalent stance in research practice—tends to favour convergence. The risks we identify for this stance are therefore structural: an instrumental orientation may systematically underestimate how theoretical commitments shape problem identification and phenomenon constitution before analysis begins; and if researchers believe they are applying frameworks to independently existing phenomena, they are unlikely to attend to the divergent ontological commitments visible from a more reflexive vantage point.

Diverging forces serve an indispensable function. Our systematic review showed that such engagement clarifies the distinctive assumptions, core commitments, and explanatory reach of individual frameworks—corresponding to the key affordance of

working *within* theories: sustained engagement with a framework's internal logic makes visible its constitutive ontology, its characteristic questions, and its implicit standards for what counts as explanation. Both contributions suggest that productive diverging engagement is a precondition for genuine dialogue: one cannot coordinate or compare theories without first understanding what makes each one distinctive.

A further dynamic concerns theories' constitutive role in delimiting phenomena. Both contributions identify a persistent tendency to treat phenomena as theory-independent entities awaiting explanation—a form of naive empiricism made visible when researchers operate inside a theory's constitutive logic, which specifies what entities populate the educational landscape, what questions are meaningful, and what forms of evidence are legitimate. The implication is epistemologically significant: theoretical plurality may not represent competing descriptions of the same underlying reality but may generate genuinely distinct phenomena, each constituted within its own framework's logic—which is why navigating theoretical diversity requires a reference framework for comparing different constitutive visions of educational reality.

A FIELD IN NEED OF EXPLICIT TOOLS

Both contributions identify the same structural issue: the absence of explicit, shared frameworks for making theoretical practice legible. In our systematic review, few papers explicitly discuss the compatibility of the theoretical approaches they engage, and still fewer articulate a reference framework for assessing it. In the companion framework, the assumptions embedded in each of the three stances—about what theories are, what phenomena are, and what research is for—largely remain unarticulated in research practice. Together, we identified a double opacity: researchers are often unclear both about how the theories they engage relate to one another and about their own epistemic stance towards them.

What both contributions point towards—and what the 3F Framework (Bikner-Ahsbahs & Johnson, 2026) addresses—is a practical means of making these dimensions explicit and comparable. The 3F Framework provides precisely the reference framework whose absence constrains theoretical dialogue, and it supports the explicit articulation of epistemic stances largely invisible in research practice.

(RE)CONCEPTUALISING RESOURCES IN AND FOR MATHEMATICS TEACHER EDUCATION

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We revisit the theorisation of resources published in the seminal paper Conceptualising resources as a theme for teacher education (Adler, 2000), briefly presenting its original foundational ideas: 'resource' as noun both and verb, and their 'transparency'; as well its forms - an expanded notion of resources as material, social

and cultural. Much has changed, particularly the explosion of digital resources and the role of resources in leveraging change. We discuss these wider functions, and reflect on what remains salient, what has evolved and what remains open to further development.

FOUNDATION: KEY CONCEPTS AND THEIR SALIENCE

In the late 1990s, a study of teachers' take-up from a professional development programme in South Africa (see Adler & Reed, 2002) identified resources *and their use* as a key area of research. The teachers were working in schools serving low-income communities, and so with limited resources. The teachers unsurprisingly lamented the need for more resources. Yet they made productive use of what they had and, at the same time, showed problematic use of some new materials. From this study, (Adler, 2000) produced a theorisation of resources, underpinned by sociocultural theory, and based upon two key concepts. First was a redefinition of 'resource' as both noun and verb, drawing attention to its use and positioning the teacher as agent. Second was a recognition that resources did not speak for themselves, that they were more or less 'transparent'—a function of their simultaneous visibility and invisibility (Lave, 1991). For a resource to be used to extend a teacher's practice, it needs to be seen, i.e., visible, and its use must be meaningful, i.e., integrated into the teacher's practice. This entails 'seeing through' the resource, appreciating how and why it can be used, and thus its invisibility. Re-defining resource as both noun and verb has continuing salience today, with a consensus that resource *use* is what matters and extensive development in the resource approach in mathematics education research (see Trouche et al., 2023). While 'transparency' has had less traction in the field, it was successful in drawing attention to the complexity of supporting meaningful resource-use, compared to the relative ease of providing access to resources.

FORM: CATEGORISING RESOURCES

In addition to the new focus then on *meaningful resource use* and, drawing on the experience of working in resource-constrained schools, Adler (2000) described and argued for a broader understanding of what counts as resources for teaching. She distinguished different physical and material resources (basic resources, e.g., buildings, electricity, desks, paper, etc.; and curriculum resources, e.g., technologies, textual materials, everyday artefacts) and included social and cultural resources (collegiality, language, time). Much in this range of resources cannot be taken for granted in resource-constrained contexts, with consequences for mathematics teaching and learning. The explosion of digital technologies resulted in the recognition of new resource forms and development of the resource approach (Trouche et al., 2023) mentioned above.

More recently, there has been attention to teachers' knowledge as a resource, as well as to ideational and meta-resources, both of which were out of view in the initial categorisation. Teachers' knowledge is viewed differently across frameworks for research on teachers' resource use (Trouche et al., ibid), revealing how the context in

which frameworks emerge matter, and so careful consideration of their use across research practices is needed. The notion of meta-resources has been systematically explored and reviewed by Gueudet et al. (2023) in which the definition of meta-resources includes ideational resources (e.g., mathematics teaching frameworks). In combination, these additions point to theoretical advances in the work on resource use.

FUNCTION: USING 'RESOURCES' IN RESEARCH

New resource-forms led to developments in the functions of theorising resource-use in research. We focus on developments related first to resource-use in projects aimed at levering educational change, and second to current research elaborating the *transparency* of ideational resources in pre-service teacher education. Venkat and Graven (2017) built from Adler (2000) to develop a model of potential resource levers for educational change in resource constrained contexts. They too distinguish material, human and cultural resources with significant insight into sustainability and scale in teacher professional development. Material and human resources sit in tension in designing interventions to lever change, and over time. Integrating ideational resources into pre-service teacher education provoked elaboration of a framework for transparency (Bretscher et al., 2025), combining situated abstraction (Noss & Hoyles, 1996) and coherence to conceptualise the notion of invisibility. The transparency framework functioned as a means of describing and categorising pre-service teachers' interpretations of an ideational resource into practice, revealing how such resources might support or hinder teacher development.

CONCLUSION

The foundational concepts of Adler's (2000) theorisation of resources in mathematics education are affirmed by our review of the (re)conceptualisation of resources and resource-use over time. New forms have emerged, instigated by new technologies and shifting interests in mathematics education research, giving rise to new functions, revealing resource tensions in leveraging educational change and describing use of ideational resources in pre-service teacher education. No doubt further developments in resource theory will engender interesting work in the field.

DOCUMENTATIONAL APPROACH TO DIDACTICS ACROSS CULTURES

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In this short description, we share the theoretical development of the Documentational Approach to Didactics (DAD) by reflecting on the 3F dimensions and our experiences of navigating between languages and cultures, and offer insights into how DAD continues to grow across different educational and research institutions.

INTERNATIONALISATION OF A THEORETICAL FRAMEWORK

The Documentational Approach to Didactics (DAD) is a theoretical framework in mathematics education research that emerged during a time when the use of online resources became widespread among teachers and students (Gueudet & Trouche, 2009; Trouche et al., 2020). It adopts a "documentational" perspective on teachers' work with resources, aiming to better understand the complexity and cultural specificity regarding their practices and professional knowledge. Although DAD originated in France, its development has been notably international and intercultural, particularly within the realm of teacher development. The audience may find it curious that East Asian scholars are introducing a theory developed in Europe. However, DAD has evolved through its application and testing in "new" contexts (Wang et al., 2023). International collaboration within and beyond the West has been a key aspect of this theoretical advancement, and it has been accompanied by the expansion of the research community.

FOUNDATION, FORM, AND FUNCTION OF DAD

The *foundation* of DAD lies in its emergence as a response to educational challenges in teachers' work brought about by the digitalisation of resources and evolving curricular demands (Gueudet & Trouche, 2009). Building on this point, *Activity Theory*—together with Cultural Historical Activity Theory and Community of Practices—provides a crucial conceptual lens, framing teachers' documentation work as goal-directed, tool-mediated activity situated within broader social and institutional systems. This perspective allows DAD to account not only for individual interactions with resources but also for the collective and culturally embedded dimensions of professional practice, thereby supporting the shift from analysing isolated teacher actions to understanding collaborative and systemic processes. Rooted in the *instrumental approach* (Vérillon & Rabardel, 1995), DAD extends the artefact–instrument distinction to the resource–document pair, emphasising teachers' active role in transforming resources through documentational genesis. It also draws heavily from the French tradition of didactics, incorporating Vergnaud's notion of schemes, Chevallard's institutional perspectives, and Brousseau's idea of milieu to conceptualise teachers' professional activity (Gueudet, 2019). Over time, these theoretical influences have converged through extensive interdisciplinary and international collaboration, providing DAD with a robust and evolving intellectual grounding that reflects both cognitive and socio-institutional dimensions of teachers' resource use.

The *form* of DAD consists of its conceptual structure, centred on the key notions of *resource*, *document*, and *scheme*, and the dynamic processes that connect them (Trouche et al., 2020). Resources are understood broadly as any artefact—material, digital, social, or cognitive—that can support teaching, while documents emerge when teachers integrate selected resources with a *scheme of utilisation* developed through their professional activity. This transformation occurs through *documentational genesis*, a dialectical process composed of *instrumentalisation* (teachers shaping

resources) and *instrumentation* (resources shaping teachers' practices). The form of DAD also includes its sensitivity to linguistic and cultural variation, highlighted through issues encountered in translating key concepts across 14 languages in the DAD Multilingual Project (https://hal.science/DAD-MULTILINGUAL).

The *function* of DAD lies in how it operates as a theoretical lens in empirical research, supporting three main purposes: understanding, designing, and rethinking teachers' work with resources (Trouche, 2019). Its primary contribution is to help researchers understand the complexity of teachers' interactions with diverse resources and the professional knowledge embedded in their documentation practices; in addition, it offers a basis for designing or supporting innovations in tasks, digital artefacts, and professional development, and for rethinking teaching by highlighting teachers' dual roles as designers and reflective collaborators. The two East Asian studies (Shinno & Mizoguchi, 2021; Wang, 2019) mainly serve the understanding function by showing how DAD makes visible culturally mediated forms of documentation work. In both examples, DAD reveals how local traditions, institutional infrastructures, and professional norms shape teachers' documentation expertise, the construction of pivotal documents such as textbooks and lesson plans, and the collective dynamics through which teachers develop and refine their resource systems.

TRANSPOSITION OF DAD ACROSS CULTURES

Applying DAD in the East Asian contexts demonstrates not only analytical uses of the framework but also ways in which these contexts generate new theoretical ideas that extend DAD beyond its original formulations. In China, the analysis of teachers' culturally situated resource practices led to the development of *documentation expertise*, a construct that highlights how professional competence emerges through institutionally structured collective work. In Japan, teachers' preparation work (known as *kyōzai kenkyū*) and lesson plan production revealed the textbook as a *pivotal document* that orchestrates teachers' documentation processes and brings the epistemological content of documents—the mathematical structure of tasks, anticipated student reasoning, and pedagogical intentions—into sharper theoretical focus. Moreover, both contexts illuminate the role of institutionally embedded collaborative settings (such as lesson study), which theorise how collective documentation deepens individual work. This invites us to consider the need for culturally responsive research and theory, as researchers interact with DAD as research resources via bi-directional processes, instrumentation (how those theoretical frameworks shape researchers' work and knowledge) and instrumentalisation (how researchers adapt and make these frameworks their own).

A DIALOGIC THEORY OF LEARNING MATHEMATICS

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In this paper, we describe a Dialogic Theory of Learning Mathematics as proposed by Skovsmose (2023). We illustrate the function of the theory using an example from a pre-service mathematics teacher at a Brazilian public higher education institution, where students of a mathematics teaching degree are working on a project that mobilises the concerns and hopes of Critical Mathematics Education.

FOUNDATION

Erosion of democracy, socio-ecological issues, and dialogues between students from different territories (Skovsmose & Faustino, 2025) are among the key concerns of Critical Mathematics Education in the 21st century. These concerns relate to the importance of theoretical advancement in a theory that supports the development of democratic aspects in the classroom at different educational levels. Every type of education brings with it a way of understanding the construction of knowledge, that is, an epistemology. Thus, it is important to question: what type of epistemology relates to an education that seeks humanisation? An epistemology that contributes to humanisation is composed of dialogical and critical elements (Skovsmose, 2023).

The dialogical elements are articulated within The Dialogic Theory of Learning Mathematics (Skovsmose, 2023, 2024), which recognises "learning as an interactive process" (2023, p. 185). While an uncritical epistemology relates scientific development directly to democracy, educational processes do not necessarily lead to democratic outcomes; indeed, scientific advancement can also contribute to barbarism. Consequently, a critical epistemology recognises that there is no inherent connection between education and democracy; therefore, all knowledge being learned must be problematised and criticised.

The foundation of the Dialogic Theory of Learning Mathematics consists of the assumption that the qualities of communication and the qualities of learning interrelate. The theory considers communication as the key reality for learning in class, where students' knowledge develops through dialogue of a certain quality that goes beyond the current situation of life. This quality is described by the form of six learning interacts (getting in contact, exploring, positioning, foregrounding, externalising, and doubting) that contribute to examining what happens in the classroom and imagining new possibilities for what can happen in the classroom (Skovsmose, 2023; Faustino & Barros, in press).

FORM: SIX LEARNING INTERACTS

We present the six learning interacts in pairs. The first one is getting in contact, and it indicates acceptance of the invitation to dialogue and can be expressed both verbally

and non-verbally. For example, when everyone becomes silent and develops a willingness to actively listen to others, participants direct their gaze towards the group members, indicating their readiness to investigate. The second one is exploring, which 'includes activities such as presenting, listening, advocating and questioning' (Skovsmose, 2024, p. 216). For example, students can present their strategies and ideas to each other in pairs or to the entire class. They can also use drawings, algorithms, and spreadsheets to explain their arguments.

The third one is positioning, which can have an epistemic aspect. For example, while students can take a stance on a mathematical concept, it can also have a political aspect when investigations open up space for using mathematics to read and write the world. Skovsmose (2024, p. 220) states, "Normally the mathematics classroom is not a space that invites positioning but rather is regulated as a narrow one-way street leading towards pre-defined mathematical ideas." The fourth one is foregrounding, which refers to students' future horizons both individually and collectively. Individually, students can reflect on how mathematics broadens or limits their future prospects, and collectively, on possibilities for future generations. The fourth learning interact opens up possibilities for dialogue in mathematics classes to contribute to the reading and writing of socio-ecological issues (Skovsmose, 2024).

The fifth one is externalising, which refers to the possibility of sharing what has been learned with other social actors, such as students from another school. For instance, Skovsmose (2023) highlights the possibility of a project leading to the production of a video that will be presented to the community. The last one is doubting, which is fundamental to teaching and learning from a critical perspective. Here, students practice doubt regarding mathematical concepts, not taking them as absolute truth, and they also practice doubt regarding how mathematics is put into action. Skovsmose states, "It is important to confront any version of ideology of certainty, and one way of doing so is to recognise doubt as a learning interact" (Skovsmose, 2024, p. 217).

FUNCTION: AN ILLUSTRATION FROM PRACTICE

The function of the Dialogic Theory of Learning Mathematics is to serve as a design guideline for preparing and conducting learning in a reflective and critical manner, opening up horizons for students and teachers, as well as possibilities for their prospective pupils. To illustrate, we describe an activity, titled "What is the best packaging?" which aimed to explore spatial geometry from a critical and investigative perspective. The activity took place with six undergraduate students in mathematics teacher education at a Brazilian public university in the state of São Paulo as part of a Supervised Practicum in Geometry and Geometric Drawing. Over the course of two class sessions, students first explored the connections between packaging and geometry, discussing mathematical concepts such as volume, shape, and spatial organisation alongside broader issues including marketing, cost, transportation, consumer behaviour and environmental impact. Students were then organised into pairs and challenged to design and defend a specific packaging format for storing

marbles within a fixed volume. The learning interacts of positioning and doubting emerged in paired interaction. One pair, Carlos and George, positioned themselves by offering different possible measurements for a box. George mobilised doubting when he realised that the box with the measurements proposed would not be feasible, because it could easily fall off a shelf. Hence, doubting led to further positioning, as they considered other possibilities.

Throughout the experience, dialogue functioned not only as a means for developing mathematical ideas but also as a pedagogical practice that students recognised as essential to their future teaching. Supported by careful planning and the professor's inquiry-oriented approach, participants were encouraged to envisage alternative possibilities for teaching geometry.

COVARIATION AND THE FOUNDATION, FORM, AND FUNCTION OF TIME

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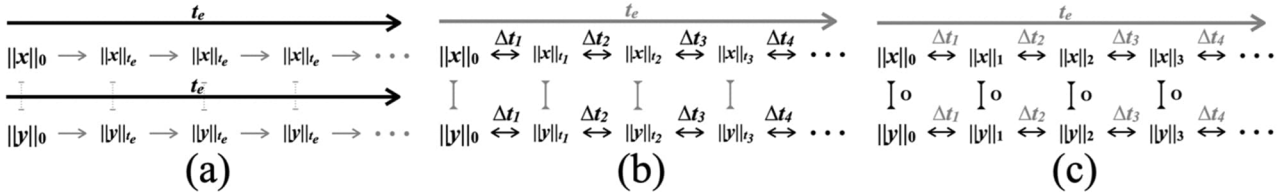
Research on quantitative and covariational reasoning has identified that individuals' conceptions of quantitative relationships vary according to the degree to which two quantities are held in mind simultaneously. These variations influence the conceptual objects individuals construct and reason with, and thus motivate the need for explanatory accounts of these variations. Drawing on Piaget's theorising of time, I theorise the role of time in relation to the extent individuals conceive a covariational relationship as a conceptual object resulting from the uniting of two quantities. In doing so, I provide different forms of reasoning that describe the covariational relationships students construct and characterise the conceptual objects they construct via these relationships.

Covariational reasoning—the mental actions involved in constructing and coordinating two quantities variations in tandem (Thompson & Carlson, 2017)—has emerged as a critical way of thinking across STEM fields (e.g., Karagöz Akar et al., 2022). A common implication of frameworks that characterise differences in students' covariational reasoning (e.g., Johnson, 2015; Thompson & Carlson, 2017) is the importance of conceiving quantities' covariation in terms of their *simultaneous* change. An individual imagining simultaneous change persistently holds in mind that the variation of one quantity necessarily occurs hand-in-hand with variations in the other quantity (Thompson & Carlson, 2017). Such a conception involves an individual constructing a *multiplicative object* that is the cognitive uniting of multiple attributes so that the object is simultaneously all of them (Inhelder & Piaget, 1964).

Despite researchers suggesting that constructing a multiplicative object constituted by quantities' simultaneous variations is important, students' development of such an

object is fleeting and underspecified. There is a need for analytic tools that provide operational distinctions between students' forms of covariational reasoning including the degree to which they consist of a multiplicative object. Responding to this need, I developed a framework (Figure 1) for the role of time in students' covariational reasoning with attention to the nature of the multiplicative objects that are constituted by their conceived covariational relationships (Moore, 2025).

Figure 1. Three forms of covariational reasoning and associated conceptions of time.



FOUNDATION

The *foundation* of the developed framework is a radical constructivist epistemology, which positions all knowledge as idiosyncratically constructed by an individual to adaptively organise their personal experiences (von Glasersfeld, 1995). This epistemology implies the knowledge of an individual is fundamentally unknowable to another individual, and that researchers address this underlying constraint by using methodologies that enable producing empirically-based, hypothetical models of student thinking. These models are not factual descriptions of how a student thinks, but instead viable models of knowing that explain a researcher's cognitive interpretations of student actions. The framework in Figure 1 emerged from such an approach.

FORM

The *form* underlying the framework stems from Piaget's genetic epistemology, and more specifically Piaget's (1970) theorising of time and covariational researchers' enactment of his epistemology (e.g., Johnson, 2015; Thompson & Carlson, 2017). Covariational researchers have provided generalised constructs of covariational reasoning, including the different ways students coordinate specific variations in one quantity with those in another quantity (e.g., quantity A *increases by decreasing amounts* as quantity B *increases by successive-constant amounts*). Complementing covariation constructs, Piaget's theorising of time presents concepts of time as inseparable from the mental operations defining space, motion, and objects. That is, Piaget described a concept of time as the product of coordinating mental operations in the construction of other concepts. Thus, my framework combines forms of covariational reasoning with Piaget's positioning of time to explicate how individuals' coordination of covarying quantities involves particular conceptions of time. Just as Piaget positioned time as an emergent feature of coordinating other concepts, the framework in Figure 1 positions a conception of time as an emergent feature of how students reason about quantities' covariation and associated relationships.

FUNCTION

The *function* of the framework in Figure 1 is an explanatory account of forms of covariational reasoning, with their distinctions being based in different conceptions of time. The first model (Figure 1a) captures a form of covariational reasoning tied to experiencing time strictly in terms of its present flow (i.e., experiential time). The second model (Figure 1b) captures a form tied to a conception of elapsed, measured time (i.e., conceptual time). The third model (Figure 1c) captures a form that involves disembedding quantities from a phenomenon and elapsed time so that their invariant relationship is taken as an object of thought. It is this last form of covariational reasoning in which an individual constructs the quantities' covariation as a multiplicative object so that it can be taken as an object of thought independent of any specific instantiation or phenomenon involving them.

To illustrate the three models of covariation, imagine an animation illustrating a road trip via a point moving along the road as the animation plays (Figure 2). Also consider the relationship between the *distance travelled from Espoo* (*DfE*) and the *distance from Helsinki* (*DfH*). The first model of covariation involves considering each quantity's variation *as the video plays*. For instance, if the animation is designed so the car travels at a constant speed, the individual could play the video and notice that *DfE* increases at a constant intensity and *DfH* decreases at a non-constant intensity. This is captured by the faded link between each quantity's magnitude in Figure 1a, as well as the foregrounding of the flow of experiential time, t_e.

Figure 2. Animation snapshot of a road trip.

In the second model, experiential time fades to the background and is replaced by durations of elapsed, measured time, $\Delta t_{\#}$. The covariation between the two quantities' magnitudes is contingent on each quantity's variation with conceptual time. For instance, an individual might consider two cars taking the same trip at different speeds as having different covariational properties between *DfE* and *DfH*. This is captured by the foregrounding of intervals of time in Figure 1b, as well as the faded link between each quantity's magnitude.

In the third model, conceptual time fades to the background and the relationship between *DfE* and *DfH* is disembedded from any specific road trip experience. For instance, an individual might coordinate how *DfH* varies for equal, successive increases in *DfE* during the road trip. Here, *DfH* and *DfE* are united into a multiplicative object and their covariation is a property of it. This is captured by intervals of time fading to the background in Figure 1c while their link is strengthened.

INCLUSIVE MATERIALISM

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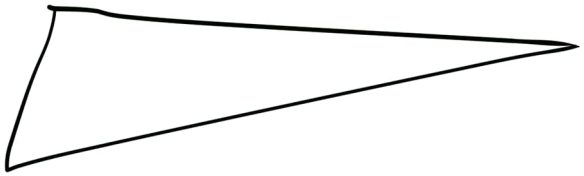
Inclusive materialism offers to read a mathematical concept as a material thing, inviting mathematics education research to engage with it.

FOUNDATION

Figure 1: Here is a triangle.



A question in mathematics education research (MER) is what a triangle is as a mathematical concept. Some researchers say that it is a collection of *abstract* things: points, lines, sets, axioms, properties and theorems. Some say that it is a *mental* thing: a 'concept image', a representation, an idea in our minds. Some say that it is a *social* thing: a word, a sentence, an artefact of disciplinary practice. Some say that it is instead a *bodily* thing: a gesture, a way of marking space, frozen movement. Some say it is a bunch of *emotions*; some say *perception*; some say *power* or *spirit*.

As our answer to this question, consider three foundational commitments.

One, I drew this triangle expending effort, tracing a digital pen over a screen, whose capacitances changed to fill pixels that emitted light that fell on my eyes, and so on… A host of matter-matter interactions occurred such that the material world was forever changed. The triangle brutely—without intellectual sophistication—appears to me simply as 'matter'. Let us call this a 'naïve materialism'.

Two, it is not as if the triangle was first in my mind or in an abstract world before I drew it on the screen. The triangle was *always material*; before I drew it, it was living a "secret life" as a material thing (Dominguez, 2025). This is a counterintuitive commitment that we may trace to Barad (2007) and call 'new materialism'. With Barad, we argue that there is no privileged human subject who births the triangle; instead, both the triangle and I were living as indeterminate, fuzzy material things. The act of drawing eventually made 'it' a determinate Triangle and 'I' a determinate Man.

Three, the previous commitment is counterintuitive because we normally think of matter as *bare matter*, as something that cannot move or evolve meaningfully without human intervention. Against this, we argue that matter itself is quite *agentic* (Barad, 2007): it births abstractions, images, ideas, language, gestures, space, movement, emotions, perception, power, spirit. Let us call this 'inclusive materialism' (IM).

FORM

How was my triangle a material thing before I drew it?

Clearly, my triangle is *my triangle*. It is a particularly obscene, carelessly drawn triangle from somewhere east of Vancouver on a sunny morning. It has bumpy lines and overhangs. I am quite attached to my triangle; I imagine it too is attached to me.

My triangle is a curious creature. It is immediately many things: affect and desire, percept and history, image and concept. With remarkable speed, the triangle settles into these *ways of existing* that seem so very different. Before the triangle, I did not exist either, at least not as a simple perceiving, desiring, imagining being with a triangle-shaped hole. The triangle creates in me a sympathetic affect, a sympathetic desire, a sympathetic image, such that my own becoming, as a somewhat accomplished artist and mathematician of triangles, is tied to my triangle's emergence.

For all this variety to appear from a material act, where even the subject is constituted *through* action, we must allow for a certain metaphysics of matter. We must allow, specifically, an imputed "secret life" of triangles. Before I drew it, my triangle and I were an indeterminate mesh of material relations; when I drew it, the mesh resolved into a bumpy, obscene triangle (it) and a careless artist-mathematician (I). Drawing brought out what was always latent in the material mesh: all its history, affect, desire.

FUNCTION

The task before us is to characterise this indeterminate mesh, this generative "secret life". If there is such a thing as a mathematical concept, it must be in this secret life from where my triangle and I were born. Where else would it be?

To do so, we must engage with the triangle aesthetically. Aesthetic, because I must constantly work towards harmonising that which appears to me qualitatively different. Here, my finger-pen dithers while my eyes trace a smooth line, while I hear plastic jarringly scrape on glass; I must see through to an underlying harmony among these multiple material sensations, arguing that they emerge from a unified secret life manifesting itself. At a different scale, the triangle emerges as a black-and-white figure on the screen, and I emerge as flesh-and-blood man; I must harmonise 'it' and 'I' towards an underlying material unity from which we have so rapidly differentiated. de Freitas and Sinclair (2014) refer to this tension between harmonisation and differentiation as an aesthetic reconstruction of the indeterminate material mesh from where the triangle appeared. Our aesthetic engagement with the triangle becomes our way to apprehend it as a concept.

To apprehend the concept this way with IM, we must ask strange questions. An IM researcher does not ask how students best understand a triangle-concept, or what the prototypical image of a triangle is. She asks instead: *what actions might make triangle differentiate differently* (i.e., into a different collection of qualitative entities), and thus: *what kinds of mathematics students emerge when such actions are performed?* She also asks: *what parts make up a triangle's secret life, and how are these parts arranged*

(e.g., are there hierarchies of gestures or affects within the concept?) She asks: *to what extent is the secret life of the triangle 'universal'?* If we change our actions, from drawing the triangle on a screen to moulding it in clay, will the concept (the secret life) stay the same or will it change? Will mathematics students and classrooms change or will they stay the same?

These are the kinds of questions that we wish to introduce within MER.

SOCIO-ECOLOGICAL THEORISING

Alf Coles[1], Kate le Roux[2], Armando Solares-Rojas[3]

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We offer socio-ecology not as a theory, but as a process of theorising, relevant to times of change. A founding process is to consider the nature of being as relational. Forming processes include considering who are agents of knowing (in any context) and what ways of knowing are acknowledged. One purpose of these forming processes is to seek awareness of who and what is marginalised. We take "functionings" to be how we produce meaning in the interaction of founding and forming processes, in the places and contexts of our theorising and we offer one brief example.

In this section, we (Alf Coles, Kate le Roux, Armando Solares) engage in a relatively new area of mathematics education, variously referred to as mathematics education and/for sustainability, for sustainable futures, the socio-ecological, the ecological. Descriptions of and motivations for this area look towards the world, commonly referring to (increasingly) visible, and multiple, intersecting social, political, economic, ecological practices that are materialising in: extreme weather events; resource depletion; health emergencies; poverty; inequality; war; forced displacement; rising totalitarianism; lack of tolerance for and silencing of difference. We join a growing number of scholars in asking, what does the world condition mean for mathematics and mathematics education? And what response is mathematics education able to offer?

Concluding a strengths-based survey of "Mathematics Education for Sustainable Futures", Makramalla et al. (2025), discuss three "moves" in the thinking around mathematics education and the socio-ecological. These take the form of considerations:

1. Considering "the nature of being as relational", inclusive of relations between individuals and communities (both human and other living organisms).
2. Considering "agents of knowing", including all humans and non-humans.
3. Considering "the nature of knowledges", and acknowledging multiplicity in ways of knowing (Makramalla et al., 2025, p. 548).

We have taken the first of these moves as our "foundings" and the second and third as the "formings" of our theorising (aiming to offer more of a process of theorising than

a product). We offer these foundings and formings in a contingent manner, and we are alert to how language unavoidably gives an unwanted fixity to conceptual thinking. We take "functionings" to be how we produce meaning in the interaction of founding and forming processes, in the places and contexts of our theorising and we offer one brief example that has been central to our thinking.

FOUNDINGS – THE NATURE OF BEING AS RELATIONAL

At its most basic, the move to consider the nature of being as relational involves recognising the socio-ecological world, and hence being itself, as constituted in processes. These processes include the biophysical, cultural, discursive, symbolic, material, historical, economic, political. No 'subject' exists in and of themselves. This is illuminated for us by Lewontin (2000), who argues that organisms, within a given environment (context), construct each other; they interact multilaterally, not deterministically. Organisms are not mere recipients of the outside world; they do not only adapt and try to survive in a fixed environment, but are appreciated as subjects with a specific constructive condition. Ontologically, disentanglements and separations that set human against more than human, some humans against others, the autonomous individual against the communal, abstract and objective knowledge against placed knowledges cannot hold. We find Mbembe's (2023) notion of 'in-common' productive as language for this co-constitutive relationality, where meaning exists in a third space not captured by the binaries of subject and object.

FORMINGS OF SOCIO-ECOLOGICAL THEORISING

We offer two "moves" which as formings (interpreted as processes of socio-ecological theorising), both noted in Makramalla et al. (2025), and relating to a "gesture of listening for entanglement" in Coles et al. (2024). These moves involve pursuing two related lines of questioning: (i) considering agents of knowing, the socio-ecological knower, and (ii) considering ways of knowing, that is, socio-ecological knowing.

(i) Knowers in a socio-ecological sense, by definition, need one another (Nymanjoh, 2023), are changing and never complete. Knowers have no need to prove their existence. The latter aligns with decolonial arguments that, refusing the need to prove one's humanity is a rejection of the very logic that denies humanity to certain groups (e.g. Fanon, 1968) - dissolving the primacy of individual "knowing".

(ii) Socio-ecological knowing is a humble, contestable, uncertain knowing *in* relation and *of* relations, involving multiple knowers. It involves collapsing categories of being, doing, thinking, and feeling. It includes knowing in contexts of extreme climate events, characterised by floods, droughts, extreme heat, and food, water, and air precarity.

FUNCTIONINGS WITH THE ATOYAC RIVER EDUCATION PROJECT

Prompted by Velasco (2017), we think of the Atoyac River Basin as a socio-ecological relation, including the education project (Solares et al., 2022), involving multiple knowers and ways of knowing. The river is the third most polluted in Mexico, this degradation is an outcome of the deeply unequal arenas in which social, economic,

political, and ecological relations have operated and continue to operate across time and spatial scales. We have reported before (Solares et al., 2022) on this project and we only have space here to highlight one aspect (scales of time). We highlight (in italics) elements of the processes of foundings and formings, discussed above.

A crucial element of this project was recognition of the past, in particular, *considering ways of knowing* such as listening to elders. When such knowing was put alongside present numerical data, it made possible new *relations* to how the river has been. And this recognition was itself crucial in terms of imagining what the river could one day be again. The project resulted in a *group (not individual)* production of a Memorial Museum to the river, which has been taken to communities around the Atoyac Basin.

FINAL THOUGHTS

We return to the questions driving our theorising, what does the world condition mean for mathematics and mathematics education? And what response is mathematics education able to offer? We believe the world condition can no longer be bracketed away from education, including mathematics education. The living world is no longer a benign and fixed background to the process of education. And, we offer the notion of socio-ecological theorising as one way that mathematics education can respond, in a manner which is contextual and involves recognition of our multiple relations.

LOOKING BACK AND AHEAD

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Appealing to Variation Theory as our meta-theoretical perspective, we provide generalisation across three pairings of theoretical advancements in this Research Forum, using the dimensions of foundation, form, and function. Then we conclude with thoughts moving forward.

GENERALISING ACROSS THEORIES: LOOKING BACK

From a Variation Theory perspective, the discernment of critical aspects happens through variation against an invariant background. For the six theoretical advancements in this Research Forum, the dimensions of foundation, form, and function provided contrast within each approach. For generalisation, we look across pairs of different approaches: (Re)conceptualising Resources and the Documentational Approach to Didactics, A Dialogic Theory of Learning and the Theorising of a Role of Time in Covariation, and Inclusive Materialism and Socio-ecological Theorising. We selected these pairs to illustrate how the 3F Framework can be used to compare and contrast theoretical approaches that have differences as well as similarities.

(Re)conceptualising Resources and the Documentational Approach to Didactics

While the foundations of the theorisation of resources (Adler & Bretscher, this Research Forum) and the cross-culturalisation of DAD (Shinno et al., this Research Forum) are different, both emerged in response to educational challenges facing teachers. The theorisation of resources has its foundations in the notion of transparency (Lave, 1991), which means that resources are both visible and invisible, like a window. Hence, resources are not just something teachers use; they also are something teachers do (i.e., teachers can source or re-source). In contrast, DAD has its foundations in French mathematics education research, and consequently also addresses socio-institutional factors, such as teachers' collective documentation. With DAD, resources are artefacts (tangible or intangible), and hence DAD focuses on teachers' use of resources. Axiologically, both approaches commit to the contextualised nature of resources, which means that resources cannot just be transmitted from one location to another. Rather, resources are shaped by context and culture.

The form of each approach puts resources at the centre. With Adler and Bretscher (this Research Forum), there is a categorisation of a range of resources, including not only tangible and intangible resources, but also meta-resources that support work with resources. For example, one meta-resource is a framework which can orient teachers' meaning making around mathematics lessons (e.g., a need for coherence between lesson goals and student participation) (Bretscher et al., 2025). In contrast, with DAD, resources are situated in a resource-scheme-document triad. Schemes are mechanisms by which teachers can transform resources into documentation, to support their instruction. The form of DAD put forward by Shinno et al. (this Research Forum) includes sensitivity to variation in culture and language, emphasising that documentation is shaped by the institutions and traditions in which people are located. To illustrate, teachers' lesson plans became documents when teachers drew on their own knowledge to make decisions about how to use a textbook resource (Shinno & Mizoguchi, 2021). Teachers' decisions were shaped by the Japanese Lesson Study tradition in which their documentation took place.

A function of both approaches is to explain teachers' work with resources, highlighting possibilities that can emerge amidst tensions. With their expansion of DAD, Shinno et al. (this Research Forum) discuss tensions and possibilities that arise when resource work is transposed across cultures. For example, transposing DAD from western to eastern cultures required communication of key ideas, which went beyond just translating words from one language to another. Yet navigating those tensions afforded new possibilities for documentation, when contextualised in Japanese Lesson Study (Shinno & Mizoguchi, 2021). In contrast, Adler and Bretscher (this Research Forum) discuss the tension between human and material resources as change levers at the systematic level, particularly in contexts where resources are limited (see Venkat & Graven, 2017).

A Dialogic Theory of Learning and Theorising a Role of Time in Covariation

The foundations of the Dialogic Theory of Learning (Faustino & Barros, this Research Forum) and the theorising of a role of time in covariation (Moore, this Research Forum) have fundamental differences. Grounded in Skovsmose's programme of research, the epistemology of the Dialogic Theory of Learning views learning as interactive communication, verbal or nonverbal. Researchers using this approach employ a critical perspective, to examine students' dialogic acts. In contrast, Moore's theorising is grounded in von Glasersfeld's radical constructivist epistemology, viewing learning as change in individuals' cognition. Because an individual's cognition is not directly accessible to others, researchers develop models of individuals' cognition based on their interpretations of individuals' observable actions. Amidst these differences, axiological commitments of both approaches position learners as capable knowers and doers.

The form of Moore's theorising constitutes a hierarchical framework, including three models of covariation, associated with increasingly abstract conceptions of time. For example, the first model for covariation is strictly in terms of elapsing time, like watching a video. Here, the attributes in the video co-vary only as the video plays. In the final model, covariation is no longer tied to elapsing time. Hence, students' covariational reasoning progresses as they develop more sophisticated models (Moore, 2025). In contrast, the form of the Dialogic Theory of Learning constitutes a network of six "interacts" which describe how individuals can engage in dialogue. For example, one interact is doubting, which means that mathematics is to be questioned, not just taken as absolute truth. Hence, learning is cultivation, the consideration of how different kinds of knowledge can relate, be disparate, or even contradict.

A function of both approaches is to explain the process of learning. Because the Dialogic Theory of Learning assumes that doubt and knowledge are inseparable (Skovsmose, 2024), the interacts are not intended to bring students to certainty, but rather to make room for new questions and possibilities. For example, Faustino and Barros (this Research Forum) demonstrated how doubting led students to consider whether measurements for a box would be feasible if the box were to be placed on a shelf. In contrast, Moore's theorising is linked to a mathematical concept, explaining advances in students' conceptions of relationships between different quantities' values (Moore, 2025). While the approaches have different foundations and forms, both centre reasoning, emphasising meaning-making, not numerical calculations, as the end goal.

Inclusive Materialism and Socio-ecological Theorising

In contrast to the other pairings, the foundations of Inclusive Materialism (Mukherjee & Sinclair, this Research Forum) and Socio-ecological Theorising (Coles et al., this Research Forum) share a similar ontological perspective. Both take a relational stance on the nature of being. For Inclusive Materialism, this means that mathematical objects, such as triangles, are also material things. For Socio-ecological Theorising, this means that humans are in relationship with other living organisms, such as a river basin, as

well as myths and other non-living, symbolic entities (see Solares et al., 2022). Epistemologically, this means that there is a multiplicity to the nature of knowing. For Inclusive Materialism, knowing triangles or fractions is knowing animate objects, that are not only occupied in the minds of humans, but have "secret lives" of their own (see Dominguez, 2025). For Socio-ecological Theorising, knowing is co-constituted. Mathematics is not separate from the living world, so to know mathematics is also to know the living world. The relational stance of both approaches also speaks to their axiological commitments. Mathematics is something more than a mental activity; it is a relationship between material things and living organisms, including humans.

The form of Inclusive Materialism and Socio-ecological Theorising both involve a process of movement. For Inclusive Materialism, which attaches the materiality of humans and mathematical objects, the movement is from the indeterminate to the determinate. For example, when a person draws a triangle or fraction, they do not give birth to the mathematical object. Rather, the object and the human are co-emerging, together with affect and desire. For example, a young student drew a fraction emerging as a piece of broken glass, with a goal of reuniting with the other fraction friends, namely the remaining pieces of broken glass (Dominguez, 2025). For Socio-ecological Theorising, there are "moves" which constitute socio-ecological knowing. With this approach, one move is to collapse categories rather than separating them into different buckets, each of which is to be analysed individually. With Socio-ecological Theorising, doing and thinking are not separate. For example, to know mathematics in a polluted community is to breathe the heavy stench of particulates that permeate the air, leaving a metallic taste in your mouth. And to know mathematics for generations in such a community is to no longer taste the air, because it becomes one and the same with the knowing. Hence, this way of knowing of mathematics is not just a difference in thinking; it feels different than knowing in a community where the air is clean.

A function of both approaches is to explain mathematics learning while problematising mathematics education researchers' stances towards the objects under study. For Inclusive Materialism, the question is not if someone understands a mathematical concept. Rather, the questions are ones of becoming, for both the mathematical concepts and the humans (see Dominguez, 2025). For example, one question can be how the human and the triangle co-emerge through actions of drawing or shaping with clay (Mukherjee & Sinclair, this Research Forum). In a similar way, with Socio-ecological Theorising, the living world is not a backdrop to mathematics education; they are one and the same. The Atoyac River Basin project reported by Solares et al. (2022) demonstrated how children's mathematical knowing encompassed listening to human relations with the river, in space and time, including elders, who knew of a time when the river was not polluted, as it had been throughout the children's lives.

Fusion and Summary

With the six theoretical advancements, we intended to provide a range of foundations, forms, and functions. Our pairings illustrated convergence and divergence in

theoretical dialogue (Scheiner & Bosch, 2025; this Research Forum). With (Re)conceptualising Resources and the Documentational Approach to Didactics, their foundations diverged, drawing on different traditions from different parts of the world, while their forms converged around the centrality of resources, and their functions converged around teachers' work with resources. With the Dialogic Theory of Learning and the Theorising of a Role of Time in Covariation, their foundations diverged around what constitutes learning (i.e., a change in dialogue vs. a change in cognition), their forms diverged (i.e., a web of interacts vs. a hierarchy of conceptual models), but their functions converged around explaining the process of learning. With Inclusive Materialism and Socio-ecological Theorising, their foundations converged around a relational stance on the nature of being, their forms converged around the process of movement, and their functions converged around the problematisation of "business as usual" approaches to explain mathematics learning.

As with any choice we make in research, our pairings are not neutral, and the pairings we presented are not the only possibilities. For example, had we chosen to pair the Dialogic Theory of Learning with Socio-ecological theorising, we could have seen different evidence of convergence and divergence across the dimensions. For example, their foundations converge around epistemological assumptions that entail collapsing, rather than constructing separate categories of knowledge. Yet, their forms diverge, with the Dialogic Theory of Learning comprising a web of interacts, and Socio-ecological Theorising comprising moves motivated by questions. And their functions converge around the need to contextualise learning within students' communities.

While we presented the dimension as nouns (i.e., foundation, form, and function), Coles et al. (this Research Forum) changed them to verbal nouns (i.e., founding, forming, and functioning). Although Coles et al. organised their contribution in a similar manner to the other advancements, the verbal nouns changed the feel of the dimensions. The verbal nouns highlighted the dynamic nature of theory, pointing to how the dimensions can be part of the act of theorising, and not only a way by which we may examine already established theories.

CONCLUDING REMARKS: LOOKING AHEAD

If mathematics education researchers are to compare and contrast different theories, an understanding of distinctive aspects of those theories is required (Scheiner & Bosch, this Research Forum). The 3F Framework, together with Variation Theory, offers a means for communication with, within, and across theories. When working with or within a theory, the dimensions of foundation, form, and function provide contrast. When working across theories, there is generalisation, as the dimensions can apply to many theories and approaches. The 3F framework provides a lens for researchers to compare and contrast theories. As illustrated in our previous section, one such use is to examine convergence and divergence in theoretical dialogue. Another use can be for the networking of theories, where comparing and contrasting may reveal hidden ideas that are shared, such as the function to explain learning (Faustino & Barros, this

Research Forum; Moore, this Research Forum). Researchers can use their comparison and contrast of dimensions of different theories to make decisions about whether to combine or coordinate different theories in their research.

Just as our research decisions are not neutral, the 3F Framework also provides a particular lens on theoretical working, which can shape discussions around theories as well as theory development itself (Bikner-Ahsbahs & Johnson, 2026). With the dimensions, we assume that theoretical roots and assumptions, as well as the structure of the theory, can shape the way mathematics education researchers understand, use, discuss, and develop theories. We do not mean to suggest that the 3F Framework is the only approach to such theoretical dialogue, and we view a need for further development in this area.

However, the relevance of the 3F Framework is not reduced to the dialogue around theories. It may also spark theoretical inquiry to answer new kinds of research questions, and in turn, can lead to modifications to existing methodologies as well as the development of new methodologies. We conclude by listing some examples:

- Related to the foundation within the axiological dimension of the Theory of Objectification, Radford (2023) has elaborated the concept of communitarian ethics. This could be the starting point for further theoretical inquiry on ethics and aesthetics within and across theories. In a similar vein, ontological assumptions could be explored to compare and contrast various kinds of ontologies of mathematical concepts within and across theories, and their relationships to resources.
- Related to the form of theories, a first question could be what kind of logic holds a theory, its concepts, phenomena, models, and claims together. A follow-up question could then be of whether a modular theory is possible, and how such a theory would inform practice.
- Related to the function of theories we could ask in what way phenomena in a specific theory are made visible that otherwise would remain hidden. Is there a complimentary theory that would make the hidden phenomena visible? What kinds of claims does a theory allow to be made and where are the boundaries? Could these boundaries be bridged (e.g., by theory networking)? How do theories function in different contexts, such as research, teaching research or teaching mathematics?

NOTE

Expanded versions of the six theoretical advancements will appear in the forthcoming book: Johnson, H. L. & Bikner-Ahsbahs, A. (Eds.). (in press). *How Theories Work in Mathematics Education Research – Foundation, Form, and Function.* Springer.

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RETHINKING KNOWLEDGE FOR TEACHING MATHEMATICS WITH TECHNOLOGY: TENSIONS, EXTENSIONS AND FUTURE AGENDA

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The integration of digital technology to promote mathematical learning remains challenging for teachers. The continuous difficulty in fully achieving technology's potential highlights the need to further examine and develop models addressing teachers' knowledge in relation to technology. In this Research Forum, we analyse existing technology-related knowledge models, exploring similarities and differences. We also consider how these models have been used in research, highlighting tensions driven by emerging technologies, and resulting extensions of the models. Finally, we discuss possible integrations across frameworks and the potential to include dimensions beyond knowledge. This discussion aims to deepen understanding of current models and suggest directions and a research agenda for future development.

INTRODUCTION

The integration of digital technology holds potential for enhancing mathematical learning, yet realizing this potential remains challenging. Teachers play a central role, with their professional knowledge strongly influencing this integration (Faggiano et al., 2021).

The persistent difficulty in fully realizing the potential of technology in education points to the relevance of further developing the existing theoretical models. One possible direction for such development lies in reconciling knowledge-based models with other approaches, through the construction of a framework that allows for joint implementation. This perspective is aligned with the work developed by Haspekian (2023), who integrates, into an instrumental approach, a specific framework that models teachers' practices into dimensions not restricted to mathematical knowledge. This is also the case of PTK (*Pedagogical Technology Knowledge*, Thomas & Hong, 2013) already incorporating teachers' personal orientations, and of KTMT (*Knowledge for Teaching Mathematics with Technology*, Rocha, 2020a) explicitly considering both external and internal contexts, the latter encompassing a wide range of teacher-specific aspects. Within this line of research, Tabach (2021) has drawn attention to the notion of competence, considering it indispensable for a meaningful and comprehensive

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 248-275. PME.

integration of technology in teaching. Emerging technologies have turned a long-standing need for the development of theoretical frameworks into a critical necessity. These technologies have reshaped the relevance of existing knowledge domains and foregrounded new ones, leading some authors to propose alternative models (e.g., I-TPACK). This calls for a critical reflection on the adequacy of current frameworks, a discussion central to the AI-TecTeachers project, which brings together most members of this Research Forum (RF).

Drawing on our ongoing discussions and collaborations in several conferences and projects, we argue that reflecting on the existing knowledge models – namely their development, articulation with other frameworks, and adaptation to emerging technologies – is crucial to deepen our understanding of technology integration.

This reflection is the focus of this RF on the topic of *knowledge for teaching mathematics with technology*, guided by the following question and sub-questions:

What do existing knowledge models for teaching mathematics with technology enable us to understand, and how can their future development be envisaged in light of ongoing theoretical, technological, and educational challenges?

(a) How should knowledge models evolve to respond to the demands posed by emerging technologies?

(b) To what extent should future models integrate domains beyond knowledge?

(c) What is the potential of integrating different theoretical frameworks to develop richer and more coherent perspectives on technology integration?

KNOWLEDGE MODELS

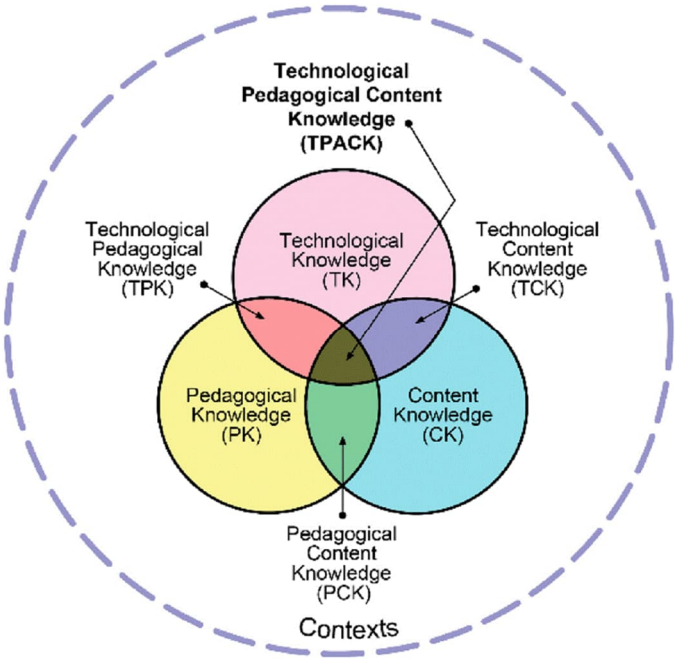
Shulman's PCK – Pedagogical Content Knowledge is a mark in the research on teachers' knowledge. Since then, several authors have developed refinements and new frameworks. The integration of technology raised new questions and led to the TPACK model (*Technological Pedagogical Content Knowledge*, Mishra & Koehler, 2006), inspired by PCK; the KTMT model (Rocha, 2020a), combining insights from research on professional knowledge with findings from studies on technology integration and emphasizing teachers' practice; and the PTK model (Thomas & Hong, 2013), incorporating instrumental genesis into a knowledge framework.

In this section we present three knowledge models with a specific focus on technology.

TPACK – Technological Pedagogical Content Knowledge

The TPACK framework emerged from Shulman's (1987) concept of PCK, which highlights the integration of content and pedagogy, i.e., understanding how specific subject matter can be organized, adapted, and taught effectively. Mishra and Koehler (2006) extended this idea by incorporating technology, recognizing that effective teaching requires not only knowledge of content and pedagogy but also knowledge about how to use technology to support and transform teaching and learning.

Figure 1: TPACK model by Mishra and Koehler (reproduced by permission of the publisher, © 2012 by tpack.org).



TPACK conceptualization (Figure 1) includes the three knowledge domains of *Content Knowledge* (CK), *Pedagogical Knowledge* (PK), and *Technological Knowledge* (TK), along with the intersections among them: *Pedagogical Content Knowledge* (PCK), *Technological Content Knowledge* (TCK), *Technological Pedagogical Knowledge* (TPK). CK refers to understanding the facts, concepts, theories, and structures of a discipline, while PK is generic and involves knowledge of teaching methods, learning processes, instructional goals, and strategies for students' assessment. TK encompasses the knowledge necessary to operate the technology. PCK is consistent with Shulman's conceptualization. TCK refers to understanding how technology and content influence each other. Teachers must not only be knowledgeable about the subject they teach but also understand how technology can transform the way that content can be presented and learned. TPK refers to understanding the features, and potential of different technologies in educational settings, as well as how teaching practices can be adapted or transformed through their use. TPACK is seen as "the connections, interactions, affordances, and constraints between and among content, pedagogy, and technology" (Mishra & Koehler, 2006, p. 1025).

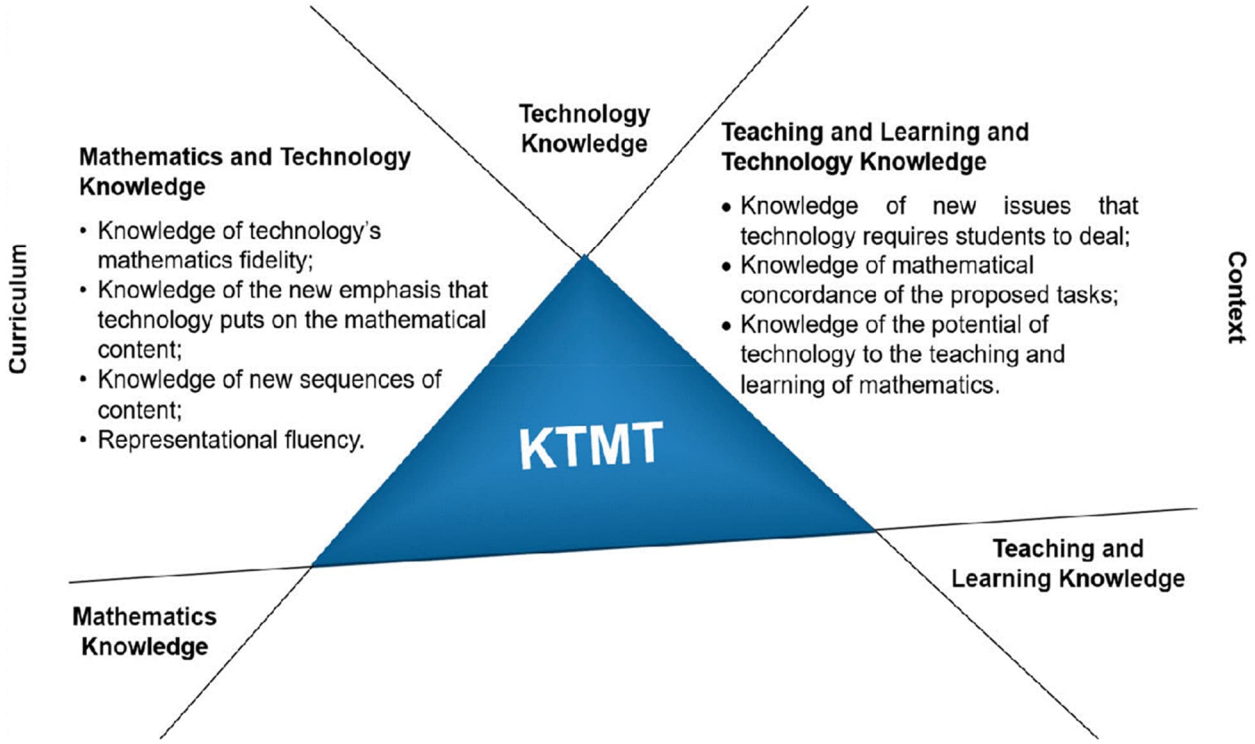
Context has evolved within the model. Initially absent, it was later acknowledged and, more recently, treated as a distinct knowledge domain, recognizing that contextual factors shape the practical application of TPACK in real classrooms (Mishra, 2019).

KTMT – Knowledge for Teaching Mathematics with Technology

The KTMT model by Rocha (2020a) builds on the work of Shulman (1987) and Mishra and Koehler (2006), while aiming to go beyond the TPACK framework. The model

(Figure 2) intends to integrate the existing research on teachers' knowledge and on technology integration into a single comprehensive model. With a particular focus on Mathematics teaching, the model includes base knowledge domains, two sets of inter-domains knowledge, and *Integrated Knowledge* (IK) in a pyramidal structure.

Figure 2: KTMT model by Rocha (2025).



The base knowledge domains are four (the last one being transversal and influential over the others): Mathematics, Teaching and Learning, Technology, and Curriculum and Context. Context includes the internal context, such as personal influences (e.g., teachers' beliefs), and external context, such as the school environment. The other domains, drawing from the earlier works of Shulman (1987), Ball et al. (2008), and Mishra and Koehler (2006), are understood basically in the same way. Inter-domain knowledge plays a central role in the model, distinguishing it from other knowledge frameworks. The KTMT model considers two sets of inter-domain knowledge: the Mathematics and Technology Knowledge (MTK), and the Teaching and Learning and Technology Knowledge (TLTK). MTK focuses on the impact of technology on mathematics, addressing how it enhances some elements while constraining others, and TLTK focuses on the impact of technology on the teaching and learning process, addressing how it enhances some approaches while constraining others. Both inter-domain knowledge areas are informed by research on technology integration, incorporating key findings of the research in the field. As so, MTK includes:

- knowledge of the technology's mathematical fidelity, meaning an understanding of how closely the output produced by technological tools aligns with the mathematical result (Dick, 2008);
- knowledge of the new emphases that technology places on mathematical content (e.g. knowing how the values represented on the coordinate axes can affect the appearance of the graph displayed by the technology) (Rocha, 2020b);
- knowledge of new sequences of content (Drijvers, 2013);
- knowledge of representations or *representational fluency*, including knowledge of how to navigate between various forms of representation and the choices that can be made to effectively illustrate or support mathematical concepts (Rocha, 2016; Zbiek et al., 2007).

TLTK includes:

- knowledge of new issues that technology requires students to deal with (e.g. what is the best strategy to look for a proper viewing window (Rocha, 2020b));
- knowledge of the mathematical concordance of the proposed tasks, i.e. the alignment between teachers' objectives and the mathematical concepts that students are actually engaging with in their learning (Zbiek et al., 2007);
- knowledge of the potential of technology to the teaching and learning of mathematics, i.e., knowledge of how to coordinate various types of tasks, leveraging the opportunities provided by technology, and allowing students to assume a more central role in their learning process (Dunham, 2000).

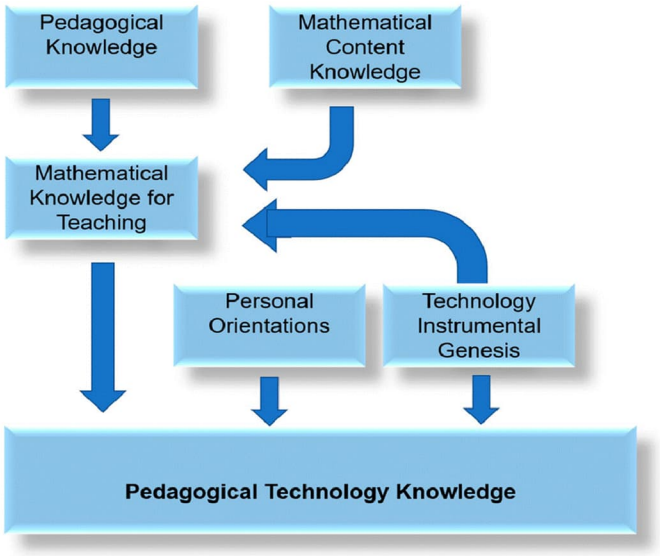
IK represents the final type of knowledge within the KTMT model, emerging from the interplay (and not intersection) among all the various knowledge domains.

PTK – Pedagogical Technology Knowledge

This model (Figure 3) adopts a very different perspective from the previous models, highlighting the use of technology and how the technological tool can be transformed into different instruments, in line with Rabardel (1995). However, it is possible to identify in this model, as in the previous ones, the influence of the work of Shulman and his notion of PCK.

Thomas and Hong (2013, p. 69) characterise the teachers' PTK as including "the principles, conventions and techniques required to teach mathematics through the technology". With its roots in the work of Shulman, this model considers as domains the *Mathematical Content Knowledge* and the *Pedagogical Knowledge*. However, instead of PCK, it considers the work of Ball et al. (2008) and their MKT – *Mathematical Knowledge for Teaching*. Besides these, the authors assume as important two domains that are not knowledge domains. That is the case of *Technology Instrumental Genesis*, assumed as highly relevant, and of *Personal Orientations* also with a strong impact recognised by the authors. All these domains contribute to PTK.

Figure 3: PTK model by Thomas and Hong (2013).



Differences and similarities across the knowledge models – discussing the models

The models presented here have several differences across them, but also several similarities. Some of these differences and similarities are related to the domains considered. We will get back to this in the section about the *Global Model*, here we discuss how the models conceived the development of the knowledge, addressing also how they are used in research.

The conceptualisation of the TPACK model is not so clear about how knowledge development is conceived. Some authors interpret the model in an integrative perspective and some others in a transformative one (Angeli & Valanides, 2009). As so, it is not clear if knowledge in two base domains (*Content*, *Pedagogy*, or *Technology*) results straightforward in knowledge in the intersection of these domains – the integrative perspective – or if it requires the development of some new knowledge – the transformative perspective. In the uses of the model in the literature, it is possible to find both interpretations.

KTMT has a pyramidal or a layer structure. As other models intending to conceptualise the knowledge needed to teach with technology, this model assumes a need for knowledge going beyond knowing how to operate the technology, valuing knowledge about the impact of technology in Mathematics and in the teaching and learning process (the two inter-domains knowledge). Instead of intersections as in TPACK, the inter-domain knowledge is a new knowledge developed from more than one *base knowledge* and integrating in its characterisation results from the research on technology integration. And the IK develops from the base and the inter-domains knowledge in a similar way. This is an on-going process with a continuous development of the knowledge in all the domains of the model. This means that although the model has a pyramidal or layer structure, the development on the three layers (the layer of the *base*

knowledge, the layer of the *inter-domain knowledge*, and the layer of *integrated knowledge*) is not sequential. As so, teachers with *Integrated Knowledge* can still continue to develop their knowledge in one (or more) of the *inter-domains* or in one (or more) of the *base domains*. It also means that the development of knowledge in one domain does not result straightforwardly in the development of knowledge of another domain (namely at an upper level or layer). It is only by the development of knowledge resulting from the interaction of the new knowledge in one domain with knowledge in some other domain that new knowledge can result (e.g. some new knowledge about technology does not result immediately in new MTK, there is a need for some interaction, with the new knowledge on technology impacting the *Mathematics Knowledge* and resulting in the development of new knowledge – the new MTK).

Regarding PTK, the main difference is that the model incorporates instrumental genesis as central in technology integration. However, this model has not been operationalized in research to analyse teachers' knowledge in practice. The few existing studies seem to be based on quantitative approaches, using the model as a basis for the development of a questionnaire. In fact, as highlighted by Tabach (2021), this is a relatively uncommon approach in existing literature.

On the contrary, TPACK seems to be the most used knowledge model. TPACK is applied in both quantitative and qualitative studies. In the latter, TPACK is often employed holistically and articulated with other theoretical frameworks, functioning more as a general guiding idea than as an analytical model in which each knowledge domain is examined in depth and separately. In contrast, the KTMT framework, due to its strong connection to classroom practice, is more frequently used in qualitative approaches that place significant emphasis on the distinct domains it encompasses. Its structure allows for a more fine-grained analysis of teachers' practices and the multiple dimensions involved in technology use.

The continuous emergence of new frameworks – from PCK to TPACK, KTMT, and PTK – shows that teachers' knowledge is not a settled construct but a dynamic theoretical object that must be repeatedly rethought as mathematical practices, tools, and classroom activity evolve, making it a persistent central concern for both mathematics education research and the PME community.

Questions for discussion:

(1) What core assumptions about teachers' knowledge and teaching practice are shared by TPACK, KTMT, and PTK, and where do they diverge?

(2) How does each framework conceptualize the relationship between content, pedagogy, and technology?

(3) What conceptual tensions and limitations emerge from a comparative analysis of these frameworks, and how do they inform potential extensions or refinements?

DEVELOPMENT OF KNOWLEDGE MODELS

In this section we start by foregrounding how established teacher-knowledge models are used with contemporary technologies, and what *tensions* this use makes visible. The focus is to show how the models operate as lenses for design and analysis, and how emergent technologies pressure their assumptions. Then, we show how the emergence of Artificial Intelligence (AI) has led scholars to explore teachers' professional knowledge specific to the use of AI-based digital technology and suggest extensions to existing knowledge models.

Enactment of different models with traditional and emergent technologies

We begin with TPACK enactments around dynamic digital tools; then bring in two PTK enactments, one in the context of computational thinking/programming at scale, and one within a Community-of-Inquiry professional development design; and close with two recent KTMT enactments with Generative AI (GenAI).

In a modelling and simulation program (Quarder et al., 2025), TPACK is specialised into four tightly coupled knowledge dimensions (theory, task, process, intervention) and woven across preparation, practice and reflection. In this enactment, TPACK both specifies what knowledge teachers need when simulations organise the modelling process, and structures how the course must interweave that knowledge so teachers can diagnose process difficulties and choose interventions that sustain autonomy and mathematical verification. The key tension is that TK cannot be handled in isolated sessions; it must be distributed across design, enactment and reflection to orchestrate simulation-based activity with integrity.

A complementary topic/tool-specific TPACK enactment follows pre-service teachers who plan, implement, video-analyse and re-design lessons on polygons with GeoGebra (Kartal & Çınar, 2022). TPACK becomes a performance lens to distinguish demonstration from inquiry (e.g. whether students "touch" the tool, use dragging to reason about invariants, or use measurement as a springboard for argument). This use of the framework surfaces an enactment gap: robust TCK/TPK plans often revert to teacher-centred routines under classroom time and management pressures, indicating the need for explicit links to evidence of practice (e.g., video analysis) to support plan-to-action translation.

A national-scale enactment (Turgut et al., 2024) operationalises PTK at scale for computational thinking (CT) in mathematics by Rasch-validating three components: mathematical knowledge for teaching with technology (MKTwT), instrumental genesis (IG), and personal orientations (PO). Distinguishing personal from professional IG, and capturing PO (e.g. pragmatic vs. epistemic value, time costs, by-hand skills) explains why uptake can stall despite positive attitudes. Three *tensions* follow: (i) a persistent gap between positive orientations and the hard work of integration reframing TK as a dual IG to be developed personally and professionally; (ii) years of experience or generic professional development are weak predictors, while

school level matters, signalling integration as situated, content-bound, and likely to need grade-specific CT articulations; (iii) large-scale measures show where work stalls but need to be paired with in-situ qualitative analyses of orchestration/design to explain why reported confidence coexists with limited peer support or code-level reasoning.

A second enactment situates PTK in a Community-of-Inquiry (CoI) professional development (Anabousy & Tabach, 2022): in-service teachers iterate plan - enact - video-reflect - redesign inquiry cycles. PTK components show strong mutual relations (notably, confidence and IG). PTK improves across components except content knowledge, with differential gain by seniority and prior technology integration level, indicating that social–organisational support is needed to consolidate PTK in practice.

We finally report on two KTMT enactments that analyse teachers' interaction with Generative AI. In a design–interaction–reflection trajectory (Bernardi et al., 2025), KTMT locate TK in the practical craft of prompting, MTK in teachers' fidelity judgments, and TLTK in the orchestration of step-wise guidance and collective critique, while Curriculum & Context domain regulates what can count as mathematically acceptable versus productively debatable. Used as an analytic scaffold, the model shows how knowledge becomes visible in enactment, and how GenAI reconfigures relations among TK, MTK and TLTK.

Another study using KTMT (Noster et al., 2024) analyses teachers' ChatGPT logs: prompting strategies are read as TK, the co-presence of correct and plausible but wrong outputs through MTK's fidelity lens; and follow-ups (eliciting, justifying, or repeating prompts) as fragile or emerging TLTK. KTMT provides a vocabulary for linking observable interaction patterns to knowledge domains and exposes task dependence and prompting choices as conditions for both fidelity and pedagogy.

The enactment cases considered offer a coherent view of what knowledge models reveal when technologies reshape mathematical activity and highlight a common set of *tensions*. TPACK, when specialised and phase-embedded, shows that TK must be woven through design, enactment and reflection and operationalised by genres (e.g., simulation-based modelling; dynamic geometry), and that teachers' orchestration of dynamic systems (not just pre-lesson plans) is pivotal. PTK, with its measurable MKTwT–IG–PO components, foregrounds instrumental genesis and beliefs, explaining why positive orientations can coexist with shallow uptake, why integration is situated rather than generic, and how communities of inquiry help convert confidence into enacted competence. KTMT, in GenAI settings, shifts attention to the fragility/negotiability of fidelity, the status of prompting as specific TK, and the need to conceptualise teaching as the orchestration of a conversational agent within mathematical norms. These tensions have direct *implications* for model development: making negotiated fidelity explicit; representing prompting strategies and verification routines within TK (and challenging what is only TK or includes other types of knowledge); theorising orchestration as managing dialogue with and around digital

systems; adopting phase-embedded and performance-sensitive designs when activity is reorganised by technologies.

These *tensions* set up the next section where we examine *extensions* addressing these pressures, review TPACK variants that integrate reliability and ethics, and consider links with instrumental and documentational approaches to understand how teachers learn to orchestrate AI-mediated mathematics responsibly and effectively.

The case of TPACK adaptations to AI

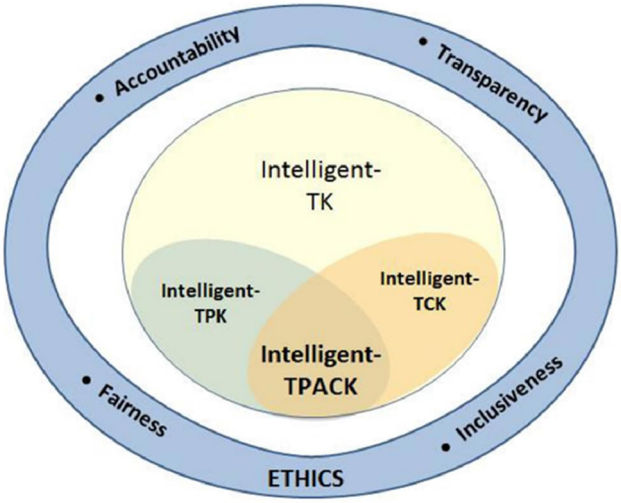
Based on the literature, we discuss how the *tensions* posed by AI-based emerging technologies are leading to *extensions* to the existing knowledge models, and outline the main characteristics of the approaches being adopted.

Most of the knowledge models developed to account for the specificities of AI-based technology are extensions of the TPACK model. In what follows, we focus on three extensions and reconceptualizations of TPACK developed to account for the unique characteristics of AI – its agency, content-generation capabilities, and inherent ethical complexities (Celik, 2023; Chiu, 2025; Lan et al., 2025).

All the authors argue that the nature of AI requires an evolution of the TPACK model. The core limitations identified across the studies include: (1) *tool agency and autonomy*: AI tools act as "partners" or "agents" that generate content, make decisions, and interact with users autonomously, shifting the teachers' role from operating a tool to managing an intelligent system; (2) *epistemic challenges*: since GenAI can produce sophisticated, human-like text and media (Chiu, 2025, p. 4191), teachers need to critically curate, validate, and contextualize AI-generated content, guarding against bias and "hallucinations"; (3) *ethical complexity*: privacy, fairness, transparency, and accountability are central to the responsible use of AI in the classroom; (4) *dynamic collaboration*: the interactive nature of AI opens possibilities for new forms of human-AI collaboration (Chiu, 2025, p. 4189), thus teachers need knowledge to strategically orchestrate human-AI synergies.

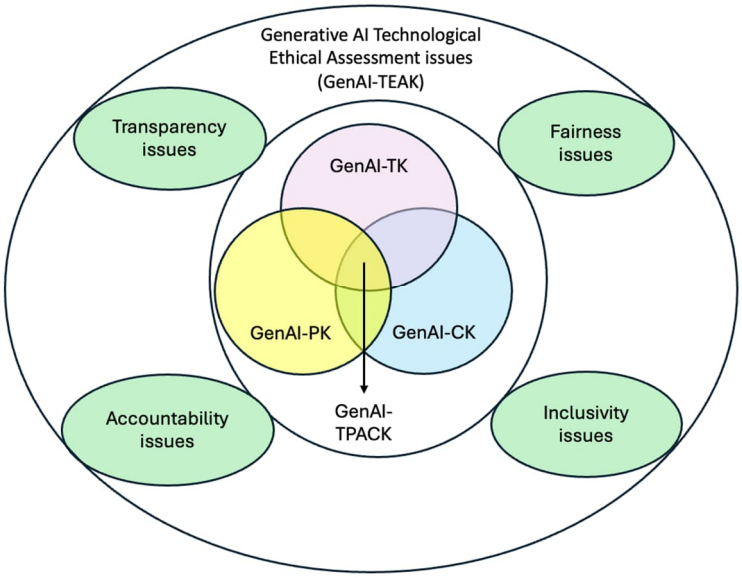
In response to these limitations, each article proposes a modified framework. While sharing a common goal, they differ in scope, structure, and methodological approach.

Figure 4: The Intelligent-TPACK framework proposed by Celik (2023).



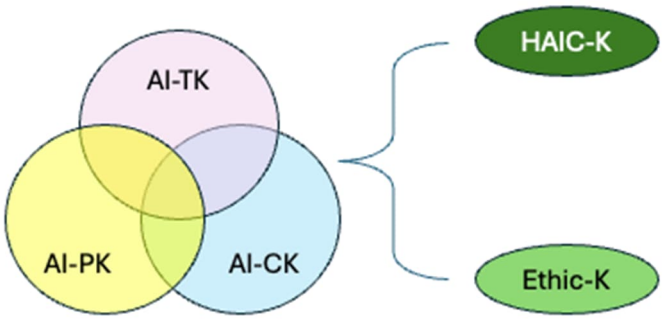
Celik's (2023) framework focuses on a broad range of AI-based tools, including chatbots, intelligent tutoring systems (ITS), AI-enhanced dashboards, and automated assessment systems. The proposed *Intelligent-TPACK framework* (Figure 4) extends the traditional model by adding a distinct fifth knowledge component: *ethics* that measures a teacher's ability to assess AI-based decisions concerning four ethical aspects: transparency, fairness, accountability, and inclusiveness.

Figure 5: The GenAI-TPACK framework, adapted from Lan et al. (2025).



Building on similar premises, Lan et al. (2025) narrow the focus to generative AI and propose the *GenAI-TPACK framework* (Figure 5), which incorporates *Generative AI Technological Ethical Assessment Knowledge* (GenAI-TEAK) as an essential component. This domain is defined as the ability to evaluate the ethical implications of GenAI tools regarding transparency, accountability, fairness, and inclusivity.

Figure 6: The I-TPACK framework, adapted from Chiu (2025).



Rather than an extension of the TPACK, Chiu (2025) proposes a radical reconceptualization, termed the *Intelligent-TPACK* (I-TPACK) framework (Figure 6) to bridge the gap between AI literacy and AI competency. The I-TPACK framework introduces five, interdependent knowledge domains: *AI-Technological Knowledge* (AI-TK) that includes prompt engineering, algorithmic awareness, and system auditing; *AI-Content Knowledge* (AI-CK) needed to critically curate, validate, and contextualize AI-generated disciplinary content, including bias detection; *AI-Pedagogical Knowledge* (AI-PK) necessary to design AI-augmented learning experiences, redesign assessments, and structure human-AI co-agency; *Human-AI Collaborative Knowledge* (HAIC-K), a novel domain focused on structuring interactions where teachers, students, and AI coordinate cognitive tasks; and *Ethical Knowledge* (EK), centralized as a core domain that permeates all other domains, addressing data privacy, algorithmic justice, and other moral dimensions. Chiu's framework is unique in its granularity, its redefinition of the original TPACK components for the AI context, and its introduction of HAIC-K. It positions teachers as "augmented pedagogy designers" who orchestrate complex human-AI synergies.

Development of knowledge models – discussion

The three studies with focus on AI provide a multi-layered understanding of the evolving teachers' knowledge base. They exhibit powerful commonalities in their foundational arguments while also offering unique contributions. Celik's (2023) and Lan et al.'s (2025) frameworks highlight the centrality of ethical knowledge as a core component. The two studies converge on acknowledging that TK serves as a necessary foundation that positively impacts the development of TPK, TCK, and ethical knowledge, which in turn predicts effective integration. Chiu's (2025) framework is the only one to formalize Human-AI Collaborative Knowledge (HAIC-K). This is a significant contribution, as it reframes AI from tool to partner, foregrounding teachers' professional knowledge for orchestrating human-AI collaboration, that is managing the dynamic interplay between human and AI in the classroom.

Questions for Discussion:

(1) How have emerging technologies prompted adaptations or extensions of TPACK, KTMT, and PTK in the context of mathematics teaching?

(2) In what ways do these adaptations highlight specific aspects of teachers' knowledge that are emphasized, transformed, or newly conceptualized within the extended or hybrid models?

(3) To what extent do the adaptations observed in knowledge models represent incremental refinements versus more fundamental conceptual shifts?

(4) How do technology-specific extensions (e.g., I-TPACK) balance the need for theoretical coherence with the risk of framework fragmentation?

COMBINING FRAMEWORKS – A POSSIBLE APPROACH?

While emerging technologies may highlight specific knowledge domains, frameworks themselves evolve continuously. Accordingly, we focus on how combined frameworks can offer more effective lenses to study the complexity of technology integration.

We address this point from two different perspectives: integrating different models in a single one and combining a knowledge model with other frameworks. Our focus is on the models and on discussing the potential of combining frameworks.

The Global Model – combining TPACK, KTMT, PTK

Rocha (2025) studied similarities and differences across TPACK, KTMT and PTK, knowledge models identified in a literature review (Rocha, 2023), and proposed the *Global Model*. Knowledge of Mathematics domain is present in all the models, although with different names: *Content Knowledge* for TPACK, a model that is not specific for Mathematics; *Mathematics Knowledge* for KTMT; and *Mathematical Content Knowledge* for PTK. Another domain present in all the models relates to pedagogical knowledge (*Pedagogical Knowledge* in the case of TPACK and PTK, and *Teaching and Learning Knowledge* in the case of KTMT). Although the names differ, in the case of the KTMT model it emphasizes teaching and learning dimensions, yet in all cases the underlying understanding is similar.

Some recognition of the relevance of contextual issues is also present in all the models, although in this case with significant differences.

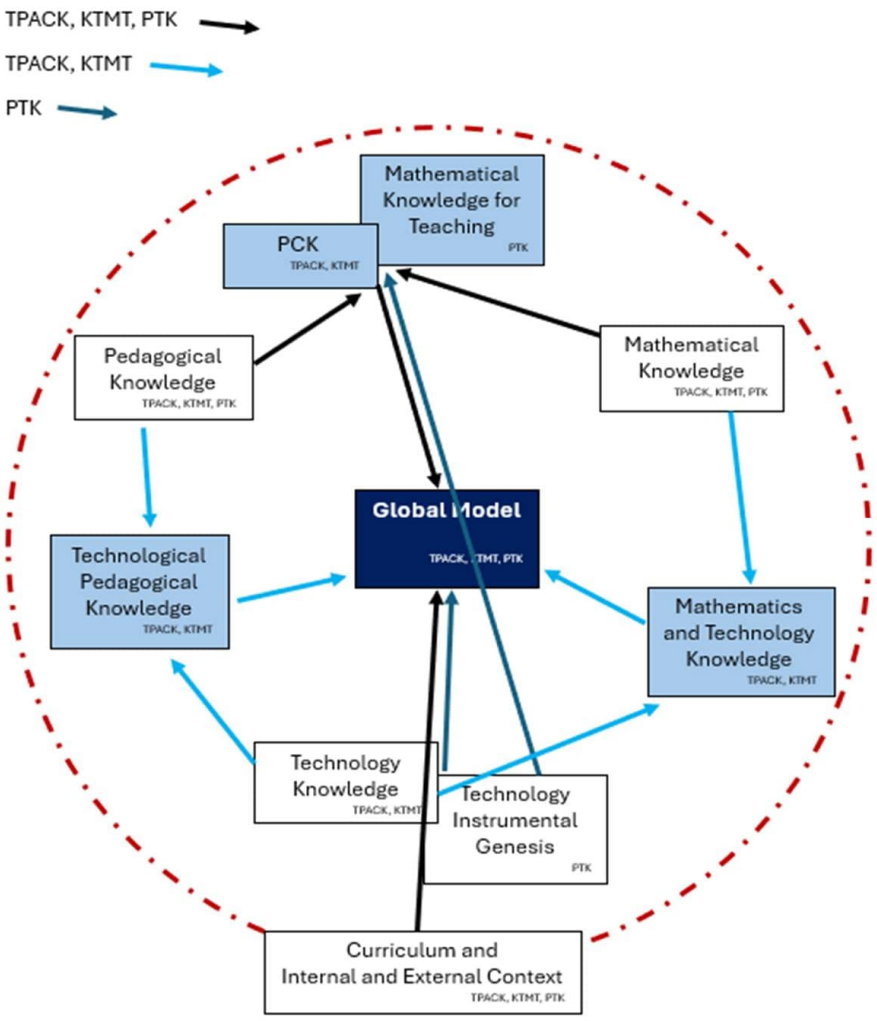
TPACK includes *Contextual Knowledge*, referring to contextual circumstances, external to the teachers but with impact on their practices. KTMT considers *Curriculum and Context*. In this case, the context includes internal and external influences on the teachers, *Curriculum* being assumed as the prescribed curriculum and, as so, one of the external influences over the teachers. So, the understanding of context is more comprehensive in the case of KTMT, including the understanding assumed in TPACK that is basically related to external influences over the teachers, but going beyond it, namely in what relates to external influences, where it also

includes the curriculum. The PTK model includes *Personal Orientations* without any reference to context. However, this domain is closely related to the understanding of internal context present in KTMT. Globally, we can assume that TPACK considers the external context of the teacher, PTK the internal context and KTMT both the internal and the external contexts.

A significant difference between the models is present in the domain related to technology. In the case of TPACK and KTMT it is considered a domain of *Technology Knowledge*. In the case of PTK it is considered *Technology Instrumental Genesis*. This means that TPACK and KTMT focus on a technical knowledge about how to operate the technology, but PTK, focusing on instrumental genesis, values the instrumentation and the instrumentalization, in line with Rabardel (1995). As so, it is not simply a matter of knowing what the technology does, but also of knowing what it does that can turn it into a useful instrument for Mathematics.

Globally, combining the three models and attending to the similarities and differences across them, results in the representation of the *Global Model* presented in Figure 7.

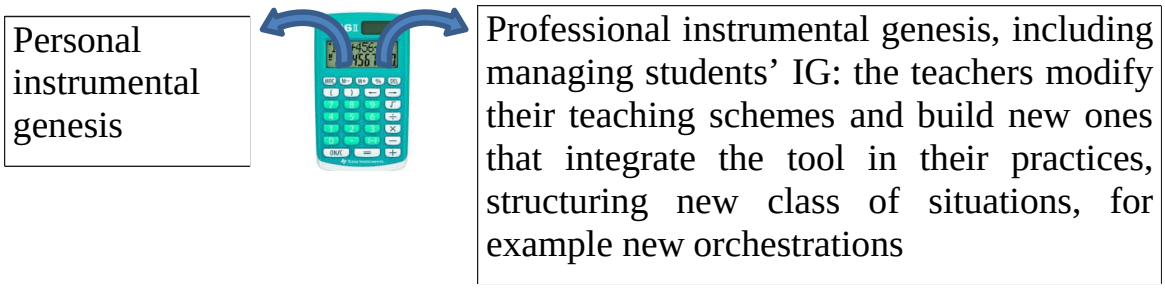
Figure 7: The *Global Model* © 2024 by Helena Rocha is licensed under CC BY 4.0. https://creativecommons.org/licenses/by/4.0/



TPACK and extensions of the Instrumental Approach to teachers

The *Instrumental Approach* (IA) used in mathematics education stems from a theoretical and cultural tradition distinct from TPACK foundations. It relies on Vérillon and Rabardel's (1995) work in cognitive ergonomics, thus, its roots trace back to Gérard Vergnaud (one of the founding figures of PME conferences). With the concept of *Instrumental Genesis* (IG), Rabardel (1995) highlighted the non-neutral impact of tools on conceptualizations, shedding light on phenomena observed in research on symbolic calculators (Artigue, 2002; Lagrange, 1999). With these tools, mathematics differs from traditional institutional content, this gap being later called *Instrumental Distance* (ID). To address both institutional and instrumental dimensions, the authors combined IA with the anthropological theory of didactics (Chevallard, 1992), giving rise to the *Instrumental approach in mathematics education* (IAME). Combined with other frameworks, IAME has led to theoretical conceptualizations extending beyond content knowledge alone. Initially centered on the students' instrumented learning, IAME progressively extended towards teachers, e.g., with the concept of *Instrumental* Orchestration (Trouche, 2004), the *Documentational Approach* (Gueudet & Trouche, 2009), or by applying the concept of IG directly to teachers (Haspekian, 2014), introducing the *Double IG* concept (Figure 8).

Figure 8: Double instrumental genesis of the teacher (Haspekian, 2014).



The integration of a new artefact raises questions falling under the professional IG such as which tasks, orchestrations, assessments to design, which common misconceptions and remedies to anticipate, or how long teaching concepts may take. These didactic reference points (*repères didactique*s) used in everyday teaching must be rebuilt. The bigger the ID, the greater the loss of such *teaching reference points* (RPs). Haspekian (2017) defined them as professional anchors that guide teacher's actions and are characterized by combining IAME with a teacher-centered framework, the Double Approach (DA) (Robert & Rogalski, 2002). This extension provides ground for drawing comparisons with the TPACK model.

In the DA framework, teaching practices are seen as a coherent multidimensional system that brings into play five components: cognitive, mediative, social, institutional, and personal. Haspekian situates the RPs in the personal component, and uses the other four to describe how they function as constituents that guide activity (Table 1). RPs thus go beyond content knowledge, they can be institutional and social (these

components strongly determine teachers' practices); didactic (related to teachers' cognitive and mediative classroom choices); or biographical (explaining individual cognitive and mediative choices within the institutional and social constraints).

Table 1: Possible RP for analysing the personal component of teachers' practices.

Institutional and social dimensions	Institutional RP: constraints related to curricula, school hours, official recommendations…
	Social RP: constraints related to the school area and students' sociocultural backgrounds
Didactic dimensions	Cognitive RP: organization of contents to be taught and cognitive pathways planned for students.
	Mediative RP: organization in space, time, discourse accompanying sessions with students.
Biographic dimension (epistemology, representations, beliefs, culture…)	Epistemological RP: teachers' personal view of mathematics, its teaching, and learning, derived from personal representations/beliefs and the cultural context
	Pedagogical RP: teachers' personal view on teaching and learning in general, derived from personal representations/beliefs and the cultural context

ID and RPs impact the decision to integrate a new artefact, each dimension potentially weighing positively or negatively. A big ID may disrupt teachers' RPs but also be of cognitive interest (by supporting learning of particular mathematical concepts, offering visualizations, etc.). The perceived didactic legitimacy of the tool may encourage its integration despite the loss of RPs. The institutional dimension may hinder the integration if the ID is too big regarding the prescribed curriculum. Conversely, this dimension may favor integration if curricula legitimize the tool use.

How do the RPs framework relate to the TPACK model?

Both perspectives share the focus on teachers' use of technology to teach mathematics, and both have the potential to nurture professional development. Yet, the issues addressed are different. Unlike TPACK, RPs do not aim to categorize knowledge, but to analyze teachers' practices in order to understand the difficulties they meet as well as the strengths they rely on when they successfully integrate technology.

This difference leads to a more structural one. While TPACK seeks to categorize knowledge according to its nature, RPs rather seek to understand their functions, i.e., their effective instantiation in teaching practices: what role do teachers' beliefs and knowledge – conscious or unconscious, pedagogical, content-related, technological, and their intersections – play in teaching? Teachers may have PCK of "differentiated

pedagogy" and strong convictions about its benefits for students, yet may not integrate it, missing RPs for its concrete implementation in classroom. Hence, the RPs lens seeks to address an explanatory, even predictive, approach to the dynamics at play, rather than a descriptive approach to a fixed and normative teacher knowledge model. Consequently, RPs are structured with categories that relate to their functions, while TPACK is structured by categories that relate to their nature.

Table 2: Identifying RPs within the TPACK, based on Tabach (2011) [left column].

Excerpts from Tabach (2011, p. 250): TPCK characteristics and operational description in teacher's utterances		Didactic RP according to the components of the Double Approach
TPCK characteristics (Mishra & Koehler 2006, p. 1029)	Type of evidence looked for – utterances which …	
An understanding of the representation of concepts using technology	Refer to algebraic concepts and their representation in a spreadsheet environment	Cognitive RPs and personal representations of the mathematical contents
Pedagogical techniques that use technologies in constructive ways to teach content	Include the teacher's pedagogical decisions which refer to the mathematics and the availability of Excel	Mediative and epistemological RPs of the contents
Knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face	Reflect the teacher's familiarity with ways students perceived concepts – correctly or incorrectly – in the presence of Excel	Cognitive RPs
Knowledge of students' prior knowledge and theories of epistemology	Reflect the teacher's familiarity with students' personal knowledge, difficulties and their theoretical sources	Cognitive and epistemological RPs of the contents
Knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones	Refer to actions taken by the teacher aimed at promoting mathematical knowledge with respect to technological affordances	Idem

The emphasis on instantiation is another key difference. RPs are not fixed pieces of knowledge but situational updates of knowledge, skills, and expertise, as well as more volatile elements such as judgments, perceptions, and intuitions specific to a teaching moment. For example, a math teacher says she cannot design assessments prior to sequences; she needs to take RPs on what students have actually engaged with during the sequence. While she certainly needs some PK or CK too, this illustrates the situational role of RPs, not given in advance nor fixed but specific to each class history.

Other researchers have also sought to move beyond the descriptive nature of TPACK to focus on more dynamic aspects. Some have done so by incorporating the concept of IG, as shown in the previous sections. Similarly, Tabach (2011) jointly used TPACK and instrumental orchestration to study the dynamic evolution of teaching practices, relating changes in teachers' expressed knowledge to changes in the observed orchestrations. In doing so, she highlighted several common points between the two frameworks. Drawing on her table of TPACK characteristics, we add the components of the DA to further the comparisons with RPs (Table 2). This shows that TPACK neither differentiate between choices and constraints, nor does it explicitly account for institutional and social dimensions that impact teachers' choices and activity. By contrast, it strongly emphasizes cognitive and mediating components, providing a holistic view of the different types of knowledge that may be needed in these components. These statements reflect that TPACK is not designed to explain teaching activity, it tackles the issue of technological integration in a different way from the RPs perspective.

Knowledge models developments by combining frameworks – discussion

Combining different knowledge models or combining a knowledge model with a different framework can be one option to develop more powerful analysis lens. However, the process is challenging and brings to the front several questions.

Questions for Discussion:

(1) What are the conceptual and practical benefits of combining knowledge frameworks for understanding technology integration in mathematics classrooms?

(2) To what extent can integrated frameworks address limitations of individual models, and where might tensions or inconsistencies emerge?

(3) What are the potential and limits of combining frameworks as a way forward?

BEYOND KNOWLEDGE

In this section we address another direction for future development of knowledge models: the integration of domains beyond knowledge, with a focus on teachers' competence.

Technological pedagogical content knowledge or knowledge for teaching mathematics with technology, specified in the discussed models and frameworks, is the basis or condition for developing learning processes. However, a teacher needs, beyond

knowledge, the ability to incorporate or apply the knowledge to the teaching process and that in turn needs an orientation concerning the goals of the teaching and learning processes. Teaching and learning are (nearly) always two sides of one coin. These domains, knowledge, teaching abilities and orientation, can be encapsulated in the concept of teachers' competencies.

In this subsection we address a direction for future development which is beyond knowledge. We ask for competence models and focus on teachers' digital competencies. Next, we refer to different ways in which teachers may adopt technology or digital media to their classroom practices. This will set the ground to delve into digital competencies for teaching. Finally, we bring two conceptualisations within mathematics education for digital competencies for teaching mathematics.

The SAMR-Model

Teaching competencies concerning digital media can also be characterised by the changes in content that teachers aim to achieve using digital media compared to technology-free teaching. According to Puentedura's (2006) SAMR model, four types of teaching can be distinguished in this regard. The four letters SAMR stand for:

S - *Substitution*: Analogue media are replaced by digital media, whereby the content requirements for learners remain the same and no extended functions of digital media are used.

A - *Augmentation*: Analogue media are replaced by digital media, whereby the content requirements for learners remain the same, but the expanded functionalities of digital media are utilised.

M - *Modification*: Traditional tasks for learners are modified in such a way that new content-related requirements emerge, for whose completion digital media, with their specific potential, are necessary.

R - *Redefinition*: Teaching is designed in such a way that learners are faced with fundamentally novel content-related demands that are accessible only through digital media.

In mathematics lessons, it must be decided by the teachers on a case-by-case basis with regard to pedagogical and didactic goals whether and how the digital media can be used for the development of students' competencies.

Digital competencies

The "European Framework for the Digital Competence of Educators" (DigCompEdu) (Redecker, 2017) provides a framework "to promote the digital competence of their citizens and boost innovation in education" (EU, 2017, Foreword). DigCompEdu emphasises six main areas of the professional activities of educators:

- Area 1: *Professional Engagement*. Using digital technologies for communication, collaboration, and professional development.

- Area 2: *Digital Resources*. Selecting, creating and sharing digital resources.
- Area 3: *Teaching and Learning*. Managing and orchestrating the use of digital technologies in teaching and learning.
- Area 4: *Assessment*. Using digital technologies and strategies to enhance assessment.
- Area 5: *Empowering Learners*. Using digital technologies to enhance inclusion, personalisation and learners' active engagement.
- Area 6: *Facilitating Learners' Digital Competence*. Enabling learners to creatively and responsibly use digital technologies for information, communication, content creation, wellbeing and problem-solving.

One aim of these activities, especially in the field of digital problem solving, is "to incorporate learning activities, assignments and assessments which require learners to identify and solve technical problems, or to transfer technological knowledge creatively to new situations" (p. 23). However, this transfer also provides the teachers with feedback "to create new, original and relevant content and knowledge" (p. 82). Note that these digital competencies are for educators in general, not specific to mathematics educators. Next, we focus on competencies specific to mathematics teachers. We bring two conceptualisations of such specifications, an earlier one by Niss and Højgaad (2011) and a more recent one by Geraniou et al (2024).

Teaching competencies based on the KOM Flower

After defining competence for mathematics students (Niss & Højgaad, 2011), it was a natural evolution for the authors to define "What mathematical competencies does a good mathematics teacher ought to have?" (p. 84). Importantly, they stress that a first and unquestionable condition is that the teacher must be fluent and knowledgeable about mathematics. The task of mathematics teacher training is to develop the following didactic and pedagogical competencies:

- Curriculum competency – being able to evaluate and draw up curricula. That is, a teacher needs to be able to evaluate the curriculum, and draw up on it the implication for planning her teaching.
- Teaching competency – being able to think out, plan and carry out teaching.
- Competency of revealing learning – being able to reveal and interpret students' learning.
- Assessment competency – being able to reveal, evaluate and characterise the students' mathematical yield and competencies.
- Cooperation competency – being able to cooperate with colleagues and others regarding teaching and its boundary conditions
- Professional development competency – being able to develop one's competency as a mathematics teacher.

The development of these competencies is a long-standing, or even a lifelong process. The starting point is the preservice teacher education, and it is an on-going process

during in-service teachers carriers. Discussing and reflecting on one's own practical work in the classroom is important, preferably in a collegial team at the respective school, and it should be regularly updated in series of professional development.

This competence model has been updated in different follow-up considerations. One recent model, integrating some of the most important competence models, refers specially to mathematics teaching competencies with Digital Media.

Digital competencies in the context of mathematics education – MDCT-Model

Geraniou et al. (2024) proposed a model of "mathematical digital competency for teacher (MDCT)". Based on their conceptual notion relevant for mathematics students, "mathematical digital competency (MDC)" (Geraniou & Jankvist, 2019), the DigCompEdu framework (Redecker, 2017), the concept of instrumental orchestration (Drijvers et al., 2020; Trouche, 2005), the documentational approach to didactics (Gueudet & Trouche, 2009) and the KOM mathematical competencies framework (Niss & Højgaard, 2011), they proposed three characteristics of MDCT:

> [MDCT1] Being able to engage in a techno-mathemat-ical discourse at a pedagogic and a meta-pedagogic level. *This means being able to orchestrate students' engagements in a techno-mathematical discourse as well as incorporating new resources and techno-mathematical development into well-known teaching situations building on the resource-document and scheme-technique dualities. In particular, the didactico-pedagogical competencies of teaching and assessment are relevant in this regard* (Geraniou et al., 2024, p. 631).

This refers to teachers' ability to orchestrate students' ideas, beliefs, working styles and thinking of mathematical objects, represented in digital representations, inside of mathematics as well as environmental problem solving situations, and to develop the understanding of the relation between mathematical objects and their representation.

Example 1. The aim of a teaching unit is to understand a parabola as a geometric object: Given are a straight line g and a point $P \notin g$. A parabola is the set of all points that are equidistant from P and g. This concept could be approached using a real instrument for drawing parabolas or a dynamic geometry system (DGS) simulation of this instrument. The design of the teaching unit must consider the relationship between the practical activity and the digital simulation, the transition to the symbolic level when determining an equation for the parabola, and the interrelationship between the geometric and symbolic representation.

This example shows that the content knowledge about parabolas, the pedagogical knowledge about adequate teaching strategies and the technological knowledge about possible resources in relation to the problem solution are the basis for this teaching unit. Beyond knowledge, it is necessary to plan, to develop, to deliver and to reflect the unit and the teaching methods. Moreover, the goal of this unit, the development of understanding and students' competencies are the means of regulations for this unit.

> [MDCT2] Being aware of which digital tools to apply within different mathematical situations and context and *being* aware of the different tools' capabilities and limitations, so as to think, and act, pedagogically with these tools, while considering the benefits and limitations of these. *This means being able to engage in a process of documentational genesis, i.e., the instrumentation-instrumentalization duality, which brings resources into the classroom as well as orchestrating students' explorations of affordances and limitations. In particular the didactico-pedagogical competency of curriculum plays a central role here* (Geraniou et al., 2024, p. 631).

This refers to teachers' ability to choose the adequate digital tool for special problem solving situations, knowing about the possibilities and constraints of this tool and to design and implement a lesson for students' use of this tool in this situation.

Example 2. To determine the (first) derivative of a function at a point P on the graph of this function – graphically, numerically and symbolically, one must consider the dynamic change of a secant through P and another point on the graph, recognise the limits of this visualisation, make the transition to the numerical level, preferably using a spreadsheet, and finally, the limit process of a sequence of points approaches a fixed point. This example illustrates the interrelationship between knowledge of the limit process and dynamic activities at the graphical, numerical and symbolic levels. It also demonstrates the understanding of pros and cons of DGS vs. spreadsheet.

The content knowledge about the derivative of function in a point and the different steps of the development of this concept is the basis for the choice of the digital tools, of the adequate representations of these tools, and of the adequate sequencing of these tools and their representations. Even more important is clarifying the limits and constraints of the tools concerning the development or construction of the mental concept of derivative in the heads of the learners.

> [MDCT3] Being able to use digital technology reflectively in problem solving and when doing (learning or teaching) mathematics. *This means being able to orchestrate students' approaches to problem-solving, involving a scheme-technique duality, where the digital resources allow students to connect instrumented techniques to mathematical schemes. Besides the teaching competency, in particular the didactico-pedagogical competency of uncovering learning plays a central role for this type of MDCT* (Geraniou et al., 2024, p. 632).

This refers to the "teachers' need to develop new interdisciplinary pedagogical approaches to integrate PCT [Programming and Computational Thinking] and mathematics" (p. 632). With the knowledge of different models of the development of concept formation teachers are able to observe and to reflect students' actions during the problem solving process, especially the use of digital media, and to decide on the adequate feedback and support concerning the on-going process.

Example 3. When learning about the altitude of a triangle in a DGS environment and starting with an acute-angled triangle, moving a triangle vertex may lead to the case where the altitude coincides with a side of the triangle and even 'disappears' if the triangle becomes obtuse-angled: the altitude is 'moved' outside the triangle. These two situations might be unexpected to the students. For the teacher, this could be a means to uncover the misconception 'the altitude of a triangle is inside the triangle' and make it a learning opportunity, leading to the observation of the location of the intersection of the lines along the altitudes in dependence of the shape of a triangle.

This example shows the interrelationship between students' actions and teachers' reactions. Teachers have to have the competence to recognize students' knowledge, especially misconceptions, about the intended goal of the activity and to lead and structure the on-going problem solving process. It is also necessary to react to unexpected students' actions and to spontaneously vary the planned teaching strategy.

Knowledge models developments – discussion

To summarise, we see in the definition of Niss and Højgaard – "*A mathematical competency* is someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations" (2019, p. 12) – a valuable description of teachers' competence. We interpret the term readiness with the orientation towards action, acting insightfully and openness for challenges before, during and after teaching processes. These are three main guidelines for teachers beyond knowledge. Knowledge is the basis for all these actions. Content, pedagogical and technological knowledge and determining the teaching goals are the prerequisites for the development of teachers' competencies concerning the teaching process. The kind of digital media and the level of integrating them in the sense of the SAMR model must be decided in relation to the teaching content, the knowledge of the learners and the environment teaching takes place. The digital competencies in the DigCompEdu framework and the teaching competencies in the KOM Flower serve as overarching guidelines for the development of teachers' competencies, the MDCT model gives criteria for the development and the evaluation of specific activities with digital media. To develop teachers' competencies in relation or beyond knowledge is a challenging task for teacher education, for teachers' professional development and should always be seen in relation to the daily work of teachers in the classroom.

Questions for Discussion:

(1) Do knowledge-based models alone sufficiently capture the factors that influence teachers' effective integration of technology, or is it necessary to incorporate other constructs?

(2) How can future models integrate non-knowledge dimensions without losing analytical clarity or theoretical coherence?

(3) What are the implications of expanded models – including knowledge and other dimensions – for research, teacher education, and the design of interventions to support effective technology integration?

CONCLUSION

This RF starts with the comparison of the existing models of teachers' knowledge specific to the use of digital technology – TPACK, TKMT, and PTK. This comparison highlights both convergences and tensions in current attempts to conceptualize mathematics teachers' professional knowledge in technology-rich environments. While these models offer valuable lenses for examining the interplay between mathematical, pedagogical, and technological dimensions of teaching, ongoing developments toward framing teachers' digital competencies signal the need to extend their scope. In particular, the emergence of AI-based tools challenges existing assumptions about the nature of mathematical activity, the distribution of agency between teachers, students, and technologies, and the forms of expertise required for meaningful classroom orchestration. Looking ahead, refining and possibly reconfiguring established knowledge models will be essential to capture how teachers learn to interpret, adapt, and critically evaluate AI-mediated mathematical practices. Such work calls for sustained theoretical dialogue and empirical inquiry across contexts and traditions. In this perspective, the present Research Forum is intended to spark discussion and stimulate further research on these issues, contributing to the collective effort to rethink mathematics teacher knowledge in times of rapid technological change.

Acknowledgements

This work was partially supported by Portuguese national funds through FCT – Foundation for Science and Technology, I.P., within the scope of the project AI-TecTeachers (https://doi.org/10.54499/2024.15124.PEX); and by the Italian Research Group GNSAGA-INDAM.

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AFFECT IN MATHEMATICS EDUCATION: STATE-OF-THE-ART AND NEW DEVELOPMENTS

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Students' emotions, beliefs, motivations, values, and attitudes, summarized under the term affect, are crucial for learning mathematics, educational choices, and future life trajectories. Over the past decade, research on affective constructs and their relations to other educational outcomes has increased in mathematics education. This research forum will present current developments and future directions in research on affect in mathematics education. Contributions addressing overarching perspectives will synthesize research from measurement, situational, and sociocultural perspectives. Contributions focusing on specific contexts will address affect in problem solving and problem posing, mathematical modelling and applications, and gender. The forum will include audience discussion and invited commentary to advance research on affect.

INTRODUCTION TO THE RESEARCH FORUM

Stanislaw Schukajlow and Pietro Di Martino

Research on affect has reached a phase in which its centrality is widely acknowledged in the field of mathematics education. However, questions of theoretical integration, conceptual clarity, and methodological coherence remain still open. Scholars have since emphasized the intertwined nature of affective, cognitive, and sociocultural dimensions in mathematical activities, students' learning, and teachers' development. Several studies have shown how affective constructs (e.g., emotions, beliefs, motivations, values, attitudes) influence the way people interpret tasks, persist through challenges, and construct meaning in mathematics. In sum, studies have shown how affect is a crucial component of the quality of teaching and learning.

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Perspectives in research on affect have progressively expanded, for example, by recognizing how affective issues emerge through interactions in classrooms, communities, and broader cultural contexts (Hannula et al., 2019). Investigations of affect provide key insights into identity, quality, and equity in mathematics education. "Affective considerations" that are used to interpret research findings have also emerged from studies whose focus and paradigms differ from those on affect. Studies on problem posing, problem solving, modelling, and applications are of special interest. These developments point to a growing convergence of research strands that still need to be fully and systematically connected. The RF will foster dialogue across researchers, theoretical frameworks, and methodologies, with the aim of articulating new conceptual connections and research directions on the topic of affect.

The portion of research papers addressing affective constructs in leading journals is considerable (Schukajlow et al., 2023). Recent reviews on specific aspects of affect, such as attitudes or self-efficacy in mathematics, have further illustrated both the progress that has been made and the complexity of the field by offering comprehensive overviews of current developments (Aquilina et al., 2025; Street et al., 2024). At the same time, this growth highlights the need to move beyond isolated constructs and local findings. Building on these insights, we consider two broader perspectives to be crucial: (a) overarching perspectives in research on affect and (b) perspectives embedded in research in contexts that are relevant for mathematics education.

Within the overarching perspectives, one of the most pressing challenges concerns the conceptual clarity of affective constructs. As discussed in the working group "Conceptual overlap in approaches to affect: Attitude, emotion, motivation, and what else?" in 2023, substantial overlap across definitions continues to cause confusion, and jingle–jangle fallacies (e.g., the distinction between self-concept and self-efficacy) represent major barriers to theoretical progress and to dialogue among researchers. This issue is closely connected to another long-standing challenge: the assessment or observation of affect. Although most studies have relied on self-reports, such as questionnaires or interviews, researchers have repeatedly argued that other methods should also be included depending on the type of approach: quantitative, qualitative, or mixed. In addition, a shift in research has emerged in recent years. Whereas earlier research primarily conceptualized affective outcomes as stable dispositions, the situational perspective highlights how specific situations shape affective experiences. A systematic review of research within the situative perspective on motivation revealed that prior research has typically addressed temporal and contextual aspects separately (Radišić et al., 2026). Studies have predominantly focused on motivation related to specific activities and classroom contexts. In the situational perspective, the sociocultural perspective plays a particularly crucial role, as it emphasizes how social norms and classroom practices influence affect. Understanding how dispositional and situational perspectives can be theoretically and methodologically integrated has thus become a key emerging challenge for the field.

Although some progress has been made in this area, most findings currently considered robust stem from Western countries. Consequently, the extent to which these results are generalizable across different cultures remains unclear.

Beyond overarching perspectives, affect research is also increasingly reconceptualized through significant mathematical contexts. Two particularly important areas in this perspective include research on problem solving and problem posing (Cai & Leikin, 2020) and research on mathematical modelling and applications (Krawitz et al., 2025). The research in these areas includes affect as an integral component of mathematical activity itself. Moreover, research on gender has been integral to affect research from the outset. For instance, Eccles' expectancy–value theory (Eccles & Wigfield, 2020) was shaped by the observation that women in the 1980s rarely chose STEM subjects or professions that required strong mathematical skills. Despite several decades of research, these issues persist, as gender differences continue to appear in central constructs such as self-efficacy expectations.

Goals and key questions of the research forum

This research forum is intended to address the two broader perspectives in research on affect—the overarching perspectives and perspectives in research in important contexts—by focusing on different aspects and involving several researchers with different approaches and from different areas of the world. The RF aims to identify conceptual tensions, methodological challenges, and promising new openings that characterize the current state of the field. This organization aims to offer participants the opportunity to engage in a well-structured, informed discussion that can help identify and share meaningful future research in the study of affect in mathematics education. A RF on affect in Helsinki carries profound symbolic significance, given the substantial contribution of the Finnish school to the genesis and development of research on affect in mathematics education. At the same time, the RF is intended as a space of critical exchange among researchers investigating affect from different perspectives, supporting reflection on emerging directions for future research.

Overarching perspectives in research on affect

- Perspectives on the assessment of affect across methodological approaches: C. Coppola (Italy), M. del Socorro García González (Mexico), Ç. Haser (Finland).
- Dispositional and situational perspectives on affect and their theoretical integration: S. Rach (Germany), J. Skott (Norway), K. Street (Norway).
- Sociocultural perspective on affect: T. Evans (New Zealand), K.-L. Yang (Taiwan).

Perspectives embedded in research in important contexts

- Affect, problem solving, and problem posing: J. Cai (USA), R. Leikin (Israel).
- Affect in mathematical modelling and applications: J. Krawitz (Germany), I. Ferrando (Spain), W. Van Dooren (Belgium).

- Affect and gender: C. Giberti (Italy), L. Gildehaus (Austria), L. Sumpter (Sweden)

Discussant: Markku S. Hannula (Finland)

PERSPECTIVES ON THE ASSESSMENT OF AFFECT ACROSS METHODOLOGICAL APPROACHES

Cristina Coppola, María del Socorro García González, Çiğdem Haser

Affect in mathematics education has been assessed in different ways, presenting advantages and limitations. While the predominant trend to assess it with quantitative instruments at a single time-point, there are new approaches that may be more effective in addressing its cognitive, emotional, situational and dynamic nature across different cultures; in tracking it over time; and for different age groups. Below, we present a brief overview of major methods, trends, and alternatives in assessing mathematics-related affect with illustrative studies.

Introduction

Mathematics education field started to explore the relationship between students' affect and their success in mathematics in the 1960s by measuring the relationship between attitude towards mathematics and students' achievement, typically operationalised through school grades (e.g., Aiken & Dreger, 1961). Despite several reports of positive correlations between mathematical affect and achievement, the direction of causality was not clear, addressing the inadequacy of the attitude instruments (Ma & Kishor, 1997) and a major persisting methodological issue: the difficulty of constructing appropriate tools for the assessment of affect. Meanwhile, the study of mathematics-related affect has evolved from relatively bounded conceptualizations toward more complex and multidimensional theoretical frameworks. Hannula (2012) proposed a multidimensional model distinguishing among types of affect, degrees of stability, and theoretical traditions. This model has led to methodological consequences: the more affect is conceived as dynamic and relational, the greater the demands placed on its empirical assessment.

Methods of assessment of affect

Researchers have been developing instruments to assess mathematics-related affect with different approaches. Quantitative approaches are typically based on standardized questionnaires and Likert-type scales, allowing the analysis of large samples through statistical techniques. Qualitative approaches focus on smaller samples and aim to provide in-depth interpretative accounts of affective phenomena through interviews, observations, and narrative data. Mixed methods designs have been adopted to combine the strengths of both approaches and mitigate their respective limitations.

Despite this diversification, research on attitude towards mathematics mostly aimed to measure attitudes through scales and correlating them with other variables, typically students' mathematical performance as Di Martino and Zan (2001) reported. The multidimensional nature of attitude also raises questions about the suitability of

common traditional instruments, such as questionnaires and Likert scales, which are not capable of capturing the interaction among the various components of attitude, a crucial aspect in models that attempt to integrate both cognitive and emotional factors (Di Martino & Zan, 2011). A recent review by Aquilina et al. (2025) confirms the persistence of this trend where reviewed empirical studies predominantly (about 81%) adopted quantitative approaches mostly using Likert-type scales.

Traditional measurement instruments limit assessing affect constructs by reducing complex and multidimensional nature to single indicators or bipolar scales; and may not fully capture the interplay between their cognitive and emotional components or the situated character. Therefore, they should be complemented with other approaches to support a more comprehensive understanding of affect.

Cross-cultural and longitudinal assessment of affect

Cross-cultural and longitudinal trends respond to the need to overcome historical limitations associated with cross-sectional and culturally restricted studies. They reflect a significant shift from static, localized, and unidimensional approaches toward a contextualized, dynamic, and structurally complex understanding of affect.

A cross-cultural investigation of differences in mathematics anxiety among engineering students in the U.S. and Colombia revealed significant differences in learning-related anxiety but not in test anxiety (Brown et al., 2020), suggesting that certain affective components may be more culturally permeable than others. Yet, there is a need to widen cultural diversity because the geographical representativeness of the field is limited despite the methodological advances (Pekrun, 2023). A longitudinal study of a cohort of students from grade 4 to 8 by using international comparison study data indicated an increase in negative emotions toward mathematics by age, and gender differences in intrinsic motivation (Sumpter & Sollerman, 2023), capturing transformations that cross-sectional studies cannot reveal. An analysis of National Assessment of Educational Progress data showed a sustained decline from 2017 to 2024 in interest, enjoyment, and confidence in mathematics among U.S. students (Campbell et al., 2025) stressing the value of longitudinal research for detecting trends that might be unnoticed in isolated analyses.

Alternative methods in assessing affect

Existing methods assessing affect assume that the respondents are aware of, willing and ready to express their affect clearly (Satyam et al., 2022) and they may not be suitable for younger participants (Hannula, 2019). Therefore, alternative ways of assessing affect have been devised, yet presenting their own limitations.

Drawing has revealed students' attitudes and beliefs when they drew a picture of, for example, themselves when doing mathematics (Quane et al., 2023) or a mathematician while working (Picker & Berry, 2000). However, drawings should be combined with additional data because students' reasons for drawing a certain object and/or its size can be misinterpreted (Quane et al., 2023).

Affect graphing enables tracking affect over time by illustrating experiences and the reasons for the changes and stability in affect when combined with autobiographies (Guillaume & Kirtman, 2010), written explanations (Andrà & Scalvini, 2023) and interviews (Satyam et al., 2022). Drawing multiple dimensions of experiences, such as motivations, emotions and perceived competence, on the same graph offers exploring how these dimensions covariate (Andrà & Scalvini, 2023). Yet, graphing has limitations of the level of structuring, respondents' subjective scaling of affect and the extent they can remember their past affect and experiences (Satyam et al., 2022).

Video diaries provide particularly younger students with the autonomy in expressing their mathematics experiences and enable access to their affect (Larkin & Jorgensen, 2016; Noyes, 2004). Stimulated recall interviews of videos can reduce the limitations and provide a deeper understanding of students' affect (Noyes, 2004).

Conclusion

The convergence of cross-cultural comparability, longitudinal analysis, integrative approaches and alternative methods outlines a promising horizon for assessing mathematical affect. Nevertheless, a key challenge remains: articulating methodological rigor, cultural sensitivity, and theoretical coherence within diverse research designs—quantitative, qualitative, and mixed—in order to understand affect as a constitutive component of mathematical experiences in specific contexts.

DISPOSITIONAL AND SITUATIONAL PERSPECTIVES ON AFFECT AND THEIR THEORETICAL INTEGRATION

Stefanie Rach, Jeppe Skott, Karin Street

Affect is an important factor in mathematics education. Whereas earlier research primarily conceptualized affective outcomes as stable dispositions, different versions of the situational perspective highlight how specific contexts and time shape affective experiences. It is an important challenge for the field to understand whether these perspectives may be theoretically and methodologically integrated, or if tensions between them may be productive even if unresolved. We refer to theoretical frameworks that address both perspectives, discuss methodological implications, and illustrate differences by presenting concrete studies from different subfields.

Theoretical frameworks on dispositional and situational perspectives

Students' and teachers' affect, for example, their beliefs, emotions, motivations, and identities, are assumed to be crucial for their educational engagement and learning. Research on affect is well established, but theoretical and methodological problems persist, particularly regarding the dispositional and situational perspectives on affect.

Different subfields of the affective domain relate differently to the question of whether their respective key constructs are most productively considered dispositional or situational. Mapping the conceptual space of this research is an important future task (Pekrun & Marsh, 2022). Key phenomena (e.g., self-efficacy, teacher identity, interest)

are not considered dispositions, but fluctuate according to situations. Even so, there is broad variation in terms of how these constructs have been operationalised, and whether theoretical assumptions are treated explicitly and with precision. We describe how conceptual developments have served to clarify theoretical underpinnings of some affective subdomains, before moving on to methodological considerations.

Inspired by constructivism, belief research was initially based on the premise that teachers' and students' *beliefs* – understood as relatively stable mental constructs residing within the individual – serve as an explanatory principle for practice (e.g., Schoenfeld, 1992). This dispositional perspective is to some extent maintained, where beliefs are still considered the default explanation for practice, but dynamic perspectives between beliefs and practice have become more prominent (cf. Pepin & Roesken-Winter, 2015).

Concerning *motivation*, recent research in mathematics education and educational psychology highlights that it arises from participation in mathematical activities rather than being an innate characteristic. This shift towards a more nuanced analysis of the situated nature is, for example, evident in the decision of Eccles and Wigfield (2020), who added the word "situated" to their "expectancy-value theory". With this relabelling, they highlight that all processes that involve the emergence and impact of expectancies and values depend on the situations in which they occur.

Identity research in mathematics education was from the outset concerned with how identities relate to social and cultural contexts. Drawing, for instance, on social practice theory (e.g., Lave, 2019; Wenger, 1998), it suggests that any response to the question who are you must be made with reference to practices, relations, and activity systems. Identities, then, are considered situated in the fundamental sense that they are constituted in cultural-historical and social circumstances.

Affective constructs can exist as trait-like (dispositions) and state-like (situational) variables. Renninger et al. (2024) show how these two manifestations interact and propose a process model of interest development that goes from situational interest occasions to the development of individual interest. On the one hand, they suggest that the interest development stages are consistent across contexts, content, and individuals. On the other hand, they emphasise that the sociocultural context plays a vital role in fostering the development of interest along these development stages. In identity research, Skott (2024) defines identity as experiences of being, becoming, and belonging. Drawing, for instance, on Dewey (1997/1938), he distinguishes between experiencing (the situated aspect) and experiences had (the more dispositional perspective) and discusses the relations between them.

The discussion above suggests that the affective domain has generally moved away from purely dispositional perspectives towards more dynamic ones that acknowledge the significance of some form of situatedness. However, different subfields of affective research attach different significance to situatedness, e.g., to what degree it is

considered a constituting element of the affective construct in question. Furthermore, theoretical underpinnings that include situational perspectives are not always adequately considered through research designs or through the way affective constructs have been operationalized methodologically speaking.

Methodological considerations on dispositional and situational perspectives

Emphasising dispositional aspects, studies may rely on methods such as cross-sectional surveys or interviews. Explicitly or implicitly, such studies may locate affect within the individual and fail to address how people relate affectively, for instance, to mathematical activity in schools and classrooms. Such methods have been challenged by researchers who call for approaches that better capture situational perspectives.

Schukajlow et al. (2023) conclude that the often-applied methodological approach of focusing on between-learners' differences in affect conflicts with the conceptual view of motivation as emergent, relational, and responsive to mathematical activity. A situational perspective is needed to account for the significance of time and context (Pekrun & Marsh, 2022).

With respect to time, researchers often use experience-sampling methods, including longitudinal designs, to capture how affect changes in the short or long term. Different conceptualizations of context are reflected in large-scale studies such as PISA; in micro-level studies that address the impact of culture, classrooms, and tasks on affect; and in techniques that enable in-depth analysis, such as case-based ethnography or microgenetic interaction analysis (e.g., Nolen, 2020).

Insights into perspectives on affect in mathematics

To illustrate the progress made in clarifying the dispositional and situational perspectives, we present exemplary studies. Addressing the context perspective, Krawitz et al. (2022) used modelling in mathematics and compared the educational systems in Germany and Taiwan. They examined whether providing students with reading comprehension prompts would improve modelling competencies needed to construct a model of a real-world situation and increase students' interest in modelling.

Combining the context and the time perspective, Parrisius et al. (2022) used a multilevel approach and data from ninth graders in 78 classrooms. They investigated whether individual students and entire classrooms differ substantially in their levels of competence and value beliefs regarding mathematics, and whether individual students' competence and value beliefs fluctuate substantially from lesson to lesson.

Heyd-Metzuyanim and Nachlieli (2024) study professional identities in the case of leaders and participants in professional development initiatives. The authors use positioning theory to document changes in how the research participants implicitly identify themselves and each other when faced with an explorative task. The study demonstrates the potential of a dynamic and situated perspective focusing on immediate interactions when seeking to understand identity-practice relationships.

In summary, while a dispositional perspective has dominated much empirical research, many educational theories already encompass situational perspectives (Pekrun & Marsh, 2022). This indicates that there is a need for greater conceptual awareness and more careful operationalisations, where methodological advances might enable enhanced investigations of existing theories by better incorporating and accounting for situational variations. At the same time, others may argue that differences in conceptual understandings of the notion of "situatedness" lead to incompatible perspectives, in which incorporating factors such as "time" and "context" is insufficient to wholly capture the situational perspective.

SOCIOCULTURAL PERSPECTIVE ON AFFECT

Tanya Evans, Kai-Lin Yang

This forum contribution advances a sociocultural perspective on affect in mathematics education, drawing on a review of 286 cross-national studies (2014–2023). We synthesise four recurring patterns: persistent between-country differences (including the East Asian learner paradox), developmental declines in positive affect and strengthening Big-Fish-Little-Pond effects, contextually patterned affect–achievement relations, and multilevel influences shaped by classroom and institutional structures. We contend that affect is constituted through participation in culturally organised practices, making cross-national research essential for theory building and for refining accounts of how affect and achievement become linked.

Affect in Mathematics Education: Insights from Cross-National Comparative Studies (2014–2023)

The past decade has witnessed a global shift toward competency-based mathematics curricula, which has increased attention to affect—students' emotions, beliefs, motivations, and attitudes—as both an outcome and a mediator of learning. Drawing on a scoping review of 286 cross-national comparative studies (Yang et al., 2025), we first synthesise key empirical findings. Across the period 2014–2023, this body of research reveals several coherent patterns regarding students' affect in mathematics.

First, systematic between-country differences persist: the "East Asian learner paradox" remains evident—students from Confucian-heritage systems often attain higher cognitive achievement yet report lower mathematics self-concept, self-efficacy, and enjoyment compared with many Western peers (Leung, 2017), although exceptions and contextual nuances (e.g., Singapore) are documented.

Second, developmental trajectories are consistent across contexts: positive affect declines from late primary to lower secondary (e.g., marked drops in high-level confidence between grades 4 and 8 across multiple jurisdictions), and the Big-Fish-Little-Pond Effect (BFLPE) strengthens with age, indicating increasing sensitivity of self-perceptions to classroom composition (Ker, 2017; Marsh et al., 2015).

Third, the affect–achievement relationship is contextually patterned and multifaceted: (a) within-country (student-level) correlations between affective indicators and achievement are typically positive, whereas between-country (aggregate) correlations can be weak, null, or even negative (Chen & Hastedt, 2022); (b) different affective constructs play distinct roles—domain self-confidence/self-concept is frequently the strongest predictor of achievement, while utility value and related measures often show weaker or inconsistent associations; (c) cultural and institutional contexts moderate effect sizes and directions (e.g., the reciprocal impact of achievement on self-concept varies across Western, Asian, and some Islamic contexts; Marsh et al., 2015) and (d) substantial within-region heterogeneity means that affect–achievement associations vary even within a specific cultural tradition.

Fourth, multilevel determinants of affective outcomes are evident. Classroom-level factors—most notably the BFLPE—fundamentally shape self-concept, though the magnitude of classroom effects varies by cultural and institutional setting. Developmental factors amplify these dynamics over time.

Achievement and Affect in Tension: Performative Participation in High-Performing Systems

Duty orientation and dual pressure as sociocultural mechanisms

A sociocultural lens clarifies the mechanisms underlying this tension. In collectivist-oriented East Asian contexts, mathematics learning is often organized around role obligation and 'vertical' academic striving, sustained by long-term, duty-bound effort and delayed gratification (Chen, 2023; Meng & Liu, 2022). Academic success serves as a primary pathway to educational mobility and social recognition, grounding motivation more in obligation and instrumental goals than in intrinsic enjoyment. Simultaneously, examination-based admission systems institutionalize dual evaluative pressures, self-referential comparison with prior performance, and norm-referenced comparison through peer competition and parental expectations, which may intensify students' sensitivity to academic judgment.

This interpretation is further supported by cross-cultural evidence. Yang et al. (2024) showed that Taiwanese students' mathematical modelling task value profiles significantly predicted modelling performance even after controlling for intra-mathematical knowledge, whereas such associations were not observed among German students. Notably, intrinsic-related task values—rather than examination-driven utility—were most strongly associated with modelling success in Taiwan. These findings suggest that the motivation–achievement link is culturally mediated and institutionally structured rather than universal.

Rethinking success and affect through performative participation

From a sociocultural perspective, learning is participation in historically organized practices (Lave & Wenger, 1991). For example, within Taiwan's performative-oriented educational culture, mathematics is structured around speed, accuracy,

ranking, and public evaluation, shaping not only what students learn but also how success is socially defined (Yang et al., 2022). Effort is celebrated, failure is attributed to insufficient diligence, and external evaluation carries moral weight. These participation structures sustain achievement while generating pressure and anxiety.

Affect, therefore, is not an internal deficit but a socioculturally constructed outcome of performative participation. Evidence from the Just-Do-Math program shows that when participation structures are redesigned, affective outcomes shift accordingly (Wang et al., 2021). Addressing the East Asian learner paradox thus requires not merely individual-level interventions, but a reconfiguration of the cultural meanings and evaluative norms through which students experience mathematics.

Beyond the Paradox: Reframing Affect–Achievement Through Sociocultural Integration

Recent cross-national evidence complicates deficit interpretations of the so-called East Asian paradox by demonstrating that sociocultural configurations may generate adaptive pathways linking affect and achievement (Gao et al., 2026). A latent profile analysis of TIMSS 2023 data across Chinese Taipei, Hong Kong, Singapore, Australia, and New Zealand revealed both convergent and divergent motivation profiles. Notably, a value-oriented inconsistent profile—characterised by low confidence alongside relatively higher attainment and utility values—was prevalent in the East Asian jurisdictions yet rare in the Western-influenced Pacific systems. While confidence remained the strongest predictor of achievement in Australia and New Zealand, attainment and utility values in the East Asian contexts appeared to compensate for lower confidence in predicting mathematics performance. Taken together, these patterns illustrate how culturally embedded valuing of mathematics can operate as a compensatory motivational resource: within East Asian collectivist traditions, strong instrumental and attainment-oriented commitments appear to sustain high performance even in the presence of comparatively low self-evaluative confidence.

Gendered distributions further highlight contextual differences: maladaptive profiles were overrepresented by girls in the Western-influenced Pacific jurisdictions, unlike in the East Asian systems, suggesting that sociocultural contexts shape both affect–achievement links and the gendered distribution of motivational risk (Gao et al., 2026).

In conclusion, we contend that, given that the sociocultural research programme now constitutes the dominant research programme in our field of mathematics education—as evidenced by the seminal work of (Inglis & Foster, 2018)—affect can no longer be treated as an isolated psychological attribute located within the individual. Rather, it must be theorised as constituted through participation in culturally and institutionally structured practices. The comparative evidence we presented makes this case difficult to ignore: affective patterns do not merely vary incidentally across contexts, but are systematically organised by local norms, values, and institutional arrangements. If affect is patterned by culture in these ways, then cross-national research is not

supplementary but indispensable to theory building. It provides the analytic leverage required to refine construct boundaries, surface culturally specific mechanisms, and develop more robust, globally responsive accounts of how affect and achievement become linked in practice.

At the same time, advancing a sociocultural perspective on affect calls for continued methodological development. Comparative research has already established affect as culturally embedded and multilevel, drawing productively on large-scale studies such as TIMSS and PISA. Future work could extend this foundation through more theory-driven cross-national designs, culturally attuned instruments, and careful attention to measurement comparability, alongside increased use of longitudinal and mixed-method approaches and broader regional sampling. Such developments could enhance theoretical clarity and contribute to the continued advancement of our field of mathematics education.

AFFECT IN PROBLEM SOLVING AND PROBLEM POSING: A COMPARATIVE AND CONNECTING PERSPECTIVE

Jinfa Cai and Roza Leikin

This article examines affective factors in mathematical problem solving (PS) and problem posing (PP) through comparative and connecting perspectives. We argue that affective experiences in PS depend on whether tasks are creativity-directed, while PP is inherently creative in nature. Using expectancy-value theory, self-determination theory, and flow theory, we argue that routine PS can generate evaluative anxiety, whereas creativity-directed PS (e.g., multiple solution tasks, insight problems) open opportunities for affect linked to interest and aesthetic pleasure. At the same time, PP, being inherently creativity-directed, extends autonomy to task definition itself, reducing anxiety while enabling competence through meaningful question generation. The PS-PP cycle creates affective regulation: posing transforms frustration into curiosity, while invested effort in self-generated problems enhances solving motivation and producing self-regulating affective-cognitive cycle.

Introduction

Problem solving (PS) and problem posing (PP) are two interconnected mathematical activities considered central to the field of mathematics education since Pólya (1945/1973), Silver (1994), and Cai (1995). Both activities involve cognitive and affective processes, yet affect has received comparatively less attention. In this article, we focus on the affective factors associated with PS and PP in mathematics.

"Affect" encompasses emotions, feelings, and character traits that influence thoughts and actions. Affect is categorized as global or local, differing in intensity and stability (Goldin, 2009): Global affect comprises moderately stable beliefs, attitudes, and values (McLeod & Adams, 2012), whereas local affect includes rapidly changing and context-dependent emotions. Consequently, Schukajlow et al. (2017) distinguished task-

specific emotions from domain-level attitudes. Importantly, affect in the context of mathematical PS can be positive (e.g., enjoyment and satisfaction) or negative (anxiety and frustration; McLeod & Adams, 2012). Cai and Leikin (2020) identified affect in PP as an area needing conceptual clarification, noting that the emotional processes accompanying problem generation remain insufficiently understood.

Both PS and PP generate local affect and influence global affective conceptions but in different ways. Our analysis of affective differences and connections between PS and PP centers on their creativity-directed nature (Leikin, 2018). Conventional PS, which involves analyzing conditions, selecting tools and strategies, and implementing them to attain the solution outcome, differs from open PS (Silver, 1995). Multiple Solution Tasks (Leikin, 2013, 2018; Silver, 1997) and insight problems (Leikin & Guberman, 2023) exemplify creative approaches within PS. PP, which comprises generating new mathematical questions from situations, is inherently creativity-directed allowing diverse question generation (Cai et al., 2013; Leikin, 2018; Silver, 1994, 1997).

We analyze the affective components of PS and PP from two perspectives. The first is comparative: How do emotions, self-efficacy beliefs, and motivational orientations manifest differently across the two activities? The second is connective: How does affect link posing and solving within their cyclic relationship?

Comparative perspectives on affect FOR PS and PP

We employ three theoretical frameworks to illuminate the affective dimensions of PS and PP: (a) Eccles and Wigfield's (2002) expectancy-value theory posits that engagement depends on expectation of success and task value; (b) Deci and Ryan's (2000) self-determination theory holds that intrinsic motivation requires three basic needs: autonomy, competence, and relatedness; and (c) Csikszentmihalyi's (1990) flow concept describes absorption arising when challenge and skill balance and individuals self-regulate task difficulty. We integrate these frameworks as complementary lenses for examining affect across mathematical activities.

In routine PS, success expectancy connects to procedural proficiency, and value is often instrumental. When PS is creativity-directed, value shifts toward intrinsic interest: Solvers experience aesthetic pleasure in elegant solutions or surprise at unexpected connections (Leikin, 2009, 2013). In PP, expectancy is more diffuse because no single correct answer exists, and intrinsic value tends to be higher because the activity involves personal expression and creative choice. In routine PP, self-efficacy builds through accumulated success but erodes quickly with failure. In creativity-directed solving, it rests on a broader base: Even when one strategy fails, success through another sustains competence beliefs (Leikin, 2018). In PP, students experience competence through creating meaningful questions regardless of whether they can solve them. Cai and Hwang (2023) noted that PP tasks have a "low floor and high ceiling," a characterization also applying to well-designed Multiple Solution Tasks. In a recent study, Cai et al. (2026) found significant differences between

reported self-efficacy and self-evaluation. For PS, the Self-efficacy Group reported higher confidence about their ability to complete the task than the Self-evaluation Group reported retrospectively about their performance, but, for PP, the Self-efficacy Group reported lower confidence than did the Self-evaluation Group.

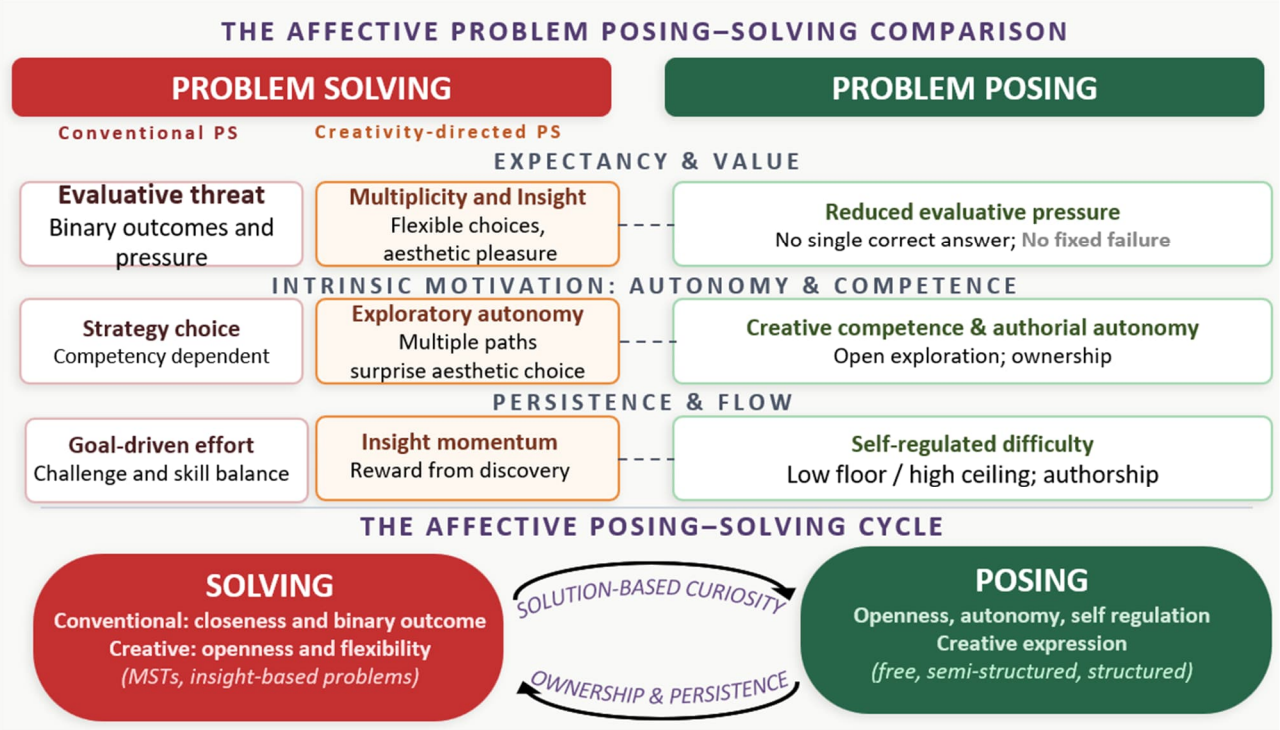
Mathematical anxiety is associated with binary outcomes, creating evaluative frames in PS (Schoenfeld, 1985). When solving involves insight-requiring tasks, frustration shifts toward enjoyment depending on PS success (Leikin, 2013; Leikin et al., 2023). PP is not associated with binary evaluation, thus reducing anxiety (Akay & Boz, 2009). However, posing is not anxiety-free. In fact, mathematicians pose problems with triggers (Kontorovich, 2020). Whereas PS addresses autonomy through strategy choice, PP extends autonomy to defining the task itself (Cai & Leikin, 2020).

The affective contrast between PS and PP does not follow a uniformly evaluative solving experience versus uniformly creative posing experience but connects to different configurations of PS and PP assignments. In closed problems, persistence depends on goal clarity and competence; in open tasks, complexity rises alongside autonomy, so motivation is not necessarily higher. Integration of PS and PP creates specific affective effects. Cai et al. (2013) found that affective investment in posing transferred to stronger solving performance. Liljedahl and Cai (2021) noted that posing embedded within solving creates conditions favourable to flow (Csikszentmihalyi, 1990) through self-regulation of difficulty.

The Connecting Perspective: Affect Across the Cycle

The comparative analysis above treats PS and PP separately. However, as Liljedahl and Cai (2021) emphasized, solutions generate new questions, and new questions reshape subsequent solving. The affective dimension of this cycle deserves attention because affect shapes and is shaped by cognition. Transitioning to posing shifts learners from convergent, evaluative modes to divergent, exploratory ones. This can transform frustration from incomplete solutions into curiosity about new questions. Conversely, satisfaction from complete solutions can fuel ambition to pose harder problems. The posing phase thus serves an affective regulatory function. The transition from posing to solving also carries affective significance. Students who invest creative effort in formulating problems are more motivated to solve them compared to externally assigned problems. Cai and Leikin (2020) described this as the affective signature of PP, involving pride, investment, curiosity, and anticipation. The posing-solving cycle, when functioning well, creates a self-regulating system: Posing calibrates challenge, solving provides feedback. Affective quality depends on classroom culture, teacher responsiveness, and learners' prior experiences. This bidirectional relationship suggests that integrating PS and PP enhances both cognitive and affective outcomes (Figure 1).

Figure 1: Affect in Problem Solving and Problem Posing



Jinfa Cai acknowledges the support of a grant from the National Science Foundation (DRL-2101552).

AFFECT IN MATHEMATICAL MODELLING AND APPLICATIONS

Janina Krawitz, Irene Ferrando, Wim Van Dooren

This contribution synthesises research on affect in mathematical modelling and applications, treating affect both as an outcome and a condition for solvers' activity. We show how task characteristics and instructional designs shape emotions and motivation, often fostering competence and interest, but sometimes eliciting frustration. We also explain how beliefs and classroom culture influence the use of real-world knowledge instead of routine procedures. Individual differences modulate these dynamics. We end by outlining future research directions.

Introduction

Mathematical modelling—the capacity to solve real-world problems with mathematical methods—has become a central competence in mathematics education, equipping students to address complex challenges. Modelling tasks embed real-world contexts and require solvers to understand the situation, simplify and structure the given information, make assumptions, mathematise, interpret and validate the solution and, if needed, refine the result in iterative cycles (Niss & Blum, 2020). Teaching modelling in school pursues distinct goals, including psychological goals that highlight the motivational importance of real-world connections (Krawitz et al., 2025).

Achieving these goals depends on specific task characteristics. At the same time, these task characteristics can also heighten uncertainty and, in some cases, negotiation, which is why affective processes arise alongside metacognitive and cognitive activity (Chamberlin, 2019).

We discuss some key insights regarding affective aspects around modelling from various perspectives. Affect is an outcome of modelling activities, but pre-existing beliefs, attitudes and values are also brought to modelling tasks. We further consider individual differences in these matters.

Affect as an outcome of modelling

Engaging in mathematical modelling shapes learners' emotions and motivation during and after task work. This is largely due to distinctive task characteristics of modelling, and to instructional approaches. Regarding task characteristics, modelling tasks typically allow multiple solutions. Their openness may affect students' perceived autonomy and control, thereby affecting their self-efficacy beliefs and other motivational and emotional experiences during task engagement. However, modelling tasks pose specific challenges, such as the need to make plausible realistic assumptions. These challenges may elicit frustration or lower self-efficacy beliefs.

Further, the relationship of the task's context to reality and its proximity to solvers' lifeworld impacts students' affective outcomes. Real-world contexts can improve affective outcomes. When solvers have a greater personal interest in the task context, they show more frequent validation sequences and more explicit, justified assumptions (Kämmerer, 2025). However, real-world wrappers do not guarantee higher interest, as studies demonstrated that students report lower interest and task-values for modelling compared to intramathematical problems.

The instructional approach matters for these affective outcomes. Numerous studies have consistently shown that student-centred formats (e.g., collaborative work), explicit strategy instruction, and structured scaffolding to support the demands of modelling (e.g., providing solution plans) enhance students' motivation. Recent empirical work nuances this picture: prompting multiple solutions reliably increases students' experience of competence during modelling, although it does not always raise interest directly, suggesting indirect pathways from instruction to affect (Achmetli & Schukajlow, 2019).

Affect as a condition in modelling

Modelling behaviour is also strongly impacted by beliefs learners bring to modelling tasks. Convincing evidence comes from research on word problems. Word problems are assumed to bring (rather simplified) real-life situations in the classroom, where learners can then practice modelling. However, learners often do not engage with using real-life knowledge, but instead routinely execute arithmetical operations without seriously considering the problem context (Van Dooren et al., 2006).

In a series of studies, Verschaffel et al. (2000) gave "tricky" word problems to upper elementary and lower secondary students, for example: "Bruce and Alice go to the same school. Bruce lives at 17 km from school, and Alice at 8 km. How far do they live from each other?" Widely replicated studies indicate that learners massively tend to ignore their real-world knowledge (answering 9 km to this problem). Such superficial problem-solving behaviour is due in part to the beliefs that learners develop gradually, implicitly and tacitly through the immersion in the mathematics classroom culture. They acquire sociomathematical norms that determine implicitly or explicitly how to behave in the mathematics classroom, and part of these norms are that word problems require a single precise numerical answer, and that learners should not engage in real-life considerations or information from outside the word problem. Several studies have shown the impact of the classroom culture. When learners are involved in more authentic settings, they perform substantially better than when answering a written question (Van Dooren et al., 2006), and learners approach the same problem differently when embedded in a mathematical test versus in an ethics test (Dewolf et al., 2011).

Influence of individual differences

Individual differences also moderate the influence of affective constructs in modelling. In classroom modelling tasks, Lu et al. (2026) report that intrinsic value positively predicts enjoyment and curiosity, while situational success expectancy—students' momentary belief that they can succeed in the ongoing modelling task—supports positive emotions and buffers against boredom and anxiety. Crucially, these affective patterns vary by learner profiles: boys report higher boredom but lower anxiety than girls, and higher prior achievement is linked to reduced negative emotions and increased curiosity, revealing heterogeneous affective dynamics across groups.

Regarding differences in students' cognitive profiles, in modelling tasks, interest gaps between talented and regular students are similar as in intramathematical problems and significantly larger than in word problems with no authentic real-world connection; also, while interest related positively to performance across task types, this correlation is attenuated for talented students (Ferrando et al., 2026). Similarly, low-achieving students show higher situational interest for open modelling problems than for closed real-world problems, and the interest-performance link depends on both openness and achievement level (Schukajlow et al., 2022).

Future research directions

While research in affect in modelling and applications has made advances in the recent years, there are still important gaps that should be addressed. First, more systematic investigations of task characteristics are needed. Because modelling tasks integrate multiple characteristics at once, the affective impact of specific characteristics remains difficult to isolate. Future studies should disentangle the individual and interactive effects of task characteristics including openness, authenticity, and cognitive demand

on students' affect, while examining whether these effects are conditioned by modelling-specific beliefs and classroom norms, and whether they differ across learners' profiles. Second, longitudinal evidence is scarce. Follow-up studies are needed to differentiate situational emotions from enduring motivational development and to understand how modelling task design and modelling interventions influence affect over time, including how trajectories vary by beliefs profiles and individual differences.

AFFECT AND GENDER

Chiara Giberti, Lara Gildehaus, Lovisa Sumpter

Here, we present the results of a narrative review, where we identified three themes based on earlier research. The themes are: how gender differences evolve, gender differences and affect in the classroom, and finding a voice. These themes are deliberated using research picked purposely to represent different geographical contexts, methods, and epistemological and philosophical framings. We end the review by discussing the consequences of gender and affect.

Setting the scene

In many countries, there is a traditional view regarding boys' and girls' participation in mathematics, where gender stereotypes still exist (e.g., Liu et al., 2020). These stereotypes relate partly to what boys and girls are considered to 'be', and mathematics as a subject (e.g., Basma & Rubie-Davies, 2025). However, one major problem in research on mathematics and affect is the dominance of Western research: although people from Western cultural backgrounds make up just 12% of the global population, a disproportionately large portion of psychological research relies on samples from these societies (Picho & Schneider, 2018). For instance, a recent meta-analysis by Flore and Wicherts (2015) found a small effect of stereotype threat reducing female adolescents' mathematics performance. However, they note that 94% of the studies in their meta-analysis were conducted in just four countries: the United States, Italy, France, and Germany. The results should be compared with a study from Uganda signalling that men are not immune to stereotype threat in mathematics (Picho & Schneider, 2018). Given that gender stereotypes are culturally-defined constructs that show both consistency and variation across cultures (e.g., Ng et al., 2012), there is scope to both challenge and confirm many of the earlier studies on mathematics, affect, and gender to build a more informed research body.

How gender differences evolve

One important question in mathematics education and affect is how gender differences evolve over time, especially given that these differences are not linked to biological sex but can be influenced. For example, there are greater gender differences in participation and performance in competitive countries and environments (Steegh et al., 2019). France exemplifies such a country, and Martinot et al. (2025) recently documented gender disparities in mathematics achievement through a four-year

longitudinal study of all first and second-grade children in France, 2.5 million students. Initially, boys and girls showed similar results, but a notable gender gap in favour of boys emerged after just four months of schooling, widening further after a year. The findings are supported by a study of Australian kindergarten children: both girls and boys entered school confident in their mathematical capabilities, but after a year, not only did girls' confidence decrease relative to boys', but girls also showed a shift in their conceptions of mathematics, viewing it as less important for future careers (Davey et al., 2025). These results highlight the early formation of stereotypically gendered conceptions that can function as an explanation for other reported gender differences in affective constructs. As an example, a comparison of Swedish TIMSS data for grades 4 and 8 shows that girls' motivation declines more than boys' during this period (Sumpter & Sollerman, 2023). This is an international trend followed by most countries (von Davier et al., 2024). What is new is that more countries, which did not show any gender differences in TIMSS 2019 and earlier studies, are reporting significant differences in favour of boys in TIMSS 2023.

Gender differences and affect in the classroom

In the classroom, students who believe they can succeed persist in challenging mathematical tasks, whereas those with low self-efficacy withdraw when difficulties arise (Watson & Rubie-Davies, 2024). Even when girls earn higher grades, they often rate their abilities lower than boys, indicating a persistent mathematics confidence gap. Higher mathematics anxiety among females reduces participation and, by taxing working memory during tests, can lower performance (Ganley & Vasilyeva, 2014). Within classrooms, affect is socially constructed through students' and teachers' beliefs, expectations, and daily interactions, with students' motivation and confidence closely tied to perceived support, feedback, and encouragement from teachers (Rojo Robas et al., 2018). Teachers' gendered stereotypes and expectations can emerge in everyday classroom practices, shaping both students' self-beliefs and teachers' practices (Watson & Rubie-Davies, 2024). For instance, the idea that girls' success in mathematics is due to hard work while male success is due to innate ability or talent (Basma & Rubie-Davies, 2025), can reinforce gender biases and lower girls' self-beliefs. Moreover, classmates' stereotypical beliefs have been found to influence girls' mathematics self-concepts more strongly than boys', highlighting that peers also play a role in shaping affective factors in the classroom (Wolff, 2021). Understanding gender and affect means viewing the classroom as a system where teacher beliefs, practices, peer interactions, and classroom norms interact.

Finding a voice: the sense of belonging

The relevance of positive identities becomes evident when considering differences in students state and trait interest: girls often report similar levels of situational interest in specific mathematical tasks or lessons as boys. However, when asked about their general interest in mathematics, girls report lower levels (Høgheim & Reber, 2019). This suggests, that girls may struggle to see mathematics as something that fits with

who they are. Such a sense of belonging refers to students' feelings of being valued and recognised as participants in mathematical activity. Many girls experience a weaker sense of belonging in mathematics, often perceiving a misalignment between dominant images of "the mathematician" and their own identities (Good et al., 2012). At the same time, research highlights positive learner identities among girls in mathematics (Radovic et al., 2017). However, these identities can be fragile (Solomon et al., 2011). Success may be associated with effort rather than with competence, which can limit students' sense of agency. Within such framing, mistakes or uncertainty can be experienced as threats to identity, rather than as integral aspects of learning. This creates ground for experiences of shame in mathematics (Jenßen, 2023), where students perceive a discrepancy between who they feel they are expected to be and how they experience themselves. Shame can lead to self-silencing or avoidance of challenging tasks, even among students who are otherwise successful. Importantly, shame is not merely an individual emotional response but is socially produced and maintained. In this context, identity is not only about how students see themselves, but also about how they are seen and addressed by others in mathematical spaces. Conversely, learning environments that support dialogue, recognise diverse forms of participation, and legitimise uncertainty can foster agency and counteract shame.

The role of affect and consequences: future perspectives

While the achievement gap in mathematics has reduced, gendered differences in students' mathematics self-efficacy and future engagement in STEM fields persist. Even in countries where no gender differences in performance are observed, such as Singapore and Finland, gender gaps in mathematics self-efficacy and self-concept persist and strongly affect teaching and learning processes (Mejía-Rodríguez et al., 2021). These results suggest that deep-seated differences in self-beliefs remain, potentially affecting their future involvement in STEM. In addition, large-scale studies (e.g., Sumpter & Sumpter, 2021) show evidence of the so-called discrimination hypothesis, thereby refuting the lag hypothesis as an explanation for the continued underrepresentation of women in mathematics and engineering education, indicating that there is room for improvement at all levels.

NAVIGATING THE SPACE OF MATHEMATICS-RELATED AFFECT

Markku S. Hannula

Research on affect in mathematics education has evolved into a complex field examining emotions, motivation, and beliefs across varied contexts and activities. This forum highlights consistent findings, such as the reciprocal relationship between affect and achievement and the situated nature of affective experiences. Future work must continue refining affective constructs, apply stronger research designs, address intersectional factors, and develop interventions that foster positive mathematical experiences for all learners.

Introduction

Mathematics education research has acknowledged the importance of affect for teaching and learning. Early research on mathematics anxiety, attitude, self-efficacy, and beliefs addressed affect mostly as a factor explaining failure. Gradually the scope of mathematics-related factors has expanded in many directions. The current research forum collected important perspectives in the field to build a more holistic understanding of the area both theoretically and methodologically.

As an attempt to build a coherent picture of mathematics-related affect as a research domain, I have suggested a metatheory to map the different affective concepts (Hannula, 2012). That metatheory consists of three dimensions: 1) the level of theorizing affect (from physiological to sociological), 2) the state vs. trait type affect, and 3) the type of affect (emotional, motivational, or cognitive). These three dimensions can be used to map affective constructs in a tree-dimensional space.

However, neither my metatheory, this research forum, nor any other attempt to synthesize mathematics-related affect theories has captured all relevant aspects in one coherent framework. Such synthetic elaborations have to choose focus and ignore some peripheral issues or else they become too complex to meaningfully communicate ideas. In the current research forum, the focus is on psychological and to minor degree sociological theorizing of affect, often explicitly elaborating the state-trait distinction. Not trying to address every aspect of affect, the elaborations can be more thorough.

State-of-the-art

While the domain of affect is theoretically complex, some of the findings seem well established regardless of the specifics of the affective concept under scrutiny. Firstly, there is a well-established positive correlation between affect and achievement. Moreover, the typical mechanism behind the correlation is some kind of reciprocal feedback system where affect influences achievement and the achievement feeds back to affect development. Many contributions in this research forum review and discuss important aspects of such reciprocal affect-achievement dynamics.

Secondly, we know that the role that affect plays in learning is not universal but significantly influenced by the context where learning is situated. All contributions in

this research forum acknowledge some kind of situatedness of affect. Affect is situated in a dynamic process, in a specific type of activity, in the educational setting, and in the sociocultural context of the educational system. Comparing across the different contributions, it is apparent that the issue of situatedness is complex. Another layer of complexity is indicated by the well-established gender differences in mathematics-related affect. It is likely that the gender interacts with situatedness, so that gender differences are emphasized in some contexts and diminished in others.

Much of the interesting development in mathematics-related affect has stemmed from research on problem solving (e.g., Goldin, 2009; Schoenfeld, 1985). In this research forum Cai and Leikin review current research in the area and Krawitz and others extend this work into the area of mathematical modelling. Other specific areas where affect has been analysed are, for example, proving (Häsä et al. 2023) and working with technology (Gómez-Chacón, 2012). These examples point out the relevance of specific affective aspects of different mathematical activities.

Affect has typically been studied using either questionnaires or interviews. As it is a personal experience, it is understandable that methods often rely on self-reports. Both interviews and questionnaires are valid approaches, each for different purposes. As discussed by Coppola and others, longitudinal and comparative designs provide stronger foundation to analyse affective dynamics and impact of sociocultural context.

Future directions

Affect terminology is sometimes confusing. This research forum is aware of semantic pitfalls, and each author uses sufficiently detailed language to describe their ideas. There is growing awareness of the space of affective constructs and explicitly locating research in that space. Such practices in communication will make the field stronger.

The current research forum revisited the gender differences in affect. In addition to gender, student ethnicity (McGraw et al., 2006) and sexuality (Moore, 2021) interact with mathematics-related affect. While the role of ethnicity is well researched in some countries, the sexuality in relation to mathematics identity is less well known and the intersections of multiple group identities are largely unexplored areas.

Reading the contributions, it is obvious that no single theoretical or methodological solution suits all needs of affect research. Therefore, when beginning to plan a study, it is important to make a thorough analysis of the phenomenon under examination and choose the right constructs for the study. Is the focus on individual experience or group level patterns? Is it enough to measure the trait type disposition or is there need to follow the dynamics of an affective state? If affect is both an outcome and a condition (see Kravitz et al.), is it the same affect in both? Such questions determine what concepts and instruments are likely to allow answering to your research questions.

Overall, advancing mathematics affect research calls for stronger research designs (e.g. longitudinal or comparative) with large samples to allow for more complex analyses. Moreover, current research has mostly focused on understanding the current

mathematics education practices rather than trying to change them. There is great need for interventions aiming to develop a healthy and productive affective relation with mathematics for all students.

Self-reports will continue to be central methods in affect research, being feasible for even studying affective states as part of the process. Think-aloud protocols and stimulated recall interviews are already frequently used for revealing state type emotions as part of the process. Alternatively, facial expressions and body posture (e.g. Hannula 2015) can indicate emotions and be analysed in detail from a video. With machine learning methods, such analysis will become faster and more feasible for large samples. In addition, physiological measures (e.g. Ketonen et al., 2023) are becoming easier to use and analyse allowing stronger triangulation of different measures of affect.

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WORKING GROUPS

PROCEDURAL FLEXIBILITY: DEFINING, ASSESSING, AND DISCUSSING CURRENT DIRECTIONS IN FLEXIBILITY

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This PME working group on procedural flexibility has emerged from an international network of researchers specialising in procedural flexibility. They meet regularly to discuss the conceptualisation and measurability of relevant constructs. The aim of this initiative is to discuss diverging approaches and move towards consolidation and shared research perspectives. Regarding future PME conferences, initiatives such as working groups and research forums are planned.

INTRODUCTION

The working group will focus on clarifying and operationalizing the construct of procedural flexibility in mathematics, with attention to its facets (e.g., potential flexibility, spontaneous/practical flexibility) and the implications for measurement. While there is broad agreement that flexibility is an important target of instruction, there is much less consensus on the definition of procedural flexibility, which has hindered cumulative knowledge building. We begin by referring to the two-component definition by Star and Newton (2009) – strategy flexibility as the knowledge of multiple solution strategies and the ability to select the most suitable according to the characteristics of the task – a definition that has been frequently used in recent works (e.g., Ableitinger, Dorner, & Star, 2025).

A central goal of the group is to advance the development of usable, valid instruments for measuring procedural flexibility across topics and grade levels. Existing measures, notably the instrument by Xu et al. (2017), offer an important foundation. However, this test is relatively long, targets only linear equations, and it remains unclear to what extent task characteristics (e.g., numerical features, representation formats, or problem structures) systematically influence observed flexibility. The instrument has been frequently used in empirical studies (e.g., Hästö et al., 2019), which underscores its importance and the need for careful extension and refinement.

Our working group will (a) synthesize definitional frameworks under the Star & Newton (2009) lens, (b) identify task features that may influence strategy choice, (c) design shorter, test instruments suitable for multiple mathematical areas, and (d) discuss issues of validity. Expected results include a consolidated view of common research desiderata in the field, the development of prototypical test items and guidelines for the (further) development of reliable and practicable test instruments for procedural flexibility in different mathematical areas.

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 308-309. PME.

WORKING GROUP STRUCTURE

In Table 1, the structure and organization of the sessions is presented.

Table 1: Structure of the two 90 min sessions.

Time	Activity
10 min	Introduction of *procedural flexibility* and WG aims by WG leaders
10 min	Introduction of the topic *definitions of relevant constructs* by WG leaders
30 min	Participants work in sub-groups to discuss different definitions of procedural flexibility, potential flexibility, spontaneous flexibility, accuracy, generic and situational strategies, and others
20 min	Plenary discussion: wrap-up, refinement of definitions and identification of necessary follow-up discussions
20 min	On the way to the second slot: Participants share experiences and challenges encountered when applying procedural flexibility assessments
15 min	Introduction of the topic *flexibility measures* and the data providing study by WG leaders
35 min	Data engagement: Participants work in sub-groups with empirical data on flexibility eligible tasks (provided by WG leaders) and identify patterns of how the choice of strategies changes depending on task characteristics
10 min	Plenary discussion of relevant ideas from the groups: Documentation of findings and determination of necessary follow-up discussions
20 min	Building on the insights gained so far, participants formulate demands for future flexibility measurements
10 min	Summary of the most important outcomes and identification of the most relevant open questions for future activities, facilitated by WG leaders

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YOUTH'S VIEW ON MATHEMATICS FOR HUMAN RIGHTS

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Young people have powerful voices when it comes to social issues and their perspectives bring something that adults alone often cannot provide. In this working group, our aim is to show how youth-created activities could sharpen our understanding of connecting human rights issues (such as dignity, democratic participation and ethical awareness) to the teaching and learning of mathematics in teacher education programs. Through our project *Human Rights Values in Mathematics Education* (ViMTE), funded by Norwegian Research Council, we invited secondary school students to create artifacts to show how they see connections between human rights values and mathematics. 16 youths made mathematical "things" to share with teacher educators and pre-service teachers, such as paintings, a rap song, a T-shirt logo, graphs, and collages. Within our conversations with youth, we learnt how they bring bold ideas for change.

In our working group, we will build on two main theoretical frameworks to explore some of the students' productions. First, Critical Mathematics Education (CME) offer frameworks to understands mathematics as a fundamentally human endeavour, shaped by social, political, and ethical contexts around us (Ernest, 2019; Skovsmose, 1994). From this perspective, mathematics classrooms foster dialogue, critical reflection, and student agency, helping learners recognize mathematics as a tool for societal transformation. CME positions the classroom as a space where students can engage in social critique, ethical questioning, and democratic participation. Second, Tibbitts' (2017) models of Human Rights Education (HRE) outline three complementary approaches that reflect different purposes and depths of engagement. The Values and Awareness model focuses on building foundational knowledge of human rights principles. The Accountability model moves further by emphasizing legal frameworks, duties of the state, and the mechanisms that protect human rights. The Transformational model is the most critical and empowering, encouraging learners to analyse root causes of injustice, reflect on power dynamics, and engage in collective action to challenge and change oppressive structures. Building on these frameworks, we propose the following objectives:

1. To examine how youth-created mathematical artifacts can illuminate connections between human rights and the teaching and learning of mathematics in teacher education programs.
2. To explore the contributions of young people's imaginative and critical perspectives through the lens of CME and Tibbitts' (2017) models of HRE

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 310-311. PME.

The following questions will guide our exploration through various activities and discussions: How can youth voices help us rethink the role of mathematics education in promoting human rights values?

- Activity A – we examine youth productions to explore how they connect mathematical thinking with human rights values. The goal is to see how youths' creative work opens new ways of understanding dignity, democracy, and ethical awareness within mathematics education.
- Activity B – By engaging with student artifacts, we consider how youth imagination and critique can inform teacher education and inspire more socially responsive mathematics classrooms.
- Activity C – through the combined lenses of CME and Tibbitts' models we create a space to discuss how mathematics can function as a tool for social change.

We propose to organise the sessions as follows:

Day 1 90 mins	1. Introduction to the guiding question and exploration of the key terms. Q & A (15 min.) 2. Exploration of current literature on issues in researching human rights (15 min.) 3. Small group working on Activity A (30 min.) 4. Whole-group reflection on implications of activity A (20 min) 5. Gathering themes and questions for session 2 and recapitulation. Q & A (10 min.)
Day 2 90 mins	1. Short summary of themes collected in the first session (10 min.) 2. Small group working on Activity B (25 min.) 3. Whole group discussion. Relating the work to our stance towards social change (10 min.) 4. Small group working on Activity C (25 min.) 5. Whole group discussion, Q & A, closing remarks (20 min.)

Acknowledgements

The themes of this WG are related to the project "Developing human rights values in mathematics teacher education: Education with and by youth", funded by the Research Council of Norway.

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EXAMINING MATHEMATICAL SENSE MAKING THROUGH POETIC APPROACHES BEYOND LINGUISTIC DATA

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We will invite participants to engage in poetic approaches and collective sharing to examine emotional relational, and material marks of mathematical sense-making. By reconstructing transcripts into poetic forms, these activities aim to humanize research by making the rhythmic, emotional, and embodied dimensions of mathematical engagement visible.

This working group is a continuation of POEME first established in PME 46. Prior POEME working group sessions (PME 46 and 47 and EARCOME 9) examined mathematical poetic inquiry through emotional relationality and linguistic patterns of repetition (Hare et al., 2024; Tiong et al., 2025). This working group attends to methods guided by an ethics of care and embodied poeticity. Participants will engage poetic approaches that illuminate emotional and relational dynamics in mathematical experiences—particularly researcher emotions and affective dimensions often marginalized by conventional coding—and practise poetic diagramming as an analytic approach foregrounding the generative agency of diagrams and gestures in mathematical sense-making.

BACKGROUND

Poetic methods offer a transformative framework of mathematical inquiry, providing a vital means to surface shared concerns, build connections across diverse identities and positions, and uncover the everyday mathematics that shapes family life. When adopting a feminist ethics of care, poetic inquiry establishes conditions for listening that resist the tendency to flatten participant experiences (Faulkner, 2019). This approach transforms researcher affect – responses of empathy, discomfort, or resonance – into vital analytic resources, where noticing one's own emotional landscape becomes a site of connection (Gerbrandt, 2026). When poetry humanizes research encounters, it can help us to listen in ways that resist pathologizing participants and instead foster empathetic, reciprocal understanding.

Building on this relational foundation, we incorporate de Freitas and Sinclair's (2012) perspective on diagrammatic agency to view diagrams and gestures as vital actants in mathematical sense-making. Consequently, lines, marks, numbers, and bodily movements are not merely static representations but dynamic forces shaping distinctive rhythms of thought. Poetic diagramming serves as a methodological bridge that makes

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 312-313. PME.

these temporal and spatial dimensions perceptible, extending poetic inquiry into the material realm. By translating gestures and drawings into poetic forms, we shift the analytic focus from what students say to how mathematical meaning and agency emerge through embodied engagement.

GOALS OF THE WORKING GROUP

Participants in this working group will consider the following questions: (1) How can poetic methods humanize mathematics education research by making relationality, emotion, everyday mathematical experiences and diagrammatic agency visible? (2) In what ways can researchers affect – including responses of empathy, discomfort, or resonance – become vital analytic resources? (3) How can poetic inquiry be extended to diagrammatic and gestural forms of data and what are some ways poetic structures are entailed in these embodied forms of mathematical sense-making?

ACTIVITIES AND TIMELINE

Session 1	We would provide an overview of the two sessions (5 min) We would introduce the roles of emotion, relationality, and care in poetic (15 min) Participants to create poems (20 min) Participants to share poems to discuss emotional and analytic insights (20 min) Participants to reflect on the poem creation and possible extensions (30 min)
Session 2	We would revisit the last session (10 min) We would explore diagramming and gesturing (15 min) Participants to reconstruct gestures, rhythms, and patterns as poetic diagrams (25 min) Participants to discuss agency and material interaction (30 min) We would seek future collaboration towards an edited book on Poetic Inquiry within Mathematics Education (10 min)

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INTERNATIONAL PERSPECTIVES ON PROOF, PROVING AND RELATED ACTIVITIES

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This is a continuation of the working group on 'international perspectives on proof and proving' at PME 43-45 (Reid et al., 2019, 2021, 2022), PME 46 (Sommerhoff & Komatsu, 2023) and PME 47 (Reid & Shinno, 2024, 2025). The aim is to foster research on proof and proving by bringing together research on proof, proving and related activities, and perspectives of international and cultural comparison. The long-term goal is to present the results of several comparative international studies on proof, proving and related activities in a PME Research Forum. The past two decades have seen a strong increase in research into proof and proving in mathematics education, as well as work addressing broader aspects of proving-related activities, such as exploring and arguing for conjectures, and other forms of mathematical justification or reasoning. Nevertheless, much of this research has been conducted in single national and cultural contexts. More recently, and perhaps in part as a result of past incarnations of this Working Group, there has been a growing interest in research that compares aspects of proof, proving and related activities in mathematics education in multiple contexts. The upcoming special issue of the journal *Educational Studies in Mathematics* (ESM) on Cultural Practices in Mathematical Argument is an example of this.

This growing research on proof, proving and related activities from an international perspective is much needed as it remains unclear whether existing research results from single national and cultural contexts are transferable, or, indeed, if the assumptions on which the studies are based are valid elsewhere. Notwithstanding the increasing amount of comparative research, comparatively little information exists about the role of proof, proving and related activities in educational contexts from an international perspective. Additional international comparisons involving a wider range of countries could shed light on the *teaching and learning of proof and proving* as concerns curriculum (including textbooks and other teaching and learning resources), student learning and achievements, teaching (including teaching practices, teachers' knowledge, and initial or continuing teacher education), and assessment. The same applies for *proving-related activities* (like argumentation) and their relation to proof (e.g., Pedemonte, 2007; Stylianides, 2008).

As part of the PME 49 Working Group sessions, there will be discussions of recent developments in related research, for example, by considering contributions from sessions at the ERME Topic Conference MATHPIE2026 on Argumentation and Proof in the Teaching and Learning of Mathematics and Interactions with Other Disciplines.

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 314-315. PME.

STRUCTURE OF THE SESSIONS

Involvement of participants from different countries and regions is essential to the functioning of the group. Over the two sessions, the following activities are planned:

Day 1: This session focuses on community building and establishing priority themes for future studies. Introduction to the working group, its aims, long-term goals and history (15 minutes); introduction of participants (25 minutes); reports from MATHPIE2026 (15 minutes); reports from other related research (10 minutes); introduction of the forthcoming ESM special issue (5 minutes); identification of themes for Day 2 (20 minutes).

Day 2: This session focuses on developing concrete collaborative directions and strengthening international partnerships. Assignment of participants to subgroups based on established Day 1 themes (10 minutes); subgroup discussions (40 minutes); reports of subgroups (20 minutes); whole group discussion of ways to expand collaborations among researchers (20 minutes).

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MAKING SENSE OF HIGH-QUALITY MATHEMATICS INSTRUCTION ACROSS CULTURES: TOWARDS A MORE INCLUSIVE CHARACTERISATION

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There have been debates over the characteristics of high-quality mathematics instruction in recent years, despite the widely held understanding that effective teaching can take many forms. More specifically, there is a lack of clarity about the necessary characteristics of high-quality mathematics instruction and what it might look like in different parts of the world. In this working group, we aim to foster discussions on various dimensions of effective teaching and to develop a more inclusive characterisation of good teaching.

In recent years, a polarising discourse, fuelled by iterations of the 'math wars', has emerged over what constitutes high-quality mathematics instruction (Chernoff, 2019; Sawatzki & Armour, 2025). The language used in these conversations promoted a particular pedagogical approach, even though it is widely recognised that effective teaching can take many forms (Kilpatrick et al., 2001). For example, in Australia, there has been a push towards explicit teaching, alongside pushback from many mathematics educators against a single dominant approach. Although mathematics education researchers generally proposed using a mix of instructional approaches, the balance required remains an open question. What is more pressing, perhaps, is the lack of clarity about the essential features of high-quality mathematics instruction and what it might look like in different parts of the world.

High-quality mathematics instruction is often characterised by its mathematical quality: the mathematics presented, the tasks used, the kinds of mathematical discourse, and a clear focus on problem-solving and problem-posing. Several frameworks, such as the *Mathematical Quality of Instruction*, or MQI (Learning Mathematics for Teaching Project, 2011), and *Teaching for Robust Understanding*, or TRU (Schoenfeld, 2013), have been developed. However, much of the narrative has been shaped by the Global North, which privileges particular ways of assessing the quality of mathematics teaching that may not be culturally relevant or even feasible given differences in classroom cultures and norms across the globe. Hence, it is critical to develop a more inclusive characterisation of high-quality mathematics instruction if we want to acknowledge and draw from the diversity of effective teaching practices, especially those from the underrepresented countries.

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Aim of this Working Group

Building on the work to characterise high-quality mathematics instruction (Learning Mathematics for Teaching Project, 2011; Schoenfeld, 2013), we aim to foster discussions around different dimensions of effective teaching and collectively examine and refine existing characterisations, based on cross-cultural analyses conducted with the participants.

The following two questions will help in framing and situating the work and discussions in this working group:

1. What are the similarities, differences, and tensions in our characterisations of high-quality mathematics instruction practised across the globe?

2. What are the potential areas for refinements or extension we make to the existing dimensions of high-quality mathematics instruction to acknowledge and draw from the diversity of effective teaching practices across different parts of the world?

Proposed activities for the two sessions

We begin the first session by setting the context and key questions to be explored in this working group (5 minutes). To answer the first question, we will provide up to 2 mathematics teaching videos (from the TIMSS Video Study or OECD Global Teaching Insights) to the conference participants (10 minutes) and focus group discussions on identifying similarities, differences, and tensions in the application of existing frameworks, such as MQI or TRU (40 minutes). We will then consolidate participants' ideas and emerging insights through a whole-group discussion (25 minutes) before synthesising and articulating the focal issues to be taken up in Session 2 (10 minutes).

The second session will begin with a recap of Session 1 and framing using Question 2 (10 minutes). We then facilitate small-group work exploring possible refinements, extensions, or re-framings of selected dimensions, considering the identified tensions (40 minutes). We then invite the participants to share and discuss their proposed directions (30 minutes). The working group will then conclude with a shared mapping of key tensions and exploratory directions for future collaboration (10 minutes).

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VOCATIONAL MATHEMATICS EDUCATION – AN EXTENSIVE, YET UNDERSTUDIED RESEARCH AREA

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BACKGROUND AND RATIONALE

Research on *Vocational Mathematics Education* (VME) is limited, yet vocational education attracts many students; in 2023 in the EU, 49 % of upper secondary students followed vocational programs (https://ec.europa.eu/eurostat). VME covers the mathematics in courses and programmes that prepare learners for high-demand occupations, such as builders, fashion designers, health workers, office managers, and machine operators. Many adolescents and adults engage in VME because an ever more digitalized and quantified society demands capabilities to skilfully perform many specialized activities of a mathematical nature. Depending on the country, VME is at lower secondary level, at upper-secondary level, at post-secondary (further education) level and/or is embedded *on-site* within industries (Sträßer & Zevenbergen, 1996). Also, the mathematical content within VME varies; it can be similar to, or slimmer than the mathematics curriculum from parallel theoretical streams, it can connect mathematics specifically to the vocations, it can aim for mathematical literacy, or it can be a hybrid (FitzSimons, 2014). Many learners in VME have a history of failure in mathematics, combined with discouragement, anxiety, apathy, or even dropping out. Also, more learners in VME than in the theoretical streams are from disadvantaged and less-represented backgrounds and have a lower SES.

Theoretical frameworks for research on VME need to create a bridge between mathematics and the crafts, and to consider that learners within VME "have a sense of belonging" to their vocation, totally different from being a student of mathematics-for-its-own-sake. Among the theories used are community of practice theory (e.g., FitzSimons, 2014), boundary crossing (Fiorentino et al, accepted), theories on authenticity for vocation-based problems (Brantlinger, 2022), or Bishop's framework of seeing learning mathematics as a process of enculturation (Dahl et al., 2025).

STRATEGY AND AIMS

We propose a new WG focusing on VME. Yet, also PME has not been focused on this very large group of students but given its high scientific reputation, the community has the opportunity to be at the forefront of establishing research and research paradigms for this large group of learners. PME offers a unique platform to initiate and sustain international research collaborations in VME. The working group aims to outline,

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discuss and study: differences across vocational education internationally, including the role of mathematics in it; differences between school mathematics for theoretical streams and for VME; characteristics of collaborations between mathematics and vocational teachers (Fiorentino et al., accepted), characteristics of teacher competences needed in VME (Voss, 2025); find, develop and compare theoretical and conceptual frameworks for researching VME.

OUTLINE OF THE SESSIONS

In a first session we propose: (1) an introduction and rationale for the WG, including discussion (30 min); (2) Activity around tables where participants are asked to map their experience with VME. WG-leaders circulate (40 min). Participants present their findings (20 min). In the second session: (1) Introduction and presentation of the WG-leaders' reflections on the submitted materials from participants at session 1, including which areas to focus on today (20 min); (2) Activity around tables with participants discussing the new task, WG-leaders circulate (40 min); (3) Discussion, evaluation and decision on the next moves. In general, we expected that this WG will provide participants with more knowledge on VME internationally and lay the ground for future research, including work on conceptual frameworks.

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VALUES FOR MATHEMATICS EDUCATION: NEXT STEPS

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GOALS OF WORKING GROUP

This Working Group is a follow up to previous PME working groups (namely, PME36 in 2012, and PME39 in 2015), which resulted in two Springer books (Clarkson et al, 2019; Dede et al, 2024) and many years of sustained RR presentations at subsequent PME conferences. This proposed Working Group is expected to have the same follow-up activities. The goals for the current Working Group are:

- To re-conceptualize existing theories/concepts on values and related constructs, by exploring emergent frameworks and personal research experiences.
- To formulate researchable questions in the field, and to explore and develop potential methodologies that address these questions.

THEORETICAL BACKGROUND

A more comprehensive understanding of how mathematics is even better taught and learnt requires the acknowledging that the human mind is tripartite in nature, involving cognition, affect, and conation, and that all three aspects can – and need to – be developed for everyone to learn mathematics and be numerate while experiencing a sense of well-being. Values constitute a key conative variable, underlying motivation and drive (Seah, 2019). DeBellis and Goldin (2006) highlight the deeply rooted nature of values, describing them as "personal truths" that inform long-term priorities.

Recently, within mathematics education research, there has been increasing interest in the development of theories and research approaches that can account for shaping decisions and behaviours and can shed further light on the complex relationship between values and affective variables (e.g. beliefs). A dynamic interplay between teaching and learning highlights the potential of values to enhance ethical and effective mathematics education (Chiu & Seah, 2024). Values have also been instrumental in the promotion of mathematical wellbeing, an affective variable (Hill et al, 2021).

PROPOSED LAYOUT OF WORKING GROUP

Session1

- Introductions and sharing of experiences about values and values research. What theoretical constructs do we build on when working with what teachers

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and students value in mathematics education? What specific expertise do we have as researchers? How can we learn from other fields (psychology, cultural studies etc)? Participants will be invited to share their views on frameworks as we come from different backgrounds. They will also be welcomed to share their own stories leading to a discussion of themes, commonalities, differences, etc.

- Brief presentation of existing ways of research and then moving to some suggestions for possible new approaches and conceptualisations (JEDI approach, applying research to practice, etc).
- Discussion in groups with a focus on the development of frameworks that support the interaction between theory and practice.

Session 2:

- Building off session 1, groups discuss: Creating a set of researchable questions in the area of values and consider i) which of these questions would you like to/be able to research? ii) what theoretical frameworks could you use/develop to support this research? iii) what methodologies are appropriate for these questions? iv) what would data consist of?
- Each group to share responses and then discuss and agree on next steps for future collaborations, including consideration of a possible joint output for participants such as a special issue for the *Journal of Mathematics Teacher Education*.

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ECOLOGICAL METHODOLOGIES FOR MATHEMATICS EDUCATION RESEARCH

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Research frameworks orient inquiry, drawing on particular principles and assumptions that guide action. They simultaneously enable and constrain how educational research is conceptualised and justified, and impact the questions we ask, along with the methodologies taken to move towards understanding. In this working group, we will explore an emerging framing in mathematics education that has been referred to as ecological (e.g., Weston, 2025) or socioecological (e.g., Coles, 2023). We use the term ecological to centre the study of relationships (Begon et al., 2012) between organisms (including humans), and their environments (e.g., natural, physical, social, cultural, political, material). There are several strands of thinking within mathematics education which have forged awareness of ecological relations, including critical mathematics education, Indigenous scholarship, ethnomathematics, feminist thinking, de-colonizing, new materialism, enactivism, socio-political research, and others (Makramalla et al., 2025).

In this working group, we build on a recent research forum at PME46 on socio-ecological issues (Amico et al., 2023) and explore what the emergence of an ecological framing could mean for the way we conceptualise and carry out research in mathematics education. The term 'ecological methodologies' is used here to capture an entangled, emergent approach to research that prioritises the living world, prompting us to consider how a range of relationships might inform the articulation of potential research and pedagogical designs. Our working group is linked to the conference theme in that we are trying to make sense of the precarity of our living world, and sense how our common and unique environments and ecological challenges can collaboratively contribute to emerging ecological methodologies for mathematics education research and pedagogies.

In addition to considering the study of relationships, an ecological framing provokes many additional considerations. One of these considerations concerns our understanding of time. Working from an ecological perspective that is grounded in the living world indicates ecological time is not linear but rather is a superposition of multiple cycles where fast and slow processes occur simultaneously, interact, and shape the present at any location. This means any point in time is a snapshot of various ongoing cycles interacting and producing a combined, layered influence. Other aspects that an ecological framing might prompt us to consider include: who or what counts as a knower; how our inquiries reinforce, generate, or collapse a range of distinctions;

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how emergence, ambiguity, and complexity could shape the boundaries of our research; how our positionality, practicing of deep listening, and recognition of participants' lived knowledge continually influence both our interpretation and the phenomena we aim to study. Throughout this working group, a guiding question is, when working from an ecological perspective, is it possible to apply an ecological lens to our existing inquiries or do we need to be asking fundamentally different questions? We will consider a variety of ecological principles and discuss how an ecological framing could impact on our methodologies both in terms of observation and research design.

AIMS OF THE WORKING GROUP

(1) To explore the consequences of an ecological framing on research design and research practices in mathematics education.

(2) To interrogate current and articulate new research practices and principles from an ecological perspective.

OUTLINE OF THE SESSIONS

Session 1 (Observing relations): (1) Sharing ecological issues from our contexts in small groups (15 mins); (2) Introduction to ecology as the study of relations, including discussion time (25 mins); (3) Focusing on relations in observing, practicing together with video data (40 mins); (4) Feedback and setting up session 2 (10 mins).

Session 2 (Ecological practices): (1) Recap of session 1 and setting up session 2 (10 mins); (2) Interrogating current research practices, in small groups using prompts from an ecological perspective (30 mins); (3) Generating new research practices in small groups (40 mins); (4) Feedback and next steps (10 mins).

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SENSING AND MAKING SENSE OF DEVELOPING SUSTAINABLE RESEARCH PRACTICE PARTNERSHIPS

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Engaging in sustainable research practice partnerships (RPPs) with teachers offers a powerful means of supporting teacher professional learning while conducting research that can meaningfully influence classroom practice and student outcomes. Coburn et al. (2013) characterize RPPs as long-term collaborations between researchers and practitioners that investigate problems emerging from practice itself. Such partnerships draw on teachers' expertise and experiences to co-construct research agendas aimed at improving student learning. Effective partnerships develop over time and are grounded in the realities of local contexts. RPPs can strengthen teachers' capacity for data-informed decision-making, supporting the productive use of research in practice (Hemsley-Brown & Sharp, 2003).

At the same time, building and sustaining RPPs presents persistent challenges. These include establishing and maintaining trust, negotiating shared goals as institutional priorities shift, and aligning research and funding timelines with the rhythms of school and partnership life (Barton et al., 2014). Recent initiatives in the mathematics education community – such as an ICME-15 Discussion Group and a special issue in *School Science and Mathematics* – signal a growing interest in research partnerships, while also highlighting the need for continued, practice-focused dialogue. This Working Group contributes to that momentum by providing a forum for PME participants to collectively examine these tensions and share experiences of designing, implementing, and sustaining RPPs. Together, we will consider whether and how long-term partnerships are feasible across varied contexts, what has made some initiatives successful or unsustainable, and what structural conditions may be necessary to support this work.

AIMS OF WORKING GROUP

(1) To identify key challenges in developing and sustaining RPPs with teachers across diverse contexts; (2) To develop strategies and recommendations for researchers, teacher educators and school stakeholders that address these challenges and support partnerships that are sustainable, mutually beneficial, and responsive to practice.

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OUTLINE OF SESSIONS

Session 1: Examining the Problem Space

This session focuses on surfacing collective knowledge about the realities of long-term collaboration in schools and interrogating assumptions about the feasibility and value of sustained RPPs.

Introduction (30min): Framing of the session goals and guiding questions. Brief overview of relevant research and illustrative examples of RPP models.
Collaborative Data Gathering (30min): Participants work in small groups to share experiences developing or attempting to develop RPPs. Guiding prompts may include: What conditions enabled or constrained partnership development? What challenges emerged over time? What opportunities or successes were leveraged? How did contextual factors (e.g., rural/urban settings, funding structures, institutional expectations) shape the work?
Synthesis and Reflection (30min): Groups report back to identify recurring themes, tensions, and questions. Collective documentation of key challenges using a shared visual space.

Session 2: Developing solutions and future directions

Building on insights from Session 1, this session shifts toward designing practical and systemic responses.

Revisiting Key Challenges (15min): Facilitators synthesize themes from Session 1 and identify priority areas for action.
Reviewing Illustrative Cases (15min): Small groups analyse brief case examples of partnership models to explore structures, supports, and strategies that have enabled sustainability.
Collaborative Action Planning (30min): Participants form self-selected groups to co-develop strategies and recommendations to address identified challenges.
Final Reflections and Closing (30min): Sharing of proposed strategies and emerging principles. Identification of opportunities for continued collaboration beyond the conference. Summary of key takeaways for advancing sustainable RPPs within the mathematics education community.

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MATHEMATICAL-PEDAGOGOCAL (MP) TASKS IN TEACHER EDUCATION: DESIGN, TEACHING AND RESEARCH

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In mathematics teacher education, the mathematics subject matter and the pedagogy of mathematics are traditionally separated between courses and departments. Despite repeated arguments for the value of integration (e.g., CBMS, 2012), traditions and institutional structures are resistant to change. The separation contributes to what Felix Klein described as the "second discontinuity" (1945) – teachers see little value in their undergraduate mathematics courses. To overcome this discontinuity, there have been efforts to introduce teaching-related tasks in mathematic courses (e.g., Wasserman et al., 2023), in order to demonstrate the potential applicability of the learned material to the work of teaching. Our goal is to compliment these initiatives by focusing on their compliment: explore the use and potential growth of teachers' mathematical knowledge via pedagogical engagement. To this end, we define a *mathematical-pedagogical task* (in what follows, *MP Task*), as a task in teacher education that uses a pedagogical setting to explore and strengthen teachers' knowledge of underlying mathematical ideas. The **goals** of this new WG are: (1) Introduce the idea of MP Task as a research and instructional tool in mathematics teacher education; (2) Explore various interpretations and diverse uses of MP Tasks; and (3) Design and mobilize new MP tasks and new collaborations.

As an example of an MP Task, consider the following:

> In class students are introduced to the concept of an inverse function and the conventional symbol f^{-1} is used. Students are confused, recognizing the symbol previously indicating a reciprocal, such as in $5^{-1} = \frac{1}{5}$. A student asks: "have they run out of symbol or what?"

This scenario was presented to secondary teachers in Zazkis and Kontorovich (2016) as a "scripting task", that is, participants were asked to write a dialogue with students that responds to the question. The results revealed that only a few participants related the symbol to the overarching notion of inverse with respect to an operation, whereas majority opted for highlighting the difference is usage with respect to numbers and functions. This presents an opportunity to enrich teachers' knowledge in follow up instruction by expanding or strengthening teachers' mathematics and exploring pedagogical responses to the situation that preserve mathematical integrity.

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Table 1: Structure of Day 1 Working Group Activities

Activities	Time	Description
Introduction	10min	Use an example and activity to introduce MP Tasks
Participant Activity	15min	Participants analyse a set of provided tasks according to the underlying mathematical ideas, using QR code
Short Presentations	30min	Three mathematics education researchers present results of MP tasks they used in teaching or research
Small Groups Activity	20min	In groups of 3-4, participants share a task from their own practice that could be developed as an MP Task
Whole Group Discussion	15min	Discuss goals of the (MP) tasks with respect to pedagogy and mathematics. Set up for Day 2

Table 2: Structure of Day 2 Working Group Activities

Activities	Time	Description
Introduction	5min	Welcome and summary of Day 1
Small Groups Activity	20min	Participants choose from a variety of mathematical topics, design MP Tasks, and analyse mathematical knowledge needed to engage with the task
Group Presentations, Discussion	30min	Groups share MP Tasks that were designed. Participants discuss the related learning goals and possible research directions.
Participant Activity	15min	Participants analyse presented extended collection of MP Tasks according to different criteria using QR code
Discussion and Next Steps	20min	Develop collaborations for future PME activities, and possibility of task compilation and dissemination

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SENSE-MAKING IN EXTENDED REALITY: EXPLORING EMBODIED MATHEMATICAL INTERACTIONS USING SMART DEVICES AND HEAD-MOUNTED DISPLAYS

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This Working Group (WG) investigates how different Extended Reality (XR) interfaces – smart devices and head-mounted displays (HMDs) – shape embodied mathematical sense-making. The WG will explore how XR technologies extend mathematical activity beyond 2D representations into gesture-rich 3D visuospatial representations. Through hands-on engagement and discussion across two 90-minute sessions, co-researchers will collaboratively analyze affordances, constraints and design implications of different XR modalities. Thereby, the WG aims to generate design principles, guide future research directions and help establish an international network for XR-focused researchers in mathematics education.

TOPIC, GOAL AND STRATEGY

Topic: embodied mathematical sense-making in XR environments, with a specific focus on interface modality (smart device versus HMD).

Goal: To examine how XR interface design reorganizes multimodal engagement, spatial reasoning, abstraction and collaboration in mathematics education and to articulate emerging design and research directions.

Strategy: The WG combines embodied engagement with comparative analysis and collaborative synthesis. Co-researchers will experience mathematical tasks using both handheld and hands-free XR tools, followed by guided discussion and qualitative and quantitative feedback to identify emerging themes and researchable questions.

THEORETICAL BACKGROUND

Embodied and enactivist perspectives consider mathematical meaning as emerging through learners' sensorimotor interaction with their environment (Anton & Abrahamson, 2025). Multimodal research has shown how gesture, artefact manipulation, speech and inscription all contribute to mathematical sense-making. Even so, most studies remain grounded in 2D or screen-based environments. XR technologies [e.g., Augmented Reality (AR) and Virtual Reality (VR)] extend multimodal interaction into spatially anchored, immersive 3D environments (Wang et al., 2025). However, XR is not a uniform modality; different interfaces afford distinct forms of embodied interaction. Handheld AR remains screen-mediated, whereas HMD-based XR offers immersive, field-of-view engagement. These interface

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differences may reorganize perceptual focus, proprioceptive anchoring, gesture–symbol coordination, cognitive load, and collaboration.

This WG investigates whether interface modality affects embodied mathematical interaction and sense-making for learners.

NEW INITIATIVE, STRUCTURE AND FOLLOW-UP

The proposed WG is a new initiative extending PME-48 discussions on embodiment and multimodality into XR contexts with varying levels of immersivity. It consolidates an emerging field examining how interface modality reorganizes embodied mathematical sense-making. Follow-up activities may include: (a) a collaborative journal article synthesizing cross-national XR studies; (b) a Colloquium on immersive and embodied mathematics learning; and (c) a PME-50 continuation WG focused on classroom implementations. Co-researchers and potential co-authors will be invited to join a post-conference writing and research network.

Table 1: Activity Plan (2 × 90 Minutes)

Duration	Tasks and Structure
15 mins	**Session 1:** Introducing GeoGebra 3D/AR for smartphones or tablets
45 mins	Activity stations: Fraction number line app prototype; differential modelling; projectile model; Earth-Sun interaction model.
30 mins	Reflection, synthesis and polling.
15 mins	**Session 2:** Introducing XR apps for HMDs.
60 mins	Immersive activity stations: co-researchers adopt three roles: Embodied Actor, Embodied Interaction Analyst and Semiotic Analyst.
15 mins	Comparative reflection, synthesis, and future research directions.

COLLABORATION, DATA COLLECTION AND OUTCOMES

Co-researchers rotate through stations, contribute written and verbal reflections, participate in live polling and collaboratively map affordances and constraints. The WG aims to: identify affordances and limitations of hand-held versus head-mounted XR; generate design principles; surface methodological considerations for studying immersive embodied interaction; develop researchable propositions concerning interface modality; and establish an international XR research network. Co-researchers will critically examine both opportunities and implementation constraints.

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PROOF ASSISTANTS AS CATALYSTS TO DISCUSS ABOUT PROOF IN MATHEMATICS EDUCATION RESEARCH

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This new working group addresses a persistent deadlock in mathematics education research on proof, as identified by Balacheff (2002) and Reid (2005), a challenge that remains still relevant today. Despite an extensive literature, learning and teaching difficulties with proof and proving are worldwide still present, particularly during the secondary-tertiary transition. Multiple theoretical frameworks and experiments – rooted in distinct epistemological, logical, cultural, and cognitive foundations – coexist, often without explicit researchers' positioning. This impedes shared understandings and the collective search for solutions to the didactic challenge of proof instruction. Proof assistants (PAs, e.g., Lean, Rocq etc.) offer a novel lens to revitalize the debate. They are increasingly present in mathematics and computer science research and are gradually implemented in educational contexts (e.g. Hanna et al., 2019) even if not initially designed for educational purposes. While all PAs share the same function of mechanically verifying proofs through formal logic, their interface designs and feedback mechanisms differ substantially. This working group uses their potentials via an unplugged simulation with a shared language to highlight the implicit elements embedded within each researcher's proof conception. It also identifies priority research directions for enhanced community dialogue within mathematics education but also between mathematics education and other domains concern by proof (e.g. computer science, represented by two members of the working group team).

THEORETICAL BACKGROUND

This working group intentionally avoids any predetermined theoretical framework to preserve openness in the debate. Participants are thus encouraged to draw upon the full spectrum of established frameworks in mathematics education research about proof, including Balacheff, Hanna, Duval, Reid, Harel & Sowder, Stylianides and others, for interpreting contrasts emerging from exemplar and PA-simulated proofs.

PLANNED ACTIVITIES

Session 1: from "exemplary proofs" to PA constraints (90 minutes)

The plan is to debate on what a "good" proof is, so to list common ground and differences depending on researchers' points of view. To achieve this aim, following a previous experience (Reid, 2005), participants pre-submit or bring their own most

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"exemplary proof" from their research or teaching (on one page). The team of the working group presents the aims and the modalities of the sessions (10 minutes).

Phase 1: Traditional mode (40 minutes)

In small groups of four to five, participants share exemplars. Each participant controls at least one colleague's proof. In each group, the participants share the justifications of their choices and their control criteria. These are synthesized in plenary.

Phase 2: What is a PA? For what? (40 minutes)

What a PA is and how it works will be illustrated by the working group team with two different PAs (a point-and-click PA and a controlled natural language PA). The *tactics* and validation processes are highlighted.

Session 2: Simulated assistant mode and debrief (2×45 minutes)

Participants rework their "exemplary proof" under an unplugged PA, with constraints provided by the working group team (limited tactics to rework the proof and simulated automated validation). Groups document the way they have to adapt or modify their reasoning processes, their proof structure, the granularity of their arguments, the language. Then, they analyze the contrasts observed between the two modes of proving (traditional mode and simulated assistant mode). They identify dimensions along which the two proofs differ, they articulate how validation operates and they examine whether the simulated mediation changes what counts as a "satisfactory" proof. Each group produces a written synthesis, identifying key analytical criteria emerging from the comparison and concrete research questions suggested by these observations. A final plenary discussion summarizes contributions into a shared set of analytical dimensions and research questions about the epistemology of proof and the use of PAs in teaching and learning.

EXPECTED OUTCOMES

The expected outcome is to produce a first shared analytical grid derived from exemplary proofs / PA contrasts for PME newsletter. A follow-up activity includes a working group at PME50 to continue the grid and the debate about the relevance of PAs in teaching and learning proof.

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DIVERSITY OF PERSPECTIVES AS A RESOURCE: CONNECTING IDENTITY THEORIES IN MATHEMATICS EDUCATION

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This Working Group emerges from a small international community of early-career researchers in mathematics education who have been collaboratively negotiating how "identity" is conceptualised and operationalised in empirical research. Over the past year, we have met regularly to examine tensions in our own studies, compare and test methodological options, and reflect on how research settings and data sources shape what identity becomes in analysis. Our projects span multiple countries, learning groups, in- and pre-service teachers. Across diverse focal concerns, including language diversity, social justice, problem solving, and emotions, we draw on sociocultural, political, and psychological perspectives through narrative and participation-based approaches. Within this breadth, identity has functioned both as a shared focus and a productive point of tension. This leads us to a common set of questions about how identity is conceptualised across topics, how these conceptualisations relate to theoretical commitments, and how the analytic choices are justified.

To extend these conversations to a wider community, the Working Group is designed around two theoretical tools. First, we use a Darragh-inspired landscape mapping (identity as narrative, participative, discursive, performative or psychoanalytic) as an organising scaffold, to visualise plurality and surface key tensions and overlaps across approaches (Darragh, 2016). Second, we use networking strategies (Prediger et al., 2008) as a meta-theoretical lens for relating perspectives through explicit moves. Networking strategies are structured in paired moves: understanding and making understandable, comparing and contrasting, combining and coordinating, and integrating locally and synthesising. Crucially, Prediger and colleagues emphasise that networking becomes most productive when grounded in a shared empirical base and paradigmatic examples, because purely abstract dialogue can obscure explanatory power and limitations.

The main goal of this Working Group is to engage early-career and senior researchers in mathematics education in the process of identity research. Through participatory, hands-on analytic work, participants will explore how different theoretical and analytical perspectives can be brought into relation, and how these choices shape the

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identity objects, warrants, and claims that become possible. Organisers will provide a shared set of short empirical excerpts (e.g., a transcript segment) that will serve as common ground for analysis and testing networking moves. Thus, our questions are:

1. What are the core ideas, underlying assumptions, and research traditions that stand behind each of the theories used in shared identity studies?
2. How are multiple perspectives connected through networking strategies, and how do these choices shape the identity objects, warrants, and claims that become possible?

OUTLINE OF SESSIONS

Session 1 (90 min): Mapping multiple perspectives

Introduction (10 min): Presentation of the goal of the Working Group and visualise the theoretical plurality identity research through Darragh's landscape mapping.

Overview of theoretical perspectives (30 min): Members of the organising team present their research topic and the theoretical perspectives followed in their work.

Landscape mapping activity (40 min): Participants will work in groups to construct potential connections and tensions based on a landscape map of the perspectives represented in the organisers' studies.

Discussion (10 min): Discussion outcomes of tensions and promising connections.

Session 2 (90 min): Testing data analysis through networking moves

Re-entry and task framing (10 min): Brief recap of Session 1 outcomes and introduce the networking moves.

Shared data and lenses briefing (15 min): Organisers will share empirical excerpts and minimal context. Organisers states the theoretical lens they would mobilise and what it foregrounds in the excerpt.

Data engagement activity (40 min): Participants, in small groups, will engage with excerpts through the aforementioned tools. They will work on interpreting the identity phenomenon foregrounded in the excerpts, choosing and justifying a networking move.

Collective synthesis (25 min): Senior researchers in the organising team synthesise patterns across groups, highlighting takeaways for operationalising identity and justifying when connecting perspectives.

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AI-BASED MATHEMATICS TEACHING SIMULATIONS: PROMISE, PITFALLS, AND RESEARCH DIRECTIONS

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AI-based teaching simulations are emerging as a significant methodological and instructional development in mathematics education (Galiç et al., 2025; Lee et al., 2025; Rotem & Chytas, 2026; Schwarts & Czaczkes, 2026). Powered by generative models, they produce adaptive, mathematically coherent student responses within defined content domains, enabling structured approximations of classroom dialogue—particularly for practicing eliciting reasoning and responding to errors. Building on traditions of rehearsing high-leverage practices in live simulations (e.g., Shaughnessy & Boerst, 2018) and digital environments (e.g., Herbst et al., 2022; Schwarts et al., 2025; Webel & Conner, 2017), AI-based simulations serve as both research instruments and pedagogical tools. They foreground analytic and instructional affordances while raising concerns about validity, disembodiment, and broader ethical and environmental implications. In this inaugural meeting of the working group, we aim to examine both the promise and the pitfalls of AI-based teaching simulations.

AI-based teaching simulations offer significant promise for mathematics teacher education and research: They create structured environments for approximating practice, enabling focused work on instructional noticing (Galiç et al., 2025; Rotem & Chytas, 2026), responsiveness (Schwarts & Czaczkes, 2026), and questioning skills (Lee et al., accepted) while making teacher pedagogical reasoning more visible and studyable. Such simulations support iterative rehearsal, design experimentation, and the decomposition of complex practice into analytically tractable components. They also open possibilities for scalable comparative research.

Yet, these environments introduce substantive pitfalls. They rely on computational infrastructures with environmental costs; they reduce and reconfigure the embodied, relational, and contingent dimensions of classroom life; and they raise concerns about epistemic integrity and bias in AI-generated interactions.

AIMS OF THE WORKING GROUP

The working group aims to (1) examine the pedagogical and methodological affordances of AI-based simulations for mathematics teaching; (2) analyze key constraints, including environmental costs, disembodiment, and issues of authenticity and bias; and (3) develop principles and collaborative research directions.

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, pp. 334-335. PME.

OUTLINE OF THE SESSIONS

Session 1: (1) Introduction to the working group's aims and overview of the literature (10 mins); (2) Trying AI-based simulation #1, focused on instructional noticing (15 mins) followed by small-group reflection and analysis (15 mins); (3) Trying AI-based simulation #2 focused on responsiveness (15 mins) followed by small-group analysis and reflection (15 mins); (4) Whole group discussion comparing simulations and synthesizing implications for studying and supporting mathematics teaching (20 mins).

Session 2: (1) Trying AI-based simulation #3 focused on supporting preservice teachers' one-on-one questioning practices (30 mins); (2) Small-group work on three critical concerns—environmental issues, disembodiment, and authenticity, interpretability, and bias in AI-generated interaction (30 mins total); (3) Discussion of key constraints and responsible research design (15 mins); (4) Whole group synthesis and discussion of future collaborations (15 mins).

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DESIGNING PROBLEM SITUATIONS TO FOSTER SENSE MAKING: A TLSOA PERSPECTIVE ON LEARNING TO THINK MATHEMATICALLY

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Developing students who engage in sense making remains a persistent challenge in mathematics education. While curricular reforms increasingly emphasize problem solving and open-ended tasks, research continues to show that students often resort to procedural execution rather than monitoring, interpreting, and evaluating their reasoning. Drawing on the work of Alan H. Schoenfeld (1992) in Learning to Think Mathematically, sense making is understood as an integral component of mathematical thinking, closely connected to metacognitive monitoring, strategic control, and the evaluation of plausibility. From this perspective, sense making is not merely understanding content, but actively regulating and interpreting one's reasoning during problem solving. At the same time, the Transformative Lesson Study incorporating Open Approach (TLSOA), developed by Maitree Inprasitha (2023), provides a structured instructional model in which carefully designed problem situations, anticipation of student thinking, whole-class discussion, and reflective forums interact to support mathematical meaning construction.

This Working Group proposes that sense making is not an accidental by-product of open-ended tasks; rather, it must be structurally embedded in the design of problem situations and supported through teachers' anticipatory and reflective practices within TLSOA. Given the evolving nature of research on sense making, metacognition, and lesson study, this topic is well suited to the Working Group format, where participants collaboratively examine, design, and refine conceptual and practical tools.

The Working Group aims to 1) Clarify and refine the notion of sense making in relation to learning to think mathematically. 2) Examine how problem situations can be designed to structurally promote sense making. 3) Explore how the TLSOA model supports or constrains students' engagement in sense making. 4) Develop a shared set of design principles for problem situations that foster sense making. 5) Initiate an international research dialogue and potential collaboration on this topic. The Working Group will address the following guiding questions: 1) What counts as sense making in mathematical activity? 2) What characteristics of problem situations provoke or inhibit sense making? 3) How does anticipation of student thinking influence opportunities for sense making? 4) In what ways does the TLSOA structure (problem

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situation, discussion, reflection) cultivate metacognitive engagement? 5) How can these insights inform teacher preparation before practicum?

DESIGN OF THE WORKING GROUP

The Working Group is designed as a collaborative process across sessions.

Session 1.1: Conceptual Framing and Shared Problematization

- Brief input on sense making (Schoenfeld) and TLSOA.
- Small-group discussion: participants share examples of problem situations that either promoted or hindered sense making.
- Collective mapping of emerging themes.

Session 1.2: Analysis and Anticipation

- Participants analyze selected problem situations.
- Identify potential points of metacognitive engagement and breakdown.
- Anticipate possible student responses and teacher moves.
- Discuss how structural features of tasks and lesson design influence sense making.

Session 2.1: Collaborative Design and Synthesis

- Small groups design or redesign a problem situation aimed at fostering sense making.
- Extract cross-case design principles.
- Draft a shared framework outlining characteristics of sense-making-oriented problem situations within TLSOA.

Session 2.2: Collaborative Observation and Reflection

- Open Class, observation and reflection.

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WHERE IS MATHEMATICS DONE? AN EMBODIED PERSPECTIVE ON EYE MOVEMENTS

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Traditionally, mathematics is understood as an abstract discipline detached from any material reality. Yet, the role of bodily interactions with the environment in understanding and doing mathematics has become increasingly evident (Abrahamson et al., 2016; Shvarts, 2025). One aspect of such interactions is selective visual focusing on different locations, enabled by the structure of our eyes and their movements. Embodied mathematics learning and eye-tracking (ET) methodology were at the focus of the multiple PME activities: embodiment (WG/WSs in 2011, 2017, 2020, 2025, & RF in 2023); and eye-tracking (WG/WSs in 2014, 2018, & a colloquium in 2023). Yet, the *crossroads* of those two aspects offer a yet underexplored critical potential for understanding the process of mathematics learning and problem-solving.

There are multiple interpretations of ET data (Schindler et al., 2025). Many of them stem from an information-processing paradigm in cognitive science, relying on eye-mind hypothesis—*you look at the information you process*—or on assumptions of overt vs. covert attention. A new interpretation comes from embodied cognitive science: eye movements are considered part of a holistic *sensory-motor system* that also include head rotation, body positions, hand manipulations, etc. All together, those sensory-motor processes manipulate the environment so as to reveal task-relevant invariants (Gibson, 1966). When mathematical artifacts are added to the human environments, eye movements reveal mathematical invariants, such as coordinates or proportionality, as eyes move along the coordinate plane axes (e.g., Shvarts, 2025) or partition proportional segments (Abrahamson et al., 2016). Such a theorization of eye movements calls for reframing the mathematics as having an interactional ecological ontology and for considering education as learning to *see* mathematics.

Considering eye movements as a part of a broader sensory-motor system fosters research designs that account for participants' movements and multimodal expressions (Heyd-Metzuyanim et al., 2023). Practically, if mathematics is "born" in interaction with the environment, ET can support the design of learning environments that foster mathematical perception. ET can also support teacher professional development: noticing students' gazes as they bodily engage with mathematical artifacts and each other, teachers learn to attend to students' perspectives (Palatnik et al., 2026).

The aims of the WG activities (Table 1): (1) *Theory:* popularizing and developing an embodied perspective on eye-movements as sensory-motor processes within broader

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body–environment interactions of solving mathematical problems; (2) *Methodology:* reflecting on the limitations and benefits of various ET research designs and working towards a framework for analysing eye movements from an embodied perspective; and (3) *Educational implications*: formulating future directions and research strategies in the embodied approach to ET that would inform mathematics education designs and teacher professionalization.

Table 1 outlines the schedule of activities

Session	Duration	Activity
1	10 min	*Brief introduction:* Getting to know each other and the WG's aims.
	15 min	*Work in pairs:* Grounding the question of where mathematics is done. Self-(observation) of problem-solving.
	15 min	*Grounding presentation:* Sensory-motor processes in perceiving and acting with the environment mathematically. An eye as a part of the body.
	30 min	*Work in groups:* Analysis of ET data: Evidence for active constitution of mathematical understanding and eye-movement interpretation (aim 1)
	20 min	*Plenary discussion:* Theoretical reflections: the observed roles of the eye movements as sensory-motor processes in mathematical problem-solving
2	10 min	*Grounding presentation:* Mobile and stationary ET; multipersonal ET.
	20 min	*Work in groups:* Range of ET data: benefits and impediments; research questions and analysis based on research design (aim 2).
	10 min	*Plenary summary*: Methodological reflections: towards an analytical framework for an embodied approach to eye movements
	20 min	*Discussion in groups:* The implications for learning environments' design and teacher professionalisation and RQs in those areas (aim 3)
	20 min	*Plenary discussion:* Educational implications, further steps.

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DEVELOPING FRACTION AND ALGEBRAIC REASONING: SUBCONSTRUCTS, SCHEMES, AND TRANSITIONS IN LEARNING AND TEACHING

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Fractions and algebra remain among the most challenging mathematical domains for students to learn and for teachers to teach. This working group will bring together researchers whose work investigates the learning and teaching of fractions through the lens of Kieren's (1981)'s five subconstructs—particularly the operator and measure subconstructs—as well as research on fraction schemes (Steffe & Olive, 2010) and the role of unit coordination (Hackenberg & Sevinc, 2024) in the development of fraction understanding and investigations to transition to Algebra basing on these constructs. We envision forming several subgroups, each focusing on one of the following areas:

- Students/teachers' fraction schemes and units coordination and their potential overlaps with Kieren's (1981) fraction subcontracts.
- The relationship or transition from fraction knowledge to algebraic reasoning, including how students/teachers (pre- and in- service) use fractions when generalizing mathematical situations or writing equations and using representations.
- Teaching practices and classroom investigations related to the use of different fraction subconstructs and their influence on learning.
- Different Methodologies (including but not limited to eye tracking, AI etc.) for investigating fraction knowledge and the relationship or transition to algebraic knowledge.

Opportunities for Participant Contribution & Organization

Participants may share empirical data, classroom examples, or methodological tools. A short pre-survey will identify subgroup themes. The WG fosters a structured yet open environment for collaboration.

Active involvement from participants is essential. They will join thematic subgroups, share materials, contribute data or theoretical tools, and engage in collaborative analysis and co-development of the WG synthesis. The WG is intentionally designed not as a series of presentations but as a collaborative working session.

This is a new PME initiative, grounded in active research communities on fractions, unit coordination, and algebra readiness. Potential follow-up activities include a Research Report proposal, continuation as a WG, and a collaborative publication. The overarching goal of the working group is to promote productive discussion about how

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to investigate these complex phenomena. Through these discussions, we hope to generate shared frameworks, methodological considerations, and future research directions that can strengthen the field's understanding of the (deep) connections between fractions and algebra.

STRUCTURE OF THE SESSIONS

Day 1: This session introduces participants to the focus of the Working Group, provides short stimulus inputs to frame collaborative work, and allows participants to identify themes for subgroup collaboration.

Introduction to the Working Group: aims, goals, and structure (10 minutes)

Participant introductions with emphasis on research interests and possible contributions (20 minutes). There will be group works related to the 4 stimulus where we carefully provide stimulus data to be shared and discussed.

Stimulus input 1: Units coordination and fraction knowledge (10 minutes)

Stimulus input 2: Quantitative reasoning and ratios (10 minutes)

Stimulus input 3: Pre-service and in-service teachers' fraction knowledge as a foundation for algebraic reasoning (15 minutes).

Stimulus input 4: Methodological approaches for investigating fractions and algebra (e.g., technological tools such as eye tracking, AI-supported analysis) (10 minutes)

Identification of emerging themes and formation of subgroups for Session 2 (15 minutes). *Note:* All stimulus inputs are intentionally brief and serve only to activate shared perspectives and generate focused collaborative work with the use of data.

Day 2: This session is devoted to substantive subgroup work, sharing insights across groups, and building plans for continued collaboration beyond the conference.

Subgroup collaborative discussions based on themes identified in Session 1 (50 minutes): Analysis of cases or data, Exchange of frameworks or methodologies, Conceptual mapping and identification of research gaps. **Reports from subgroups** to the whole group (20 minutes). **Whole-group discussion**: Opportunities to expand cross-national collaboration, Identification of shared outcomes and next steps, Decisions regarding contributions to the PME Newsletter and potential future proposals (20 minutes).

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EARLY MENTAL MATHEMATICS INTERVENTION MODELS AND DATA IN LOW ATTAINMENT CONTEXTS

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Access to early mathematics remains a challenge in many parts of the world. A range of literature posits early mental mathematics, underpinned by, and aimed at, strong number sense, as key to improved mathematics outcomes. This new Working Group (WG) brings together academics working on early mental mathematics learning interventions within schools and pre- and in-service teacher education to understand commonalities and contrasts in intervention models, research methodologies and outcomes. The WG will engage PME participants through collaborative discussion of intervention models, research methodologies and empirical data at multiple levels.

BACKGROUND TO WG FOCUS

Early mental mathematics has been widely written about as underpinned by, and enabling, number sense building that provides strong foundations for mathematical learning (Chu et al, 2018). However, in many parts of the world, most children fail to meet even core grade level outcomes in mathematics at early primary level (UNICEF, 2022). This WG aims to advance understandings of intervention models and their research methodologies that seek to improve early mental mathematics outcomes through work in schools and pre- and in-service teacher education.

A **key goal** is to understand:

1. Overlaps/contrasts in intervention models seeking to improve teaching/ learning of mental mathematics in low attainment contexts, and their data sources.
2. Research methodologies across multiple levels (teacher educator experiences, teacher perspectives and learning outcomes), seeking to understand experiences and outcomes linked to these intervention models.

STRATEGY AND STRUCTURE

Session 1: Early number interventions models, materials and data

Introduction (20-30 min): Overview of intervention models and materials implemented by WG team, background on their development with discussion for low attainment contexts and how they interconnect, drawing on several case studies.

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Data Engagement (40 min): Participants will discuss models, materials and data from five research groups linked to their pre- and in-service teacher education interventions on mental mathematics. We will focus discussion on overlaps and differences in models and data. Groups will engage with how interventions and tools differ from their own country experiences in teacher education.

Discussion (20-30 min): Whole group discussion of models/data sources and insights emerging across groups. WG leaders will facilitate discussion in each group, ensuring alignment with overall goals. Mike Askew will move across all groups.

Session 2: Multi-level data analysis of change in mental mathematics outcomes

Introduction (25 min): Several projects by co-ordinators in this WG have data linked to intervention outcomes at multiple levels: teacher educator experiences, pre-service teacher reflections and learning outcomes. Brief sharing of these will orient and stimulate group discussions to follow.

Data Engagement (40 min): Participants will work in sub-groups to analyse selected datasets across these levels linked to a project. They will share insights into what multi-level data makes visible and offer insights on ways to strengthen and connect multi-level analysis. WG leaders and Mike Askew will facilitate discussions.

Discussion (25 min): Whole group discussion of models and insights; concluding synthesis of insights and suggestions for continued collaboration from Mike Askew.

EXPECTED OUTCOMES

This WG will identify aspects of early mental mathematics intervention models associated with teaching/learning improvements in low attainment contexts and propose future research directions. Participants will gain insights into materials that address gaps in mental mathematics in different education systems. The WG will foster international collaborations on mental mathematics among researchers and hopes to support future PME Research Report/Research Forum contributions.

Acknowledgements

This work is supported by funding from the European Commission under the Regional Teachers Initiative for Africa.

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MATHEMATICAL LEARNING IN INTEGRATED STEM INQUIRY: DESIGN AND ASSESSMENT CHALLENGES

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This Integrated STEM inquiry has become a prominent approach for engaging learners in authentic, problem-based experiences that connect science, technology, engineering, and mathematics. However, from the perspective of the psychology of mathematics learning, significant theoretical and methodological challenges remain, particularly regarding how mathematical learning is designed, enacted, and assessed within integrated STEM contexts.

This working group is grounded in three complementary theoretical perspectives. First, *sociocultural theories of learning* (Sfard, 2008; Cobb & Yackle, 1996; Vygotsky, 1978) emphasize that mathematical understanding develops through social interaction, discourse, and the use of cultural tools. Integrated STEM inquiries often involve collaborative design, multimodal representations, and shared problem solving, making them rich sites for examining socially mediated mathematical thinking. Second, *design-based and engineering design learning theories* (English, 2016; NRC, 2014) highlight iterative cycles of problem definition, testing, and refinement. While these cycles support sense-making and modeling, they also complicate the identification of stable mathematical objects and learning trajectories. Third, *complexity and systems perspectives on learning* (Bakker et al 2021; Barab, S., & Squire, K., 2004) suggest that mathematical learning in integrated STEM is nonlinear, distributed across tools, peers, and time, and not easily captured through traditional pre and post assessments.

Prior research has documented positive outcomes of integrated STEM for engagement and attitudes, yet evidence for robust mathematical learning remains mixed. A persistent gap concerns the centrality of mathematics: mathematics is often positioned as a supporting tool rather than an explicit object of inquiry. Furthermore, assessment practices frequently lack alignment with the mathematical reasoning demanded by integrated tasks, overlooking learning manifested through discourse, representations, and design decisions. The working group discussions on integrated STEM education at IGPME conferences in 2017, 2018, 2019, 2024, and 2025, questions remain about the benefits of this approach to mathematics teaching and learning. This working group aims to address these theoretical tensions by foregrounding mathematics as a core analytical lens in integrated STEM inquiry.

This working group is designed as a highly interactive forum for collaborative sense-making and theoretical exchange. For Session 1, the goal is to establish a shared theoretical foundation for integrated STEM inquiry and surface key challenges in

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identifying, supporting, and assessing mathematical learning within interdisciplinary contexts. For Session 2, the goal is to collaboratively examine concrete examples of integrated STEM inquiry and explore ways to theorize and assess mathematical learning within these settings. The following are the plans for Sessions 1 and 2.

Session 1 Framing Mathematical Learning in Integrated STEM (90 min)

- Brief introduction and overview of working group goals (10 min)
- Interactive presentation on theoretical perspectives (e.g., sociocultural theory, discourse, interdisciplinary learning) and current research gaps (20 min)
- Small-group discussions (3–5 participants) sharing experiences and challenges in integrating and assessing mathematics in STEM inquiry (30 min)
- Whole-group synthesis to identify key themes and priority issues (20 min)
- Reflection and preparation for Session 2 (10 min)

Session 2 Collaborative Analysis and Assessment (90 min)

- Recap of Session 1 themes and session goals (10 min)
- Small-group case analysis (e.g., tasks, student work, video excerpts) to identify mathematical thinking and learning processes (35 min)
- Group discussion on assessment strategies and theoretical alignment (25 min)
- Whole-group knowledge building: design and assessment principles, future research directions (15 min)
- Closing reflection and next steps for collaboration (5 min)

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SEMINARS

DOCTORAL PROGRAMS IN MATHEMATICS EDUCATION: STRUCTURES, CULTURES, COLLABORATION

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This Seminar examines doctoral programs in mathematics education and didactics of mathematics from cross-national and cross-institutional perspectives. It conceptualizes doctoral learning as a culturally situated system shaped by supervision, research communities, and professional socialization. Through structured participant exchange and collaborative analysis, the aim is to identify shared patterns, tensions, and possibilities for collaboration across contexts. Outcomes include comparative insights, new research questions, and pathways for continued PME engagement.

TOPIC AND FOCUS

Doctoral programs in mathematics education (didactics of mathematics) vary widely across regions and institutions in terminology, structure, and scholarly traditions. This Seminar adopts a comparative perspective to examine similarities/differences in doctoral program structures, learning cultures, and developmental expectations across national and institutional contexts, with attention to how mathematics education researchers are prepared. The topic is timely within PME given increasing international mobility, evolving doctoral expectations, and growing interest in doctoral preparation in the field (Herbst, 2023). The goal is to co-develop a shared cross-national understanding of doctoral learning cultures and to identify directions for collaboration and future research through structured participant exchange and collaborative analysis.

THEORETICAL BACKGROUND

The Seminar draws on sociocultural and community-based perspectives that conceptualize doctoral preparation as a socially, culturally, and institutionally situated learning process (Lave & Wenger, 1991). Prior research highlights supervision, participation in research communities, and professional socialization as central to doctoral learning, positioning doctoral programs as learning cultures rather than solely formal curricula (Herbst, 2023). Building on this work, the Seminar creates space within PME to examine convergences and tensions across doctoral learning cultures and to consider implications for supervision, program design, and researcher development across contexts.

PARTICIPANT ENGAGEMENT

Participants will contribute brief contextual perspectives on doctoral programs in their institutions or countries and engage in structured small-group and whole-group

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comparative discussions. Activities are designed to surface shared challenges, divergences, and priorities in doctoral learning cultures and to support collective sense-making across contexts. This Seminar is a new initiative within PME, informed by recent calls to articulate key features of doctoral preparation while recognizing cultural and institutional diversity (Herbst, 2023). Follow-up activities may include comparative research projects and future PME activities (e.g., Research Reports).

PROPOSED LAYOUT

Seminar session (90 minutes): Mapping Doctoral Program Structures and Cultures, Comparative Insights, and Collaborative Futures

- (15 min) Two large number lines (1–10) on floor; current students and professors position themselves to indicate overall doctoral program ratings, followed by discussion of group differences and potential nostalgia bias.
- (10 min) Using a digital Word cloud tool and a QR code, participants share specific, one or two-word, strengths/weaknesses of their programs.
- (10 min) Researchers provide short summary of strengths/weaknesses, and essential features of mathematics education doctoral programs from results of two recent papers.
- (35 min) Think-Pair-Share: heterogenous groups based on "culture of" components from reflexive thematic analysis. Participants describe their experience in writing first, pair with a colleague to compare notes next, followed by a whole group discussion with an emphasis on cross-cultural collaboration, academic mobility, and comparative research.
- (10 min) Theoretical, guided discussion over nostalgia bias and how the possible effect contributes to our approach to teaching, research, and supervision within our current roles. Short reflection on the impact on professional identities.
- (10 min) Reflection and exit ticket for feedback, insights, and next steps with link and QR code for Google form: the most influential experience during your program; ideas on how to strengthen supervisory and mentoring systems; ways to ensure equitable access to research and professional opportunities.

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INNOVATING PEER REVIEW FOR RESPONSIVE MATHEMATICS EDUCATION IN A CHANGING WORLD

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Our focus is intentional innovation within peer review. We envisage practices sensitive to and enabling of mathematics education scholarship that is responsive in a complex, changing world. We open dialogue about the nature of contextually relevant research, the traditions and purposes of responsible peer review, and the ethical role of editors.

ACTIVITY OF THE SEMINAR

Peer review is a longstanding practice, where we negotiate and establish what counts as legitimate mathematics education scholarship. Thus, there is opportunity for intentional innovation towards peer review that is sensitive to and enabling of contextually relevant research that is responsive, in complex, changing, and uncertain times. We aim to open dialogue about what matters in our scholarship in these times, and how the peer review process can serve these needs. We identify the following questions as being of interest to researchers offering diverse mathematics education interests, experiences, and contexts, and to current (and future) peer reviewers.

Round 1 questions (the focus of small group sharing and plenary dialogue)

- What are the most significant changes in the contexts of our scholarship? (*Contexts* includes specific mathematics teaching, learning, and research settings, and framing education and societal contexts.)
- What new possibilities for scholarly contributions might these changes implicate?
- How can our scholarship contribute to societal and educational settings?
- What challenges does this pose for the peer review process?

We sense the need for critical reflection in the field of mathematics education research on the nature of research problems; who participates in scholarship and in what way; theoretical choices and interdisciplinarity; methodological approaches; and the position of 'local' knowledge in transnational research communities. We will have prepared examples from our work to share, if necessary.

Round 2 questions (the focus of small group sharing and plenary dialogue)

- What are the criteria generally used for peer review in the field of mathematics education? Why are they important? To whom?
- What peer review practices serve mathematics education that is relevant and responsive in context, in complex, changing, uncertain times?

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- What might be the responsible role of (journal) editors in the process?
- Where is there space for dialogue within/around peer review processes?

We will listen for participants' additional points of concern around research design and writing, peer review writing and receiving, and editorial experiences and handling.

MOTIVATION FOR THE SEMINAR

Our current editorial work within the relatively new area of mathematics education and the socio-ecological, in juxtaposition with our prior experiences editing journals, books and proceedings, prompted us to convene this working group. Locating mathematics education in considerations of present and future liveability, for all, in a world of dynamic, interrelated local and global socio-ecological issues poses challenges to dominant academic traditions that value fixity, categorisation, and closure. We are learning that socio-ecological mathematics education needs "to go visiting" (Haraway, 2016, p. 4) with others, for "we require each other in unexpected collaboration and combinations" (ibid.). We find that this requires of mathematics education an openness to new relations, between knowledges, ways of knowing, peoples, and contexts, realised in/through ethical empirical, methodological, and theoretical practices.

As editors handling research writing within these new compelling contexts, we find ourselves mediating between new voices and ways of knowing, and the traditions of what and who counts as legitimate in mathematics education scholarship (which tend to be represented in the practices of peer review). This experience has surfaced questions around peer review processes in mathematics education more broadly, such as what 'counts' as a scholarly contribution (empirical, methodological, theoretical practices, and their relations), for whom it is a contribution (scale-ability), and how this is communicated. We feel deeply troubled, not only by the knowledge implications of our 'gatekeeping' as editors (e.g., Niss, 2018; Andrade-Molina et al., 2020), but also the ethics of our roles. Indeed, Haraway (2016) warns of aiming for a 'safe' imagined future, encouraging us to be "truly present in entwined relation of places, times, matters and meanings" (p. 1), and to risk exploring new directions, in an orientation that is both responsive, and responsible (able to respond). We look forward to exploring these new directions with others.

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NATIONAL PRESENTATIONS

FINNISH STUDENTS' MATHEMATICS ACHIEVEMENT IN INTERNATIONAL LARGE-SCALE ASSESSMENTS

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In this National Presentation, I will outline the key features of the Finnish education system and the national curriculum. I will then present highlights of Finland's mathematics results in large-scale international assessments, PISA and TIMSS studies.

FINNISH EDUCATION SYSTEM

To set the ground for assessing mathematics education in Finland, I introduce some key aspects of Finnish education system, focusing mainly on basic education.

Finnish education policy is grounded in the view that education is a fundamental right and seeks to provide equal opportunities for learning to all citizens, independent of their age, nationality, residence, financial situation, or native language (Ministry of Education and Culture, n.d.). Pre-primary and basic education is free of charge for students including learning materials and equipment, daily warm school meals, health and welfare services and school transportation under certain conditions, such as long or dangerous commute (Finnish National Agency for Education, n.d.).

The Finnish education system is decentralized, where local authorities, most commonly municipalities, have a high degree of autonomy, meaning that they also have significant responsibility for organizing education. Municipalities can determine how to primarily use the funding as they consider best. Finland has local school policy, meaning that a student can attend their local school – that is the school to which they are assigned based on their place of residence (Finnish National Agency for Education, n.d.). The official languages of Finland are Finnish and Swedish, and therefore, there are both Finnish and Swedish speaking public schools. About 5 percent of the Finnish population is Swedish-speaking. According to Basic Education Act (628/1998), instruction for students who speak Sámi and live in the Sámi homeland should be provided primarily in the Sámi language. Instruction should also be provided in sign language for students with hearing impairments if necessary. Finnish, Swedish, or Sámi are taught as the mother tongue, depending on the student's language of instruction, but also at the guardian's request, Romany, sign language, or another of the student's mother tongues may also be taught as the mother tongue. There are relatively few private schools in Finland. According to Ministry of Education and Culture (n.d.), the legislation governing school operations applies equally to all schools, regardless of who operates them. In practice, this means that also private schools must follow the national core curriculum. In addition, private schools are not

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allowed to make profit, and no tuition fees are charged in basic education (Ministry of Education and Culture, n.d.).

Compulsory education begins when a child turns 6 with one-year preprimary education. Basic education lasts for nine years. The age of compulsory education was extended to upper secondary education in 2021, and it ends when either the student turns 18 or when the student has, prior to that, successfully completed a qualification in upper secondary education (Finnish National Agency for Education, n.d.). However, there is no compulsory schooling in Finland, so guardians may decide that a student subject to compulsory education fulfills this obligation in a manner other than by participating in the education referred to in the Basic Education Act and the laws governing post-basic education. Upper secondary education is divided into general upper secondary education and vocational upper secondary education and training. Both lines of upper secondary education give eligibility for higher educational studies. Higher education system in Finland consists of universities and universities of applied sciences.

There are no dead ends in Finnish education system. For example, streaming in lower secondary education mathematics was officially abolished in 1985. However, later research shows that informal streaming has continued in many schools, particularly in mathematics (Ouakrim-Soivio & Kupiainen, 2023). The key difference is that even the slowest-paced groups still follow the National Core Curriculum, ensuring that students do not face dead ends if they later wish to change their study path. Also, it is possible to apply to university or university of applied sciences after either general upper secondary or vocational upper secondary education.

By the Basic Education Act (628/1998), in basic education, the school year includes a total of 190 school days, and the school year starts on 1 August and ends on 31 July. It is split into two terms: the autumn term and the spring term. Each education provider decides how long the holidays are and when they take place. Start for teaching in autumn term varies, but in many schools, it starts a few days before mid-August. Teaching for the school year finishes on the final working day of either the last week of May or the first week of June. All other holiday schedules are determined by the education provider (Ministry of Education and Culture, n.d.).

The Finnish government sets the overarching goals in basic education and determines how instructional time is distributed across subjects. The government's decree on the distribution of lesson hours (422/2012) stipulates the minimum amount of instruction to be provided during basic education, divided by grade level groups (1–2, 3–6, and 7–9) for each subject or subject group, expressed in weekly annual lesson hours (alh). The minimum number of mathematics lessons in basic education is a total of 33 alh in grades 1–9. Following the National Board of Education's reform, the number of mathematics lessons increased by one weekly lesson in 2025. Instruction is generally

distributed among the grades as follows: grades 1–2 (7 alh), grades 3–6 (15 alh), grades 7–9 (11 alh) (Finnish National Agency for Education, n.d.).

The Finnish National Core Curriculum (FNCC)

The Finnish National Agency for Education (EDUFI), operating under the Ministry of Education and Culture, defines the subject-specific objectives and content areas of education through the Finnish National Core Curriculum (FNCC). In addition to the content areas related to the objectives, the FNCC emphasizes transversal competence in all subjects. Transversal competence refers to an entity consisting of knowledge, skills, values, attitudes and will, such as thinking and learning to learn, cultural competence, interaction and self-expression or multiliteracy. Every education provider, including municipalities with public schools and private schools separately, must prepare local curricula based on the FNCC. The FNCC describes the content and the objectives of learning on relatively general level. Although the FNCC offers recommendations for instruction and assessment, decisions regarding teaching practices and student evaluation are largely left to teachers (Finnish National Agency for Education, n.d.).

The most recent FNCC came into force for Grades 1–6 in 2016 (Finnish National Agency for Education, n.d.). Its implementation for Grades 7–9 was gradual, beginning with grade 7 in 2017, followed by Grade 8 in 2018 and Grade 9 in 2019. The FNCC describes the content of subjects in three periods: for grades 1 to 2, grades 3 to 6, and grades 7 to 9. In local curricula, the content is generally described in greater detail and organized by grade level, as determined by the education provider.

Mathematics as a school subject

Mathematics curriculum for basic education emphasizes students' positive attitude towards mathematics and positive self-image as learners need to be supported (Finnish National Agency for Education, 2014). By the FNCC, learning mathematics is viewed as a purposeful and sustained effort in which students take an active role in their own progress. Mathematics teaching should help students recognize how the subject is relevant both in their daily lives and in society at large. Overall, instruction and learning aim to strengthen students' ability to use and apply mathematics in diverse ways.

By the FNCC, in grades 1 to 2, key content areas in mathematics are numbers and operations, geometry and measuring, data processing and statistics and in addition, thinking skills. In grades 3 to 6, the content areas are the same as in grades 1 to 2, but in addition, probability and algebra are included. In lower secondary school – grades 7 to 9 – content areas are numbers and operations, algebra, functions, geometry, data processing, statistics, and probability, and in addition, thinking skills and methods.

In grades 1 and 2, mathematics instruction provides a variety of experiences that help lay the foundation for the development of mathematical concepts and structures. Instruction is based on topics and problems that are familiar and interesting to students.

Students are taught to express their mathematical thinking through concrete materials, speech, writing, and drawing and interpreting pictures. Mathematics instruction creates a solid foundation for understanding the concept of numbers and the decimal system, as well as for learning arithmetic skills.

The FNCC describes that the central purpose of mathematics education for students in grades 3–6 is to deepen students' understanding of mathematical concepts and structures and to develop their ability to present their mathematical thinking and solutions to others in various ways and using different tools. In addition, it is essential that students be able to solve a variety of problems independently and in groups, as well as compare different solutions. Students study the concepts of numbers and the decimal system and deepen their understanding of them. In addition, students develop fluency in arithmetic skills.

The primary goal of mathematics instruction in grades 7–9 is to strengthen students' overall mathematical proficiency and deepen their understanding of mathematical concepts and the connections between them. Instruction guides students toward mathematical modeling and problem-solving and encourages them to discover and apply mathematics in their own lives. Mathematics instruction involves goal-oriented, precise, targeted, and long-term activities. Presenting students' solutions and discussing them are encouraged, and students' teamwork skills are developed.

Although programming is one of the transversal competences among other information and communication technology competences, most of the learning content of programming is embedded in mathematics. Starting from grade 1 or 2, students start to familiarize themselves with the basics of programming formulating and testing step-by-step instructions. By the end of basic education, well-performing students should be able to apply the principles of algorithmic thinking and to program simple programs (Finnish National Agency for Education, 2014).

Student assessment

According to the Basic Education Act, students' learning, competence, work and behavior must be assessed in a comprehensive manner. Learning, competence, and work are assessed by subject. Behavior is assessed separately. As described in the FNCC, the purpose of student assessment is to guide and encourage learning and to develop students' ability to engage in self-assessment (formative assessment) and to determine the extent to which a student has achieved the objectives set for the subjects (summative assessment) (Finnish National Agency for Education, 2014). Assessment is based on equal treatment of students. Every student must be informed of what is expected to be learned and how learning, competence, work, and behavior will be assessed. The end-of-year assessment given at the end of grades 1–8 and the final assessment of basic education in the end of the 9th grade must be based on the objectives set out in the core curriculum for basic education and specified in the local curriculum (Finnish National Agency for Education, 2014).

The chapter in the FNCC about student assessment was revised in 2020. By clarifying guidelines, more consistent assessment principles were established nationwide to support the assessment work of schools and teachers. By the end of each school year, students are evaluated with grading scale which ranges from 4 to 10, where 4 means failed and 10 means excellent. Grading mark 8 indicates a good level of proficiency. On lower grades, 1 to 3, students are assessed verbally and starting in the 4th grade, they receive numerical marks.

Before reformation of the guidelines of assessment, the FNCC described only competences for mark 8 ("good") at the end of basic education (end of grade 9). After the reform, descriptions of competencies for marks 5 ("pass"), 7 ("satisfactory"), and 9 ("very good") were added. In addition, descriptions of competencies for marks 5, 7, and 9 were added for the end of elementary school (end of grade 6). Assessment of proficiency for the other marks – 4, 6 and 10 – is based on the learning objectives for mathematics and in relation to the criteria for marks 5, 7, 8, and 9 (Finnish National Agency for Education, 2014).

Finnish school teachers

In elementary school, Grades 1 to 6 are normally taught by primary teachers (class teachers), who teach various subjects. In lower secondary school, grades 7 to 9 are taught by subject teachers, who have subject-specific expertise. Qualified Finnish teacher, both primary teachers and subject teachers, have completed a master's degree at a university. The main difference between primary teachers' and subject teachers' qualifications is the scope of educational studies. Primary teachers need to have completed interdisciplinary studies in the subjects and thematics areas taught in elementary school (60 credits) and pedagogical studies (60 credits) in the university (Decree 986/1998, Section 4). Subject teachers need to have studied at least one of the subjects they are qualified to teach as a major (at least 120 credits) and educational studies (teacher education courses, 60 credits) and other subjects to taught as minor (at least 60 credits) (Decree 986/1998, Amending Decree 865/2005, Section 10). Subject teachers usually have qualifications to teach two to three subjects. Most commonly, mathematics subject teachers also have a qualification to teach either physics or chemistry or all of the three, but also other subject combinations are possible. In 2019, as many as 95% of Finnish teachers were formally qualified (Vipunen, Education administration's reporting portal, 2025).

Learning materials

Teachers have considerable freedom to modify and choose learning materials in ways they find appropriate, and they can evaluate these materials in relation to the objectives, content areas, and assessment criteria set out in the FNCC (Finnish National Agency for Education, n.d.). According to the Finnish Publishers Association, 95 percent or more of basic and upper-secondary education schools use learning materials produced by a professional publisher. Most publishers producing materials for basic and upper

secondary education rely on author teams of three to seven people—typically experienced teachers and teacher educators. This approach helps ensure that the authors possess strong subject knowledge, understand suitable pedagogical methods, and can write in a way that matches students' linguistic and conceptual development. Since government-driven school inspections were discontinued in 1985, competition among publishers is viewed as the mechanism that maintains the quality of learning materials. Although only a few publishers produce Finnish-language educational materials, they are generally regarded as high-quality (Ouakrim-Soivio & Kupiainen, 2023).

According to Lepik et al. (2015) Finnish teachers rely heavily on mathematics textbooks in their teaching. Textbooks are a particularly important source of exercises and homework for students. Mathematics textbooks often include teacher's guides, which Finnish teachers also use extensively. Hemmi et al. (2018) describe how teacher guides often offer clear objectives for each lesson and support for different themes, such as taking into account students' thinking and mathematical progression. In addition, guides give ideas for differentiation, especially ideas on how to support low-performing students (Hemmi et al., 2018).

Assessment at the educational system level

The FNCC together with the local curriculum are important part of the steering of education. Curriculum has central part in setting out and implementing both national targets and goals and tasks considered important locally (Finnish National Agency for Education, 2014). In Finland, there is no school inspections, high-stake tests or mandatory standardized national tests during basic education. The only high-stake test in Finnish general education is the matriculation examination in the of general upper secondary education. However, education providers are required to evaluate the education it provides and participate in external evaluations of its activities (Basic Education Act 628/1998, Section 21). The Finnish Education Evaluation Centre (FINEEC) is responsible for the national evaluation of education. They evaluate basic education with learning outcomes evaluations and different thematic and system evaluations. It is noteworthy that the evaluations implemented by FINEEC are not annual and they are sample based. Only Finnish/Swedish, mathematics, and English are assessed on a regular basis by FINEEC, while most other subjects are evaluated only once during each ten-year curriculum cycle. Latest national mathematics evaluation was the assessment of learning outcomes at the end of basic education in 2021 (Metsämuuronen & Nousiainen, 2021).

International large-scale assessments are also important part of tracking the Finnish education system. They play an essential role when examining the state of the Finnish education. The Ministry of Education and Culture decides on participation in international large-scale assessments and provides funding for them. These are mainly implemented by Finnish Institute for Educational Research in University of Jyväskylä

in Finland. All assessments are conducted in both Finnish and Swedish. The size of the basic sample, which is generally representative of Finland as a whole, is approximately 5,500–7,000 students, but it may be larger if oversamples are taken from different subgroups, such as students with an immigrant background or Swedish-speaking students, for closer examination. Mathematics achievement in basic education is assessed in two international assessment studies: Programme for International Student Assessment (PISA) and Trends in Mathematics and Science Study (TIMSS). They both have a slightly different focus and target group. In the following sections, these are introduced in more detail.

BACKROUND AND FRAMEWORK OF PISA AND TIMSS

Programme for International Student Assessment (PISA)

PISA is a joint research program of OECD member countries. It provides information on the state of education and educational outcomes within an international comparative framework. PISA examines how 15-year-olds master key skills essential for the future, what factors influence these skills, and how proficiency changes over time. The PISA program consists of PISA surveys conducted every three years. However, the eighth cycle, PISA 2022, had to be postponed by one year from its original schedule due to the COVID-19 pandemic. The study began in 2000, and Finland has participated in every cycle. The number of participating countries has increased cycle by cycle. A total of 81 countries and regions participated in the 2022 assessment. There are three core domains assessed in every PISA cycle: mathematical literacy, reading literacy and science literacy. In each cycle, one of these is the main domain, while the other two are minor domains, from which data is collected to track trends. Mathematical literacy has been the main domain in 2003, 2012 and 2022. In PISA 2022 Mathematics framework, mathematical literacy is defined as follows:

> Mathematical literacy is an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to know the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by constructive, engaged and reflective 21st Century citizens. (OECD, 2023a, p. 22)

The assessment of mathematics achievement is based on the idea that it is important for young people to understand mathematics so that they are prepared to act as members of modern society and contribute to it. The framework for mathematical literacy was updated in PISA 2022 to meet the demands of today's society, placing greater emphasis on the role of mathematical reasoning and problem-solving. From its very first cycles, the PISA study has emphasized the application of mathematical knowledge in various real-life situations. These also include less structured environments where students must determine what information is relevant and how it should be applied. Problem-solving requires the ability to identify and formulate a real-

life situation mathematically, to use and manipulate these mathematical concepts and phenomena, and finally to evaluate and interpret mathematical results in relation to the real-life situation.

With the revised framework, the PISA study aims to emphasize that assessing mathematical literacy requires not only examining problem-solving processes but also mathematical reasoning (both deductive and inductive), which is an important part of being mathematically literate and integral to problem-solving processes. Although mathematical reasoning and problem-solving are overlapping processes, mathematical reasoning offers a broader perspective than on solving practical problems, which should be examined alongside problem-solving processes. Therefore, in PISA 2022 mathematical literacy consists of two related aspects: mathematical reasoning and problem solving. Problem solving is assessed through three processes (which includes the mathematical processes that describe what individuals do to connect the context of the problem with mathematics and thus solve the problem): 1) Formulating situations mathematically, 2) Employing mathematical concepts, facts and procedures, and 3) Interpreting, applying and evaluating mathematical outcomes (OECD, 2023a).

In addition to mathematical reasoning and problem-solving processes, mathematical literacy is analyzed in terms of mathematical content. The four content categories are: change and relationships; space and shape; quantity; uncertainty and data (OECD, 2023a). These reflect the traditional classifications of mathematics content found in many national curricula, such as number properties, functions and algebra, geometry, and statistics and probability. The selection of content considers the types of knowledge and skills a 15-year-old has had the opportunity to acquire at school or in their free time, as well as the types of competencies young people will need in their daily lives or future studies. In addition, a distinctive feature of PISA items is that all items are set in various contexts that may relate to students' personal lives, such as school, family, and hobbies, or they may address situations related to working life or society, such as managing costs and revenues or the spread of diseases (OECD, 2023a).

In PISA 2022, mathematics items were developed for the first time specifically for a computer-based assessment. This made it possible to use features of the digital environment, such as spreadsheets, in the items. Although PISA has been administered electronically since 2015 in Finland, the mathematics items in earlier cycles were trend items originally designed for paper-based testing.

Trends in Mathematics and Science Study (TIMSS)

TIMSS is conducted in both 4th and 8th grades to assess mathematics and science achievement every fourth year since 1995. TIMSS study is administered by the International Association for the Evaluation of Educational Achievement (IEA) and directed by the TIMSS & PIRLS International Study Center established in Boston College's Lynch School of Education and Human Development. In Finland, the TIMSS study is conducted by Finnish Institute for Educational Research in University

of Jyväskylä. In addition to assessing students' proficiency, the study examines the school- and home-related factors underlying their performance. The TIMSS assessment is based on the curricula of the participating countries. Finland participated in TIMSS for the first time in 1999 with 7th grade students. After a break of a few cycles, Finland participated in TIMSS 2011, which included not only 7th graders but also 4th and 8th graders. Since then, Finland has participated in every assessment cycle with both 4th and 8th grade students, except in 2015, when only 4th graders were included. In TIMSS 2023, there were a total of 59 countries or benchmarking systems in grade 4 and 44 countries or benchmarking systems in grade 8 study.

Since TIMSS 2019, the study has been implemented in digital form. The study will also collect comprehensive background data. In addition to the cognitive assessment, students complete a questionnaire on their learning in mathematics and science. The questionnaire examines students' attitudes toward these subjects, their social background, and other factors and background information related to their study of mathematics and science. Also, teachers who teach mathematics and science to the participating students have taken Teacher Questionnaires, which are used to gather information about their education, work background, attitudes, professional development, the use of digital devices in teaching, and teaching and assessment practices. School principals responded to School Questionnaire designed to gather information on school resources, time spent on instruction, emphasis on environmental sustainability, safety, and the principals' education and professional background. In grade 4 study, also parents or guardians were asked to complete a questionnaire that addressed the child's early learning experiences, as well as issues related to the child's home background, such as the parents' or guardians' education, occupational status, or the number of books and digital devices at home, and the parents' or guardians' views on their child's schooling. In addition, in every cycle, Encyclopedia (e.g. Reynolds et al., 2024 for TIMSS 2023) is published to present information about national contexts, education policies and curricula for mathematics and science instruction in the education systems and benchmarking systems participating in TIMSS.

In the TIMSS studies, the curriculum is a key conceptual framework for understanding how education is implemented and what factors influence on students' opportunity to learn. TIMSS curriculum model comprises three levels: the intended curriculum, the implemented curriculum, and the attained curriculum (Mullis et al., 2021). Intended curriculum refers to the education system level. In Finland, FNCC and the local curricula represent this level. What happens within the school community constitutes the implemented curriculum. The implemented curriculum describes how the planned curriculum is carried out using the available teachers, resources, and classes. The attained curriculum encompasses student learning outcomes in a broad sense: knowledge, skills, processes, and attitudes. The implemented curriculum, with its specific approaches, naturally influences student learning outcomes. In addition,

students' home backgrounds and their own characteristics, such as attitude, influence their studies and learning outcomes (Mullis, 2017).

The TIMSS mathematics framework consists of two dimensions (Mullis et al., 2021). The content dimension defines the areas of mathematics to be assessed, and the process dimension defines the cognitive processes that students are expected to use when solving problems. For example, in grade 4 the three content domains are Number, Measurement and Geometry, in addition to Data. The most comprehensive topic within Number domain is whole numbers and mathematical operations. In addition, this content domain assesses proficiency with fractions and decimals, as well as basic concepts of algebra, such as the use of unknown variables in simple equations. About half of the geometry domain is related to geometric shapes, such as calculating the perimeters and areas of polygons, and the other half involved measurement, in which students' ability to measure lengths with a ruler and perform calculations using various units of measurement are assessed. Data domain consists of reading and presenting data, and in addition, interpreting and combining data and comparing data sets (Mullis et al., 2021).

By the TIMSS Framework (Mullis et al., 2021), in grade 8 study the four content domains are Number, Algebra, Geometry and Measurement, Data and Probability. Number domain consists of three topics: whole numbers, fractions, and decimals, as well as ratios, proportions, and percentages. Algebra contents are assessed through patterns and relationships by using various expressions, equations, and functions. In geometry, students are expected to analyze two- and three-dimensional shapes and their geometric relationships. Also, students should calculate perimeters, areas and volumes. Data content area assesses students' ability to interpret and use various visual representations of data, in addition to their understanding of probability (Mullis et al., 2021).

TIMSS Framework describes three cognitive domains for both grade 4 and 8: knowing, applying and reasoning (Mullis et al, 2021). In knowing domain, items assess students' ability to recall, recognize, and apply mathematical facts, concepts, and procedures. In applying, items assess students' ability to apply their skills and conceptual understanding in various real-life or purely mathematical situations. The tasks in reasoning process domain assess students' ability to think logically and systematically, which is necessary for formulating and justifying solutions to problems and for drawing conclusions (Mullis et al, 2021).

MATHEMATICS ACHIEVEMENT IN PISA AND TIMSS IN FINLAND

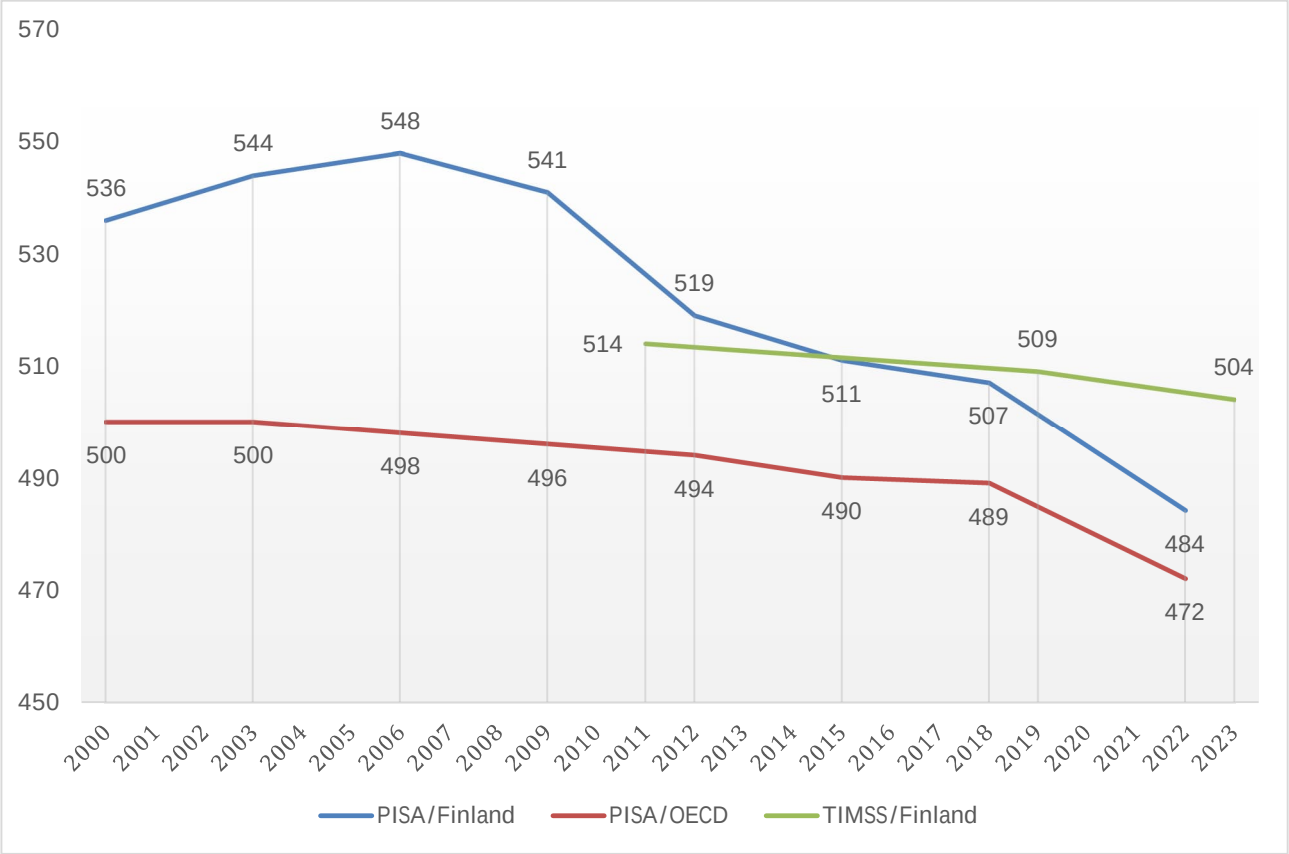
The Finnish education system rose to prominence in the early 2000s when the first results of PISA study were released. At that time, Finland was one of the top-performing countries in mathematics among participants, but its scores have been declining for nearly two decades (OECD, 2023b). Mathematics performance has declined in Finland also according to national evaluations implemented by FINEEC

(Metsämuuronen & Nousiainen, 2021) and the other international large-scale assessment TIMSS (von Davier et al., 2024) both in elementary school and lower secondary school. In this section, I will focus on the results of international large-scale mathematics assessments in lower secondary education, i.e. PISA and TIMSS in grade 8. Although, national reports are partly based on the international reports of the PISA 2022 study (OECD, 2023b) and the TIMSS 2023 study (von Davier et al., 2024), they contain more results for Finland and more details from Finnish perspective. Therefore, the results presented in the next subsections are based on Finnish national reports from the PISA 2022 study (Hiltunen et al., 2023) and the 8th grade TIMSS 2023 study (Hiltunen et al., 2024).

Overall trend in mathematics achievement

In PISA 2003, Finland had 544 points in mathematical literacy (Figure 1). The mathematics average scale score for Finland was at the highest in 2006, 548 points. Since then, Finland's score has declined in every cycle, reaching 484 points in the most recent PISA assessment in 2022. The average score in mathematics was 64 points lower than in the peak year. Finland's decline in scores was greater than in any other country that participated in PISA during these two cycles. The difference in previous cycle, PISA 2018, was also considerable -23 points. However, the average score in mathematics in Finland was still above OECD average. It is also worth noting that in most countries, students had poorer performance in mathematics in 2022 than in 2018 and OECD average had considerably declined. Previously, the largest change in the OECD mathematics scores average had been 4 points, but in 2022 the average had fallen by 17 points compared to PISA 2018.

Figure 1. Trends in Finnish mathematics achievement in PISA and TIMSS (grade 8), and OECD averages in PISA



Finland was also among the top-performing countries in TIMSS when Finnish 8th graders participated for the first time in 2011. Although a decline in mathematics performance is visible in TIMSS as well, it has been less pronounced than the drop seen in PISA. In TIMSS 2011, Finnish students scored an average of 514 points, whereas in the latest cycle, TIMSS 2023, the average score was 10 points lower. This long-term decrease is statistically significant.

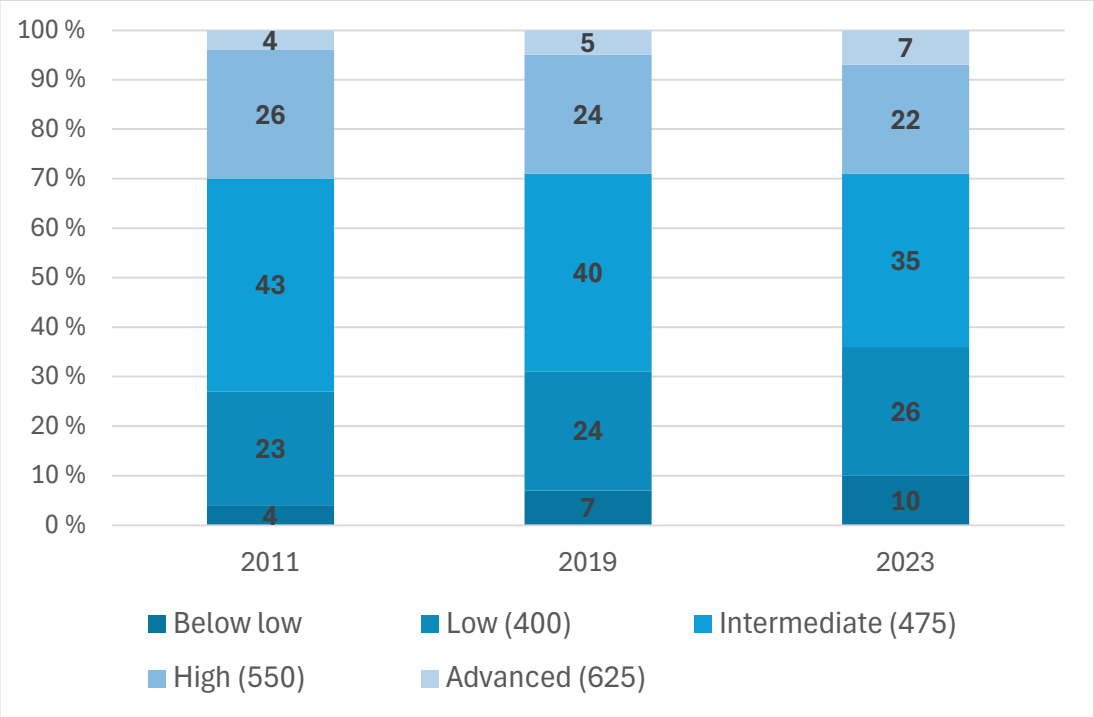
Variation in mathematics proficiency

An important observation emerging from TIMSS results is the increasing variation in students' performance. The standard deviation of scores, was 82 points in Finland in 2023. The variation in performance has clearly increased in mathematics, as the standard deviation in 2019 was 73 points—10 points lower. In 2011, the standard deviation of scores was only 65 points, which was the lowest among the participating countries at that time. Variation in performance can be also looked at through share of students on different proficiency levels defined in TIMSS.

Student shares at the extremes of the achievement distribution have increased. In 2011, 4 percent of Finnish students scored below the low achievement level, while in the

2023 study, the corresponding proportion was 10 percent (Figure 2). On the other hand, the proportion of students reaching an advanced achievement level has increased during the same period. In 2011, 4 percent of Finnish students were at the advanced level, and in 2023, 7 percent. When examining the combined proportions of students reaching high and advanced achievement levels, we see that in 2011, a total of 30 percent of students fell into these categories, and in 2023, 29 percent. In 2011, the combined proportion of students performing below the low level and those at the low level was 27 percent, while in 2023, this proportion was 36 percent. Changes at the extremes of achievement distribution can be seen also by looking at the percentiles. In 2023, the 5th percentile was 37 points lower than in 2011. In addition, the 95th percentile in 2023 was 20 points higher than 12 years earlier. This means that the lowest-performing students performed even weaker than before, while at the same time the highest-performing students performed even better.

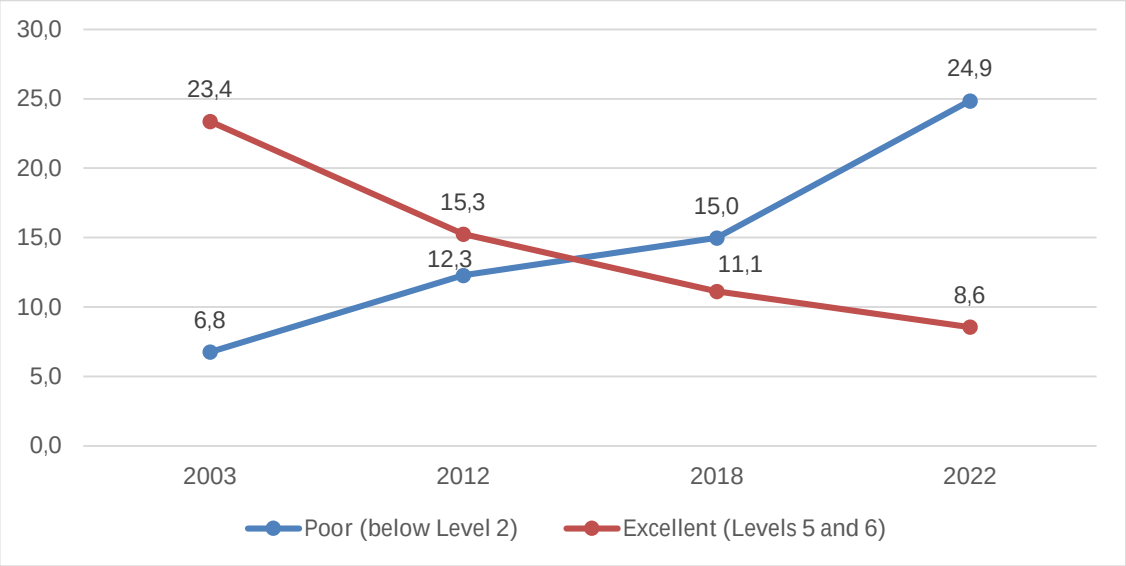
Figure 2. Distribution of Finnish students across different proficiency levels in mathematics in 2011, 2019, and 2023 (Hiltunen et al., 2024)



In PISA 2022, the mathematics scale is divided into eight (below level 1c, 1c, 1b, 1a, and 2–6) proficiency levels. Level 2 has been considered the baseline level of proficiency students need to participate fully in society (OECD, 2023a). Therefore, it is important to address these students as their mathematics skills may not be sufficient for further education or labor market after basic education. In PISA, there can be observed similar, but more pronounced, trend for lowest-performing students as in TIMSS (Figure 3). According to Finnish national PISA report, in 2003, there were less

than 7 percent of low-performing students in Finland who did not reach proficiency level 2 in mathematics. In 2022, nearly a quarter of Finnish students performed below proficiency level 2. At the same time, the proportion of high-achieving students, reaching level 5 at least, has decreased. In 2003, nearly a quarter of Finnish students reached level 5 or 6, but in 2022, less than 9 percent of students reached them. In contrast to the TIMSS results, PISA study thus shows that the proportion of high-achieving students has declined.

Figure 3. Percentage of high and low performers in mathematics in Finland 2000–2022 in PISA (%) (Ministry of Education and Culture, 2023)



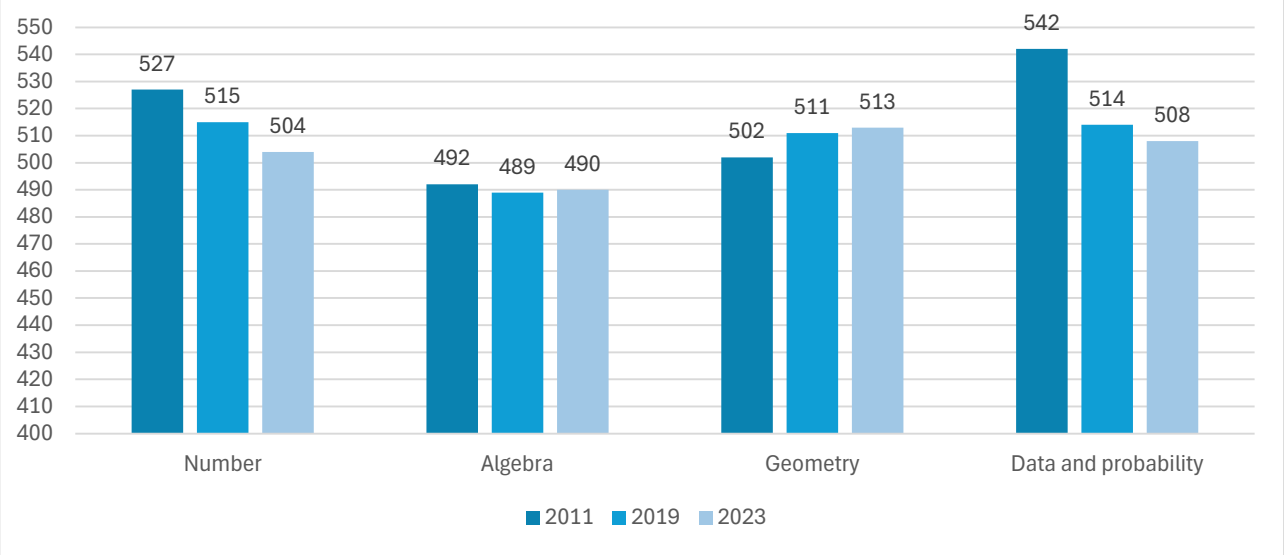
Achievement in different mathematics content areas

In next, we will look at the assessment results in different content areas in mathematics. According to PISA 2022, Finnish students' performance in four different content areas was very consistent, and no content area stood out as being particularly stronger than the others. However, there are some changes between the cycles that are worth looking at. In 2012, Finnish students' achievement in geometry (content area space and shape) was the weakest and quantity, which involves working with numbers and applying basic arithmetic operations, was the strongest content area. Over the ten years, the performance in quantity contents had declined the most (–47 points), whilst in geometry the change was the smallest (–22 points).

According to TIMSS 2023, Finnish 8th graders performed the best in geometry (508 points), while the weakest content area was algebra (490 points) (Figure 4). Performance in algebra was the weakest also in TIMSS 2011, where there has not been any significant change between the cycles. The biggest decline has been in data and probability where average scale score has decreased 34 points in 12 years. The next biggest change was in number content area where the average scale score has decreased

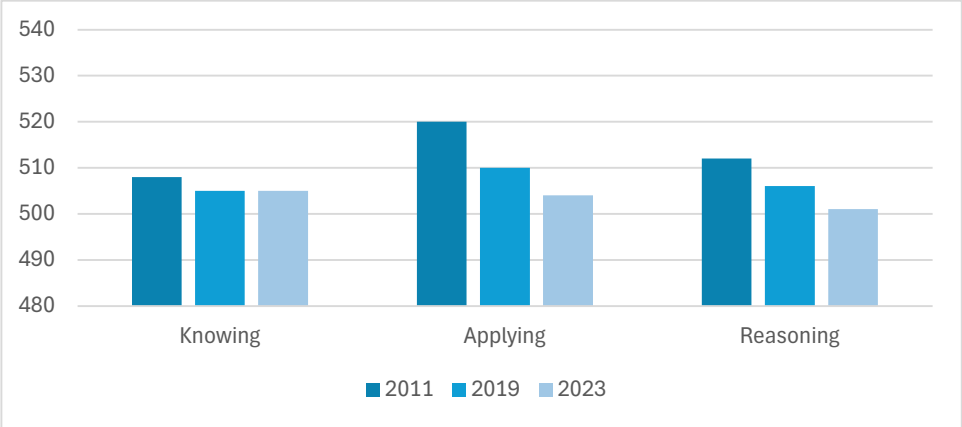
23 points. In geometry, Finnish students' performance was statistically significantly even better in 2023 than 12 years ago. The average in 2023 is 11 points higher than in 2011. Therefore, it can be said that the changes in proficiency in the content areas of PISA and TIMSS support each other in the sense that proficiency in the content area of numbers (quantity in PISA) has declined significantly in both studies over the past decade. While in PISA the change in geometric content was the smallest, in TIMSS performance in geometry have gotten even better.

Figure 4. Finnish students' mathematics achievement in TIMSS by content area in 2011, 2019 and 2023 (Hiltunen et al., 2024)



According to PISA 2022, on average, students were equally proficient in different mathematical problem-solving processes (formulating, employing, interpreting) and reasoning, differences in averages were only 4 points if any. Due to the changes made in PISA 2022 Framework, it is not possible to look at trends in different process areas. In TIMSS, the changes between different cycles in mathematics' process areas are more subtle than in content areas (Figure 5). There are no statistically significant changes in knowing. The biggest difference (–16 points) is in applying between 2011 and 2023. There is also 11 points decline in reasoning, which is statistically significant. It is not straightforward to compare the process categories in PISA with those of the TIMSS study. However, given the applied nature of PISA, the question arises as to what extent the steeper decline observed in PISA is due to students' weaker applying and reasoning skills.

Figure 5. Finnish students' mathematics achievement in TIMSS by process area in 2011, 2019 and 2023 (Hiltunen et al., 2024)



Boys' and girls' performance in mathematics

Internationally, on average, boys have performed better in mathematics than girls. For example, in TIMSS 2023 grade 8 only in four countries and in PISA 2022 in six countries, including Finland, girls had statistically significantly higher average scale score. In Finland boys and girls have performed relatively equally. In the TIMSS studies, there has been no statistically significant difference in mathematics proficiency between Finnish eighth-grade girls and boys. In PISA as well, gender differences have remained moderate. Finnish boys outperformed girls in the early 2000s, and in PISA 2009 and 2012 no differences were observed. Since PISA 2015, girls have performed slightly better than boys. The 5-point advantage for girls in 2022 was exceptional, given that Finland was the only OECD country in which girls scored higher than boys.

However, there is greater variation in boys' math proficiency than in girls'. The standard deviation for Finnish girls was 78 points and for Finnish boys 85 points in TIMSS 2023, and 84 points for girls and 94 points for boys in PISA 2022. Especially, boys' share in low achievers has pronounced in PISA. More than one-fifth (22%) of girls and more than one-quarter (27%) of boys in Finland were classified as low achievers, meaning they scored below proficiency level 2. The gap between boys' and girls' shares at the excellent proficiency level was much smaller: 9.5 percent for boys and 7.6 percent for girls.

Students' attitudes towards mathematics

In Finland, students' attitudes towards mathematics have been poor. In TIMSS 2023, compared to other countries, Finland had the smallest proportion of students who like learning mathematics very much (7%) and the largest proportion of students who do not like learning mathematics (69%), while the international averages were 21 percent and 46 percent, respectively. Observations have been similar also in earlier TIMSS cycles. In Finland, girls liked mathematics less than boys. Only 5 percent of girls liked

mathematics very much, and as many as 74 percent did not like it. Among boys, 8 percent liked mathematics very much, and 63 percent did not like it. In addition, differences in average scale score between these two groups were 77 points for girls and 49 points for boys. Thus, the relationship between girls' liking of mathematics and their proficiency was stronger than that of boys.

In PISA, an index reflecting self-efficacy was constructed based on nine statements presented to the students. The higher the index value, the greater the students' self-efficacy in mathematics. Finnish students' self-efficacy was close to the average of OECD countries. The index value was -0.37 in OECD countries and -0.35 in Finland. Girls' self-efficacy was lower than that of boys, both on average across OECD countries and in Finland.

In TIMSS, students' confidence in mathematics was examined with eight statements and students were classified in three categories. Internationally, on average, more than half (55%) of eighth-grade students reported feeling "not confident" in mathematics, and only 13 percent of eighth-grade students reported feeling "very confident" in mathematics. Finnish students' confidence in mathematics is at the international average. Thirteen percent of Finnish students were very confident in mathematics, while 57 percent were not confident in mathematics.

Based on PISA 2022, students in Finland experience the least mathematics anxiety compared to students in other countries that participated in the study. Although mathematics anxiety is relatively low, experiencing mathematics anxiety was associated with mathematics proficiency to a greater extent in Finland than in average. Mathematics anxiety was also the second strongest predictor of variation in mathematics proficiency among background variables related to mathematics learning, after self-efficacy in mathematics, and its explanatory power was higher than the OECD average.

In TIMSS, students were asked to indicate their level of agreement with eight statements about the value of mathematics. Seventeen percent of Finnish students strongly valued mathematics. Just under half (46%) valued mathematics to some extent, and 37 percent of students did not value mathematics. Finnish students' value of mathematics was lower than the international average, as internationally, on average, about one-third (34%) of students strongly valued mathematics, 41 percent valued it somewhat, and about one-quarter (24%) did not value mathematics (von Davier et al., 2024). In Finland, there was no difference between girls and boys in the extent to which they valued mathematics, but among girls, valuing mathematics was more strongly connected to mathematics proficiency. The difference in mathematics scores between those who strongly valued mathematics and those who did not was 64 points for girls and 41 points for boys, with the former group scoring higher.

Mathematics learning environment

In TIMSS, 8th grade students were asked to evaluate instructional clarity in mathematics lessons. This scale included items that asked students to indicate their agreement with different statements about their teachers' provision of mathematics instruction, such as answering questions clearly or giving helpful feedback. A clear majority of Finnish students found mathematics instruction to be at least moderately clear, as 36 percent of students rated the clarity of instruction as high and 43 percent as moderate. About one-fifth of Finnish students (21%) found the clarity of instruction to be low. According to the international average, 48 percent of students fell into the "high clarity of instruction" group and 15% into the "low clarity of instruction" group.

According to PISA, students' perceptions that their teacher provided support during math classes were associated with better mathematics proficiency. The support provided by the teacher also appeared to be related to the students' experience of math anxiety: if students felt that the teacher provided support, their experiences of math anxiety were also less pronounced. However, students felt they received less support from their teachers in 2022 than in 2012. For example, in Finland, 78 percent of students felt that their teacher provided extra help when they needed it. The average for OECD countries was 70 percent. In 2012, that figure was as high as 85 percent in Finland.

Disorderly behavior has negative relationship with average mathematics achievement. In TIMSS 2023, students were asked to indicate how often disruptive behavior occurred in their mathematics classes. Seventeen percent of Finnish students reported that there was no disruptive behavior or that it occurred only in a few lessons, 71 percent said there was disruptive behavior in some lessons, and 12 percent reported that disruptive behavior occurred in most lessons. Compared internationally, there is somewhat disruptive behavior in mathematics lessons in Finland, but the less students report it happening on most lessons than on average. The international average for students who tell it happens on most lessons, is 17 percent, while 22 percent of students internationally report there is no disruptive behavior or it happens only in a few lessons.

PISA also examined disciplinary climate in mathematics classes, and the findings were similar to those of disruptive behavior in TIMSS. The level of disciplinary climate in mathematics classes was perceived as relatively low. Girls, in particular, perceived the disciplinary climate weaker than boys did. Among the other Nordic countries, only in Sweden was the disciplinary climate, based on student responses, weaker than in Finland. In Finland, the disciplinary climate also had a stronger than average relationship with mathematics anxiety. On the other hand, disciplinary climate during mathematics lessons was perceived as better than in 2012. In connection with disciplinary climate, the study also examined how disruptive students found the use of digital resources during mathematics classes. In Finland, students found the use of digital resources distracting significantly more often than in most other PISA countries

(OECD, 2023b). However, the question did not take into account the extent to which the use of digital resources was the students' own choice, that is, whether the teacher had instructed them to use, for example, computers and educational apps, or whether students were using digital resources during class for purposes other than learning. Since August 2025 revised Basic Education Act has prohibited students from using mobile devices during class. Students may use a phone for educational purposes, for health reasons, or as a learning aid with the teacher's permission.

Teachers in Finland are known to be highly educated. According to TIMSS 2023 data, nearly all students (93–94%) had a teacher who held a master's degree. In Finland, 26 percent of students had a math teacher who did not major in mathematics. These teachers may have majored in education or some other unspecified subject major, such as physics or chemistry. Even though, nearly all teachers held a master's degree, students whose teachers did not major in mathematics had an average score in mathematics that was 16–21 points lower than that of students whose mathematics teacher had majored in mathematics. TIMSS Teacher questionnaire does not indicate what other courses teachers have taken in addition to their major, but in Finland, a subject teacher is qualified to teach secondary subject after completing at least 60 credits in that subject (Finnish National Agency for Education, n.d.).

The TIMSS study also examined teachers' participation in professional development and future professional development needs. Teachers' participation in professional development has generally been quite low in Finland. In 2023, the proportion of students whose math teacher had participated in professional development had decreased from 2019 in nearly all of the professional development topic areas surveyed. Among the mathematics teachers who participated in the TIMSS study, participation in training related to addressing students' individual needs was the only area where participation had increased from 2019 to 2023. However, there still seems to be need for such trainings. From 52 to 63 percent of the students had a mathematics teacher who felt a need for training that addressed the integration of technology into teaching, addressing students' individual needs, or improving students' critical thinking or problem-solving skills. The least need for training was felt regarding mathematics content.

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MULTIDISCIPLINARY RESEARCH ON TEACHING AND LEARNING MATHEMATICS: STATE OF THE ART IN FINLAND

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This study describes Finnish research on mathematics education and learning and show that multidisciplinary approaches combining traditions of mathematics didactics and educational psychology have characterized majority of studies. This may be due to the structure of teacher education. There are no barriers between the disciplines because scholars of mathematics didactic and educational psychology typically work in same departments. One consequence of the multidisciplinary is also the methodological diversity of Finnish research on mathematics learning and teaching.

INTRODUCTION

A special feature in Finnish research on mathematics teaching and learning is that it is not separated from general education and psychologically oriented learning research. Main reason for that is the structure of teacher education in Finnish universities. The training of primary school teachers, so called class teachers teaching all subjects, and mathematics subject teacher is partly conducted in same units. Mathematics subject teachers make their mathematics studies in mathematics departments together with other mathematics students who do not plan to be teachers. After a few years of mathematics studies, they spend a year in teacher education departments and university training schools in which they get their pedagogical/psychological and didactics education, and practical teaching training.

This means that mathematics didactics is mainly taught in departments of teacher education. Here I want to mention that I use the term mathematics didactics in the neutral way typical for the European tradition. General didactics refers to evidence informed methods for teaching and mathematics didactics methods special for mathematics teaching. The term can be seen more of less equivalent to the term pedagogical content knowledge in mathematics (Grigaliūnienė et al., 2025; Vollmer & Klette, 2023).

From the organization point of view the structure of teacher education means that professorships and other academic staff responsible to teach and research mathematics teaching and learning in schools are typically working in teacher education departments. In addition, same didactics and learning research scholars are typically teaching both, primary and secondary schools teacher students. Mathematics didactic

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teachers and researchers often work together with general education researchers, education psychologists and special education experts.

One of the consequences of the above-described situation is that didactic oriented and learning sciences research oriented traditions on mathematics learning and teaching in Finland are not as separated as in many other countries (see Gauvrit, 2012; Simplicio et al., 2020).

On one hand we have important research areas originally initiated by mathematics didactics experts, but which were also well informed about the state of the art in educational psychology and learning sciences research. On the other hand, research projects led by educational psychologists or learning scientists typically collaborate with mathematic didactic experts and have access to high level mathematics expertise, which is crucial particularly for studies dealing with more advanced mathematical topics. This close relationship between the traditions has made it possible to combine new theoretical and methodological development in international research on mathematics didactics and findings from psychology and learning sciences.

MAIN TRENDS IN FINNISH RESEARCH ON MATHEMATICS TEACHING AND LEARNING

From mathematics didactic towards educational psychology

Some of the strong traditions in Finnish research on mathematics learning have their origin in mathematics didactics but have later created deep theoretical and methodological connections with learning sciences and (educational) psychology.

Research on beliefs and affects is an example of research tradition which has its origin in mathematics didactics but in which researchers have started to use also theories and methods of educational psychology and learning sciences.

Erkki Pehkonen initiated in Finland a very influential research tradition focusing on beliefs in mathematics learning. The research tradition originated from practical observation of mathematics teachers' beliefs (Pehkonen, 1994) but in later works the belief concept was analysed within deeper psychological frames (Furinghetti & Pehkonen 2002). The importance of teachers' beliefs related to mathematics teaching was supported by the results of a recent study (Niemi et al., 2025) analysing various predictors of students' motivation to learn mathematics. However, against the expectations it was the teachers' transmissive and not constructivist beliefs which were related to higher motivation among students. Another resent study presented opposite findings, teachers' constructive beliefs about mathematics learning resulted higher success expectancy in the class than transmissive beliefs (Haataja et al., 2024).

These studies on non-cognitive aspects of mathematics learning were continued by Markku Hannula's widely recognized works on affects, emotions, and attitudes in mathematics learning and teaching (Hannula, 2002; 2006; 2020) in which specific mathematics didactic themes and more general pedagogical and psychological themes

were integrated. Hannula's studies have been very influential in the field of research on mathematics learning and instruction. However, in studies in which mathematics learners are in the focus of the empirical analyses he has largely and critically discussed existing theories of emotions and affects and proposed theoretical models going beyond simplified models of emotions in mathematics learning (e.g. Hannula, 2012).

Another research field Pehkonen started in Finland was *research on creativity and mathematical problem solving* (Pehkonen, 1997; Pehkonen et al., 2013). The origin of this tradition was in the mathematics and mathematical education but gradually the authors integrated ideas of cognitive psychology and constructivism in the framework used in describing processes of problem solving and learning environments supporting learning mathematics trough problem solving (Pehkonen et al., 2013). This field of research which was later continued by many Finnish researchers of mathematics learning and teaching who have focused on inquiry learning in technology assisted environments (Hähkiöniemi et al., 2013), collaborative mathematical problem solving (Salminen-Saari et al., 2021), and mathematical word problem solving (Pongsakdi et al., 2019). In recent reviews (Vessonen et al., 2024: 2025) Finnish researchers presented an extensive synthesis of the effects of mathematical word problem solving. On average, interventions were effective and the study provides detailed analysis of the aspects of interventions which resulted in best learning improvement.

From educational psychology towards mathematics didactic

In some topics the origin of the Finnish research has been in educational sciences and educational psychology, but it has later had substantial impact also in research and teaching of didactics of mathematics.

Research on conceptual change in learning mathematics is an example of opposite development. Theories of conceptual change have been developed in educational psychology (Posner et al., 1982; Vosniadou, 1994). In early studies conceptual change theory was applied as a psychological explanation for challenges to learn counterintuitive physics concepts and to difficulties to overcome misconceptions. Our study on the development of the conceptual understanding of real numbers was one of the first studies which applied conceptual change theory in mathematics learning (Lehtinen et al., 1997). In this study the cognitive science based theory of conceptual change was connected to the history and philosophy of mathematics and later to mathematics didactics. The results showed that the change from discrete to continuous concept of numbers requires a radical conceptual change in students. Interestingly when the first results about conceptual change in mathematics were presented in mathematics education conferences, it caused some critical discussion among mathematic didactics community because the claims were violating the idea of the systematic construction of mathematical knowledge. This early study was a beginning for series of studies analysing conceptual change in the extension on number concepts (e.g. McMullen et al., 2015; Kärki et al., 2025). The studies have shown that the so-

called natural number bias is strongly limiting primary students' understanding of rational numbers. The study about primary school teacher students rational number knowledge showed that the natural number bias is still visible among university students (Merenluoto & Lehtinen, 2004). Additional evidence about the radical conceptual change related to the extension of number concepts comes from a study which showed that university mathematicians still remembered the experience when they started to understand the change from rational to real numbers (Merenluoto & Palonen, 2007).

Parallel research in mathematics didactics and educational psychology

Interestingly some topics have currently raised researchers' interest simultaneously in mathematics, didactics of mathematics, and educational psychology.

Research on teaching interaction in mathematics classrooms is a topic in which both Finnish researchers of mathematics didactic and educational psychologists have recently conducted innovative research. Hiltunen et al. (2026) analysed mathematical justification in small group and whole class discussions. The main finding was that in the whole class discussions mathematical justification was often more superficial than in small group discussions. Small group discussions and problem-solving during mathematics lessons has been the topic in Haataja's (Haataja et al., 2019; 2021) research program. The unique feature of these studies was that mobile eye-tracking technology was used to analyse teachers' and students' mutual eye contacts when teachers were scaffolding group work. The results highlighted the importance of both students' and teachers' gaze behaviour for the functioning of the teaching interaction as well as for teachers' authority and agency. Teacher noticing and professional vision are theoretical approaches which Finnish educational researchers have mainly applied in studies taking place in mathematics classrooms. Pouta et al. (2021) studied how experienced teachers and student teachers notice and interpret primary school students' difficulties in solving fraction tasks and how relevant support they were able to give for students struggling with fraction tasks. The results showed that student teachers had better verbal interpretations of students' difficulties, but experienced teacher were more skilful in supporting students learning. In addition, they studied how experienced teachers and student teachers differed in their ability to simultaneously support individual students and to be aware of the whole mathematics classroom situation (Pouta et al., 2026).

Research on flexible and adaptive mathematical skills is an example of Finnish research on mathematics learning and teaching which started simultaneously by mathematics scholars in mathematics departments and learning researchers in teacher education departments, and only later resulted in collaboration between these traditions in research but also in organizing in-service flexibility related training for mathematics teachers. In the Department of Mathematical Sciences Peter Hästö's and his team's research focused on the use of flexible strategies in algebra, particularly in solving

linear equations among students of lower and higher secondary education. They found that there were quite strong interindividual differences in students' capacity to innovate and use flexible strategies. Flexibility was a strong predictor of students' speed and accuracy in solving linear equations (Hästö & Palkki, 2919). In the study aimed at using mathematical flexibility test scores as predictors of matriculation exam results there were some relationships between flexibility and matriculation results but the effect was not so strong as expected when students' accuracy as flexibility tasks was used as covariant (Hästö et al., 2019). The international comparison study by Star et al., 2022 showed similarities and differences between the three included countries (Finland, Spain, and Sweden). In all countries, flexibility and accuracy in equation solving increased from middle school to secondary school. However, students tend to rely on standard algorithms and flexibility, particularly spontaneously applied flexible strategies, was quite rare in all countries.

The research group on mathematics learning and instruction at the Department of Teacher Education at the University of Turku approached the learning of flexible mathematics from another perspective. Their aim was to understand what kind of representation of numbers make it possible for students to see the opportunities for flexible strategies to solve arithmetic problems with natural numbers (McMullen, Brezovszky et al., 2016) and rational numbers (McMullen et al., 2020). Adaptive number knowledge refers to rich network of connections between numbers and between numbers and arithmetic operations. An interesting finding was that particularly adaptive rational number knowledge was strong predictor of later mathematical development including algebra learning. The results show that students with high achievement in routine school tasks on rational numbers are divided in two different profiles. Half of the students had high achievement in all routine conceptual and procedural tasks but only average achievement tasks measuring adaptive number knowledge. The other half had otherwise the same achievement profile but clearly higher achievement in adaptive number knowledge tasks. Later comparison of brain activation in tasks based on acquiring routine solutions with tasks requiring adaptive number knowledge indicated differences in brain activation (Ylinen et al., 2025). Both behavioural and brain imaging studies indicate that adaptive number skills sills are not only more routine skills but a different representation of numerical knowledge.

Based on basic research on adaptive number knowledge and flexible arithmetic the group created computer games aimed at supporting the development of adaptive number knowledge and flexible arithmetic strategies in natural numbers (Lehtinen et al., 2015; Brezovszky et al., 2019) and in rational numbers (Kärki 2022). Based on randomized experimental studies, computer games have successfully enhanced adaptive and flexible skills. An explanation is that these games enable the type of intensive experimenting and training with numerical relations (deliberate practice) which is difficult to organize in traditional classroom teaching.

In addition, later the research group in University of Jyväskylä, have found evidence of the importance of adaptive number knowledge in solving challenging problems involving mathematical patterns (Skyttä et al., 2025, 2026).

Relying on the findings of research on flexible and adaptive mathematics, the three above-described groups created together an online in-service training "Toward Flexible Mathematics" (www.flexibility.fi). Currently almost ten thousand Finnish teachers teaching mathematics from early education to upper secondary education have studied in the program. Scientific studies on the experiences and impact of the program are just in the beginning. The first published study has focused on the question if participating teachers are creating new practice related knowledge (teaching scripts) when participating in the course (Moberg et al., 2025).

Studies on mathematics learning in educational psychology and special education

In the Finnish research on mathematics learning and teaching there are also topics which are mainly studied among education psychology, early education, and special education.

Research on early development of mathematical skills has been a strong tradition in the (neuro)psychology, early education, and special education in Finnish universities. The main focus has been in early and primary education. In their highly cited longitudinal study following same students from preschool to second grade in primary school Aunola et al. (2004) were able to show high stability of the differences in mathematical skills during the three years follow-up period. Stable interindividual differences existed already before preschool and the children's counting skill was best predictor of later mathematical achievement. Also, other studies have focused on predicting later mathematical development with early numeracy. Results of the longitudinal study by Aunio and Niemivirta (2010) confirmed earlier findings and showed that the acquisition of counting and relational skills before formal schooling predicted acquisition of basic and applied arithmetical skills in grade one, even if demographic factors such as parents' education was controlled. The study also showed the importance of attentional skills on mathematical development.

A recent study on the early development of mathematical skills among 3 and 4 years old highlighted the important role of family background such as number of children in the family, family income, and the level of parents' education as predictors of the interindividual skills (Aunio et al. 2025). Mothers' education was a significant predictor also in a study focusing on the development of pupils' arithmetic fluency from first to third grade. The main task of the study was, however, to analyse how the domain specific initial skills, such as symbolic magnitude processing and number sequence skills and domain general dispositions such as working memory, non-verbal reasoning, and sustained attention do predict the development of arithmetic fluency during first three school year. In addition to the family background, domain specific

skill were the most important predictors of the mathematical development during first school years, and domain general factors had only minor effects.

The relationship between early mathematical and language skills has been analysed in several studies. In the study of Koponen et al., (2007) both mathematics specific skills (counting, and number concept) and language skills (letter knowledge, phonemic awareness) measured in the kindergarten predicted mathematical skills in the grade four in primary school. In the recent intervention study in which special needs children (comorbid fluency problem) were supported in developing calculation strategies and flexible ways to use them, the results showed that this improved significantly children's calculation fluency (Pulkkinen et al., 2026).

Oher research groups have also focused on the relationship between mathematical and language skills. In their study on the relationship between mathematical and language skills, educational aspirations, interest and mathematical achievement Korhonen et al., (2016) found that academic self-concept was a domain general predictor of mathematics and language related educational aspiration whereas interest was domain specific.

Spontaneous focusing tendencies in the development of mathematical knowledge and skills is a research field which started at the Department of Teacher education at University of Turku (Hannula & Lehtinen, 2005) and is currently widely studied in many countries around the world (Bojorque et al., 2021; Hannula-Sormunen et al., 2023; Li et al., 2025). Early longitudinal studies showed that among young children there are interindividual differences in children's tendency to spontaneously focus on numerical aspects of various situations (SFON), this tendency is relatively stable over time, and it is strong predictor of later mathematical development (Hannula-Sormunen et al., 2015). Intervention studies showed that SFON tendency can be enhanced, and it has subsequently positive impact on the development of mathematical skills (Hannula, Mattinen et al., 2005; Hannula-Sormunen et al., 2020). Later studies have analysed if similar inter-individual differences exist in children's focusing on other mathematically relevant aspects of situations. Individual differences in spontaneous focusing on quantitative relationships appeared to be a strong predictor of later learning of rational numbers (McMullen, Hannula-Sormunen et al., 2016; Van Hof et al., 20169.). Studies also showed that children have substantial individual differences in their tendency to spontaneously focus on numerical order. The findings revealed that this tendency had a unique contribution to children's numerical ordering skills (Harju et al., 2024).

Many studies conducted in psychology and special education in Finland have focused on mathematical learning difficulties and dyscalculia. In their large longitudinal study, Zhang et al. (2020) found that basically the same general and domain specific cognitive skills such as language skills, spatial skills, and counting skills predicted low mathematical achievement and more stable mathematical learning difficulties. In a

more recent study, the mathematical difficulties among students at the end of primary school (grade 6) was predicted with various measures conducted when children were in kindergarten. Both arithmetic and cognitive skills (arithmetic operations, rapid automatized naming, number concepts, spatial skills, counting) but also indicators of behavioural problems (attention-orientation) were significant predictors of mathematical difficulties 6 years later (Psyridou et al., 2024). Neuroimaging studies comparing children with developmental dyscalculia with normally developing children indicated that brain areas dealing with numerical information are not responding to numerical processing demands to the same degree as in typically developing children (Price et al., 2007). Studies on mathematical learning difficulty include also computer assisted interventions for children with low numeracy skills. Räsänen et al. (2016) compared different games based on neuropsychological hypothesis of mathematical difficulties.

Educational issues studied within mathematics departments

Some mathematics scholars mainly studying various topic of mathematics have also contributed to the research on the learning and teaching of mathematics. In addition to the above-described learning flexible mathematics in secondary education (see Hästö et al. 2019) mathematicians' pedagogical studies have mainly focused on university level mathematics learning.

University level mathematics learning and teaching is not very widely studied in Finnish universities but there are a couple of researchers who have focused on teaching and learning advanced mathematical knowledge on tertiary level education. Some of the professors and lecturers working in mathematics departments have been deeply interested in pedagogical questions related to teaching mathematics, and a few have also conducted empirical research. In some mathematics departments it has been possible to make PhD thesis about themes related to mathematics teaching. The studies have focused for example on large pedagogical and cultural changes in university level mathematics teaching by applying the ideas of cognitive apprenticeship (Rämö et al. 2019). Viholainen et al. (2023) have studied university mathematics students' epistemic beliefs, motivation, and study habits. Oikkonen (2022) has elaborated further David Tall's framework of the three worlds of mathematics (embodied, symbolic, and formal) by combining it with subjective-social and objective dimensions. The framework was used in studying the transition from the secondary mathematics education to university mathematics studies.

METHODOLOGICAL APPROACHES IN FINNISH RESEARCH ON MATHEMATICS EDUCATION

One consequence of the interdisciplinarity of the research on mathematics learning and teaching is also the methodological diversity of this study field in Finland. The methodological diversity is characterizing all phases of research including research designs, data collection methods and data analysis.

Some researchers of mathematics didactics have been involved in the large international studies such and TIMSS and PISA, which focus on mathematical students' achievement. In addition to basic reporting of the results, data from these studies have been used for example in analysing longer trends in mathematics achievement (Kupari, 2004) or the immigrant students' achievement in mathematics tests of PISA (Pulkkinen et al., 2024). Finnish researchers have also been participating in the methodological development o6f the measures as parts of the international teams (e.g. Bos et al., 2011). Some studies have applied international comparison designs to analyse various aspects of mathematics teaching (Hemmi et al., 2013; Krzywacki et al., 2023)

Large scale national samples have also been used to deepen the analyses beyond the mere achievement of students, for example to study top achievers' learning trajectories in mathematics on the basis of longitudinal data of achievement, attitudes, and socioeconomic background (Niemi et al., 2023).

Mixed methods approaches have actively been used in Finnish research on mathematics learning and teaching. Many of the major findings are based on the combined use of quantitative and qualitative methods to analyse the data (e.g. Hannula & Suriakumaran 202). Intensive qualitative analysis of individual processes combined with summarizing quantitative analyses has made it possible to make long term predictions on the basis of deep situational analyses (e.g. Hannula & Lehtinen, 2005). Advanced methods for quantitative data analyses include complex statistical modelling technics (e.g. Lehikoinen et al., 2025; Psyridou et al., 2025) and neural networks (e.g. Psyridou et al., 2024).

Experimental designs have been used in resent studies on the use of educational technology, particularly educational games in mathematics education (e.g. Kiili et al., 2018). A challenge for using controlled experimental design in school context is that it is difficult to randomly select students to experimental and control conditions if the aim is to test methods in classroom teaching. One solution is to use large scale group randomized design which was use in Brezovszky et al. (2019) study on the effect of an educational computer game on adaptive number concept.

Innovative test development has been important particularly in studies which have opened new research fields such as the use of flexible strategies in solving linear equations (Hästö et al., 2019) or in studying spontaneous focusing on quantitative relationships (McMullen et al., 2016). On the other hand, new test development for electronic data collections systems and learning analytics have required innovative methods development (Hakkarainen et al., 2025).

Analysis of classroom videos have been used to analyse mathematics related dialogues (e.g. Hähkiöniemi et al., 2022). Recently many studies on teaching-learning processes in the mathematics classrooms have made use of mobile eye-tracking, which makes it possible to get detailed information about teachers' focusing of visual attention in

various classroom situations (e.g. Hannula et al., 2022). This intensive data can be also used in stimulated recall interviews to uncover the interpretations and aims teachers have when teaching (e.g. Pouta et al, 2021). Mobile eye-tracking in mathematics classrooms have mainly been used to analyse teachers' visual attention. However, the research group of Markku Hannula has applied mobile eye-tracking tools simultaneously for teachers and students in order to catch the various moments of reciprocal attention (Heyd-Metzuyanim et al., 2023). Some studies have also applied brain imaging methods such as magnetic resonance imaging (fMRI) to analyse brain processes when students have solved mathematical tasks (e.g. Price et al., 2007; Ylinen et al., 2025).

CONCLUSIONS

As mentioned in the beginning of the article, the particular structure of the Finnish teacher education has provided excellent opportunities to overcome borders of different research traditions of mathematics teaching and learning. Thus, the interdisciplinary nature characterizes Finnish research in the field. Of course, there are theoretical and methodological differences and the mathematical topics dealt with in studies focusing on various levels of education are very different. Independently of these differences the different research traditions share enough common ground, which has made it possible to share theoretical ideas, methods, and in some cases create interdisciplinary research groups including researchers originally coming from different traditions.

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PME 49
Helsinki, Finland
2026

PUBLICATION REPORTS

EXAMINING SIXTH-GRADE STUDENTS' RATIO REASONING SKILLS THROUGH SCENARIO-BASED DIGITAL GAMES

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Ratio is an essential concept for students to understand proportions and develop proportional reasoning skills. Traditionally, students learn ratios from textbooks, which offer limited interactivity. Using a convergent mixed-methods design, this study compares the ratio reasoning skills of sixth-grade students who engaged with scenario-based digital games (SBDGs) (experimental group) with those who did not (control group). The SBDGs aimed to enhance students' ratio reasoning skills through context-rich ratio tasks. A ratio-reasoning test with 20 items was administered as a pre-test and a post-test. The results showed that the experimental group performed significantly better than the control group on the post-test. We also observed that the use of SBDGs enhanced students' cognitive, affective, social, and metacognitive development.

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, p. 393. PME.

PROTOTYPING MATHEMATICS TEACHER LEARNING AND IMPLEMENTATION TRAJECTORIES IN INQUIRY-BASED MATHEMATICS INSTRUCTION

Tuval Avishai, Alik Palatnik, Yifat Ben-David Kolikant

The Hebrew University of Jerusalem

This study examines in-service mathematics teachers' learning as they work toward inquiry-based mathematics instruction. Using empirically grounded prototype construction methodology, we identify three Teacher Learning and Implementation (TL&I) trajectories: Seamless Transition to Open Inquiry, Gradual Ascent Toward Open Inquiry, and Preference for Structured Inquiry. We then map each prototype to a corresponding System of Amplifiers and Filters (SAF) configuration, a framework comprising teachers' beliefs, prior knowledge, professional phase, and learning environments. The TL&I–SAF alignment suggests a framework for anticipating teacher learning and implementation challenges, thereby supporting more individualized, data-informed professional development.

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INTERPRETING MATHEMATICAL ARGUMENTATION: A CROSS-NATIONAL STUDY OF PRESERVICE TEACHERS

Michal Ayalon[1], Francesca Morselli[2], Gözde Kaplan-Can[3], and Samaher Nama[1]

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This study examines how preservice mathematics teachers from Israel, Türkiye, and Italy interpret a written whole-class episode involving mathematical argumentation. Using a framework that distinguishes structural and dialogic aspects, as well as cognitive, social, and affective dimensions, we identified the lenses through which participants made sense of students' reasoning and interactions. While all groups employed a wide set of lenses, ranging from mathematical justifications to interactional dynamics and emotional conditions, their emphases differed across countries, suggesting the influence of local curricular and professional contexts. The findings highlight the need to support novices in noticing the multifaceted nature of argumentation in classroom discourse.

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RHYTHM IN AN EPISTEMIC SPACE. EXPLORING THE DERIVATIVE WITH AUGMENTED REALITY

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Learning with Augmented Reality (AR) is a new and promising approach for learning calculus. However, investigations of how the hybrid combination of the real and the virtual makes AR experience beneficial are missing. To address this need, a case study is conducted that conceptualizes and coordinates three perspectives of the AR space, a spatial, an epistemic and a temporal perspective. The aim is to reconstruct how a learner who explores the derivative with AR generates an epistemic AR space. Methodically, local analyses of each episode from the three perspectives are combined with a global analysis across all episodes. Results show that the AR space is created by spacing and synthesizing actions, rhythmically conducted with an epistemic purpose, thus, producing an epistemic ecology where the key is exploratory.

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A SYNTHESIZED COGNITIVE MATHEMATICS LEARNING TAXONOMY

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This paper is based on a recent work that synthesizes 21 cognitive taxonomies relevant to mathematics education (Gun & Bossé, 2025). Using networking strategies to align task-demand, learning, process, and problem-solving frameworks, we constructed a six-level Synthesized Taxonomy (ST) that links features of mathematical tasks (action) with evidence of students' cognitive activity (actor). An illustrative coding example demonstrates how the same mathematical idea can be operationalized across ST levels, supporting comparisons of task design and student work. We close by critiquing what the ST does not yet capture (e.g., metacognition and sociocultural dimensions) and outlining research pathways for empirical validation and refinement.

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LEARNING THROUGH DESIGN: TEACHER EDUCATORS' LEARNING IN DEVELOPING PCK ASSESSMENTS FOR FRACTIONS

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This study investigates how teacher educators transform content-focused assessment items into pedagogical content knowledge (PCK) items, with a focus on fractions. A qualitative design-based approach was used to explore how lecturers collaboratively redesigned fraction assessment items to address learner thinking. Data from written reflections and online discussions were analysed using the Mathematical Knowledge for Teaching framework. The findings highlight that adapting items to assess PCK requires careful alignment between distractors and learner misconceptions, strong understanding of fraction concepts and pedagogical reasoning. Overall, the iterative design process created opportunity for professional learning for teacher educators.

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TEACHERS' USE OF EVIDENCE OF STUDENTS' LEARNING IN INSTRUCTIONAL DECISIONS: A SCOPING REVIEW

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This paper reports on a scoping review of research on math teachers' use of evidence of students' learning for informing instructional decisions. Following PRISMA guidelines, 103 English-language papers published between 2017-2023 in high-impact mathematics education journals were analyzed. The review focused on two lines of inquiry: formative assessment and teacher noticing. Despite theoretical resemblance, papers were rarely associated with both lines of inquiry, greatly favoring teacher noticing. In contrast, the empirical research methods were largely similar across both lines of inquiry. Studies typically involved small- to medium-sized groups of teachers, prospective and practicing, elementary and secondary teachers. Most studies were situated outside classroom settings, with few non-interventional classroom studies.

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SYSTEMATIC REVIEW OF RESEARCH ON MATHEMATICS PCK: INSIGHTS FROM A TOPIC-SPECIFIC APPROACH

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One of the most important characteristics of mathematics teachers is their knowledge enabling them to provide learning experiences that promote students' understanding, defined as pedagogical content knowledge by Shulman in 1986. The current review captures the results of 237 studies published between 2013 and 2022, thematically addressing two components - knowledge of instructional strategies and representations and knowledge of student learning difficulties. Approached from a topic-specific perspective this review found 83 unique topic-specific learning difficulties that were mentioned in PCK measures including procedural and conceptual difficulties or combinations of them. Teaching strategies were found to be reported less frequently and some strategies such as the use of manipulatives and visualizations were mentioned in relation to many mathematical topics.

2026. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M.S. Hannula & P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: General Contributions*, p. 400. PME.

A MODEL FOR CHARACTERIZING CRITICAL EVENTS IN TEACHER EDUCATION: NOVICE MATHEMATICS TEACHER-EDUCATORS' NOTICING

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Research on teachers' noticing has mainly focused on classroom teaching, while little is known about how mathematics teacher-educators notice and make sense of events during teacher-education sessions. This study examines critical events in teacher-education sessions (CETEs) as identified by novice mathematics teacher- educators (NMTEs). The study is based on 72 written reports of CETEs produced by twenty-four NMTEs *participating in a university-based teacher-educator preparation course. A* three-dimensional analytic model was derived from the data analysis *to characterize CETEs in terms of: event participants, event focus, and* three dimensions of teaching and learning *(cognitive, affective, and social). This paper discusses the theoretical and empirical construction of the model and demonstrates its potential contribution through examples from the research.*

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BECOMING A MATHEMATICS TEACHER WHEN ONE'S RELATIONSHIP WITH MATHEMATICS HAS BEEN NEGATIVE: A STUDY ON MATHEMATICAL REDEMPTION AND THE SPRINGBOARD EFFECT

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In this study, we seek to answer the question: why would someone want to become a mathematics teacher when their relationship with the discipline has been predominantly negative? To do so, we build on the phenomenon of Mathematical Redemption, analyzing it in depth through a qualitative methodological perspective centered on emotions, supported by the Cognitive Structure of Emotions Theory. Seventeen Mexican preservice teachers participated, and four participants were subsequently interviewed through a structured interview, selected via purposive sampling. The results indicate that Mathematical Redemption can be understood as a specific case of the Springboard Effect. The latter is proposed as an operational framework to identify and analyze Mathematical Redemption in future mathematics teachers.

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INVESTIGATING HOW HIGH SCHOOL STUDENTS LEARN ML: THE CASE OF DECISION TREES FOR CLASSIFICATION

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The growing application of artificial intelligence and machine learning necessitates their integration into high school curricula, with decision trees being particularly accessible for secondary students. This study examines the construction processes of classification trees by thirty high school students (aged 15–18) who participated in a summer program at a university. Data, including student worksheets, screen recordings, and dialogues, were analyzed using a four-strand analytical framework: conceptual, strategic, technical, and reflective. Findings show that students demonstrated conceptual understanding and technical proficiency in constructing decision trees for classification. Their strategies, however, were often unsystematic, and reflective reasoning remained substantially challenging. Working on performance metrics emerged as a key catalyst supporting the development of both strategic and reflective reasoning. Overall, the results suggest that high school students can learn to construct and interpret decision trees using appropriate digital data analysis software but require additional instructional support to develop strategic and reflective competencies.

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ZERO AS A SEMANTIC BOUNDARY: RETHINKING THE MENTAL NUMBER LINE

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Representing zero uniquely challenges numerical theories as it denotes absence yet functions as a fundamental symbolic component. We examined how adults compare zero to positive single-digit numbers in symbolic (Arabic) and nonsymbolic (arrays of items) formats. While classic effects were replicated for positive numbers, the symbolic distance effect was driven solely by boundary values (0, 1) and vanished without them. Conversely, nonsymbolic zero followed the standard nonlinear distance pattern of other quantities. These findings suggest that the mental number line reflects a flexible relational mapping shaped by contextual and semantic boundaries, rather than a uniform spatial continuum.

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EXPERT AND NOVICE MATHEMATICS TEACHERS' VISUAL ATTENTION TO STUDENTS WITH DIFFERENT CHARACTERISTICS DURING INSTRUCTION

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Despite growing interest in teachers' professional noticing, research on how mathematics teachers attend to students with varying achievements and other aspects during instruction remains scarce. This case study employs mobile eye tracking to compare an expert and a novice mathematics teacher's visual attention across three groups of students—strong, intermediate, and struggling. Results show that the expert teacher distributed attention evenly, whereas the novice teacher focused primarily on strong students, followed by intermediate and struggling ones. Both teachers shifted focus within the same groups, but only the expert made significant transitions between strong and struggling or intermediate students, while the novice often returned to strong students after focusing on intermediate ones.

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EFFECTS OF AN INSTRUCTION ON UNDERSTANDING THE ROLE OF VARIABLES IN PROOFS FOR GENERAL VALIDITY

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A universal statement is generally valid, i.e., it is valid for all numbers of a given set, and no counterexamples exist. Proofs for such statements are generally valid if they establish the validity of the statement and the inexistence of counterexamples, for example, by using variables. We investigate the effects of an instruction about the role of variables in proofs and the role of first-year university students' variable understanding on their evaluation and understanding of general validity. The instruction has a positive effect on understanding the general validity of proofs and is particularly supportive for students with a low understanding of variables. The results show the potential of explicit instruction on the role of variables in proofs to reduce systematic differences based on prior knowledge.

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FIVE PURPOSES FOR CROSSOVER MIXED ANALYSIS IN MATHEMATICS EDUCATION RESEARCH: A CASE STUDY

Heather Lynn Johnson, Evan McClintock, Courtney Donovan

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With Crossover Mixed Analysis (CMA), researchers use one research tradition (i.e., qualitative, quantitative, mixed) to transform data collected via another tradition. This case study investigates the use of CMA to examine undergraduate students'(N=673) graph reasoning and graph selection. Our theoretical perspective is pragmatic; we examine the utility of CMA for mathematics education research. Results revealed five purposes for CMA: validate a measure of students' graph reasoning, establish that levels in a theoretical framework form a scale, test a theoretically grounded model relating the constructs of graph reasoning and graph selection, form associations between constructs, and narrate significant associations. Using CMA, researchers can advance results emerging from qualitative studies of students' reasoning.

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MULTIPLE MOBILE EYE-TRACKING AS A REFLECTIVE-PRACTICE RESOURCE IN EMBODIED MATHEMATICS TEACHER EDUCATION

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This report examines a mathematics teacher PD utilizing mobile eye-tracking (MET) for reflection. Five pre-service teachers built a body-scale icosahedron, an embodied task requiring coordinated seeing, acting, and reasoning. They then participated in an online stimulated-recall session, reflecting on selected episodes from the multiple point-of-view videos overlaid with gaze. Using a professional-vision lens, we analyzed how participants highlighted, coded, and elaborated on activity's perceptual–interactional features. The MET-enhanced reflection supported noticing of attention allocation, coordination, and learner experience, and often aligned with researchers' microgenetic analysis. We discuss design implications for scaffolding teacher reflection on embodied mathematical sense-making.

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THE CAUSAL ROLE OF INHIBITION IN MIDDLE-SCHOOL GEOMETRY

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While geometry is crucial for educational success, it remains challenging to teach and learn effectively. This study examines the causal relationship between inhibition and middle-school geometry problem-solving among typically developed adults. Two experiments manipulated inhibition levels using Stroop tasks (color-word in Exp1; numerical in Exp2) while solving geometry problems. Results showed that higher inhibitory demands improved performance: faster response times (RT) without accuracy costs in Experiment 1, and both faster and more accurate responses in Experiment 2. Stronger effects in the numerical Stroop context suggest that domain-specific inhibition may be particularly relevant to geometry problem-solving, with important future implications for mathematics education.

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THE IMPACT OF CHINESE MATHEMATICS TEACHERS' COMPETENCE ON STUDENTS' AFFECTIVE LEARNING OUTCOMES AND THE MEDIATING ROLE OF INSTRUCTIONAL QUALITY

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This Publication Report presents a recent study examining the impact chain from teacher competence to student affective outcomes, mediated by instructional quality, in junior secondary schools within China. Analyzing data from 117 teachers and 3471 students, the study found no significant direct effects, aligning with Western research. However, weak indirect pathways emerged: MPCK—cognitive activation/classroom management—interest/task value. The findings challenge the cultural universality of prevalent models, highlighting how systemic factors (e.g., exam culture and large classes) attenuate expected relationships. The presentation will extend the published work by offering a critical, culturally-embedded reinterpretation of the "competence-instruction-affect" chain and its implications in a Confucian heritage context.

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ORAL COMMUNICATIONS

CONTEXTUALIZING THE TRANSITION TO MIDDLE-SCHOOL USING TIMSS DATA

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The transition between elementary and middle school mathematics has been linked to declines in motivation, achievement and engagement, which may relate to curricular and pedagogical discontinuities (Gueudet et al., 2016; Kaur et al., 2022).We elucidate these discontinuities by comparing Finnish elementary- and middle-school teachers' views about instruction practices, social learning environment, and teacher profession using TIMSS 2023 data (Fishbein et al., 2025).

We combined data from teacher questionnaires of grade 4 (n=423) and grade 8 (n=400) teachers, identified 99 items with identical questions and compared the means of the two groups by independent samples t-tests. Due to the large number of tests, we only consider differences at significance level $p<0.001$ of which there were 46. In the second phase of the analysis, we analyzed these 46 items qualitatively to identify common themes.

Based on the surveyed teachers' views, the following differences between elementary and middle-school emerged. In instructional practices, use of digital devices and homework differed the most, organizing mathematics instruction somewhat, whereas teachers' assessment practices content contextualization were similar. Regarding the social learning environment, parents', peers' and teachers' achievement expectations varied, factors limiting teaching differed somewhat, and safety and behavior were similar. Related to the teacher profession, work-load and pressure experience differed whereas work satisfaction and professional development activity were similar. Details about the categories will be given in the presentation.

The identified differences between elementary- and middle school teachers are related to at least some of the curricular and pedagogical discontinuities across the transition. We will utilize the results in future design research on supporting students' transition in mathematics.

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PILOTING THE IMPLEMENTATION OF MENTAL STARTER INTERVENTION IN THE NIGERIAN CONTEXT

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Performance in Nigerian public primary schools is critically below curricular expectations, with many pupils graduating without the necessary foundational numeracy skills. Studies have linked this poor pupils' performance to inadequate preparation of primary school teachers. It is this situation that motivates this study to focus on an intervention to support pre-service teachers' competence in the teaching of mental mathematics. Mental mathematics involve helping learners to move from unit counting to more fluent and strategic working with number. For instance, to calculate 278 + 309, one can strategically rewrite the sum as 287 + 300. This can fluently be answered as 587. This cognitive skill is fundamental to developing mathematical proficiency and fostering problem-solving abilities in young learners. We draw on the Mental Starter Assessment Project (MSAP) developed in the South African context. Each unit of MSAP include eight structured starters lessons, with guided tasks and individual practice. Thus, the study is guided by the following research question: How does the implementation of structured mental starter lessons enhance pre-service teachers' professional learning within the Nigerian context?

The study adopts Strom and Viesca's (2023) teacher learning-practice framework, emphasising a shift from "being" to "becoming," where teacher learning is viewed as an ongoing, emergent, and non-linear process of transformation rather than a fixed state. Twelve pre-service teachers (PSTs) participated in the study. Each PST implemented two instructional units—jump strategy and rounding and adjusting—over a six-week period. Lesson Starters (LS) 3 in each unit was observed and followed by structured feedback, with a subsequent observation conducted during LS 7. The study reports initial findings on LS 3, analysed across implementation of strategies, responsiveness to learners' thinking, adherence to the teacher guide, and pacing. Preliminary findings from the observation of LS 3 revealed that the PSTs have weak alignment with the teacher guide, limited strategic questioning, and extended pacing. In response, structured feedback was provided to support improved pedagogical practice. The upcoming observation of LS 7 will assess whether PSTs' enactments reflect changes in response to feedback and whether these shifts translate into improved learner outcomes. Their teaching develops non-linearly through enactment, reflection, and feedback, which acts as a space for refining practice.

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RELATIONAL THINKING IN SOLVING NUMBER SENTENCES: A TEXTBOOK ANALYSIS

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Relational thinking is a type of thinking that deals with finding relations between quantities in a number sentence. According to Kızıltoprak & Köse (2017), relational thinking is mostly concerned with finding or examining the given quantities rather than knowing the result of operations. Studies on students' conceptions of equivalence show that exposure to non-standard number sentences is critical for fostering relational understanding (Singh & Kosko, 2017). Relational thinking therefore depends not only on cognitive readiness but also on structured opportunities to engage with equations that require structural comparison.

This study examines relational thinking as a curricular affordance embedded in elementary mathematics textbooks. A content analysis was conducted on two Grade 1–6 textbook series. Number sentences were classified according to their relational potential: basic operational, flexible operational, basic relational, and full relational (structures requiring compensatory reasoning or involving two unknowns), drawing from the continuum proposed by Singh and Kosko (2017).

Findings indicate that operational structures overwhelmingly dominate across grade levels. Most textbook exercises follow the traditional format $a + b = c$, reinforcing a computational view of equivalence. Although relational-type equations occasionally appear, they are not systematically emphasized. The predominance of operational forms may limit opportunities for students to develop a structural conception of equivalence. Furthermore, no clear progression in relational complexity was observed from Grade 1 to Grade 6, suggesting limited curricular scaffolding toward more advanced relational reasoning. The findings therefore raise a central issue for mathematics education: relational thinking may not fail to develop because students are incapable, but because curricular materials insufficiently demand structural reasoning.

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PRE-SERVICE MATHEMATICS TEACHERS' MOVES FOR SUPPORTING STUDENT REASONING

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Teacher actions that support students' reasoning are critical for the conceptual understanding of mathematics. Different frameworks have been developed to evaluate how teachers and teacher candidates promote mathematical reasoning. For instance, Ellis et al. (2019) developed the "Teacher Moves for Supporting Student Reasoning (TMSSR)" framework consisting of four categories (Eliciting, Responding, Facilitating, Extending) to analyze teacher's instructional moves. Each category includes low and high potential moves in terms of supporting students' reasoning. In this study, we aimed to reveal the moves of two teacher candidates in terms of supporting student reasoning via examining their two teaching practices on different subjects within the scope of the Mathematics Teaching Practice course. The data sources were teacher candidates' lesson plans, audio recordings of the lessons, self-evaluation reports, and the evaluations of the mentors. Four researchers coded the transcribed lesson recordings individually and collectively according to the TMSSR framework (Ellis et al., 2019).

Preliminary findings showed that the teacher candidates mostly have response- and guidance-oriented moves during their teaching practices, while giving less attention to deepen students' mathematical reasoning. Further, teacher candidates exhibited low-potential moves such as eliciting answers, validating correct answers, cueing, and providing information. The teacher candidates' moves were mostly in the "Responding to students' reasoning" category which might be due to the introductory nature of the lesson topics, time constraints, and the tendency to stick to the lesson plans. The findings inform implications for mathematics teacher education. Analyzing teacher moves demonstrated that teachers' tone of voice and facial expressions are important and it requires the joint use of multimodal data. Furthermore, the cultural background of the teachers should not be neglected. In some cases, it was difficult to decide on the coding, and an intermediate coding category was needed. For instance, when responding to students' answers, some teacher candidates' moves included both the correction of the students' wrong answers and the emphasis of the underlying reason, which does not exactly match with any code in the framework. Further results will be discussed during the presentation.

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AFFORDANCES IN DIFFERENT FORMATS: WHAT DO YOUNG STUDENTS USE AND WHY?

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Today, students are increasingly encountering knowledge assessments in digital formats, and the digital format has in many places replaced the paper-and-pencil format. However, the affordances and constraints of the format alter students' opportunities to demonstrate mathematical knowledge and skills, and equivalence in results does not necessarily imply equivalence in the processes used by students (Grapin & Sayac, 2022). The affordance theory aims to capture such affordances and constraints (e.g., Gibson, 1979). The present pilot study aims to explore what Swedish students, aged 9 to 10, experience as an affordance (or constraint) in the paper and digital formats, and how they make use of those affordances when solving mathematical tasks. Data generation consists of interviews with and video recordings of seven students in a lab setting. The students worked alone on four mathematical tasks, choosing the format for each. The tasks were selected from the Swedish national tests for grade 3, and an interview guide was developed based on them. The data then consists of students' solutions in their chosen format and the reason they chose that format. The deductive analysis was conducted by examining what affordances the students considered the different formats offered and, based on this, drawing conclusions about whether the formats gave the students different opportunities to solve the tasks (Komatsu & Jones, 2020). The preliminary results show that students identify different affordances in the two formats, such as "easier to erase on a computer", or "easier to calculate using an algorithm on paper". Many students find digital formats suitable for tasks like selecting options or writing short answers. However, for problem-solving tasks requiring presenting solutions, paper offers better opportunities to work and communicate. Hence, the students express that their ability to demonstrate knowledge varies across formats. One implication is that when assessing students, the chosen format impacts the validity (e.g., Grapin & Sayac, 2022) as well as the possibility for students to demonstrate their knowledge fairly.

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MATHEMATICS ACHIEVEMENT ACROSS SCHOOL SYSTEMS IN ONE CHILEAN REGION: A MOTIVATOR FOR CHANGE

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System-level difficulties in mathematics learning exist in Chile, particularly for the most economically and culturally disadvantaged (Gónzalez-Pizarro et al. 2024), with the COVID-19 pandemic resulting in many students having little face-to-face teaching for almost two years (Hofflinger et al., 2024). This study examines the performance of each school system—i.e., public (PU), private-subsidised (PS) and private-paid (PP) — in one Chilean region, overall and by topic, for the 2022 (post COVID-19) 1.5-hour Years 4 and 10 national mathematics tests. It compares average school performance (by topic) and high/low performance by system. Research questions: (i) What was the performance on the 2022 Year 4 and 10 national mathematics tests, overall and by topic, for each system? (ii) How did high/low school performance compare for each system? Publicly available summary data for each school (Year 4, n=500; Year 10, n=163), reporting a score of 1-10 for each topic, was analysed to give (a) *average school score* (i.e., broad overview of performance), (b) *highest average school score* (and percentage of schools with score) to compare results for highest performing schools by system, (c) *lowest average school score* (and percentage of schools with score), to explore whether being a 'low performing school' is similar for each system and (d) *percentage of schools with average scores* greater than 5, (/10) overall, and by topic. The PP system performed better in every topic, suggesting a systemic issue. However, highest performing Year 4 schools performed similarly, regardless of system, but there are few schools in this group. Lowest performing schools in the PP system scored higher, than *lowest average school score* in PU and PS systems. However, the highest performing school in PU system at Year 4 performed similarly (i.e., average school score) to PP system, so some schools do perform differently to what might be expected based on school system. But having the *lowest average school score* (5) in PP system is different to being the lowest performing PU (2) or PS (2.5) school. Further results will be discussed in the presentation.

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EXPLORING ADAPTIVE SUPPORT THROUGH AN INSTRUCTED AI ASSISTANT FOR MATHEMATICAL MODELLING

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Mathematical modelling is cognitively demanding, with barriers arising at various stages of the modelling process (Niss & Blum, 2020). To overcome these barriers while preserving the learner's autonomy, adaptive support is essential (e.g. Stender & Kaiser, 2015). Generative artificial intelligence (genAI), with its interactive nature and capacity for immediate feedback, has the potential to assist learners in solving mathematical problems (Pepin et al., 2025). This raises a central question: Can genAI be applied to mathematical modelling in a way that provides effective, adaptive support?

Against this background, the study's specific aim is to examine how a modelling-specific AI assistant supports pre-service teachers during mathematical modelling processes. To this end, an AI assistant was developed, which is designed to act as a mentor following theoretical principles for supporting modelling processes, such as strategic assistance. Five groups of pre-service teachers (consisting of two to three members each) worked on modelling problems with the modelling-specific AI assistant. The chat histories and video recordings of their work processes were then analysed using qualitative content analysis.

The preliminary findings show that the AI assistant provided multiple forms of support. On the one hand, the AI assistant reliably initiated the discourse by enquiring about the group's particular challenges and the current status of their work. It provided general strategic support related to mathematical modelling (e.g. encouraging the formulation of assumptions) and explicitly highlighted its proneness to errors to enhance the user's awareness. Conversely, an inherent tendency to offer detailed information at times overrode the intended scaffolding approach (despite system-level instructions), thereby shifting the interaction to content-related support.

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REVERSING THE DIRECTION OF LOGICAL IMPLICATION IN A GEOMETRIC EXPLORATION TASK

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Exploration tasks, which require students to identify all mathematical objects satisfying a given property, can bring the structure of logical implication to the forefront. Such tasks require students to prove p⇒q while also considering q⇒p to ensure completeness. This approach is expected to clarify how students understand assumptions and conclusions in proofs and can contribute to improving students' reasoning and proof activities in classroom instruction. Building on the one-directional reasoning framework of Dawkins & Norton (2022), this study develops a framework that incorporates both directions of necessity and sufficiency and analyzes how a pair of high school students manipulated and reversed the direction of implication while working on a geometric exploration task. For this exploration, designed to make both assumptions and conclusions in proofs salient, students constructed equilateral triangles externally on two sides AB and AC of a triangle △ABC (△ABD and △ACE), ensuring they did not overlap with △ABC. The given property q was that the four points D, B, C, E are concyclic. Students were asked to determine all possible shapes of △ABC (property p) such that q holds. Initially, the students reasoned in the direction of p⇒q, generating candidate triangles and attempting to verify concyclic deductively. However, during trial-and-error exploration, one student suggested drawing a circle and placing the four points on it. This marked a turning point: instead of deriving q from candidate p, the students instantiated q diagrammatically and explored which triangles satisfied this assumption. Their reasoning thus reversed from p⇒q to q⇒p. The reversal appears to have been motivated by a desire for exhaustiveness. Simply listing familiar triangle types did not guarantee all solutions were identified, whereas assuming concyclic constrained the search space and enabled a more systematic exploration. The diagram acted as a mediating representation, facilitating the shift in reasoning direction. While both directions of implication emerged during the activity, they were not explicitly integrated into a coherent necessary-and-sufficient structure. Arguments based on the assumption of q were not fully re-examined to confirm that each identified p indeed implies q. This case suggests that, although the reversal of implication can arise spontaneously in exploration, bidirectional reasoning does not automatically lead to logical integration of both directions.

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PROPORTIONAL REASONING AS COGNITIVE REGULATION: A MULTIPLE-CASE STUDY WITH SECONDARY STUDENTS WITH ATTENTION DIFFICULTIES

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Proportional reasoning (PR) is frequently derailed by the over-use of additive strategies and the overgeneralization of proportional models to non-proportional situations. This literature distinguishes the transition from additive to multiplicative thinking and warns about the 'illusion of proportionality' (Tourniaire & Pulos, 1985). Inhibition and working-memory demands are implicated in PR, where learners must suppress additive heuristics and stabilise multiplicative structure (Cragg & Gilmore, 2014). We adopt the claim that PR operates as cognitive regulation, coordinating executive function (EF) and metacognitive monitoring during problem-solving. The research questions are: (1) how do attention and EF relate to students' strategy use? (2) how does graduated scaffolding influence coherence, transfer, and correction of overgeneralized proportionality? This multiple-case study followed three secondary students (13–15) with attention difficulties as they solved proportional-reasoning tasks under a five-level scaffolding sequence (unsupported → representational scaffolds → invariance prompts → strategic cues → worked examples). Strategy use (additive/mixed/multiplicative/overgeneralized) and coherence (0–3) were coded. Initially, the findings showed that all relied on additive/mixed strategies; two shifted to multiplicative after Level-1 scaffolds, and the third after Level-3 prompts, with coherence rising from 0–1 to 2–3; two students showed transfer. Attention lapses aligned with additive regression, while externalising invariants appeared to lower working-memory demands; on a non-proportional lure, all three first overgeneralized, then two corrected after invariance prompts. We conclude that proportional reasoning functioned as a regulatory mechanism: representational scaffolds reduced EF load and, together with brief metacognitive cues, supported inhibition of impulsive additive responses; instruction should cue invariance early, fade scaffolds strategically, and use short reappraisal prompts, with future work scaling sample size, adding controlled EF measures, and contrasting dynamic vs. static representations.

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AN INTERDISCIPLINARY APPROACH TO SUPPORT STUDENTS' CONCEPTUALIZATIONS ON LINEAR FUNCTIONAL RELATIONSHIPS

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Mathematical and scientific thinking are inherently interconnected ways of understanding and explaining the world around us. However, students struggle to transfer and apply mathematical ideas, concepts, and formulas to unfamiliar situations in physics (Michelsen, 2006). Meaningful connections between mathematics and physics can be strengthened through a quantitative literacy framework in relation to quantitative reasoning (Brahmia et al., 2023). We further argue that an interdisciplinary approach can be beneficial for connecting ideas across two disciplines by bringing kinematics quantities and functions together.

In this study, we focus on the following research question: How do students reason on linear functional relationships within an instruction involving interdisciplinary mathematics and physics tasks focusing on quantitative literacy? To focus on the research question, we developed design-based research in which the first author conducted four classroom teaching experiment sessions. The participants of the study were 10 science high school students. In all the teaching sessions, the students discussed examples of the linear functional relationship in the context of position-time and velocity-time relationships.

Results indicated that during the initial session of the classroom teaching experiment, none of the students were able to represent the position–time and velocity–time relationships as functions. On the contrary, by the fourth session, students could explain why the relationship between kinematics quantities indicates a function while writing the function rule and explaining where the constant term and the coefficient in that rule come from. Based on the results, we claim that an interdisciplinary approach grounded in quantitative literacy helped elaborate on students' conceptualization of linear functional relationships. Potential implications and directions for subsequent research will be discussed in the oral communication.

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HIGH-LEVERAGE NOTICING: FROM STUDENT CONTRIBUTIONS TO PEDAGOGICAL OPPORTUNITIES

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Teaching mathematics requires teachers to navigate a continuous flow of student contributions. Yet, not all contributions hold the same pedagogical potential. Professional noticing (PN) conceptualises teachers' work as attending to, interpreting, and deciding how to respond to students' mathematical thinking (Jacobs, Lamb, & Philipp, 2010). However, distinguishing which student thinking are pedagogically worth pursuing remains a practical and analytic challenge in classroom settings.

The High-Leverage Noticing (HLN) Model addresses this challenge by framing noticing as identifying moments in which student contributions hold pedagogical potential. Drawing on work that characterises certain instances of student thinking as Mathematically Significant Pedagogical Opportunities (MOST) (Leatham et al., 2015), the model specifies how such criteria can inform each component of the noticing triad. In the HLN Model, attending is oriented toward identifying the mathematical point embedded in student contributions. Interpreting involves examining the mathematical centrality and developmental accessibility. Deciding conciders whether such moments create intellectual need within the flow of instruction and advance understanding.

The model was implemented in a practicum module with eight preservice middle school mathematics teachers. Reflective journals, lesson plans, and classroom episodes were analysed to examine how structuring noticing influenced engagement with student mathematical thinking. Findings indicate a qualitative shift in preservice teachers' noticing skills. Although attending statements decreased overall, this reflected greater selectivity rather than reduced engagement. Participants increasingly focused on contributions holding pedagogical potential and sustained their analysis across attending, interpreting, and deciding. By structuring PN through MOST, the HLN Model reorganises noticing by foregrounding an intentional stance toward student contributions that meet these criteria.

Acknowledgement

This study was supported by TÜBİTAK-BİDEB (2214-A International Research Fellowship Programme).

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FEELING AND MAKING SENSE OF QUADRATIC FUNCTION THROUGH MODELLING FREE FALL

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Introduction: This study analyses how high school students construct meaning from free fall, examining how the sensorimotor experience fosters the perception of nonlinearity, generating the need for mathematical modelling using Tracker software.

Theoretical Framework: We articulated Objectification Theory (Radford, 2023) for the phenomenological dimension; Anthropological Didactics Theory (Chevellard, 1999) for the reconstruction of mathematical organizations; and the Theory of Registers of Semiotics Representation (Duval, 2006) for cognitive conversion.

Methodology: A qualitative and interpretive approach was used with eight students working in small teams. Data included video recordings, written productions, and Tracker files. The analysis focused on identifying gestures and verbal expressions that demonstrate objectification processes and register conversions.

Results and Discussion: Sensorimotor experience focused on "feeling" rapid acceleration. When linear fitting failed, to map the curvature, students consciously switched to a quadratic fit in Tracker, recognizing the functional necessity (Chevallard, 1999) of a t^2term to match their physical observation. Video analysis showed this conversion facilitated attributing physical meaning to the acceleration parameter (g) within the symbolic model.

Conclusions: Meaning-making emerges from articulating sensory experience with symbolic representations. Bodily interaction fosters awareness of nonlinearity, while digital modelling allows for progressive objectification in formal mathematical terms. Physically, free fall provides a robust context where position depends quadratically on time ($y \propto t^2$); therefore, digital modelling allows students to move from "feeling" motion to formalizing the quadratic nature of acceleration.

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CO-DESIGNING A GEOGEBRA-BASED ADAPTIVE LEARNING ENVIRONMENT FOR LINEAR FUNCTIONS

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Despite their increasing use, digital mathematics learning environments (DMLEs) often rely on a one-size-fits-all approach that overlooks inherent learner heterogeneity (Awang et al., 2024). Adaptive learning addresses this challenge by tailoring, for instance, task difficulty and scaffolding strategies to evolving student needs, making it a promising approach for future mathematics learning (Engelbrecht & Borba, 2024). However, to ensure that DMLEs actually reflect classroom realities requires involving pedagogical agents, such as teachers, in their development. Co-design offers a promising solution by directly embedding pedagogical expertise into adaptive DMLE design (Voogt et al., 2015). In our case study, we utilize the five iterative phases of the ADDIE (*analysis*, *design*, *development*, *implementation*, *evaluation*) model to co-design a GeoGebra-based adaptive DMLE on linear functions with one Austrian in-service secondary mathematics teacher experienced in DMLE development. Audio recordings, field notes, and material prototypes from four co-design sessions were analysed using reflexive thematic analysis to link ADDIE phases with practical design rationales. Results demonstrate how the teacher's pedagogical expertise grounded the adaptive DMLE in classroom realities. Going beyond theoretical knowledge, the teacher shaped the design by setting time constraints, focusing the mathematical scope on slope and intercept, and defining adaptive difficulty and scaffolding strategies to target common student misconceptions. A subsequent teacher-led pilot refined the prototype's linguistic precision and usability and ultimately yielded a practice-informed strategy for future classroom evaluations. Practically, this study contributes a prototype and concrete adaptive specifications for a classroom-ready adaptive DMLE on linear functions. For the research community, it offers a transparent, replicable account of structured collaboration, illustrating how bridging theoretical design with everyday classroom realities yields more effective educational technology.

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HOW MATHEMATICS TEACHER EDUCATORS RELATE PRIOR EXPERIENCES TO THEIR EXPERTISE

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Research on mathematics teacher educators (MTEs) has expanded significantly in recent years. Recent scholarship conceptualises MTE work through the lens of expertise, understood as what MTEs do and why when educating teachers, with particular attention to their pedagogical communication (Barbosa, 2025). If MTE expertise is distinctive, understanding how it is constituted matters. Prior experiences may be consequential, yet how MTEs connect these experiences to the expertise they enact remains underexplored (Goos, 2025). This study asks: How do MTEs relate their prior experiences to the expertise they enact in mathematics teacher education?

Using an interpretivist qualitative design, I conducted semi-structured interviews with four MTEs working across different course types in an undergraduate mathematics teacher education programme. Thematic analysis of the interview transcripts identified five themes: experiences in school settings, academic trajectory and research, professional experience as an MTE, professional discussions, and literature and material selection. These themes depict a network linking past and present practices: school experiences serve as contrastive or inspirational lenses; academic training structures sequencing and criteria; accumulated professional experience refines sensitivity; peer discussions provide resources and intersubjective support; and literature guide task design and pedagogical reorganisation.

I propose two constructs to interpret the findings. Biographical-disciplinary capital synthesises school and academic experiences into a repertoire that anchors the initial constitution of expertise. Sociomaterial mediation captures how expertise is updated through professional practice, peer discussions, and engagement with artifacts and literature. These dimensions interact dynamically and recursively, suggesting that MTE expertise is continuously constituted and reorganised. The findings indicate that supporting MTE development requires making prior experiences explicit. Future research should compare different forms of biographical-disciplinary capital and sociomaterial mediation across settings to refine these constructs.

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HOW DO NORWEGIAN PRE-SERVICE MATHEMATICS TEACHERS ENACT HUMAN RIGHTS RESPONSIBILITIES?

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In the context of global challenges, injustice and threats to democracy, it is important that human rights values to be included in mathematics teaching and learning. Norway provides an interesting case, since the curriculum explicitly requires such values – democratic participation, human dignity and ethical awareness – to be incorporated into all subjects, including mathematics. When these values are built into teaching, mathematics can be more meaningful for learners and youth can contribute to a socially just democratic society. The question then arises of how pre-service teachers (PSTs) understand these values and incorporate them into their teaching. We used a participatory design to first understand the perspectives of youth and then those of PSTs. We applied Robinson et al.'s (2020) framework in which educators' *responsibilities* are categorised in terms of: knowledge and values (learning about human rights); attitude and environment (to create an environment for teaching through human rights values); and agency and action (to educate for human rights). Our research question for this OC is: how do PSTs express their responsibilities about, through and for human rights in response to the perspectives of youth.

16 lower-secondary students in Oslo discussed mathematics and human rights over one year. They produced artefacts (e.g., artwork, a rap song, film) expressing their ideas with the explicit purpose of sharing these artefacts with PSTs. Next, PSTs at a Norwegian university participated in focus groups and activities in their mathematics teaching courses to: (1) discuss their ideas about mathematics and human rights; (2) view and discuss the artefacts; (3) design mathematics teaching activities in response to the artefacts. Our analysis of data from one group with four PSTs shows the artefacts prompt shifts in their thinking, with comments such as: "people are basically just portrayed as numbers, that they don't have any significance", and the real world "becomes so overwhelming that it spills over and […] blinds the person". The artefacts also influence PSTs' design of mathematics teaching activities, e.g. by integrating poverty, wealth tax, or media statistics, to include human rights values in mathematics.

Acknowledgements: ViMTE project funded by Research Council of Norway, additional team members: T. M. Foyn, K. H. Hauge, B. Herbel-Eisenmann, S. Kacerja, M. Loeb.

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STUDENTS' VIEW OF MATHEMATICAL PROBLEMS IN THE CONTEXT OF DESIGNING A MATHEMATICAL ESCAPE ROOM

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Research highlights a gap between experts' and students' views of mathematical problems. Experts characterize problem solving as involving the absence of a ready-made procedure and the activation of productive thinking (Perkins, 2000), whereas students often see a problem as a story with numbers to combine to obtain an answer, paying little attention to context (Di Martino, 2018; Lesh & Doerr, 2003). This view tends to become increasingly stereotyped over schooling (Di Martino, 2018).

Drawing on the "view of mathematics" component of the attitude construct (Di Martino & Zan, 2010), we investigate upper secondary students' views of mathematical problems within a project in which they collaboratively designed a mathematical escape room for lower secondary students. We address two questions: (RQ1) What are students' views at the beginning of the project? (RQ2) Does designing a mathematical escape room foster greater attention to context? If so, how? To answer RQ1, we conducted an inductive qualitative content analysis of students' responses to an open-ended prompt; many proposed stereotyped problems. Regarding RQ2, analysis of the designed escape rooms showed that such stereotypes largely persisted—unsurprising, given the limited scope of the intervention. However, preliminary findings from a similar project including an explicit reflection on narrative–problem coherence suggest increased attention to context. Designing a mathematical escape room thus appears promising, provided that attention to context is explicitly discussed with students.

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INTERACTIONS OF MATHEMATICAL MODELLING AND COMPUTATIONAL THINKING IN DATA-BASED PROJECTS

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Mathematical modelling is the process of translating problems from a given context into mathematics. Similarly, computational thinking (CT) refers to thinking processes needed for translating problems (and solutions) from a given context into the domain of computer science. Therefore, CT and mathematical modelling share practices, such as generalising from special cases (cf. Kallia et al., 2021). However, their interaction in mathematics learning remains underexplored. This study addresses the following research question: Which CT practices appear in data-based modelling projects of upper secondary school students, and how are they deployed in relation to mathematical modelling? Data were collected during a mathematical modelling week with seven data-based projects involving 40 upper secondary school students. Groups documented their modelling processes through computational essays (Odden et al., 2023), combining programming code and explanatory text. These essays were deductively coded using qualitative content analysis based on the CT model given by Kallia et al. (2021) to reconstruct deployed CT practices. Preliminary results highlight two CT practices mainly deployed by students: *(1) algorithmic thinking* to construct computational models as implementations of mathematical models, and *(2) generalisation* to construct adaptable and transferable solutions. Additionally, mutual influence of modelling and CT was observed in learners' transitions between domains: they often transitioned from computational results to mathematical models by visualising the results of the computational model with a mathematical representation. These visualisations were used to evaluate intermediate results in the mathematical domain rather than the real context. Interpretation of these results is limited, as they rely on modelling artefacts rather than observed processes. Overall, the preliminary results suggest that CT and mathematical modelling are supportive of each other, due to the productive back-and-forth between algorithmic implementation and mathematical reasoning.

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DO STUDENTS' ESTIMATES AND USE OF THE SAMPLE PROPORTION IMPROVE AS GRADE LEVEL PROGRESSES THROUGH SECONDARY SCHOOL?

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Understanding sampling is key to statistical reasoning; in particular, the sample proportion relies on the sampling process to infer information about a population (Ben-Zvi et al., 2015). However, the literature reflects difficulties associated with its understanding (Batanero et al., 2020). In order to complement previous research, the aim of this study is to analyse the responses of 104 Spanish students (aged 12 to 15) in a task adapted from the literature that requires the estimation of the sample proportion and its application in a resampling context. We used the content analysis (Neuendorf, 2018) to identify and classify the students' answers. Regarding the population proportion estimate, 35.6% of students apply proportional reasoning. Of those students, 34.3% provide the correct estimate. One quarter of students (25%) provide an estimate without a strategy, and of these, only 7.1% provide the correct value. The remainder apply additive reasoning (12.5%), respond imprecisely (12.5%), state they are unable to provide an estimate (1%), or do not respond (13.5%). In the total population estimate, only 6.7% apply proportional reasoning, with 42.9% of their responses being correct; meanwhile, the use of strategies involving additive reasoning increases (46.2%), not using a strategy (27.9%), imprecise responses (8.7%), or non-response (10.6%). Correct answers were also scarce when calculating an expected value using the sample proportion. Our results highlight the need to reinforce the understanding of sampling from an early age, considering proposals that promote different levels of algebraic reasoning in a stochastic sense (i.e., Burgos et al., 2022).

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NUANCED TEACHING AND TEACHING PERSPECTIVES: REFINING OUR LENSES FOR OBSERVING CHANGE

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In our work offering professional development (PD) intended to broaden and improve mathematics graduate teaching assistants' (MGTA) teaching practices (Pilgrim et al., 2025), we aimed to understand whether and how MGTAs' teaching practices changed over time. We videorecorded various lessons that the MGTAs facilitated six times over the course of two years while they participated in a PD program that is focused on student engagement in active learning and group problem solving. Initially, we solely used observation tools to track change (e.g., Toolkit for Assessing Mathematics Instruction (TAMI), Hayward et al., 2017). However, these tools did not offer enough insight for us to fully understand the nuanced changes the MGTAs were making to their teaching practices. We then turned our attention to additional data sources (interviews, artifacts and reflections from PD seminars, post-observation debriefs) and a different lens – the Theory of Planned Behaviour (TPB, Azjen, 1991), which explores attitudes, intentions, norms, perceived behavioural control, and resulting behaviours. This combination of theoretical framework and additional data allowed us to notice the oftentimes nuanced changes in how the MGTAs thought about and aimed to teach. It also allowed us to understand that the MGTAs were persistent in and dedicated to their growth in teaching. One participant, Ayla, is an example of the deeper understanding we were able to achieve. Close to the end of the PD program, we observed Ayla in her classroom and saw that students were not engaging with each other, despite Ayla's prompts. In exploring her journals, interviews, and debriefs through TPB, we came to understand that she had persevered in her beliefs and attitudes toward supporting students engaging with each other and that the changes she was making were subtle yet indicative of profound change. In this O.C., we share further examples of nuanced changes that we observed in MGTAs' classrooms through deeper analysis with TPB.

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A DISCIPLINARY DIVIDE: AN ANALYSIS OF K-12 EDUCATOR ATTITUDES TOWARD INTEGRATED STEM

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Shifting STEM learning experiences from siloed content explorations to interdisciplinary integration requires understanding the affordances and challenges in schools. English (2016) indicates that meaningful STEM integration requires "boundary-crossing" where disciplines are mutually reinforcing and Stohlmann et al. (2012) argue that successful integration hinges on teachers' perceptions of feasibly integrating content. As such, this study examined teachers' perceptions towards the feasibility of integrating STEM content by examining attitudes toward integration between elementary and secondary STEM teachers and how STEM backgrounds influence perceived feasibility. A survey was completed by 50 K-12 educators; 20 were elementary teachers and 30 were secondary STEM teachers. Of the 30 secondary, 14 taught science and 17 taught mathematics and one taught both. The statistical analysis included Mann-Whitney U tests to compare the independent groups.

Findings suggest there may be an integration gap between elementary and secondary mathematics teacher that does not exist between elementary and secondary science teachers. A statistically significant difference was found between elementary and secondary mathematics teachers' perspectives on the feasibility of integration whereas there were no statistically significant differences ($p=0.01$) between elementary and secondary science teachers in the same metrics. This suggests the barrier to integration may not necessarily be between elementary and secondary contexts but may be specifically related to mathematics classrooms.

The results may challenge the assumption that content specialization leads to better STEM integration. While secondary science teachers seem to successfully incorporate other STEM disciplines, including mathematics and data sciences, secondary mathematics teachers believe they face unique structural and pedagogical hurdles that lower their perception on the feasibility of integration. Essentially, mathematics teachers may feel structurally and pedagogically isolated.

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HOW IS MATHEMATICAL REASONING RECONTEXTUALISED IN SWEDISH AND ENGLISH CURRICULUM DOCUMENTS?

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Mathematical reasoning is an aim for student learning in policy documents worldwide, but the concept is often ascribed different meanings (e.g., Reid, 2022). Our overarching project aims to understand how mathematical reasoning is construed and recontextualised within and between discourses in the curriculum chain in Sweden and England. By comparing the curriculum chains in Sweden and in England, we hope to contribute to a general understanding of processes of recontextualization. Here, we compare how curriculum documents in Sweden and England construe reasoning, in relation to three reform frameworks based on mathematics education research. Previous research on conceptualisations of reasoning within research has resulted in categorisation of a range of meanings, for example, generalising, conjecturing, identifying patterns, comparing, justifying, and proving (e.g., Jeannotte & Kieran, 2017). Similarly, we found 39 named types of reasoning across three frameworks: Adding it Up, NCTM Standards, and the KOM-project (Morgan et al., 2024). Five types of reasoning were frequent and common to all frameworks: argument, (logical) chain, explanation, justification, and proof. The data in this study are the current mathematics curriculum documents and official commentary and guidance materials relevant for lower secondary school. Our analyses identify parts of the texts describing processes and objects construed as types of reasoning and compare the results between the two countries. We focus particularly on the *objects* that were given as goals of reasoning processes or named as types of reasoning, for example, argument or explanation. The preliminary results show that while *argument* and *justification* are common in both frameworks and curriculum documents, the curriculum documents also emphasise other aspects. For example, *response/refutation* (Sw. "bemötande") is common in the Swedish curriculum and *generalisation* in the English curriculum. We will continue with more detailed analyses and based on these, discuss possible reasons for differences between the recontextualizations of reasoning in the two countries.

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CONCEPTUALISATION OF A VIGNETTE TEST TO ASSESS MATHEMATICS TEACHERS' LESSON PLANNING SKILLS

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Lesson planning is a complex cognitive process that must align learning objectives, curricular requirements, and learners' prior knowledge. Despite its importance for the teaching profession, (Mathematics) teachers' lesson planning skills (LPS) remain under-researched. To address this research gap, a text vignette test was developed to assess the LPS of lower secondary Mathematics' teachers in Germany. Drawing on Cevikbas et al. (2024), we conceptualise those skills within professional teacher competence as situation-specific, mediating between cognitive dispositions and performance (Blömeke et al., 2022). The test consists of two text vignettes (in randomised order) representing typical planning situations in lower secondary mathematics, and 23 items which are linked to one of the two vignettes and refer to five cognitive demands teachers need to meet during lesson planning: *pedagogical adaptivity*, *clarity of learning objectives*, *structuring the lesson*, *content transformation* and *ICT (= information and communication technology) integration*, adapted from prior research (e.g., König et al., 2021). To assess test quality, the instrument was administered to a subsample of N = 198 pre-service lower secondary Mathematics teachers at the start of the German induction phase as part of an ongoing, large-scale, longitudinal study. Findings indicate a reliable measurement (EAP-Reliability = 0.589, $\alpha = 0.670$), and substantial agreement in double-coded open response items (20% of the data, $\kappa = 0.67$, 95% *CI* [0.54, 0.80]), although some poorly fitting items were identified. A revised version of the test is currently being implemented in the longitudinal study and additional validation is required. Overall, initial findings suggest that the test may be promising to assess LPS and provide insights into the development of these skills, with potential implications for teaching quality.

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TEACHERS' KNOWLEDGE IN GAI-SUPPORTED MATHEMATICS EDUCATION

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The growing use of Generative AI (GAI) tools such as ChatGPT in mathematics education raises new questions about teachers' knowledge. Existing models, such as the Knowledge for Teaching Mathematics with Technology (KTMT; Rocha, 2020), provide robust accounts of how teachers integrate technology into pedagogy and content. However, these models were developed before the advent of GAI and do not fully account for its distinctive demands, such as fluent error or non-determinism.

In this study, we reconsider KTMT in GAI contexts by extending three interrelated knowledge domains: AI Knowledge (AIK), AI Mathematical Knowledge (AIMK), and AI Pedagogical Knowledge (AIPK), capturing respectively teachers' understanding of how chatbots operate, their ability to evaluate AI-generated mathematical content critically, and their capacity to adapt AI output to their teaching and students' learning.

With the aim of examining the mobilization of these domains, a qualitative pilot study was conducted in Portugal, involving four pre-service mathematics teachers (PMTs) in a workshop in which they used ChatGPT to solve a mathematics problem and to design a lesson. Data from discussions, chatbot interactions, and surveys were analysed through episode coding based on the extended model. All three domains emerged and interacted dynamically: PMTs drew on AIK when reasoning about how prompting shapes AI output, on AIMK when evaluating mathematical correctness and identifying conceptual omissions, and on AIPK when revising AI-generated lesson plans according to their own pedagogical principles. These findings offer preliminary evidence for the need to reconsider existing teachers' knowledge models in light of emerging technology and provide initial empirical support for the effectiveness of the proposed extended version of the KTMT model in the GAI context.

Acknowledgement

This study was supported by Portuguese funds through FCT, I.P., within the scope of the project AI-TecTeachers (2024.15124.PEX). The research of ML Bernardi is funded by a PhD fellowship within the framework of the Italian "D.M. n. 118/23"- PNRR, M4, C1, Inv. 4.1 – PhD Project "Tech4Math-Math4STEM" (CUP H91I23000500007).

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MATHEMATICS EDUCATION IN A RURAL SCHOOL IN SOUTHERN BRAZIL

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In Southern Brazil, many students in rural schools come from family farming communities. This study examines how mathematics education (ME) is enacted when schooling explicitly connects curricular content with students' cultural and generational knowledge. Grounded in ethnomathematics (D'Ambrosio, 2018), research on culture and popular knowledge in ME (Knijnik, 2006), and critical/Freirean perspectives, we ask: What is revealed in the pedagogies and practices of a rural school when analysed through a lens attentive to culture, power, and participation? The study is qualitative and ethnographic, conducted in a rural school organized through the pedagogy of alternation (periods at school alternating with periods in family contexts). Participants were 66 students and 8 teachers across disciplines. Data include participant observation (Feb–Jul 2023), field notes, transcribed interviews, school documents, and photographs. Content analysis examined how mathematical knowledge was selected, legitimized, and connected to rural life. Four findings emerged. (1) The school mobilized contextualized mathematics linked to family farming, especially financial decision-making and everyday quantitative reasoning. (2) Mathematics-related activities supported dialogue, intersubjective relations, and critical participation, positioning students as agents who question and justify decisions affecting their communities. (3) Alternation enabled school–family–community integration, creating recurring pathways for community knowledge to enter the classroom and for curricular mathematics to circulate back into practice. (4) Family farming was treated not only economically but also as a locus of sustainability, shaping which problems, data, and criteria counted as "good" solutions. These results indicate that this school format supports ethnomathematical and critical approaches, with implications for task design, teacher education, and institutional structures that connect community knowledge and school mathematics. Internationally, it offers an empirically grounded account of sustainability as a curricular organizer, informing debates on epistemic justice, rurality, and the socio-ecological turn in mathematics education.

Acknowledgment: Grant 2021/11937-0, FAPESP

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ENHANCING NUMBER SENSE THROUGH SUBITIZING AND GROUPITIZING

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Many students struggle with grouping and the multiplicative aspect of place value, key aspects of number sense. This study examines how number sense develops through a teaching sequence based on subitizing and groupitizing. It addresses the following research question: how do students' responses to number sense tasks change, particularly in the ways they structure and coordinate quantities in additive, multiplicative, and equivalence thinking? Number sense is defined as a flexible understanding of numbers and operations beyond procedures (Reys, 1994). This study builds on a developmental hypothesis in which number sense progresses from subitizing and groupitizing to more structured forms. Subitizing and groupitizing are considered as tools for structuring quantities as wholes, enabling the construction of flexible mental representations essential for understanding numeration and supporting number sense development. The participants were eight students aged 8–9 who were struggling with mathematics. A pre- and post-test assessed number sense through additive, multiplicative, and equivalence. Tasks were presented on a screen within a 30-minute session, with limited exposure time to encourage reliance on mental structuring rather than counting one by one. Qualitative analyses of the students' responses focused on correctness, organization of quantities, and coordination of representations. The analysis shows consistent changes in students' number sense across participants. In additive tasks, responses shifted from sequential counting to increasingly structured groupings, reflecting a growing ability to treat quantities as wholes. In multiplicative tasks, students produced structured arrays more often, reflecting emerging coordination of quantities. In equivalence tasks, responses improved but remained fragile. The time-limited nature of the intervention makes it difficult to determine whether these remaining difficulties reflect persistent obstacles or could have been overcome. However, these changes suggest that subitizing and groupitizing appear to underpin students' structuring of quantities across levels of thinking, supporting the development of mental representations essential to number sense and emphasizing the role of visuospatial abilities in numerical understanding (Gunderson et al., 2012).

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THE CONTRIBUTION OF GENERATIVE AI (CHATGPT) TO PRE-SERVICE MATHEMATICS TEACHERS' PROBLEM-POSING SKILLS AND TPACK

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The rapid emergence of generative artificial intelligence (GenAI) has opened new possibilities for supporting teaching practices, particularly in areas where teachers experience difficulty, such as mathematical problem posing (PP). Despite its central role in mathematics education, PP remains a challenging task for many teachers (Cai & Jiang, 2017). This study examines how interaction with ChatGPT supports pre-service mathematics teachers' PP practices and contributes to their technological pedagogical content knowledge.

Fifteen pre-service high school mathematics teachers participated in a preparatory course focused on teaching advanced mathematics. After solving several mathematical problems, participants were asked to generate and refine new problems by interacting with ChatGPT using structured prompts. Data included 30 prompts, corresponding AI responses, and teachers' reflections. A qualitative analysis using constant comparative methods identified key themes in both ChatGPT's suggestions and teachers' responses.

Findings indicate that ChatGPT functioned as a pedagogical partner that supported multiple dimensions of problem posing. Its contributions included improving clarity and mathematical precision, increased cognitive demand, the incorporation of real-world contexts, encouraging inquiry-based thinking, and suggesting differentiation strategies for diverse learners. From a TPACK perspective, the use of ChatGPT supported the integration of technological, pedagogical, and content knowledge by enabling teachers to design richer mathematical tasks and reconsider instructional decisions.

The results suggest that integrating AI tools into teacher education programs can provide meaningful opportunities for developing both pedagogical skills and adaptive expertise in technology-rich environments.

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STUDENTS' FUNCTIONAL THINKING IN GRADE 7

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Previous research has identified functional thinking as a key component relevant to students' mathematical development involving functions, generalising relationships, and expressing them (e.g., Blanton et al., 2019). Functional thinking encompasses progression levels, with the main ones being recursive patterning, corresponding relationships, and covariational relationships, with the latter one as the most sophisticated (e.g., Deng et al., 2025). At the level of covariational relationships, students tend to be more limited in their functional thinking, suggesting a need for tasks beyond pattern-related ones to support the transition to complex levels of functional thinking, where covariational relationships are essential. One way to investigate this progression is to analyse students' arguments with respect to levels of functional thinking, where students' arguments are anchored in different mathematical properties (e.g., Lithner, 2008). Against this background, this study explores students' arguments regarding the progression of functional thinking, with particular emphasis on the sophisticated levels.

The present study aims to present results from semi-structured observations conducted in a lab setting with Swedish grade 7 students (N=17) while working on tasks involving functions, emphasising the covariational relationship. Drawing on transcript data, students' arguments were analysed to identify how mathematical properties related to functional thinking were manifested across different progression levels. The results indicate that the mathematical properties associated with functional thinking emerge at various progression levels. Students' limited generalisations, particularly in pattern-related tasks, indicate challenges in transition beyond specific cases. One implication is that pattern-related tasks alone may be insufficient to support the generalisation of relationships between quantities, meaning that a variety of tasks is needed when assessing students' functional thinking.

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EXPLORING PLACE VALUE REPRESENTATIONS IN MATHEMATICS WORKBOOKS

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Place value is a concept that learners find difficult to understand. The use of visual representations (VR) could assist in developing grade 3 learners' understanding of place value when using workbooks. Visual representations in workbooks are central to developing place-value understanding in mathematics. Workbooks are key curriculum tools used in many countries. The research sought to explore: What are the similarities and differences between the use of visual representations in place value activities in Singaporean and South African workbooks? As a South African primary mathematics teacher educator I wanted to analyse use of representations in our government provided workbooks and chose the Singaporean workbooks for comparative purposes as their learner performance contrasts significantly with South African learners. In international rankings they rank towards the top while South Africa at the bottom.

I used an adapted version of the Visual Representation Framework by Fotakoupoulou and Spiliotopoulou (2008) to analyse the VRs. The categories used to analyse the two workbooks were: a) type of VR, b) relation of VR to content, c) relation of VR to reality, d) function of the VR, e) dimensionality of VR. The study builds on constructivism which suggests that tools and signs are central to facilitating learning (Daniels, 2005). The workbooks are considered the tools, and the place value VRs serve as signs that assist learners in the learning process.

The findings suggest that the South African workbook provides more place-value VRs, particularly, Diene's blocks and flard cards. While the Singaporean workbooks provide fewer VRs particularly, photos, flard cards, and place value charts. Similarly, the VR in both textbooks had an exemplifying function. The adapted VRF by Booysen et al., (2024) has exemplifying (type a), this is a worked example and can be found in South African workbook. While exemplifying (type b) function is a practice example and is the most dominant function in Singaporean workbook. I argue that analysis of the similarities and differences of VR across countries is useful for teacher education because brings the attention to the function of the VR in mediating learning.

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STUDENTS-WITH-GENAI CO-CREATING MATHEMATICAL MODELS: TOWARDS A NEW PEDAGOGICAL PARTNERSHIP?

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The research investigates how student agency mediated by Generative Artificial Intelligence (GenAI) manifests in sociocritical mathematical modeling activities in higher education. The central problem lies in the theoretical gap regarding the active role of GenAI as a non-human agent in the production of mathematical knowledge, particularly in contexts that value critical thinking and social responsibility. The study is grounded in the theoretical framework of humans-with-media (Borba, 2021), which conceptualizes the human-technological collective as the fundamental unit of knowledge production. From this perspective, GenAI is not treated as a mere instrument, but as an agent that participates in, influences, and is influenced by human actions. Additionally, the concept of machine agency, distinguishing between good agency and bad agency, is mobilized to categorize the observed interactions analytically. Methodologically, the research is qualitative in nature and was conducted with engineering students enrolled in a Differential Equations course at a Brazilian federal university during the first semester of 2025. The students developed mathematical models in groups, with explicit authorization and documentation of GenAI use. The results indicate that GenAI exercised good agency by functioning as a cognitive scaffold, simplifying complex phenomena and overcoming operational obstacles, such as code generation and graph construction, thereby expanding the space for students' critical analysis. Conversely, it exhibited bad agency by inducing epistemic dependency and producing mathematically incorrect responses at more advanced levels, requiring students to act as critical "filters," a competence they did not always possess. It is concluded that agency within the humans-with-GenAI collective is necessarily negotiated and conditioned by students' prior knowledge, which transforms the tool from a potential source of misinformation into a formative resource. The study contributes to refining the analytical categories of AI agency in mathematics education. It highlights the need for further investigations into pedagogical strategies that promote critical and formative uses of artificial intelligence.

Acknowledgment to CNPq, funding agency of the Brazilian Government 409575/20224-0.

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RESEARCH CYCLES TO UNDERSTAND HOW STUDENTS LEARN SOLIDS OF REVOLUTION

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This study builds on prior research on students' understanding of integrals and solids of revolution (SoR), which has documented students' reliance on procedural knowledge and difficulties with geometric and conceptual interpretations (e.g., Mofolo-Mbokane et al., 2013). More recent studies have emphasized challenges in coordinating graphical, geometric, and symbolic representations, as well as limitations in students' visuospatial reasoning (e.g., Blanco et al., 2019). The present study proposes a theoretical model of how students may construct SoR and discusses subsequent classroom testing and subsequent improvements of the theoretical model. Grounded in APOS theory, this study investigates students' understanding of SoR using the theoretical model (called a genetic decomposition; GD) as a design tool for instructional activities and interview tasks. The research questions are: 1) how well can the GD model students' constructions of SoR, and 2) how the GD can be refined or improved based on data? The study was conducted in a first-year university calculus course. GD-based activities were designed and implemented. They emphasized geometric reasoning, partitioning, and Riemann sums, as described in the GD. Data were collected through semi-structured interviews with 11 students using think-aloud protocols and then analysed using the GD. Preliminary results indicate that most students were able to construct key components of the GD, particularly in visualizing rotations and correctly representing solids. Compared to earlier research, students demonstrated improved coordination between geometric and symbolic reasoning, suggesting the effectiveness of GD-informed instruction. However, significant difficulties remain. Some students relied on memorized procedures without connecting them to the underlying geometric meaning, showing the need for further research to study rotations about non-standard axes. These findings provide empirical support for the GD as a viable model of student learning of SoR. They also highlight the need for the GD to clearly consider infinitesimal elements and for instructional tasks to explicitly require students to construct and interpret these elements to connect Riemann sums to integrals.

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USING CHATGPT FOR MATHEMATICAL MODELLING: PRE-SERVICE PRIMARY TEACHERS' REFLECTIONS

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The rapid emergence of generative artificial intelligence (AI), particularly ChatGPT, has increased discussions about its role in mathematics education and teacher preparation. While recent research highlights the pedagogical potential and mathematical risks of AI-generated content (Noster et al., 2024; Schorcht et al., 2024), little is known about how pre-service primary mathematics teachers (PSMTs) incorporate ChatGPT into their developing professional roles. This study examines how PSMTs conceptualize ChatGPT as a tool for mathematical modelling (MM) and which competencies they consider necessary for its responsible use. The study involved 19 primary pre-service mathematics teachers in their second or third semester at a German university. During a 13-week Education for Sustainable Development (ESD)-MM seminar, one 90-minute session was dedicated to ChatGPT. During this session, the PSMTs worked in small groups to develop and critically evaluate an ESD-MM task by using ChatGPT. Data include a collaborative in-class presentation and individual written reflections. A qualitative thematic analysis was conducted.

The findings show common acceptance of ChatGPT as support tool, accompanied by explicit concerns. While all participants described ChatGPT as a supportive tool for idea generation, none considered it a substitute for the teacher. While 79% thought it was generally suitable for modelling task development, this approval was dependent on certain conditions. Specifically, 21% questioned its ability to generate authentic MM problems due to its limited openness and modelling depth. Risks frequently mentioned included mathematical discrepancies (84%) and insufficient consideration of learners' context (79%). Participants stressed the importance of strong mathematical knowledge and critical evaluation skills. Additionally, 95% of participants highlighted the importance of didactic competence. ChatGPT is perceived as valuable for inspiration, yet its use is understood to increase—rather than reduce—professional responsibility. Further results will be discussed in detail in the presentation.

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MEDIATING REFLECTION: CRITICAL-FRIEND DIALOGUE AS A METHODOLOGICAL TOOL IN MATHEMATICS TEACHER EDUCATOR SELF-STUDY

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Reflection is widely recognized as central to mathematics teacher educators' (MTEs) professional practice (Beswick & Goos, 2018). In self-study research, it serves as both object and method, yet it is often treated as an individual cognitive activity, leaving under-theorized the dialogic structures that mediate it. This study examines critical-friend dialogue as a methodological tool in MTE self-study, asking: How does critical-friend dialogue function as a mediational structure that reorganizes reflective practice? Drawing on Schuck and Russell (2005), we conceptualize critical friendship not as collegial exchange but as a structured methodological practice. Through questioning and re-voicing, the critical friend surfaces implicit assumptions and opens reflective accounts to challenge. We therefore frame dialogue as mediation, a process through which reflections are articulated, interrogated, and reorganized. Rather than foregrounding learning outcomes, we analyze how the dialogic structure itself operates and which interactional moves enable or constrain criticality. The study adopts a collaborative self-study design involving two MTEs from different countries who each taught a 14-week technology-integrated mathematics teacher education course, one emphasizing digital resource design, the other digital task design. Across the semester, both engaged in systematic written reflections and weekly recorded critical-friend dialogues. The dialogues followed a recurring reflective cycle: one MTE articulated intentions, described enactment, and identified tensions; the critical friend posed probing questions and requested clarification to deepen the reflection. Analysis focused on how dialogic moves mediated reflection, particularly moments of re-voicing intentions and surfacing implicit rationales. Preliminary findings indicate that critical-friend dialogue functioned as a methodological mediator by externalizing tacit reasoning, structuring reflective depth, and revealing tensions in the dual role of the critical friend as both supporter and challenger. The study suggests that the methodological value of critical-friend dialogue emerges from the interactional quality of dialogue.

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WHAT'S LEFT FOR US TO DO? HUMANS' ROLE IN MATHEMATICAL MODELLING IN THE AGE OF GENERATIVE ARTIFICIAL INTELLIGENCE

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This study investigates how Large Language Models (LLMs) engage with mathematical modelling tasks and what their growing capabilities imply for mathematics education. Through an analysis of eight diverse modelling problems, ranging from classical estimation tasks to project-based modelling and regression, we examined whether LLMs can solve such problems independently and which aspects of the modelling process still require human involvement.

Across most problems, LLMs produced solutions that appeared mathematically sound and convincing at first glance. Closer inspection, however, revealed important limitations, including a lack of transparency, unrealistic assumptions, and insufficient attention to contextual factors. While LLMs handled computational components effectively, they often neglected realism and human considerations, resulting in plausible yet flawed solutions. In some cases, particularly when a definitive answer could be found online, LLMs bypassed modelling altogether, as in estimating the container transshipment area of the Port of Hamburg, where the model retrieved the actual area rather than constructing an estimate based on assumptions. Tasks with unclear or multiple objectives also proved challenging. For example, in problems such as optimizing bus stop placement, LLMs handled generic formulations well but struggled when local context was required, at times proposing bus stops in very remote areas with no nearby housing. Extrapolation posed a further difficulty: in a regression task predicting patients' temperature normalization from medication dosage, the LLM produced a reasonable model but failed to comment on uncertainty or the validity of predictions beyond the observed data range, even suggesting that temperatures would normalize within seconds at sufficiently high doses.

These findings have important implications for mathematics education. The availability of powerful AI tools does not diminish the value of teaching modelling but calls for a reconsideration of how modelling is taught and what we expect students to accomplish in real-world contexts. To preserve meaningful engagement, modelling tasks should be designed so that essential steps such as making assumptions, justifying choices, and interpreting results cannot be bypassed. Overall, the results highlight the continued necessity of human judgement throughout the modelling cycle, with LLMs best understood as assistants to human modelers rather than replacements.

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ANALYZING A STEP FUNCTION GRAPHING TASK

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Many studies point out that students lack a complete understanding of the concept of a function, with covariational reasoning being the least developed aspect. This contrasts with the claim that "ideas of continuous covariation are epistemologically necessary for students and teachers to develop useful and robust conceptions of functions" (Thompson & Carlson, 2017, p. 423). Therefore, we have selected a task through which we aim to examine the current state of covariational reasoning among Slovak secondary school students. Inspired by The Parking Fees scenario and task (Ronau et al., 2014), it presents a real-life situation in which students are asked to sketch a step function graph. Our aim was to determine their level of covariational reasoning and identify any other difficulties relating to the concept of a function.

Twenty-four 18-year-old high school students solved nine tasks as part of a review of functions. This task was the first task in the series, and it took them the longest to solve (10–15 minutes). We anticipated the following difficulties: a) choosing the proper scale for the graph axes ($\sum$=2); b) sketching a linear function instead of a step function ($\sum$=6); c) sketching isolated points ($\sum$=10); d) sketching a graph that resembles a staircase (with segments parallel to the y-axis) ($\sum$=2); e) problems with the endpoints of individual segments in a step graph ($\sum$=5). After analysing the students' solutions, we found that our expectations had been met (see the number of students above). Additionally, 19 students had difficulty sketching the interval from 0 to 20 minutes, when parking is free. One student explicitly stated: "I don't know how to sketch the first 20 minutes in a graph.". Four students identified price as the independent variable instead of time, and two had difficulty determining function values at the interval endpoints where the value changes or a jump occurs. These findings support the claim that students have limited ability to work with different function representations when expressing invariant relationships between two co-varying quantities in applied contexts (Thompson & Carlson, 2017). The results highlight the need for targeted pedagogical interventions to improve students' understanding of functions.

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A COMPARISON OF GROUP AND INDIVIDUAL PROBLEM-POSING PERFORMANCES OF PRE-SERVICE ELEMENTARY MATHEMATICS TEACHERS: THE CONTEXT OF PAPER FOLDING

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Problem posing is the process of generating a new problem based on a given context or reformulating an existing one (Stoyanova & Ellerton, 1996). This process functions as an assessment tool by revealing individuals' mathematical understanding and eliciting critical skills such as creativity (English, 1997). Paper-folding activities, which embody geometric principles and have an experiential nature, provide a rich context for problem-posing processes. This study aims to compare the group and individual problem-posing performances of pre-service elementary mathematics teachers within the context of paper folding activities and to collect their views on these problem-posing activities. Conducted within a holistic single-case design, the study involved nine pre-service teachers whose spatial visualization skills were at or above the average level. The participants engaged in paper folding based problem-posing activities both individually and in two groups of four to five members. The problem-posing situations were designed based on the free, semi-structured, and structured classification proposed by Stoyanova and Ellerton (1996), and the generated problems were evaluated using a rubric developed in the context of this research. In addition, individual interviews conducted after the implementation were analyzed to examine the participants' understanding and performances on the types. The results did not indicate a clear overall superiority between group and individual performances. However, according to the problem type higher scores were obtained in the free problem-posing for both individual and group working, whereas in the structured problem posing, group and individual scores were close to each other. The interviews revealed that group working supported the emergence of diverse perspectives and creative ideas, whereas working individually was associated with decision-making and responsibility-taking. Overall, the findings suggest that, in addition to the types of working, factors such as problem type influence problem-posing performance.

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EXPLORING CHILDREN'S USE OF GESTURE TO DESCRIBE GEOMETRIC FEATURES OF A HIDDEN OBJECT

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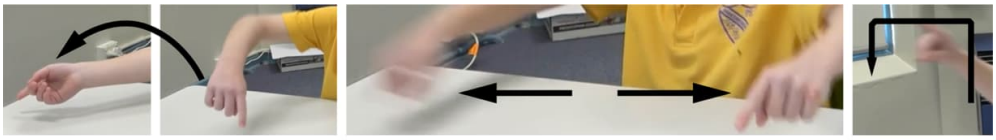
Embodied modes play an important role in children's mathematical learning, assisting in making representational connections along the pathway to abstract thinking (Nahdi et al., 2024). Ehrlich et al. (2006) consider the interplay between gesture and verbal representations in relation to geometric features–noticing children's verbal *and* gestural responses. Theoretical perspectives including Ehrlich et al.'s (2006) geometric feature categories and the van Hiele (1999) model were taken to explored young children's descriptions of an unseen object to answer the research question: *What geometric features are revealed through children's verbal and gestural responses?*

Deductive micro-analysis was employed to explore qualitative data from video-recorded task-based interviews with 106 4-to-6-year-olds. The 'feely bag' task was designed to elicit verbal and gestural responses. The most prevalent mode of communication was verbal, 98 of the 106 children, 39 of these were speech-only. There were 57 children who used gesture, most, 93%, used co-speech gesture, e.g., *'I could feel the block was like [G], and that [G] then I think it was like that [G]'* (Figure 1).

Figure 1: Sequence of co-speech gestures in place of words



1. Start and end position of arch | 2. Draws horizontal sides of block | 3. Draws sides

Speech-only children referred to the *perceptual whole* most often (90%), compared with co-speech gesture children who referred to *perceptual features* (62%) most often. Children's verbal and gesture representations generally aligned to Level 1 or 2 of the Van Hiele model. A distinct difference was noticed at Level 2, with 32 gesturers and only 4 speech-only. Co-speech gesturers seem to communicate 'more' mathematics. Van Hiele's model does not specifically include the interpretation of embodied modes in relation to geometric thinking, therefore further embodied research is needed.

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DIMENSIONS OF TEACHER AGENCY IN THE USE OF MATHEMATICS TEXTBOOKS

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Classrooms are dynamic spaces, with teachers exercising agency as curriculum designers who mediate between official mandates and enacted practice through resources like textbooks. Thus, this study is guided by the following research question: *how does teacher agency manifest in the ways mathematics teachers use textbooks in their instructional practice?* Building on Priestley et al. (2015), teacher agency is the capacity to act, shaped by many factors and sustained from past experiences, projective orientations toward future goals, and a practical-evaluative dimension in which teachers make situated judgments based on available possibilities. This qualitative study conducted online semi-structured interviews with two Brazilian public secondary-school mathematics teachers, selected because their daily textbook access occurs within a tightly policy-regulated context that creates productive tensions for observing teacher agency. By analyzing teachers' discourses, it was possible to observe how the official curriculum is enacted through tightly prescribed routines aligned with standardized testing, thereby shaping teachers' actions. Past experience informs textbook use, as it relates to teachers' prior knowledge and professional trajectories. From both projective and practical-evaluative dimensions, time pressure steers planning and in-the-moment decisions toward test-oriented coverage, contributing to a diminished sense of individual autonomy in textbook use (Carvalho & Amaral, 2025). The analysis showed that standardized assessments position the official curriculum as a structuring force shaping how teachers enact their agency in textbook use. Thus, agency emerges as a negotiated construction rather than an individual trait, as accountability pressures narrow the boundaries of what teachers can imagine and enact beyond performance metrics.

Acknowledgements

Thanks to São Paulo Research Foundation (FAPESP) for the grant #2025/09749-1.

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SEMIOTIC REGISTER USE AND COORDINATION IN MIDDLE SCHOOL STUDENTS' FRACTION PROBLEM SOLVING

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Mathematical understanding relies on the coordination of multiple semiotic registers rather than on the manipulation of symbols alone (Duval, 2006, 2017). According to Duval, mathematical thinking requires transformations within and between representational systems such as symbolic, graphical, numerical, and verbal forms. Fraction concepts, in particular, demand flexible movement across registers, as students must interpret part–whole relations, proportional reasoning, and operational procedures through various representations. Despite extensive evidence that students experience difficulties in transitioning between representations, limited research has examined how register coordination emerges during problem-solving processes. This study therefore investigates semiotic register coordination in middle school students' fraction problem solving across paper-and-pencil and digital contexts. This qualitative action research study was conducted with sixth-grade students attending a public middle school. Following an initial readiness test, a heterogeneous group of ten students was selected, and the implementation was carried out over four weeks in both paper-and-pencil and digital learning environments. In the digital setting, students worked in pairs using tablets equipped with tools such as fraction bars and number lines, while data were collected through think-aloud protocols, screen recordings, and written documents. Content analysis was employed to examine students' use of semiotic registers and the transformations between them. Findings indicate that in paper-and-pencil contexts, students predominantly relied on symbolic procedures and tended to remain within a single register. Register coordination was limited, and transitions between representations were rarely observed unless explicitly prompted. In contrast, the digital environment encouraged more frequent use of visual and graphical registers, as available tools facilitated engagement with multiple representations. However, students often used these representations operationally rather than reflectively, and difficulties emerged when coordinating transformations across registers. Overall, the findings suggest that while digital environments can promote exposure to diverse representations, meaningful register coordination depends on deliberate instructional support that foregrounds the reasons for transitioning between representations, rather than merely providing representational tools.

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MATHEMATICS TEACHER EDUCATORS' EXPERIENCES WITH AND VISIONS FOR AI IN TEACHER EDUCATION

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The rapid diffusion of artificial intelligence (AI) is reshaping how mathematical ideas can be explained, represented, and assessed. Yet research has largely focused on technical affordances and students' AI use, leaving mathematics teacher educators' (MTEs) visions and experiences underexplored. This gap matters because MTEs shape norms of pedagogical reasoning and professional judgment in teacher preparation and influence how AI will be taken up in future classrooms. Accordingly, this research adopts a human-centered AI literacy perspective emphasizing pedagogical purposes, ethical responsibility, and professional agency (UNESCO, 2024). The study examines how MTEs (a) describe their experiences with AI, (b) conceptualize its role in teacher education, (c) identify opportunities, and (d) articulate concerns and conditions for responsible integration. An international qualitative exploratory design used an online open-ended survey with 31 MTEs from 17 countries. Data were analyzed via thematic analysis using a hybrid inductive–deductive approach; a checked subset indicated substantial agreement ($\kappa = .78$). Most participants reported professional AI use (87%), yet most had no structured training (71%), relying on self-learning/peer exchange. AI use focused on preparation and design rather than replacing teaching. MTEs framed AI as (1) a pedagogical assistant, (2) a catalyst for innovation, (3) an object of critical study (AI literacy/critique as content), and (4) a threat/ethical dilemma (bias, profit-driven logics, erosion of human-centered teaching). Opportunities included time efficiency, richer resources, rapid feedback, differentiation, and support for problem posing/solving; some used AI outputs as artifacts for critique to strengthen PSTs' evaluative judgment. Key risks involved overreliance, reduced autonomy and reflective/cognitive skills, ethical issues (privacy, bias, transparency, integrity), and inaccurate outputs, alongside equity/access concerns. In line with UNESCO (2024), AI literacy is treated as a multidimensional competence (conceptual understanding, pedagogical design, ethical awareness, reflective judgment), and participants stressed verifying and contextualizing AI outputs given risks to epistemic trust. Rather than "whether," the key issue is how to integrate AI to augment professional judgment: (a) embed AI literacy systematically; (b) design tasks requiring PSTs to validate and contextualize AI outputs; and (c) develop institutional guidance and professional learning addressing ethics, transparency, and equity. Overall, the study highlights both the promise and the peril of AI, calling for ethically grounded integration that augments rather than replaces human professional judgment.

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MAKING SENSE OF TIME: LEARNING EXPERENCES TO HELP YOUNG LEARNERS CONSTRUCT MATHEMATICAL UNDERSTANDING OF TIME

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Understanding time is challenging for children aged 7-8 as it is abstract and not directly perceptible (Burny et al., 2009). As a result, many struggle to comprehend units of time and how these units relate to a given duration. Children often focus on reading individual clock times rather than understanding the interval between them, and many rely on informal strategies such as "counting numbers" on the clock face. Crossing hour boundaries (e.g., from 9:50 to 10:20) is particularly problematic, as children may subtract smaller numbers from larger ones without considering the hour change. Van de Walle (2020) advocates the use of an empty timeline to support children in solving elapsed-time problems, however, Primary Two students were observed to have difficulty sketching the timeline accurately to find duration. Timelines are often treated as embellishments rather than scaled representations in where equal spacing reflects equal units, hindering children's coherent understanding of duration. This exploratory case study investigates whether a carefully sequenced instructional design can strengthen Primary Two students' understanding of telling time and calculating duration. The intervention comprised three components: (1) students constructed one-hour (60-minute) rectangular paper strips marked at 5-minute intervals to reinforce the equivalence of 1 hour and 60 minutes while connecting the timeline to analogue clock reading; (2) students worked in groups to manipulate analogue clocks to show given start and end times, and use the one-hour (60-minute) paper strips to scaffold timeline drawings to find duration between the two time points (3) students completed individual practice tasks to assess their mastery of finding duration. Qualitative insights are gathered through classroom observations, teachers' reflection and student work samples. Preliminary findings indicate that students shifted from counting labels or performing unstructured numerical subtraction to coordinating clock representations with unit-based reasoning along a timeline. They were able to identify start and end times more reliably, justify their answers using the hour strips and explain durations in terms of "hours and minutes" rather than "what numbers I subtract." Spatial proportionality in their timelines also showed marked improvement. Our findings offer educators ways to enhance student understanding of time and duration.

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GEOMETRIC AND MAP-BASED SPATIAL ABILITIES ACROSS DOMAINS

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Spatial ability is crucial for learning both geometry and cartography. In psychology, object-based mental rotation and self-based spatial perspective taking have been identified as distinct yet related constructs rather than as a single unitary ability (e.g., Hegarty & Waller, 2004). However, prior psychological research has rarely examined how these two spatial abilities manifest across different academic disciplines. The study aimed to examine the extent to which geometry-based mental rotation and map-based spatial perspective taking are mutually explanatory, or whether they represent two distinguishable spatial competencies that may influence one another, which are cores to understanding cross-domain transfer and for informing instructional design.

To address this issue, forty-three university students completed two within-subject tasks: a geometric mental rotation task and a map-based spatial perspective-taking task, with angular disparity varied across multiple rotation angles. Reaction time and accuracy were analyzed using repeated-measures ANOVA. Reaction time results revealed a task effect, with geometric mental rotation eliciting longer response times than the map-based task, as well as a significant interaction between task and angular disparity, indicating a linear increase in angular cost for geometry but a reduced angular sensitivity for maps. Accuracy results showed no overall task difference, indicating comparable correctness across domains, yet revealed a significant interaction with angular disparity. Accuracy in the geometric task decreased as angular disparity increased, reflecting the difficulty of constructing mental images, while orthogonal orientations did not lead to additional declines in accuracy, whereas the map-based task showed an accuracy decrease at 90°, with no clear differences between larger angular disparities, suggesting reorientation processes. Together, these dissociable patterns in speed and accuracy indicate that geometric mental rotation and map-based spatial perspective taking draw on spatial resources but are not mutually explanatory abilities. Extending previous studies, the present findings show that the distinction between geometric and map-based spatial abilities is not attributable solely to angular disparity, but to the structural constraints governing spatial transformation—coordinate-axis alignment in geometry versus the directionality of alignment processes in map-based tasks. From a geometric perspective, instruction may benefit from leveraging coordinate-axis alignment to support reasoning about rotation and transformation.

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CULTIVATING PRE-SERVICE TEACHERS' PERCEIVED BELIEFS REGARDING MEASUREMENT AND GEOMETRY IN SCHOOL MATHEMATICS

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Geometry and measurement are essential strands in school mathematics, fostering logical reasoning and practical application in daily life (NCTM, 2000). Despite their importance, there is relatively little research on pre-service teachers' beliefs about these domains, even though beliefs developed during teacher preparation strongly influence teaching practices. This research investigates the second-year pre-service mathematics teachers' perceived beliefs regarding measurement and geometry in school mathematics. Over a four-month period, the study employed reflective practice through the transformative lesson study incorporating the Open Approach (TLSOA Model), integrated into a 15-week. This process included Co-Plan, in which the Lesson Study (LS) team collaboratively designed problem situations that focus to challenge cognitive conflict and affective interruption; Co-Do, in which participants engaged in problem-solving and shared their ideas; and Co-See, in which participants engaged in reflection. Data were gathered from 27 participants' written reflections on five problem-solving scenarios, critical analysis of problem situations in the textbook, and micro-teaching, with sequential reflections collected at the two-month mark and upon the conclusion of the four-month study. The qualitative data were examined using thematic analysis, guided by a framework of perceived beliefs with a specific focus on Emotional Experiences model (Inprasitha, 2001).

The findings reveal that initial belief systems (AS1) reflected an instrumentalist view emphasizing procedural certainty and teacher-centered transmission. Cognitive and affective interruptions (I), including astonishment, empathetic disruption, and realistic discomfort, subsequently destabilized these foundations. Cognitive evaluative schema (CES) facilitated reflective meaning-making through ambiguity tolerance, intellectual satisfaction, and moral-professional commitment. Ultimately, revised belief systems (AS2) demonstrated three interconnected transformations: repositioning measurement and geometry toward a problem-solving view of mathematics; reframing the teacher's role as a problem-situation designer; and endorsing problem situations that enable students to generate, communicate, and refine their own mathematical ideas.

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PROSPECTIVE MATHEMATICS TEACHERS' LEARNING THROUGH NARRATIVE AND IMAGINATION

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Inquiry-based (IB) teaching is a challenge for teachers to implement particularly if their experience with learning mathematics did not align with it. This paper reports on a pilot study to explore an approach to support prospective secondary mathematics teachers' (SPTs') learning of IB teaching. The aim was to gain insights on how they made sense of IB teaching through creating a narrative based on their imagination teaching a mathematics lesson. Theoretical framing of the study includes: (1) Narrative as a way of representing experience. It helps to make knowledge more engaging, helps us think and do things more effectively, and aids in the process of meaning making. (2) Imagination as the ability to think about what might be possible. It enables students to pick up new ideas, try them out, and find a place for them amidst the things they have already learned (Egan, 2005). (3) Inquiry-based teaching as approaches that support students' individual and collaborative construction of knowledge through inquiry/exploration of mathematics concepts and finding their own path to solutions. Participants were 16 SPTs enrolled in a mathematics education course. They had a 3- or 4-year undergraduate degree in mathematics but no school experience with IB learning. In the course, they were exposed to theories of IB leaning, tasks, and discourse. They explored examples of tasks but not examples of complete IB lessons. The assignment used for this study required them to imagine themselves conducting an IB lesson of a mathematics concept of their choice from their G7-9 curriculum. The guideline included: The lesson must include an introduction, a task they created, their intervention in at least three groups of their students with different experiences while working on the task, and whole-class discussion. The narrative must have a temporal flow as lived and include dialogues. Data sources were the four narratives of the four SPT groups. For analysis, each narrative was categorized based on the structure of the lesson, each category was coded to identify factors of inquiry-based teaching which were compared across the groups. Findings indicated that two groups had fully IB lessons, one was mostly IB, and one was partly teacher-centered and partly IB. The fully and mostly IB groups had appropriate IB lesson introductions**,** tasks, teacher-student interactions and lesson conclusions**.** The partial IB group used teacher-centered intervention/interactions and lesson conclusion directed to what they wanted students to know. The findings suggest that the narrative-imagination approach has potential to help the SPTs to make sense of and develop a holistic image of IB teaching in a way that they could apply in their teaching. But the SPTs whose tasks focused on exploring patterns/relationships of concepts were more successful in imagining IB lessons.

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NONLINEAR DYNAMICS IN EARLY ARITHMETIC DEVELOPMENT: PROBLEM-SOLVING AS A BIFURCATION MECHANISM IN CHILDREN WITH INTELLECTUAL DISABILITIES

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Research on early arithmetic development in children with Intellectual Disabilities (IDs) has primarily relied on linear prediction models, despite evidence that learning may involve qualitative shifts (Silver, 2013). This study examines whether the relationship between problem-solving and arithmetic attainment exhibits nonlinear characteristics. Participants were 64 children with IDs aged 6-9 years attending special and inclusive settings in Greece. Linear and cubic models were compared, followed by cusp catastrophe modelling to test whether problem-solving functions as a bifurcation variable in predicting arithmetic attainment. Age, fine motor skills, developmental benchmarks, and parental attitudes toward mathematics and literacy were specified as asymmetry parameters, given their documented role in early numeracy development (Susperreguy et al., 2020). Cubic models outperformed linear alternatives. Asymmetry parameters significantly shaped the developmental landscape, influencing the likelihood of nonlinear transitions. Qualitative analysis of children's responses across early numeracy tasks (counting, measurement, classification, probability) identified four problem-solving profiles: naïve, transitional, emerging strategic, and advanced. The transitional profile was characterized by instability and strategy switching and aligned with the bifurcation region identified in the quantitative model. This suggests that apparent inconsistency may reflect developmental reorganization rather than deficit. Findings support a nonlinear account of early arithmetic development in children with ID and position problem-solving as a mechanism capable of reorganizing mathematical performance.

ACKNOWLEDGMENT

This research is conducted in the operating framework of the University of Thessaly Innovation, Technology Transfer Unit and Entrepreneurship Center "One Planet Thessaly", under the "University of Thessaly Grants for Scientific Publication Support" action and is funded by the Special Account of Research Grants of the University of Thessaly.

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CULTIVATING SPATIAL ABILITY THROUGH DIMENSIONAL PUZZLES FOR EARLY CHILDHOOD STUDENTS

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Spatial ability represents a critical skill for the cultivation of spatial sense and the facilitation of meaningful mathematics learning. Spatial ability refers to the capacity to comprehend visual-spatial relations among objects (Zhang, 2016). Early childhood learners require systematic cultivation of spatial ability to establish a foundational essential for learning at the primary education level.

This research was designed to analyse spatial ability in early childhood learners through their engagement in solving two- and three-dimensional puzzles. The study employed qualitative research methodology, incorporating video analysis and analytical narrative interpretation. The target group comprised forty kindergarten-level early childhood learners from Khon Kaen University Demonstration School, selected through purposive selection. The learners received instructional intervention based on the I-WISDOM model through a ten-lesson learning unit focused on rice and farmer. Data analysis utilized the spatial ability analytical framework established by Linn and Petersen (1985).

The research findings revealed that early childhood learners demonstrated spatial ability comprising three dimensions. First, spatial perception: learners accurately identified visual-spatial relationships depicted on various faces of cubes and coordinate grids. Second, mental rotation: learners effectively manipulated and arranged images in both two-dimensional representations (coordinate grids) and three-dimensional forms (cubes) with accuracy and efficiency. Third, spatial visualization: learners analysed and executed complex visual-spatial operations involving detailed and multistep procedures. Additionally, some learners experienced difficulties and systematic errors in performance execution, indicating areas requiring further developmental attention and instructional support.

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SCAFFOLDING THROUGH PROBLEMS

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While research on teachers' problem posing predominantly engages teachers in researcher-designed tasks or professional development settings, this study focuses on teachers' practice of problem posing - reformulation or generation of problems - within their own instructional contexts. We adopt an integrated lens of Schoenfeld's (2011) goals, resources, and orientations and Ball et al.'s (2008) Mathematical Knowledge for Teaching (MKT) to analyse how pedagogical intent and content knowledge shape their problem posing. We argue that orientations and goals inform the pedagogical intent behind tasks, drawing on MKT domains to notice affordances and adapt problems. In this context, this study examines: how teachers' problem posing practices are shaped by their knowledge resources, goals, and orientations? Drawing on a broader study, we focus on narrative accounts of two purposefully selected secondary teachers, Medha and Anay, each with over a decade of experience. Using in-depth semi-structured interviews, we conducted a case-based narrative analysis guided by an integrated lens of Schoenfeld (2011) and Ball et al. (2008). Both cases point to an implicit 'inventory of problems' that teachers draw upon and iteratively transform. Medha's narrative, centred on combinatorics, shows how systematically she modifies or adds constraints to the problems, enabling a progression from intuitive counting to formal applications and strategic problem solving, while foregrounding anticipated student errors. In contrast, Anay's narrative, focusing on deriving the sine rule, illustrates how problems are reformulated to preserve the 'joy of discovery' and student engagement, sequencing problems from numeric to algebraic, while supporting higher-order reasoning. Both cases mobilise integrated use of MKT domains, particularly Specialised Content Knowledge (SCK), Knowledge of Content and Students (KCS), and Knowledge of Content and Teaching (KCT). What distinguishes these cases is that Medha's reformulations emphasise conceptual structuring, whereas Anay's are strongly oriented by fostering student agency and valuing curiosity. Notably, the problems remain close to textbook forms - it is the teachers' deliberate sequencing and reformulation that expands their pedagogical potential. These narratives thus offer resources to elicit deeper reflections in professional development contexts.

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UNDERSTANDING TIME IN MULTIGRADE CLASSROOMS: A DESIGN-BASED STUDY IN MATHEMATICS EDUCATION

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Teaching time is a challenge in mathematics education due to its abstract nature, which is invisible and cannot be directly manipulated. Classic and contemporary studies show that students face difficulties in constructing notions of duration, sequencing, periodicity, and measurement, especially when teaching focuses on reading the clock. Recent curriculum research also shows fragmented attention to school time, with little coordination between grades and limited integration of physical, linguistic, and sociocultural dimensions (van Bommel & Walla, 2025). These difficulties are intensified in multigrade classrooms, where age and cognitive diversity require flexible and inclusive teaching approaches.

This study aims to design, implement, and document a didactic proposal for teaching time in multigrade classrooms, aimed at broadening the understanding of time as a mathematical quantity and as a social and cultural phenomenon. The theoretical framework is based on the psychology of mathematics education, which analyzes the transition from subjective conceptions of time to conventional measurement systems, as well as on recent studies that question the centrality of the clock as the only teaching tool for addressing this quantity.

The research was conducted using a qualitative, design-based approach. It was implemented in multigrade schools in Mexico and Colombia, with the participation of two teachers and 50 students from different grades.

The documentation of the process included video recordings of sessions, notes, photographic records, student productions, and interviews with teachers.

Preliminary results show that students were able to distinguish between estimation and measurement, recognize the need for more accurate instruments, and establish relationships between natural phenomena, social practices, and time measurement systems. Interaction between students from different grades encouraged peer mediation processes and progressively more elaborate explanations.

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A STUDY ON INSTRUCTIONAL INTERVENTIONS FOR PRE-SERVICE MATHEMATICS TEACHERS' PERFORMANCE IN PROBLEM POSING

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This study builds on prior problem-posing research, particularly using posing tasks to understand student thinking (Cai, 2022) and the link between teacher questioning competence and professional knowledge/beliefs (Chapin et al., 2013). Professional development in problem posing helps teachers apply questioning strategies (Li et al., 2019). Yet, effective methods for cultivating PMTs' problem-posing skills and integrating problem evaluation into instruction remain lacking. This study addresses this gap through a problem-posing intervention in a Mathematical Proposition Techniques course.

In a 5-week P-PBL intervention, 38 senior PMTs improved problem-posing performance—greater diversity, richer content, and deeper integration. They designed advanced problems (e.g., probability, limits, integrals, vectors) and adopted multiple geometric perspectives (e.g., projection theorem, Monge's circle theorem). However, problem reasonableness declined; many novel, challenging problems lacked appropriateness, revealing a creativity-feasibility imbalance. The instructor focused on posing strategies but not on evaluating question quality.

Teaching problem-posing strategies alone is insufficient; fostering evaluation awareness is equally important. Future instruction should integrate problem-evaluation interventions to improve reasonableness and educational value. These findings inform teacher preparation programs, especially for developing PMTs' ability to pose meaningful, sound problems.

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DIFFERENCES AND COMMONALITIES IN STUDENTS' FOUR THINKING SKILLS: RATIO AND PROPORTION

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According to the PISA framework for 21st-century competencies, critical, reflective, systems, and creative thinking are essential capacities for students in contemporary society (OECD, 2018). However, limited research has examined the distinctive characteristics of students' performance across these four domains. To address this gap, this study explored the differences and commonalities among these thinking skills within ratio and proportion. Three thinking-skills assessment questionnaires were developed and randomly administered to 150 students from six classes. Students' responses were coded and analyzed to identify patterns in their reasoning.

In critical thinking, students more readily evaluated statements containing explicit numerical relationships than abstract claims requiring definition-based reasoning. In reflective thinking, they demonstrated greater strength in comparing procedural efficiency than in judging contextual effectiveness. In systems thinking, students typically relied on a given variable (n) to construct algebraic expressions, favoring symbolic abstraction over multiple representations. In creative thinking, responses were often shaped by familiar formulas and task-provided elements. Across all four domains, students' reasoning was grounded primarily in visible numerical relationships or explicitly stated information, with comparatively limited engagement in abstract principles or integrated structural reasoning.

This pattern suggests that thinking processes become more visible when explicitly embedded into task structures, implying that structured and scaffolded instruction more effectively promotes higher-order thinking. Further teaching experiments are needed to compare infused (explicit) and immersed (implicit) approaches (see Tiruneh et al., 2018) in fostering the development of the four thinking skills in mathematical literacy.

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AGENCY-DRIVEN INDICATORS OF MATHEMATICAL RESILIENCE: MACHINE LEARNING AND THE HOLISTIC RESILIENCE INDEX (HRI)

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Mathematical resilience among socio-economically disadvantaged students is a key concern in equity-oriented mathematics education. This study examines (1) which PISA 2022 variables predict mathematically resilient students and (2) how these predictors can be interpreted through the Holistic Resilience Index (HRI). Mathematically resilient students are defined as those in the lowest quartile of the PISA Economic, Social and Cultural Status (ESCS) index and the highest quartile of mathematics achievement, following OECD's classification of disadvantaged high achievers (OECD, 2023). The HRI is a four-dimensional interpretive framework: perseverance (effort, self-efficacy), stress resistance (buffering adversity), emotional control (mathematics anxiety), and sense of belonging (feeling safe, cooperation, student–teacher relationships). It is used as a post hoc lens rather than as a model component, building on Cheung et al. (2024), who used a similar Random Forest–SHAP approach without an overarching framework. Approximately 60 PISA 2022 variables, including student attitudes, school climate, and learning environment indicators, were entered into a Random Forest classifier ($n \approx 690{,}000$; 79 economies, with PISA student weights). An optimal subset of 29 variables was identified via cross-validated feature selection based on F1 performance, and SHAP values were used to interpret predictor importance and direction. The model achieved an F1 of 0.591 on a held-out test set. Results reveal a clear pattern across HRI dimensions. Sense of belonging has the strongest impact: feeling safe is the top positive predictor, followed by cooperation and student–teacher relationships. Mathematics anxiety is the strongest negative predictor, and frequent bullying victimization is also strongly negative. Effort, self-efficacy, and curiosity positively support perseverance and stress resistance. Student gender is salient but non-malleable and is interpreted as reflecting differential exposure to contextual conditions. Overall, agency-driven, school-based factors outweigh fixed structural conditions, suggesting that relational safety and supportive learning environments are central to fostering mathematical resilience.

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LEARNING TO NOTICE MATHEMATICAL SENSE-MAKING: A FIRST-CYCLE DBR STUDY WITH LESSON STUDY

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This paper reports on the first cycle of a design-based research (DBR) project investigating how teachers in under-resourced South African schools support mathematical problem solving as sense-making. The study addresses the research question: How does participation in lesson study support teachers' attention to and facilitation of students' sense-making processes during problem solving?

The study is situated in a resource-constrained context in Cape Town, South Africa, where structural inequalities shape classroom practice. Mathematical problem solving is conceptualized as sense-making—students' active construction of meaning through talk, gesture, representation, and interaction, rather than answer-focused procedures (Schoenfeld, 1985). This framing positions teacher noticing—including attention to embodied and social forms of meaning-making—as central to instructional practice.

The intervention integrates lesson study as the pedagogical structure for teacher learning within a DBR framework involving iterative design, enactment, and refinement. Four Grade 7 teachers from four nearby primary schools participated in collaborative cycles of planning, teaching, observation, and reflection. Data analysis followed grounded theory techniques, involving iterative coding of teacher discourse and practices to trace shifts in pedagogical attention during Cycle 1.

Findings show three key shifts in teacher learning. First, teachers moved from prioritizing correct answers to attending to students' reasoning processes and the emergence of multiple strategies. Second, teachers developed increased sensitivity to students' embodied and social sense-making practices, including gestures, diagrams, and peer interactions, making these features more explicit in their instructional decisions (Choy, 2025). Third, teachers increasingly framed struggle as productive, delaying intervention to allow students to explore and articulate their thinking. However, challenges persisted in selecting cognitively demanding tasks and supporting student reflection within limited instructional time. As a first-cycle DBR study, these findings are interpreted as initial design conjectures, suggesting that lesson study can support the development of teacher noticing of multimodal sense-making in resource-constrained contexts, while highlighting areas for further design refinement.

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MAKING SENSE OF THE WORLD: INTERWEAVING MATHEMATICS AND POLITICAL APPORTIONMENT

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This oral communication describes a classroom design in which Grade 8 students (13-year olds) engage mathematically with the political practice of apportionment. Rather than teaching proportional reasoning through textbook procedures, the study immerses students in real-world electoral data, asking: What does students' interweaving of apportionment with their own perspectives look like, and what themes emerge? The study draws on métissage (Campbell-Chudoba, 2023): a theory of identity formation through the interweaving of personal experience, values, and the world. Mathematics is not context-free; learners engage with it through their own histories, reshaping both themselves and their understanding. Student responses are also understood through story making (Jørgensen, 2024): a relational mode in which personal narratives intertwine with mathematical and political reasoning. Data was collected from 59 Grade 8 students across two Western Canadian classrooms. Students worked on (1) the 2024 British Columbia provincial election (allocating 93 legislative seats) and (2) a student council scenario (distributing 12 seats across three grade levels), with a final question asking whether fairness and equality are the same. Five themes emerged. First, some students distributed extra seats equally across parties, prioritizing uniform treatment even at the cost of proportionality. Second, others argued the party with the most votes deserved more seats: that popularity should translate into power. Third, another group advocated for smaller parties, reasoning that extra seats would amplify underrepresented voices. Fourth, in the council scenario, some students argued older grades deserved more seats due to experience, reframing representation as a matter of voice and maturity. Fifth, students distinguished fairness from equality, drawing on personal analogies such as food portions, height restrictions, and accessibility seating to articulate that fairness depends on context, not uniformity. Students did not simply compute; they weighed considerations and drew on lived experience. The focus was less on calculation and more on developing a general sense of what proportional reasoning is about. The métissage framework illuminates how students interweave justice with mathematical tools through story making, indicating that proportional reasoning can be experienced as relevant and meaningful in socially situated ways.

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WHAT DID CHATGPT "NOTICE" ABOUT PRIME NUMBERS? INSIGHTS FROM AN AI-GENERATED LESSON PLAY

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The introduction of Artificial Intelligence (AI) into education has the potential to usher in an era of new possibilities for reimagining teacher learning. For instance, Gurl et al. (2024) have explored the use of ChatGPT as a lesson-planning assistant and have found it useful, provided that teachers critically reflect on the AI agent's output, which may contain errors and misconceptions that are difficult to notice when using it. Furthermore, these AI-generated lesson plans tend to be more teacher-centred in their suggested approaches. Given the importance of lesson planning, it is critical for teachers to think more deeply about the mathematical concepts they teach, anticipate their students' thinking, and consider different teaching approaches in response to student thinking. A lesson play—" an elaboration on a lesson or part of a lesson in the form of a script"—is a way for teachers to "imagine a classroom", "learners' reactions", and "their possible explanations" in response to a critical incident (Zazkis et al., 2013, p. 271). A lesson play can thus make visible what teachers *notice* about content, student thinking, and teaching approaches—critical aspects of teaching that serve as useful foci for teachers (Choy, 2016). In this presentation, I investigated the characteristics of a ChatGPT-generated lesson play by examining what it "noticed" about the mathematics content, student thinking, and teaching approaches. The presentation is framed by the following question: What did ChatGPT notice when prompted to write a lesson play? The lesson play presented and analysed in this paper was generated by ChatGPT (GPT-5.2) to address two main pedagogical aims: explain what prime numbers are and how to determine whether a given number is prime, using a prompt developed by Zazkis et al. (2013, p. 89). Analysis began with a line-by-line parsing to identify key segments of the discourse. I then analysed the discourse patterns embedded in the generated lesson play and coded it using Choy's (2016) foci for noticing. I will present findings on the characteristics of this AI-generated lesson play, invite feedback, and discuss possible directions for future research.

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FEELING AND UNDERSTANDING FRACTIONS: RELATIVE UNIT AND MUSICAL TIME IN TEACHER TRAINING

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In mathematics education, the teaching and learning of fractions is often reduced to procedures that do not consolidate meanings about unity, equal division, and equivalence. This paper is part of the context of "Sensing and making sense of mathematics," showing how musical time (experienced physically, auditorily, and visually) offers a privileged terrain for sensing relationships (duration, repetition, half/double) and, based on this, giving meaning to fractional structures.

We present the results of a research program that explores cognitive links between music and mathematics through rhythmic experiences. We propose the notion of relative unit as a central construct: a reference of measurement that is fixed and transformed according to the rhythmic pattern, allowing durations to be compared, rescaled, and relationships verified (Liern, 2011). Through tasks, we analyze multimodal evidence from students and describe a trajectory of construction and use of relative unit at four levels: identification (discrimination of durations), categorization (regularities and comparisons), association (bridges between registers and explicitation of the unit), and signification (justification of equivalences, reasoning about equal distribution, and resolution of situations with fractions). The results show that the transition from performing the rhythm to a meaningful understanding of fractions is strengthened when the task requires changes in unit and verification procedures (Conde, 2013). Based on the results, we formulated a research agenda aimed at teacher training (Parada & Fiallo, 2022), which contains: (i) re-signifying the relative unit, (ii) designing tasks that articulate sensory experience and mathematical justification, (iii) orchestrating mediations (gestural, notational, and technological), and (iv) interpreting multimodal evidence as indicators of fractional thinking. We propose lines of work for initial and continuing training through reflection and action, with didactic designs and rubrics for analyzing multimodal evidence.

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MAKING ROOM FOR MATHEMATICAL PROFICIENCY

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A large body of mathematics education research highlights a shift away from schools' conventional lecture-based mathematics instruction toward innovative and flexible teaching practices that support diverse learners and improve mathematical confidence, understanding, and achievement. Research also identifies various classroom approaches that can promote this shift and develop students' conceptual understanding. Yet teachers need to be intentional in applying these methods, ensuring that they yield the intended outcomes. This is not an easy task, given teachers' prior experiences as learners, university training, and current teaching practices. Teachers may need to let go of some ties to further enable students to sense and make sense of mathematics, deepen their understanding, and build long-lasting mathematical knowledge. Inspired by this understanding, this research sought to identify and investigate potential classroom scenarios that favour meaningful teaching practices. Grade 4 and 5 mathematics classrooms were observed seeking opportunities to intentionally develop students' mathematical proficiency, namely, conceptual understanding, strategic competence, procedural fluency, adaptive reasoning, and productive disposition. Kilpatrick et al.'s (2001) model of mathematical proficiency was used as the theoretical framework for data analysis. The research methodology was grounded in action research (Stringer, 2007), involving me, the classroom teacher, the assistant teacher, and the elementary school pedagogical coordinator as research collaborators. We gathered to discuss, plan, and follow up on classroom practices and tasks aligned with the research theoretical framework. Classroom interventions were implemented to prompt, nurture and develop students' mathematical proficiency. Data was collected through field observations of classroom activities and from students' activity materials. Preliminary results show that students will naturally ask questions and create opportunities for teachers to explore and foster students' mathematical proficiencies. However, it is crucial to adjust original lesson plans or activities as needed to allocate sufficient time to delve into those opportunities. This oral communication will present examples of how classroom activities and situations were explored to support students' development of mathematical proficiency. Research findings indicate that teachers can be more flexible and adapt ingrained, stagnant teaching models to better assist students and meaningfully promote their mathematical learning.

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EPISTEMIC POSITIONING AND AGENCY IN AI-MEDIATED MATHEMATICS LEARNING AMONG GIFTED STUDENTS

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Within sociocultural perspectives, tools are understood as mediators that reorganize thinking processes rather than merely supporting performance (Vygotsky, 1978). Recent discussions in mathematics education therefore focus on epistemic agency, referring to learners' role in directing knowledge construction (Scardamalia & Bereiter, 2006). Although artificial intelligence is often assumed to diminish learner agency due to its epistemic authority, little is known about how students conceptualize AI and how these conceptualizations relate to perceived control during problem solving. This study examines how mathematically gifted students position artificial intelligence epistemically and how these positions relate to perceived procedural agency in AI-supported mathematical activity. Guided by an interpretive perspective on epistemic positioning, the study employed a qualitative interpretive design to analyse students' written responses. The study involved 53 mathematically gifted students participating in an enrichment program. Data consisted of written metaphor responses and reflections on control during AI use. Metaphors were analysed qualitatively as indicators of epistemic positioning and categorized into seven roles (Instrumental Tool, Cognitive Assistant, Navigator, Collaborative Partner, Epistemic Authority, Cognitive Extension, and Opaque Intelligence). The analysis resulted in a set of epistemic agency positions describing how responsibility and control were distributed between students and AI, ranging from primarily human-dominant agency (occasionally negotiated), supported but learner-regulated agency, and guided agency, to distributed and negotiated agency, AI-dominant or externally validated agency, shared cognitive agency through externalization, and epistemic opacity. Results indicate that metaphorical positioning and perceived agency were not aligned along a single continuum. Students who described AI as a *teacher* or *expert* frequently reported maintaining control over questioning, decision-making, and evaluation processes, suggesting that attributing epistemic authority to AI does not necessarily reduce learner agency. Instead, agency appeared dynamically negotiated during interaction. These findings suggest that AI-mediated mathematical activity reorganizes rather than transfers epistemic responsibility between learner and technology.

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MATHEMATICIANS' ASSESSMENT OF POTENTIAL WRITING CONVENTIONS FOR SYMBOLS IN STUDENTS' PROOFS

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Writing proofs is challenging for first-year students, for example because they have to adhere to writing conventions that are often only implicitly given. These conventions can be understood as non-universal practices and vary across different proof contexts, for example blackboard or textbook proofs (Lew & Mejía-Ramos, 2019). It is therefore worthwhile to examine which conventions instructors apply to student proofs submitted as homework. Given that students frequently struggle with technical language, particularly with the use of symbols (Guce, 2017), we ask: *What writing conventions do instructors hold regarding the use of symbols in students' written proofs?*

To address this question, five instructors at a German university were asked to think-aloud while evaluating a handwritten and mathematically valid student proof on matrix eigenvalues containing (among others) four potential breaches of convention in the use of symbols in text. The potential breaches are *grammatically incorrect use of a symbol*, *a symbol at the beginning of a sentence*, the *use of a relation sign to connect a non-symbolic and a symbolic expression*, and the *use of an implication arrow as a logical connective*. In a subsequent discussion about breaches which were not evaluated, we asked whether the instructors would revise their feedback to make conventions explicit that are often applied implicitly.

Using qualitative content analysis on our data, we found that the use of symbols in text was scarcely evaluated. However, after discussing the breaches of conventions, most instructors considered them to be relevant confirming previous studies on such breaching experiments (Lew & Mejía-Ramos, 2019). The lecturers also gave several reasons why some conventions should be followed, such as avoiding misunderstandings, improving readability, and aesthetics. Interestingly, many lecturers who still regarded some of the included breaches as irrelevant nevertheless identified contexts in which they would find them disturbing, for instance in computer-written student proofs or in scientific publications.

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THE PERSPECTIVE OF THEORY OF OBJECTIFICATION IN ALGEBRAICAL PROBLEM POSING BY GRADUATE STUDENTS

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We intend to analyze students' knowing through problem posing tasks under a sociocultural lens. It considers the objectification and subjectification processes emerging during a problem-posing activity, framed by the theory of objectification that prioritizes collective, semiotically mediated work. The Theory of Objectification (TO) have been settled as a sociocultural approach to mathematics education that conceives learning as a historical, cultural, and collectively mediated process (Radford, 2021). It has been adopted as lens because it allows us not only to analyze the processes of *knowing* during the activity and the *becoming* (how students collectively assist and care for each other and learn through joint work). The study adopts a qualitative, interpretive approach. Data were collected from audio and video recordings, written paperwork and researcher's logbook during a workshop with 20 graduate students working in small groups on a problem-posing task based on an algebraic growing figure from audio and video recordings, written paperwork and researcher's logbook. The results indicate how objectification (*knowing*) unfolds through joint work. Students collectively negotiated the meaning of the figure, mobilizing mathematical ideas such as functions, equations, generalization, and arithmetic progression, as well as producing multiple interpretations (e.g., parked cars, a cross, a flower). Participants used speech, gestures, drawings, and material actions (e.g., modeling clay) to pose problems and discuss mathematical ideas. At the same time, these interactions involved processes of subjectification, as students listened to each other, negotiated interpretations, and showed concern for how their problem would be understood. Students debated whether the situation was best represented by $f(x) = 4x - 1$ or $f(x) = 4x - 3$. This disagreement was solved through an intense, collective dialogue between them. This highlights the importance and power of considering problem posing in the classroom as a collective and dialogical activity. Therefore, in this process, learning through problem posing involves a deep intertwining between culture and human action, including ways of thinking, acting, feeling, and positioning oneself in the world.

Additional information

This work was supported by CNPQ, Brazil.

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MODELING TEACHING, AFFECTIVE FACTORS, AND MATH ACHIEVEMENT: A MULTILEVEL PISA 2022 ANALYSIS

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This study contributes to research modelling mathematics performance by integrating perceived teaching quality with students' motivational and emotional variables (Li et al., 2021; Liu et al., 2024). Mathematics achievement is conceptualized as a dynamic system in which these dimensions interact, with affective factors playing a central mediating role. Unlike prior work that typically examines teaching practices (Liu et al., 2024) and student characteristics separately (Du et al., 2021; Namkung et al., 2019), our model combines them within a single explanatory structure, revealing pathways of influence. The aim is to test, using a multilevel model, the direct and indirect effects of teaching-related variables (teacher support, cognitive activation) and students' motivational and emotional characteristics (engagement, self-efficacy, anxiety), accounting for the mediating roles of self-efficacy and anxiety and for school-level variance.

The analyses draw on PISA 2022 mathematics data from 4,875 students aged 15 in French-speaking of Belgium. Two-level multilevel models (students/schools) were estimated, retaining only significant relations. Results show that self-efficacy is the strongest positive predictor of performance, whereas anxiety exerts a substantial negative effect. Teacher support and student engagement influence achievement indirectly by strengthening self-efficacy and reducing anxiety. Cognitive activation displays a non-linear effect: beneficial when accompanied by sufficient student engagement and teacher support, but potentially counterproductive when intensified without adequate support. Differences between schools further confirm the value of a multilevel approach. This modelling shows that teaching practices influence performance primarily through motivational and emotional mechanisms. These results suggest that more effective mathematics learning environments rely both on supportive and cognitively stimulating teaching practices and on strengthening students' motivational and emotional resources.

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COMPUTATIONAL THINKING AND MATHEMATICS EDUCATION: A BIBLIOMETRIC ANALYSIS

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Introduction

Computational Thinking (CT) is increasingly described as a set of epistemic practices rather than merely programming skills (Li et al., 2020), its relation to modelling and reasoning in mathematics education has been noted (diSessa, 2018), yet its structural position within mathematics education research remains unclear (Rich et al., 2024).

Method

This study used a bibliometric design with a structured Scopus TITLE-ABS-KEY search to build two datasets (computational thinking publications and mathematics–computational thinking intersections) and analyzed them via VOSviewer keyword co-occurrence mapping (van Eck & Waltman, 2010).

Results

CT research is primarily structured around K–12 education, STEM, programming, and teacher education, whereas mathematics education is anchored in modelling, reasoning, and problem solving; within their intersection, CT appears mainly in pedagogical and programming contexts, with modelling and problem solving serving as linking nodes, and although algorithmic thinking and debugging have gained recent visibility, CT remains structurally connected yet not conceptually central to the broader mathematical thinking literature. These findings provide a macro-level descriptive mapping of CT's positioning within mathematics education research.

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REFLECTIONS OF ALGEBRAIC HABITS OF MIND IN NON-ROUTINE ALGEBRA PROBLEMS

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In recent years, the emphasis on mathematical thinking processes in mathematics curricula has made it a more visible goal for students to be able to use their knowledge flexibly in different problem situations and to produce generalisations (Poindexter, 2011). As a specific form of mathematical habits of mind, algebraic habits of mind constitute a framework that can be developed through classroom practices and discussions and that encompasses process actions such as organising information, predicting patterns, identifying repeated information in patterns, defining rules, describing change, and verifying rules (Driscoll, 1999; Eroğlu & Tanışlı, 2017). The purpose of this study is to examine middle school mathematics teachers' algebraic habits of mind and how these habits are reflected in the problem-solving processes of Grade 7 students with whom they have worked together for three years, in the context of a non-routine river-crossing problem. In the research conducted with a qualitative case study design, written solutions and clinical interview data were collected from two teachers and their students, and the data were analysed according to the components in Driscoll's framework. The findings show that participants mostly simplified the problem and structured the initial situation, and that they predicted the pattern by organising information and identifying repeated information using representations such as arrows, diagrams, and tables. However, it was also determined that difficulties experienced particularly in the components of organising information, predicting patterns, and identifying repeated information markedly reduced performance in the later stages of the solution, namely defining rules, describing change, and verifying rules. This situation may lead to an inability to implement the generalisation–verification cycle consistently and may hinder the development of algebraic thinking. Consequently, regularly integrating non-routine algebra problems into instruction and employing multi-representation-based guidance that makes early algebraic habits of mind visible seem critical for strengthening generalisation and rule verification.

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INVESTIGATING THE INTERPLAY: TEACHER KNOWLEDGE, INSTRUCTIONAL QUALITY, AND STUDENT LEARNING

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Research in ISCED 1-3 contexts show that instructional quality is positively associated with student learning and that teacher knowledge is positively associated with instructional quality, although establishing a clear connection among the constructs remains challenging (Hill et al., 2008). Having validated instruments for community college settings (ISCED 5) to measure (a) mathematical knowledge held by instructors who routinely teach algebra (Mesa et al., 2026); (b) the quality of instruction in algebra classrooms, Evaluating the Quality of instruction in Postsecondary Education (EQIPM; Ko, 2025); and (c) student performance on the Algebra and Precalculus Readiness-Community College (APCR-CC) assessment (Peralta, 2017), we report on an anomalous case. Ellie, an instructor whose standard score on the Quality of Instructor-Content Interaction construct of EQIPM was 0.26; the average gain from the beginning to the end of the term score on the APCR-CC for her students was 2.42 (the second highest in the sample of instructors); her knowledge standardized score was -0.16, allowed us to find consistency in the measures obtained, that is, the high scores in EQIPM and APCR-CC suggest that Ellie's instruction supported students' performance as the APCR-CC items required the foundational work presented in Ellie's lessons. Likewise, the contrasting knowledge measurement score does suggest weakness in Ellie's knowledge not captured by EQIPM or assessed in the APCR-CC. We believe that this detailed qualitative analysis helps us provide a better understanding of how these measures work together in the community college context.

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PRE-SERVICE ELEMENTARY SCHOOL TEACHERS' ATTITUDES TOWARDS MATHEMATICS AND GENERALIZATION WHEN SOLVING FUNCTIONS PROBLEMS

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This study is focused on the attitudes towards mathematics and generalization of Spanish pre-service elementary school teachers. Positive attitudes toward mathematics improve academic performance and reduce anxiety about the subject (e.g., Akin & Kurbanoglu, 2011). Algebraic sense involves relationships between variables. Attitudes is one element of socio-affective sense; and generalization of functions are connected to algebraic sense (Carraher & Schliemann, 2007). Teachers must help elementary students manage their attitudes, particularly to what algebraic sense concerns. Our specific research objectives are to describe: (a) their attitudes toward mathematics, (b) the generalization evidenced in problems involving functions, and (c) the relationship between their attitude toward mathematics and their success in generalization. Seventy three pre-service teachers completed an individual questionnaire, with (a) three closed questions about attitudes —"how do you feel about mathematics?", "what has your previous experience been like solving mathematical problems?", and "does liking mathematics depend on the teacher?" —; and (b) two function problems requiring generalization. The data analysis was mainly quantitative.

Most pre-service teachers recognize the influence of the teacher as a determining factor in their attitude toward mathematics (95.83%). The majority express an ambivalent attitude: they find mathematics interesting but, at the same time, perceive it as difficult. There were a minority of pre-service teachers who expressed the generalization correctly, with a tendency to express it through algebraic symbolism, even when it was required verbally. We found no significant differences between generalization and attitudes, nor between generalization and the specific problems provided.

Acknowledgement

This study is funded by the research projects PID2020-113601 GB-I00, and PID2024-157106NB-I00; financed by MCIN/AEI/10.13039/501100011033, and FEDER, UE.

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ADULT READERS' ENGAGEMENT WITH DATA VISUALISATIONS WHEN INTERPRETING STATISTICS IN NEWS MEDIA

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Ideally, schools should foster students' abilities to act as informed and active citizens. This includes the development of data literacy, which mathematics education is well positioned to support (OECD, 2025). Within this competence, data visualisation literacy (DVL) constitutes a specialised form of visual understanding essential for interpreting graphical data. However, to design teaching for DVL in real-life contexts, it is necessary to understand how data visualisations (DVs) are interpreted by citizens outside educational contexts (Börner et al., 2019).

With a focus on DVL, this pilot study explores experienced readers' strategies and what they take away from a text. Two adults, an HR coordinator (R1) and a medical science researcher (R2), read a newspaper article presenting statistical data in seven DVs integrated with the verbal narrative. Data were collected through eye-tracking and interviews, and the ways in which the readers constructed meaning were analysed using an analytical framework that qualitatively distinguishes between visualisation preferences, decoding strategies, and readers' propensity to adhere to textual guidance. Our findings show that although both readers regarded the statistical content as important, they used the DVs differently. While R1 prioritised the verbal narrative, R2 based their interpretation chiefly on the DVs, drew their own conclusions from them, and recognised the narrative as one of several possible alternatives. This pattern was evident in gaze patterns and further supported by interview data. Although these findings do not allow conclusions about which DVL abilities are required of a literate citizen, they clearly demonstrate the benefits of developing DVL skills and provide sufficient reason to consider the development of DVL in mathematics education for all students. Further studies will examine the gap between current teaching and the data literacy demands of contemporary public information.

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IDENTITY+: FROM MATH TEACHER TO AI EDUCATOR

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Math teacher identity, shaped through practice, discourse, and participation in professional communities, also involves teachers' background and motivations (Akkerman & Meijer, 2011; Ntow & Adler, 2019). The process of developing an *AI Educator* identity, one who teaches students about AI, foregrounds teachers' epistemic authority to frame what counts as legitimate knowledge in their classrooms. Thus, becoming an AI Educator entails reauthoring one's professional role in relation to knowledge, agency, and instructional responsibility. Our team developed a validated research instrument to measure teachers' AI Educator identities (Jiang et al., under review). We applied this instrument to research the shift of math teachers' AI Educator identity over the course of a professional development workshop.

The sample comprises 11 secondary mathematics teachers who attended a workshop on how to teach a curriculum module that contextualizes AI learning from a disciplinary perspective. During the workshop, the teachers experienced the content firsthand and learned about the corresponding pedagogical aspects. To measure the impact of the workshop on the change in AI Educator identity, a pre- and post-survey was administered containing six identical questions.

Results demonstrated a significant increase in the overlap between teachers' existing identities and that of an AI Educator. Teachers saw greater alignment between AI and math education, reported increased confidence in their ability to successfully integrate AI teaching into their math classes, and anticipated allocating more class time to AI topics. Results suggest teachers came to view AI education as advancing core mathematical learning goals, not displacing them, signaling a shift in what they perceive as legitimate mathematical content. During the presentation, the details and implications of these results, their limitations, and next steps will be discussed.

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INVESTIGATING STUDENTS' CREATIVE MATHEMATICAL REASONING: THE CASE OF THE TRAPEZOID

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Developing students' capacity for meaningful mathematical reasoning is a central goal of mathematics education, requiring not only procedural fluency but also the ability to construct explanations, generate novel strategies, and justify thinking. Within this perspective, Creative Mathematical Reasoning (CMR) involves constructing new and plausible solution paths grounded in mathematical principles rather than imitation (Lithner, 2008). Despite its importance, classroom practices often privilege imitative reasoning, limiting opportunities for meaningful sense-making. Moreover, as reasoning is interactionally constituted through classroom discourse, little is known about how CMR emerges within such interactions. This study addresses this gap by examining the following research question: How does students' CMR emerge within classroom interactions during engagement with a trapezoid-based proof task?

This study was conducted within a design-based research framework in a high school classroom in Türkiye with 14 students aged 15–16. Over a seven-week sequence of classroom observations on proof-based geometry tasks, the present analysis focuses on one week involving a trapezoid task, during which students engaged in classroom interaction. Data were analysed qualitatively by identifying episodes of students' reasoning from transcripts, which were then coded using Lithner's (2008) framework to examine instances of creative mathematical reasoning, particularly in terms of novelty, plausibility, and mathematical foundation.

The findings indicate that the task, together with interactional support, enabled students to move beyond procedural activity and construct novel solution pathways, particularly through geometric dissection and reconfiguration. These ideas were developed through classroom discourse, where students articulated, justified, and refined their reasoning in response to peers and teacher prompts. As a result, students' reasoning integrated novelty, plausibility, and mathematical foundation, drawing on geometric principles such as area conservation and properties of parallel sides. Overall, the study highlights the role of proof-based geometry tasks and classroom interaction in fostering creative and mathematically grounded reasoning.

Acknowledgment: This study was funded by the Scientific and Technological Research Council of Türkiye (TUBITAK) ARDEB 3501 under Grant Number 222K223.

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DEVELOPING AN INSTRUMENT FOR ASSESSING (PRE-SERVICE) TEACHERS' BELIEFS IN ERROR SITUATIONS

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How teachers deal with learners' errors depends on a variety of factors, including their beliefs. Therefore, beliefs towards errors are considered to be part of teachers' professional error-competence (Wuttke & Seifried, 2017). From a behaviourist perspective, errors represent deficits in the learning process, while from a constructivist point of view, errors are seen as valuable learning opportunities. Regarding error handling, using student-centred activities can be related to a strength-based framing of students, whereas in deficit-centred framing often a teacher-centred instruction is preferred (Scheiner, 2026). Although different Likert-based instruments on teachers' beliefs in error contexts already exist, these either only cover a single aspect of errors (e.g., error causes or a constructivist view on errors) or do not originate from a teaching context (e.g., employees). Therefore, this study aims to develop and evaluate a standardised instrument that comprehensively and operationally captures the various factors that influence teachers' beliefs regarding errors in mathematics teaching.

For this purpose, Likert-based items were deductively developed using existing items and related literature. The subscales cover a deficit- and a resource-oriented view on errors, beliefs on deficit- and strength-oriented error handling and beliefs on teaching-related and learner-related error causes. The items were reviewed by 9 experts, revised and tested in a pilot study (N = 100 pre-service teachers). The items' factorial structure was evaluated using exploratory factor analysis and reliability analysis.

In a first analysis the subscales *resource-oriented view on errors* (α = .766), *strength-oriented error handling* (α = .696) and *deficit-oriented error handling* (α = .767) showed satisfactory or sufficient reliabilities, partly after excluding single items. On the other hand, for the remaining three subscales only insufficient reliabilities were achieved or exploratory factor analysis revealed a deviating factor structure. These results indicate that some items already represent reliable subscales for assessing teacher beliefs in error situations, while others need further adjustments.

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USING CONTRADICTIONS TO ACCESS AND EXPAND STUDENTS' MATHEMATICAL THINKING IN WHOLE-CLASS DISCUSSIONS

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The aim of this presentation is to exemplify and discuss how contradictions can be used as a didactical tool for accessing and expanding students' mathematical thinking in whole-class discussions. The study draws on learning activity theory and the El'konin–Davydov curriculum (Davydov, 2008). From this perspective, contradictions are understood not as individual errors, but as productive tensions that emerge within collective mathematical activity (see Radford, 2021). A contradiction arises when students' existing, often empirical, ways of reasoning are insufficient to resolve a mathematical problem, creating a need for a new way of understanding the object of learning. Such contradictions become visible through students' verbal contributions as well as through their embodied actions and interactions with mediating tools. For teachers, these moments provide opportunities to gain insight into students' current ways of thinking and to support the development of more conceptual ways of thinking.

The presentation is based on a design research study: Problem Situations, Learning Models and Teaching Strategies. Video-recorded data from collaboratively designed research lessons, focusing the positional system, were analysed with a focus on whole-class discussions in which contradictions surfaced in relation to central mathematical ideas. The analysis focused on how contradictions were expressed through students' talk and action with shared tools, as well as how teachers responded to and sustained these moments in instruction. The findings implicates that when contradictions are recognised and maintained in whole-class discussions, they can support collective exploration of underlying mathematical relationships. By posing provocative questions or juxtaposing different ideas teachers can use contradictions to make students' thinking visible and open for collective reflections in the classroom. From a didactical perspective, contradictions thus function not only as access points to students' thinking, but also as resources for creating a shared space for evaluating and developing mathematical ideas.

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MOTIVATIONAL MECHANISMS SHAPING GENDERED PARTICIPATION IN MATHS-INTENSIVE FIELDS

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This presentation reports two conceptual replications examining how gender-segregated secondary schooling relates to students' mathematics achievement and participation in mathematics-intensive university pathways in New Zealand. New Zealand provides a distinctive context: 11.4% of students attend single-sex schools, and 83.6% of these are state or state-integrated (MoE, 2025), unlike the United States and Australia where single-gender schooling is more often private or Catholic. This setting allows a strong test of generalisability within a shared national system.

Study 1 replicates Docherty et al. (2020) using enrolment data from the University of Auckland (2016–2020), extending earlier work at the University of Canterbury. Logistic regression showed that girls from single-sex schools were overrepresented in Engineering: 42.0% of female Engineering students at Auckland had attended girls' schools, compared with ~33% in other programmes. The difference from Canterbury's 56.0% was small ($d = 0.35$), supporting the robustness of the association. No comparable pattern was observed for boys.

Study 2 extends Smith and Evans (2024) using TIMSS 2019 data and hierarchical linear modelling, framed by situated expectancy–value theory. Confidence positively predicted mathematics achievement across genders. Importance value showed no main effect once confidence was controlled; however, for girls only, school gender type moderated the value–achievement relation. At high levels of valuing mathematics, girls in single-sex schools outperformed their co-educational peers.

Together, the findings suggest that single-sex schooling in New Zealand is associated with girls' participation in mathematics-intensive pathways and, under specific motivational conditions, mathematics achievement, highlighting important contextual boundary conditions in gendered mathematics trajectories.

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MATHEMATICAL WELL-BEING AND STEM CAREERS INTEREST: AN EXPLORATORY PROFILE ANALYSIS

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While mathematics education is widely recognized as a critical foundation for students' engagement and success in STEM disciplines, students increasingly suffer from negative affective experiences, such as math anxiety and low engagement. Previous research on students' STEM aspirations has mainly focused on cognitive and motivational variables (Ferdinand et al., 2024), leaving students' holistic mathematical experiences underexplored. This study highlights the critical role of mathematical well-being (MWB), an emerging, multidimensional construct capturing students' subjective experiences of feelings and functioning in mathematics learning (Campbell, 2025). Specifically, this exploratory study investigates whether distinct MWB profiles are associated with variations in students' STEM career interest (STEM-CI). Drawing on a sample of 155 secondary school students in Hong Kong, the study employed a profile-based exploratory approach. MWB was assessed using a comprehensive 22-item scale (Zhang et al., 2025), alongside a validated STEM-CI survey (Kier et al., 2014). Participants were categorized into low, moderate, and high MWB profiles based on tertile distributions. A one-way ANOVA revealed a significant, gradient effect of MWB on STEM-CI: students in the high MWB group (M = 3.53) demonstrated significantly greater interest in STEM careers than those in the moderate (M = 2.96) and low (M = 2.81) MWB groups. Post Hoc comparisons confirmed significant differences between the high MWB group and the other two cohorts. These findings highlight the importance of multidimensional MWB in shaping students' STEM aspirations, which offering valuable insights for educators and policymakers aiming to broaden the STEM pipeline.

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MIXED-METHODS ANALYSES IN PROBLEM SOLVING: IDENTIFYING STUDENTS' PROFILES

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This oral communication presents preliminary findings on arithmetic problem solving and is based on a cohort study conducted in France during the transition from elementary to secondary school. It aims to identify students' profiles to be followed in a longitudinal study and to inform teacher professional development. The research question is: To what extent does a mixed-methods approach support the construction of students' profiles in arithmetic problem solving, and what convergence exists between qualitative and quantitative analyses in this process? This study draws on Vergnaud's Theory of Conceptual Fields (2009), Brousseau's Theory of Didactic Situations (1997), and categorizations of arithmetic problems, which guided both the design of the diagnostic test and the interpretation of students' responses, leading to the construction of students' profiles. A sequential mixed-methods design (QUAL → QUAN) was implemented (Bikner-Ahsbahs et al., 2015). The literature review (problem solving, profiles, mixed-methods), together with teachers' interviews and classroom observations, informed the design of a diagnostic test and an analytic grid, which were refined through a didactic a priori analysis and a pilot study (n = 74). The analytic grid combined quantitative and qualitative criteria (calculations, errors, reasoning, representations, etc.), used to code the data. The test was administered in January 2025 to 327 students (aged 10–12) across 13 teachers. Data were analysed both qualitatively (observations, interviews, and students' work) and quantitatively using R (R Project for Statistical Computing) through Multiple Correspondence Analysis (MCA) and Partitioning Around Medoids (PAM) clustering, and convergence between the two analyses was used to validate the students' profiles. Five distinct profiles were identified. This study shows that a mixed-methods approach produces robust and didactically meaningful students' profiles, with strong convergence between qualitative and quantitative results. In the cohort study, these profiles will be used to track students' development over time and to support the design of teaching resources. We will illustrate the method and the characteristics of the students' profiles using a sample problem and its analysis.

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FOSTERING STUDENTS' QUANTITATIVE AND COVARIATIONAL REASONING IN FUNCTIONS USING BUILDING BLOCKS

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Quantitative reasoning, particularly covariational reasoning, which involves coordinating how quantities change together, is central to understanding functions (Carlson et al., 2002). In this study, we report on a teaching experiment on students' quantitative and covariational reasoning using a building-block-based task. Our research question is: "In what ways do students engage in quantitative and covariational reasoning regarding functions while working with a physical building-block-based task?" We adopted a teaching experiment methodology with six low-achieving 11th-grade students. Data included activity sheets, observations, and audio recordings from an 80-minute session. The task involved a physical stage lighting model constructed from building blocks that students could manipulate by turning a handle to rotate gears and adjust a lamp's height. Data were analysed iteratively using episode-based analysis grounded in covariational reasoning frameworks (Carlson et al., 2002). We identified four recurring forms of reasoning in the first phase of the teaching experiment. First, students coordinated changes between quantities, identifying an inverse proportional relationship between gear teeth and rotations. Second, by measuring physical variations, students coordinated amounts of change and represented these relationships algebraically and graphically. In the third episode, students reasoned about rates of change. However, when attempting algebraic representations, they experienced conversion challenges and relied on teacher guidance to bridge conceptual reasoning with formal representations. Finally, moving beyond pairwise comparisons, students coordinated multiple quantities simultaneously, connecting rotations, height, and illuminated area to interpret illumination effectiveness. Overall, such tasks can provide opportunities for engaging students in quantitative and covariational reasoning, even among low-achieving students. The building-block-based design provided a shared manipulable representation that made the simultaneous variation of quantities visible and accessible for students. However, teacher guidance played a role in connecting intuitive to formal representations. In future iterations, we aim to refine task instructions to reduce teacher funnelling and promote more independent exploration, thereby strengthening the interpretive potential of the task designed with building blocks.

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EXPLORING THE CONSTRUCTION OF MATHEMATICAL MEANING THROUGH CORPUS-BASED LANGUAGE ACTIVITIES

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This study builds on research that conceptualises language as a central resource for constructing mathematical meaning rather than merely expressing it (Radford, 2002; Planas et al., 2018). Linguistic choices can provide insight into how students articulate and stabilise mathematical meanings, especially when language is explicitly foregrounded in instruction. The study adopts a Data-Driven Learning approach, in which students engage in guided exploration of a corpus, understood as a structured collection of authentic texts (Corino et al., 2022). Our corpus consists of excerpts from secondary school textbooks related to the concept of function. Corpus-based activities are designed as language-focused tasks in which students identify recurring linguistic patterns and use them to complete mathematically meaningful statements, to foster reflection on mathematical meanings starting from linguistic forms. An eight-hour research activity was conducted with 38 Grade 10 students, combining (a) pen-and-paper tasks integrating procedural questions with explanation, and (b) corpus-based activities supported by automated formative feedback and collective discussion. A qualitative analysis focusing on recurring patterns where correct procedures coexisted with difficulties in linguistic articulation, and on lexical choices indicating conceptual uncertainty. The results reveal a recurrent asymmetry: while many students successfully perform procedures, they struggle to articulate relational meanings. Variation in lexical choices and non-responses indicates uncertainty in expressing relationships between mathematical objects. Corpus-based activities make these tensions more visible. Although students often recognise linguistic patterns, they encounter difficulties when using them to construct mathematically meaningful statements. The combination of immediate feedback, guided exploration, and collective discussion can create a space in which mathematical meanings are examined and progressively stabilised.

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EXPERIMENTAL EVIDENCE FOR THE EFFECT OF TEACHER MATHEMATICS ANXIETY ON TEACHING BEHAVIOR

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Primary school teachers are affected by mathematics anxiety (MA), which involves feelings of fear, apprehension, and tension when engaging with mathematics (Ashcraft, 2002) and is more common among female teachers (Artemenko, 2021). While correlational research suggests that teacher MA may transfer to teacher behavior in the classroom, experimental evidence remains absent. Drawing on interpersonal theory (Leary, 1957), this study conceptualized teacher behavior along two fundamental dimensions: warmth (responsiveness, communion) and directivity (guidance, agency), recognizing that both are critical for student outcomes. In a preregistered randomized experiment, 172 preservice primary school teachers (80% female, 70% aged 20-29 years) were asked to imagine teaching either a simple (teaching the area of rectangles; low-complexity condition) or a complex mathematical topic (teaching Euler's polyhedron formula; high-complexity condition) and to describe how warm and directive they would behave toward the class in either scenario. Given that anxiety involves preoccupation with anticipated threats and increased internal focus, it was hypothesized that MA would impair teachers' sensitive attunement to students, manifesting in reduced warmth. Further, to maintain control and to avoid revealing perceived limitations in their expertise, it was expected that MA would be associated with greater directivity, particularly among women who are faced with the negative stereotype about their group's mathematics-related capabilities. Manipulation checks confirmed significantly higher MA in the high-complexity condition ($F(1, 141) = 20.08$, $p = < .001$, $\eta^2 = .13$). Latent change score analysis revealed that preservice teachers exhibited significantly less warmth ($\beta = -.53$, $p = .021$) and, regardless of gender, more directivity ($\beta = .62$, $p < .001$) in the high-complexity condition relative to the low- complexity condition. These findings provide the first causal evidence that MA may influence interpersonal teaching behavior. They underscore the importance of addressing MA in teacher professional development.

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DISCURSIVE CHANGES IN A TEACHER'S NARRATIVES IN A NEWLY FORMED LESSON STUDY COMMUNITY

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Lesson study is a fundamental system supporting teacher professional development in Japan (Fernandez & Yoshida, 2012). However, in lower secondary schools, subject specialization, declining student populations leading to single-teacher departments, and weakened district study groups limit opportunities for mathematics-focused lesson study. To address this issue, we established a cross-school lesson study community comprising mathematics teachers, university faculty, and representatives from the board of education. Within this project (Sato, 2025), the present study focuses on the first lesson study cycle and examines changes in the narratives of Mr. Fujita, who taught the research lesson, as he participated in the lesson study community.

This study adopts Sfard's (2008) commognitive framework, which conceptualizes learning as participation in discourse. From this perspective, professional development is analysed as discursive change, particularly as changes in the narratives through which teaching is structured and connected across lessons.

The community was launched on October 10, 2026. Following the kick-off, four additional online meetings were held to refine the lesson, resulting in approximately seven hours of discussion in total. Mr. Fujita taught the research lesson on spatial geometry for Grade 7 on February 17, 2027. Data consists of (a) transcripts of meetings and post-lesson discussions, and (b) four revised versions of Mr. Fujita's teaching plan. The analysis focused on identifying changes in his narratives across both spoken interactions and written planning.

Teachers from different schools used different words when discussing mathematics lessons. Clarifying the processes reified by these words became essential for constructing endorsed narratives within the community discourse. Through feedback from community members, Mr. Fujita became increasingly attentive to his word use, reflecting gradual discursive changes in how he framed his teaching.

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USING BIBLIOTHERAPY TO RELIEVE MATHEMATICS ANXIETY IN COLLEGE STUDENTS

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Mathematics anxiety is widely recognized as a consistent detriment to mathematics learning and performance (Luttenberger et al, 2018). Its presentations through avoidance, dissociation, memory issues, and panic appear to be very pronounced in developmental mathematics students and those taking general education college mathematics courses. Bibliotherapy is a personal growth tool that involves reading and responding to stories. It has four components that ensure its efficacy—relatability, normalization, modelling and reframing (EdTech Books, n.d.).

In this mixed methods study, we consider the research question: can a bibliotherapy story intervention about overcoming mathematics struggle (and specifically written for each type of course) aid in reducing mathematics anxiety for students in developmental algebra, mathematics for liberal arts, and algebra for elementary school teachers.

The AMAS (Worden, 2023) nine-question survey was administered as a pre- and post-instrument to a sample of 33 developmental students (17 matched), 27 liberal arts students (15 matched) and 14 pre-service teachers (12 matched). Students answered the same questions about their views and personal experiences with mathematics. Several weeks into the semester, students listened to and responded to their course's specific 3-minute bibliotherapy story, by reflecting on how the story was resonant, relevant, and able to provide them with a possible map to success in mathematics.

We saw a statistically significant decrease in the matched mean AMAS score across all three types of students, with the developmental group showing the largest decrease ($p=0.006$). 42.8% of students associated mathematics with anxiety, stress and confusion at the beginning, versus only 17.2% on the post-survey. There was a significant shift in seeing mathematics success as a process of perseverance, patience, and growth. We recommend further investigation into bibliotherapy's effects, as well as its use as a significant tool for reducing math anxiety among college students.

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THE IMPACT OF TEACHING THROUGH PROBLEM-SOLVING ON STUDENTS' ATTITUDES TOWARD MATHEMATICS

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Teaching Through Problem-solving (Takahashi, 2021) situates a carefully designed problem as the starting point for instruction, through which targeted mathematical concepts emerge. Student-generated solutions, including partial responses, are deliberately selected and sequenced to surface key structures and relationships. Whole-class discussion is structured to compare strategies and progressively formalize mathematical ideas while maintaining high cognitive demand. Rather than adding problems to instruction, the problem itself functions as the driving engine of mathematical learning. In enacting this approach, the teacher in this study also incorporated selected practices aimed at enhancing student engagement and thinking, such as the use of vertical non-permanent surfaces to support collaborative problem-solving (Liljedahl, 2021).

This study is part of a larger project and focuses on how teaching optimization using derivatives through a Teaching Through Problem-solving approach in two upper secondary classes (N=27) influence students' attitudes toward mathematics. Measuring mathematical attitudes can be performed at different levels and through various instruments. This study used the widely recognized Attitudes Toward Mathematics Inventory (Lim & Chapman, 2013), including its short form, which was revised and adapted for Swedish upper secondary students. We aimed to create a version suitable for students who struggle in mathematics and show low motivation, administering it as a pre- and post-test to capture potential changes in attitudes. Preliminary results across four analytical categories show varying patterns of change. In the category *The tendency to participate in or avoid mathematics activities*, positively worded items showed a slight increase in looking forward to lessons and a small decrease in finding challenges appealing. Negatively worded items revealed clearer shifts, including reduced reluctance to answer questions and less nervousness about solving problems. These results will be discussed and illustrated further in the presentation.

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THE OTHER WHEN INTERPRETING UNDERSTANDING IN MATHEMATICS: A PHILOSOPHICAL VIEW

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Interpreting students' mathematical understanding on the basis of their shared classroom activity remains an ongoing challenge, one that also involves specific aspects related to otherness (Abtahi et al., 2018). Among these are issues concerning students' intentionality and responsibility towards others, as well as the transformative effect of access to the understanding of the other (Guillemette & Dematte, 2022; Metz & Simmt, 2015). As a theoretical contribution, we examine the role of the other in interpreting mathematical understanding, drawing on Levinas's phenomenology of sociability and Ricœur's dialogical hermeneutics. Specifically, we broaden the ontological significance of understanding—as an internal cognitive phenomenon—and incorporate the responsibility for the other in the functional approach to interpreting understanding, defined in terms of the uses made of mathematical knowledge. We recognise the existence of a meaning inherent in understanding that is prior to—and irreducible to—intentionality and rational knowledge, and that points directly to otherness: understanding emerges when the self is called into question by the critical presence of the other, who continually addresses and concerns me. Thus, we propose an interpretation that provides a shared context appropriate to the call of the other, in which dialogue elicits responsibility, understood as the student's selfless willingness to approach the other and offer different responses to otherness—distancing, representation, and consent. This (ethical) responsibility enables, through dialogical consent with the other, the transformation that leads to an inclusive return to the student's mathematical understanding. Our interpretative proposal complements phenomenological methods of guided introspection developed for empathic access to students' mathematical experiences (Metz & Simmt, 2015). Moreover, it establishes integrative links between hermeneutics and ethics (Guillemette & Dematte, 2022) within the specific context of understanding in mathematics.

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EXPLORING MATHEMATICAL CONNECTIONS IN ARGUMENTATION WITH A RATIONAL-QUESTIONING GPT

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Although National Council of Teachers of Mathematics (2024) stresses the need to both develop and test generative artificial intelligence (GenAI) tools, current research predominantly focuses on testing them. The motivation of this study is based on the idea that rational questioning (Zhuang & Conner, 2022) expected of teachers during argumentation can also be provided to students through a GPT specifically developed for this purpose. This study aimed to develop and test a customized GPT for rational questioning, addressing the research question: How does such a GPT affect students' mathematical connections? Data were collected using a pattern generalization task designed to investigate mathematical connections involving participants' numerical understanding of the derivative concept. The study involved a university student in a mathematics education programme who had sufficient mathematical background to complete the task and experience using GenAI tools. The participant first worked individually and was then given the opportunity to use the customized GPT to complete or refine the solution. Data were analysed using Habermas' (1998) theory of rationality and García-García and Dolores-Flores' (2021) mathematical connections framework. Findings indicate that the customized GPT supported the participant in making correct mathematical connections but exhibited limitations in sustaining rational questioning, as its performance was influenced by the participant's questions. Future studies should explore the sustainability of GPT's questioning and test it with varied tasks and diverse participants.

Acknowledgment: This work was supported by The Scientific and Technological Research Council of Türkiye (TÜBİTAK) under the 2224-A Grant Program.

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HOW DO PRE-SERVICE EARLY CHILDHOOD TEACHERS ENVISION THE TEACHING OF MATHEMATICS?

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Early childhood education is critical for the development of mathematics skills, but research on the knowledge and conceptions of early childhood education (ECE) teachers about mathematics teaching and learning is scarce. Building upon Alsina and Delgado-Rebolledo's (2022) model of the knowledge of the ECE teacher to teach mathematics, we investigated how do pre-service ECE teachers envision the teaching and learning of mathematics in a Kindergarten classroom. In particular, we aimed to uncover their conceptions about what mathematics are taught in this educational level, and how. To do this, we asked 18 fourth-year pre-service ECE teachers from a Chilean public university to *draw a Kindergarten classroom in which the Mathematical Thinking Nucleus of the Chilean ECE curriculum is being developed*. Participants' drawings were analyzed using a filter integration method (Makramalla, 2018), considering a series of questions tapping on aspects such as: What mathematical concepts are evident in the classroom? How are these concepts presented? What objects/materials are in the classroom? Where is the class taking place (indoor/outdoor)? What are children expected to do? The answers to these questions allowed us to characterize the classrooms drawn. The analysis was done using Atlas.ti, double-coded by two of the authors and solving disagreements through discussion. The coding process led us to infer a gradient of five conceptions related to mathematics and its teaching and learning in ECE: classrooms focused on 1) numbers and arithmetic presented in a traditional format; 2) the names of numbers, figures, and objects, allowing for their manipulation; 3) the association between numbers and their related quantities, using concrete materials; 4) problem solving through guided play; and 5) the identification and generalization of visual patterns. We will discuss what kind of mathematical knowledge and how to teach it emerges in each of these classrooms, as well as the role of the teacher within them.

Acknowledgements. This research was supported by the Chilean Agency of Research and Development, through the funds Fondecyt 3250855, Milenio NCS2021_014, and CIAE CIA250005.

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PRESERVICE TEACHERS' FLEXIBILITY IN NON-ROUTINE PROBLEM SOLVING

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Solving non-routine problems often requires modifying or switching strategies, as initial approaches may not lead to a solution. This ability is referred to as flexibility in problem solving. Previous research has examined flexibility within and across problems differing markedly in surface features and structure, primarily focusing on students (e.g., Elia et al., 2009). Yet, non-routine problems in which the need for a change in strategy is not immediately apparent due to superficial similarities and subtle structural differences have received little attention in flexibility research. Teachers, as students' role models, are expected to solve such problems flexibly. However, it remains unclear how pre- and in-service teachers deal with non-routine problems that are superficially similar yet non-isomorphic, thereby requiring flexibility.

Therefore, this qualitative exploratory study aims to examine elementary school preservice teachers' strategy use and flexibility using a design based on such problems. A total of 48 participants worked individually on one of three series of systematically varied non-routine mathematical problems in videotaped semi-structured think-aloud interviews. Each series consisted of three problems that shared surface features (i.e., context, numerical values) and varied structurally in subtle ways, potentially obscuring these differences. Data were analyzed using a process-oriented qualitative content analysis focusing on participants' use, modification, and switching of strategies.

Overall, only a minority of participants exhibited flexibility exclusively across the problems of a series, primarily through modifying strategies. In many cases, flexibility emerged only after unsuccessful attempts, often by switching to a trial-and-error strategy. Furthermore, participants rarely engaged in further attempts once a solution had been reached, indicating limited flexible exploration of alternative strategies.

The findings reveal that series of problems with subtle structural variations can make preservice teachers' levels of strategic flexibility in problem solving visible in ways that might otherwise remain less apparent. In teacher education, flexibility should therefore be explicitly addressed, particularly by supporting preservice teachers in reflecting on their strategy use and their responses to unsuccessful attempts, fostering modifications of strategies, and considering further attempts beyond initial solutions.

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TEACHERS' ENACTMENT OF ASSESSMENT RUBRICS

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This study stems from an ongoing research project focusing on the transition from school to university, with particular attention to students' development of mathematical competencies. In preparing students for this transition, we foreground helping them grow toward independent and responsible learners. To support students in this objective, we design evidence-seeking activities with criterion-referenced rubrics—focusing on two mathematical competencies: "argumentation" and "representing and communicating"—and teachers' feedback throughout. Current research has examined the positive effects of rubrics with evidence-seeking activities (Nieminen et al., 2021) on students' learning. To meaningfully engage in evidence-seeking, teachers' own ways of identifying and articulating evidence become crucial, as they implicitly legitimate evidence in the discipline. Research in this field typically does not analyse how teachers enact these practices; this study addresses that gap. The analysis of teachers' use of rubrics is framed within the Documentational Approach to Didactics (Trouche et al., 2018), which studies how teachers enact—that is, appropriate and adapt—resources designed by others. On this basis, the research question is (RQ):

How do teachers enact the designed rubrics in their assessment practices as outlined by the kinds of evidence they identify and articulate?

Data came from a one-hour video-recorded session in which three teachers watched together three videos of student group presentations from a prior classroom implementation, assessed them using the rubrics, and discussed the evidence for their assessments. To address the RQ, we analysed the transcript through a stepwise process: examining how teachers appropriated rubric language, what they treated as evidence for their judgments, and which elements they adapted autonomously.

The findings provide insight into teachers' documentational work as a dynamic interplay of appropriation and adaptation, constituting an initial exploration of teachers' enactment of rubric-based assessment. As future work, the activity will be implemented in classroom settings.

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REALITY AND GEOMETRY IN THE CURRICULUM

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This study builds on research in curriculum theory and Mathematics Education (Shimizu & Vithal, 2024) that examines the relationship between school knowledge and reality, particularly in Geometry. Previous studies highlight that Geometry can become more meaningful when connected to students' lived experiences, contributing, for instance, to their understanding of space and geometric objects (Kageyama, 2018). At the same time, critical perspectives on reality, such as those proposed by Kosík (1976), warn against reducing reality to its immediate appearance, while Freire (2000) emphasizes the need for education to foster critical consciousness and transformation. Within this framework, Geometry is understood not only as a tool for describing the physical world but also as a means of interpreting and critically engaging with it.

The study presents results based on a qualitative documentary analysis of the Brazilian curricular document Base Nacional Comum Curricular – BNCC (Brasil, 2018). Focusing on the Mathematics section, the analysis examined how reality is mobilized in relation to Geometry across different stages of schooling.

The results indicate that the BNCC consistently establishes connections between Geometry and reality, particularly through spatial experiences such as observation, location, and recognition of shapes in the environment. However, the analysis also reveals that the relationship between reality and Geometry is predominantly framed in functional terms. From a critical perspective, this may limit the potential of Geometry to promote deeper understandings of reality, as the social, cultural, and historical dimensions of real-world contexts are not always made explicit.

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INDICATIONS OF SUBSTITUTE-RELATIONAL CONCEPTION IN YOUNG STUDENTS' TALK ABOUT THE EQUAL SIGN

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For many years, research on students' understanding of the equals sign (see, e.g., Kieran, 1981) focused on the operational (equal sign as a trigger of "to do" something) versus the relational (the equal sign describes the relation between numbers) aspect of the symbol. The conclusion has been that students need to embrace the relational understanding to manage algebraic expressions, particularly equations. In more recent research studies, students' understanding of the relational aspect of the equals sign has been more nuanced. For instance, Jones et al. (2013) discuss the relational conception in terms of a sameness-relational conception or a substitute-relational conception, and advocate making students aware of the substitute-relational conception when teaching the concept of the equal sign. Their study was made on students in the first years of secondary school (aged 11-12 years). However, the equal sign is typically taught to students from very young age. Our question therefore is: Can students, already in the early school years have the substitute-relational conception of the equal sign?

From a previous project (Gunnarsson & Persson, 2024), we have data with 12 students in grades 1-3 (7-10 years old) when they talk about the equal sign. The students were interviewed in pairs and shown different types of expressions with equal signs and discussed these. Hence, triggered by the study of Jones et al. (2013), our data was re-analyzed, now focusing on the different relational aspects of the equal sign by using codes from Jones et al. (*ibid*.).

In this short presentation, the result is described in brief, concentrating on how the young students' talk includes utterances about the different relational aspects of the equal sign. Many of the students give utterances that can be categorized as a sign of a relational conception of the equal sign. At least some of the students give utterances that can be indications of a substitute-relational conception.

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BRIDGING INDIVIDUAL DIFFERENCES IN MATHEMATICS: DESIGN AND EFFICACY OF INTELLIGENT PERSONALIZED HOMEWORK BASED ON COGNITIVE DIAGNOSIS

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As uniform homework overlooks individual cognitive differences, leveraging intelligent technologies for personalized design has become an essential reform. Specifically, empirical findings indicate that cognitive diagnosis models (CDMs) serve as the foundation for designing effective personalized interventions (Wu et al., 2022). However, since most existing studies are confined to constructing theoretical frameworks or macro-implementation strategies, systematic empirical evidence regarding their effectiveness in authentic teaching contexts remains scarce.

This research aims to empirically investigate the efficacy of personalized homework underpinned by intelligent cognitive diagnosis in providing precise learning support. Consequently, the study selects "Quadratic Functions" as the focal case, specifically because its abstract complexity often precipitates cognitive gaps, rendering uniform standardized homework ill-suited for students with varying cognitive needs. To accurately diagnose mastery, a cognitive diagnostic assessment was developed, the results of which directly informed the design of personalized homework assignments.

The sample comprises 59 Grade 9 students from two classes. An independent Kruskal-Wallis test confirmed baseline equivalence in pre-test performance. Subsequently, the classes were assigned to experimental condition (personalized homework) and a control condition (standard assignments) for a six-week intervention, followed by a post-test. The findings reveal that the experimental group achieved a superior cognitive progression rate (46.66% vs. 31.03%). Furthermore, they demonstrated a positive academic trajectory (Cohen's d=0.35) contrasting with the control group's decline, validating the intervention's efficacy. Moreover, the long-term impact of diagnostic-based homework warrants longitudinal investigation, given the importance of sustained cognitive development. The presentation will further detail specific implementation strategies and pedagogical implications.

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COMPARING GRADE 5 MATHEMATICS FROM TÜRKİYE AND THE USA THROUGH A SYNTHESIZED MATHEMATICS LEARNING TAXONOMY

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This study employs Gun and Bossé's (2025) Synthesized Mathematics Learning Taxonomy (ST) as its framework. The ST distinguishes between Knowledge Level (mathematical task (action) and mathematical learning) and Cognitive Level (mathematical processes (actor) and problem solving) taxonomies. The ST recognizes the levels of mathematical understanding: (1) Knowledge and Routine Procedures; (2) Conceptual Understanding and Relational Thinking; (3) Strategic Problem-Solving; (4) Analytical and Extended Abstract Thinking; (5) Real-World Application and Synthesis; and (6) Creative Problem-Solving and Innovation. Our research question is: How do the ST levels of 5th-grade mathematics learning outcomes under the Türkiye Century Maarif Model (TCMM2024) compare with the standards of North Carolina (NC2025)? From recent studies, gaps in the literature remain: most studies have analyzed curricula through at most one learning-level lens; no studies have yet compared NC2025 with TCMM2024; and few studies focus in detail on a single grade comparison. The ST addresses all three gaps simultaneously. Two researchers employed document analysis.

The findings are distilled into two themes: (a) considering the ST levels by outcomes and standards, (b) ST levels by themes and domains, and (c) a comparative analysis. (Data and results are hyperlinked here.) This study found that while TCMM2024's learning outcomes were heavier in ST Levels 2 and 4, NC2025's standards focused on Levels 2 and 3. TCMM2024 saw its highest share of Level 4 outcomes in the Algebraic Thinking with Operations, Numbers and Quantities, and Geometric Shapes. NC2025 demonstrated a relatively smaller set of Level 4 standards, with no Level 5 standards. Little difference was found in overall rigor. TCMM2024 more often anticipated students' abstract thinking, whereas NC2025 focused more on students' solving modeled word problems. This all leads to two implications: First, in TCMM2024, researchers can examine how learners transition from relational explanations (Level 2) to generalized propositions (Level 4). Second, in NC2025, researchers can study how learners' strategy selection, representation shifts, and reasonableness checks stabilize over time (Level 3) and when structural justification emerges (Level 4).

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EXAMINING TEACHERS' DESIGN OF MODEL-ELICITING TASKS IN VIRTUAL FIELD TRIP CONTEXTS

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Virtual Field Trips (VFTs) are often positioned as powerful learning environments that enhance students' engagement and sense of presence (Dede, 2009). Their interactive structures support active learning processes by enabling learners to investigate, respond, and make decisions within contextualized environments (Freeman et al., 2014). Grounded in place-based learning principles, such environments offer opportunities to situate mathematical thinking within socially and spatially meaningful contexts (Asakle, 2014). However, contextual immersion alone does not ensure mathematical sense-making. The Models-and-Modeling Perspective (MMP) (Lesh & Doerr, 2003; Lesh & Lehrer, 2003) offers a framework for conceptualizing modelling as the development of explicit, shareable, and revisable mathematical systems. From this perspective, learning is not the production of answers but the construction and refinement of models through iterative cycles of expression, testing, and revision (Lesh, 2006). Model-Eliciting Activities (MEAs), grounded in six interrelated design principles (Lesh et al., 2000), operationalize this view by requiring tasks to support model construction, documentation, self-assessment, shareability, reusability, and grounding in realistic contexts. This study investigates: (1) How do secondary mathematics teachers design ThingLink-based VFT tasks when aiming to integrate mathematical modelling? (2) To what extent and in what ways do these designs align with MEA design principles?

Adopting a qualitative design within a modelling-oriented professional development program, the study involves 12 secondary mathematics teachers selected through purposive sampling based on sustained engagement in the design process. Teachers were invited to design place-based VFT tasks using ThingLink, integrating mathematical modelling into digitally represented spatial environments. Data sources include teacher-designed task artefacts, embedded prompts within ThingLink environments, and written design rationales. Using the six MEA principles as analytic lenses, the study examines whether teacher-designed VFT tasks support model construction, documentation, shareability, reusability, and iterative refinement.

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FROM THE MATHEMATICAL OUT-OF-SCHOOL LAB TO THE CLASSROOM – EXPLORING THE UPTAKE OF CONTENT IN SCHOOL MATHEMATICS LESSONS

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Previous research on out-of-school learning particularly emphasises the importance of meaningfully linking outer-school learning, such as learning in out-of-school labs (OSLs), and inner-school learning to achieve long-term effects (e. g., Haidari et al., 2024). Against this background, we explore how the content of a mathematical OSL project is taken up in school mathematics lessons (research objective 1) and to what extent students are able to connect their experiences made during the OSL project with the classroom content (research objective 2). For this purpose, we used a qualitative approach and analysed data from the larger research project "MerLab" (Mathematik erleben im Lehr-Lern-Labor; experiencing mathematics in an OSL), funded by the Reinhard Frank-Stiftung. The data analysed here were obtained from written classroom observations of two 90-minute mathematics lessons of a 6th grade class and semi-structured interviews with six students in pairs from that class (four female, two male; M_{age} = 10.83, SD = 0.41) after participating in a day-long mathematics OSL project entitled "Discovering symmetries in a game and sports museum" at a large German university. The OSL project aimed at enabling students to engage with different types of symmetries by working in small groups on various hands-on tasks and with digital elements focusing on activities of constructing and analysing symmetrical figures as well as verbalizing symmetry properties. To analyse the data, we used thematic analysis (Braun & Clarke, 2006). With regard to research objective 1, we observed that the mathematics teacher explicitly took up selected assignments and excerpts from the lab booklet the students used during the OSL project in her mathematics lessons. Furthermore, she implicitly drew on the vocabulary introduced during the OSL project. Regarding research objective 2, we observed that the students primarily remembered the exploration of different figures for symmetry during the OSL project. In addition, two of the interviewed student pairs pointed out that they had reused an excerpt from the lab booklet in the school mathematics lessons.

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AN OPENNESS SPECTRUM IN MATHEMATICAL MODELLING

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Mathematical modelling is often treated as inherently "open", as it typically involves making assumptions, selecting variables and representations, and iteratively interpreting and validating results (Stender & Kaiser, 2015). Although modelling literature distinguishes openness of the initial state, goal state, and transformation (Wiehe et al., 2025), modelling tasks are often defined in binary terms, i.e. as open or closed. This raises the question of whether the theoretical framework can be expanded to conceptualize degrees and types of openness along a spectrum that explicitly accommodates less-open modelling tasks. The current study raises the research question of whether and how a modelling task can be defined and coded along an openness spectrum?

The study follows an iterative design-based process combining construct synthesis, rubric construction, reliability testing, and validation, through task coding. Openness indicators were first derived by synthesizing modelling and open-task literature (e.g., Wiehe et al., 2025). They were then tested through independent coding of a corpus of 10 modelling tasks sampled from literature (e.g., Wess et al., 2021) using a designed rubric. Inter-rater reliability was examined, and descriptors that generated systematic disagreement were revised, enabling comparisons across task sets while preserving interpretability. Analyses of the tasks based on the designed rubric revealed multiple openness levels within tasks used in existing research, as well as different combinations of task characteristics that converge to the same overall openness degree. The study contributes theoretically by reframing openness in modelling as a spectrum that includes less-open and closed-ended modelling tasks; methodologically, by developing a rubric that yields an openness profile and overall quantitative degree of openness; and practically, by enabling clearer task design and comparison through distinguishing openness from guidance and scaffolding.

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CONCEPTUALIZING PHASES OF JUNIOR HIGH SCHOOL STUDENTS' PROGRAMMING-BASED MATHEMATICAL PROBLEM-SOLVING

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The integration of programming into mathematics classes creates new opportunities for fostering both computational thinking (CT) and mathematical thinking (MT) in embodied digital environments. As CT represents a problem-solving methodology (Wing, 2006), it shares cognitive connections with mathematical problem-solving. However, prior studies have either focused on general problem-solving processes (Pólya, 1945) or universal CT practices (Shute et al., 2017), neglecting the specific trajectory of mathematical problem-solving in programming contexts. This study views programming as an embodied digital practice, aiming to characterize the cognitive trajectories of programming-based mathematical problem-solving.

The research questions are: (1) What distinct cognitive phases and transition paths emerge in embodied programming-based mathematical problem-solving? (2) How do CT and MT function and interact across the phases? Data were collected from a 16-lesson mathematics course at a Chinese junior high school, including 35 students' programming artifacts, recordings of coding processes and discussion transcripts.

Episodic segmentation and thematic analysis identified three core phases encompassing six sub-processes, which accounted for over 95% of students observed problem-solving behaviors. The mathematization phase includes symbolic abstraction and relational abstraction, dominated by the abstract nature of MT. The algorithmization phase involves module decomposition and logical sequence design, guided by MT-CT synergy. The automation phase focuses on execution validation and system optimization, driven by CT's practical dimension. State-transition analysis further revealed that 42% of the problem-solving processes followed linear paths, while complex problem-solving exhibited non-linear recursive loops (58%). These findings demonstrate how programming serves as an embodied cognitive mediator to support the meaningful integration of computational and mathematical thinking.

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NOTICING THE SERVICEABILITY OF STUDENT WORK IN A SIMULATION OF PROBLEM-CENTERED INSTRUCTION

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We contribute to research on teacher learning from digital simulations. Problem-based instruction (PBI) uses problems as instruments to launch students into activity leading to new mathematical ideas. In PBI, teachers must notice whether student work has features serviceable for progressing toward instructional goals, though other features of student work (e.g., its correctness, its compliance with norms of the situation used to frame the work, or its responsiveness to the question asked) might also be noticeable (Herbst et al., 2023). To the extent that PBI is desirable, teacher learning environments should support development of disposition to notice serviceability. Our simulation series is situated in a lesson in which a problem of creating a circle tangent to two intersecting lines at two given points is used to develop the theorem that tangent segments are equidistant. We report on data from two simulations. One of them engages teachers in teaching the entire problem-based lesson and is used pre- and post-intervention. In this simulation the teacher needs to, among other things, select student work for a classroom discussion. The other, intervening simulation focuses teachers' attention on noticing student contributions through sorting samples of student work into bins. All simulations were completed by a group of 29 practicing teachers who had, on average, 16 years and 11 years of experience teaching high school mathematics and high school geometry respectively. We looked for evidence that individuals' noticing of serviceability in the intervention was associated with increased selection of serviceable work in the full lesson. A Principal Component Analysis allowed us to identify a variable to account for teachers' sorting activity that indicated noticing of serviceable work. Teachers who increased selection of serviceable work from pre- to post- performance in the full lesson tended to also score at higher values of that identified variable though teachers who did not show increase their selection of serviceable work were more widely distributed across the range of this variable. The analysis follows a pupil-mediation paradigm for research on teaching applied to teacher learning by identifying the strength to which participants' noticing of serviceability had a strong effect size predicting who would improve their serviceability scores.

Additional information: Work supported by grant 220020524 from the James S. McDonnell Foundation.

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HOW DO I DIVIDE TO GET THE RESULT I WANT?

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Szkudlarek et al. (2022) showed that young children have a sense of division before meeting the division sign. We are interested in what children can do with regard to multiplication and division in an environment that does not explicitly indicate either are involved, yet require both to be used. Our specific research question is: What issues do young children find with division when engaged with an app called *Stick and Split*?

Video recordings were taken of five pairs of 5-6 year olds and five pairs of 6-7 year olds using the app for the first time. Each pair had two 30-minute sessions. The app shows a target number above rods of certain lengths. The goal is to make a series of rods of the target length by either sticking rods of equal length together, or splitting rods into equal parts. The tasks started with rod lengths being factors of the target number to tasks where, for example, 24 had to be made with varying length rods (Figure 1). Analysis developed themes which occurred across the pairs of students.

Figure 1. Screen shot of *Stick and Split*

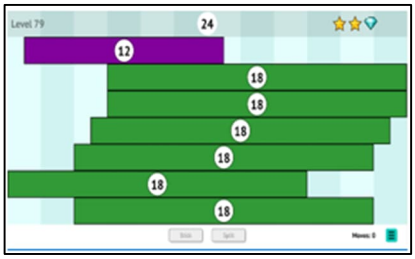
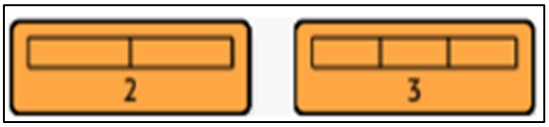


Figure 2: Split options



On tasks which required splitting of rods, even though students knew which rod length they wanted, they were unsure how many equal parts to divide a longer rod to achieve that. For example, one pair tried to split a 6 rod into three 2-rods. On clicking 'split' the app presented the option to split into two or three equal parts (Figure 2). The split-into-two option was chosen which, to their surprise, caused two 3-rods to appear instead of the three 2-rods they were expecting. We observed that students repeatedly confused the desired result with the divisor. We inferred that students showed awareness that rods can be split equally to their wanted length, but that they struggled to know which divisor would achieve that. This supports the notion that young students can have a sense of division by knowing that one number is a factor of another, but not know so easily which divisor is required to obtain that factor.

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CHILDREN'S EARLY-STAGE PROPORTIONAL REASONING IN INTRODUCTORY LESSONS ON MULTIPLICATION

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Given its educational importance, proportional reasoning has been widely studied, and its structure, key aspects, and developmental indicators have been extensively identified. Among these, unitizing, covariance, and invariance are consistently highlighted as central (e.g., Lamon, 2020). However, promoting the development of proportional reasoning in school settings remains a challenge. In particular, how teachers notice students' thinking and provide appropriate support in everyday classroom interactions continues to be an important issue. This study examines the kinds of early-stage proportional reasoning that children demonstrate through interactions with teachers and peers in a second-grade classroom (ages 7–8) where multiplication is introduced. Drawing on Jeannotte and Kieran (2017), proportional reasoning is conceptualized as a discursive activity in which narratives about two covarying quantities in proportional relationship are generated and developed through processes of exploration and validation. The term "early-stage" is used to inclusively capture children's reasoning, including approaches based on counting or additive relationships. Three lessons were designed and implemented on November 17, 18, and 21, 2022, in a class of 33 pupils at a university-affiliated elementary school in northern Japan. The lessons were grounded in a familiar context in which children recalled visiting a local temple shop and took on the role of shopkeepers explaining drink prices to customers. The unit price was fixed at 130 yen, regardless of drink type. Tasks were designed to promote unitizing with composed units ("○ bottles for □ yen"), supported by a photograph of price table used in the shop that highlighted both the constancy of the unit price and the relationship between the number of bottles and total cost. The lessons were video-recorded, transcribed, and analyzed qualitatively. Ten episodes on early-stage proportional reasoning were identified and examined from structural and process aspects (Jeannotte & Kieran, 2017). Children compared known information and justified their methods, leading to increasingly flexible use of composed units to make claims. The underlying idea of invariance ("all unit prices are the same") was gradually made explicit in their narratives; analysis of the price table served as an important opportunity. In both cases, the teacher's support—through the visual and coherent structuring of children's narratives—became evident.

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AUTHENTICITY OF MATHEMATICAL LITERACY OPPORTUNITIES IN TEXTBOOKS: A COMPARATIVE STUDY OF TAIWAN AND THE INTERNATIONAL BACCALAUREATE

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A central goal of mathematics education is to support students in connecting mathematical concepts to real-world situations. Yet many educational materials lack authenticity or are merely dressed in contextual language, leading students to disconnect mathematics from lived reality and fail to develop mathematical literacy (Incikabi, 2025). Taiwan, a high-achieving East Asian country long emphasizing formal mathematical knowledge, launched a new curriculum in 2019 promoting mathematical literacy for real-world application. However, whether this emphasis is reflected in the potentially implemented curriculum—textbooks—remains unclear. This study adopts the eight-aspect authenticity framework and scoring system proposed by Incikabi (2025) to examine the authenticity of a current Taiwanese mathematics textbook and to compare it with an International Baccalaureate (IB) textbook, selected for its emphasis on mathematical literacy. The trigonometry unit, given its strong connection to real-life applications, was chosen for analysis. A document analysis was conducted on 59 and 41 items drawn from Taiwanese and IB textbooks respectively. The framework evaluates eight aspects: event, question, language use, solution requirements, existence of information, realism of information, specificity of information, and affective purpose. Each item was rated on each aspect using a 3-point sufficiency scale (0 = insufficient, 0.5 = partially sufficient, 1 = sufficient). Based on total scores and the presence or absence of specific aspects, items were classified into three levels of authenticity: good-fit, stereotyped, and poor-fit. Only 39% of the Taiwanese textbook items qualified as authentic tasks, compared to 61% in IB textbooks. However, the number of good-fit items is comparable (TW: 13; IB: 15). In both textbooks, authentic items generally demonstrate relatively high sufficiency in language use, solution requirements, and existence of information. Nevertheless, more than half of the good-fit items in both textbooks are rated only as partially sufficient in realism of information and affective purpose. A notable discrepancy emerges in specificity of information: while only 13% of IB items fall below sufficiency, 46% of Taiwanese items do so. These findings highlight a gap between the curriculum's literacy-oriented goals and their realization in textbooks, and provide important implications for pedagogical practices and textbook design.

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REGIONAL VARIATIONS IN ACADEMIC RESILIENCE: MACHINE-LEARNING EVIDENCE FROM NORDIC AND EAST ASIAN COUNTRIES IN PISA 2022

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PISA 2022 results indicate that socioeconomic status (ESCS) leads to a math achievement gap of four to five years of schooling, a disparity intensified in the post-pandemic era (OECD, 2023). However, certain disadvantaged students demonstrate exceptional persistence and motivation, achieving success far beyond expectations—a group defined as "Academically Resilient Students" (ARS) (Xenofontos & Mouroutsou, 2023). As resilience factors are deeply culturally embedded and pathways to success vary across systems (Agasisti et al., 2018), comparing East Asia and the Nordic region enables the identification of regional feature heterogeneities. This study defines ARS as students in the bottom 25% of the national ESCS index who rank in the top 25% globally for math performance (PVMATH1). Utilizing Random Forest algorithms to analyze 61 student-level features across East Asia (Taiwan, Hong Kong, Japan, Korea, Macau, and Singapore) and the Nordic region (Denmark, Finland, Iceland, Norway, and Sweden), model performance was evaluated using Receiver Operating Characteristic - Area Under the Curve (ROC-AUC). In intra-regional identification, the ROC-AUC reached 79.03% for the Nordic region (N=7,804; ARS ratio: 16.61%) and 83.13% for East Asia (N=8,606; ARS ratio: 45.34%). Inter-regional classification demonstrated even greater power; across the total sample of 16,410 disadvantaged students, the ROC-AUC for distinguishing between regions was 95.04%, rising to 96.36% for the ARS cohort. This indicates that inter-regional cultural structures exhibit higher discriminability and separability than intra-regional resilience characteristics. Although intra-regional performance is slightly lower, the models remain highly effective in defining academic resilience, providing robust evidence for its strong context-dependency. By leveraging big data mining to identify cultural heterogeneities, this study demonstrates that academic resilience is not a universal trait but a context-dependent construct, necessitating highly culturally-sensitive educational interventions.

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EXPLORING ELEMENTARY STUDENTS' MATHEMATICAL COMMUNICATION SKILLS WITHIN THE PROBLEM-SOLVING CYCLE

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Mathematical literacy tasks require students not only to compute but also to communicate how they make sense of contextual information. In the PISA mathematics framework, these demands relate to the modelling cycle—formulating, employing, and interpreting/evaluating—and to communication as a key 21st-century skill (OECD, 2023). In Taiwan, the tools for assessing students' mathematical communication in literacy-oriented tasks are still scarce. This study investigates how Taiwanese sixth graders' mathematical communication difficulties vary across stages of the cycle. Mathematical communication is defined as three sub-skills: accessing/retrieving (A), integrating/interpreting (I), and representing/expressing (E).

Research questions. (1) How do students' A/I/E performances vary across stages of the cycle? (2) What recurring communication error types characterize each stage? We developed a questionnaire based on contextualized quantitative literacy situations, using a two-dimensional blueprint crossing problem-solving stages with communication sub-skills. Data were collected from 696 sixth-grade students. Responses included selected and constructed items and were coded using scoring guidelines (double scoring with adjudication). Analyses combined (a) a partial credit approach with IRT to estimate A/I/E abilities and (b) stage-focused profiling with qualitative summaries of recurring error types. The findings provide evidence for a stable "A-advantage" profile in literacy-oriented tasks; that is, students perform strongest in A (67.95), while I (53.68) and E (55.00) remain major learning needs. Stage comparisons reveal differentiated bottlenecks: (a) during formulating, students most often struggle to integrate information into a solvable mathematical problem; (b) during employing, weak integration is frequently accompanied by procedure-first attempts (e.g., number-plugging) with limited justification; (c) during interpreting/evaluating, expressing becomes the weakest, reflecting high demands in communicating meaning and reasonableness through representations and explanations. Error patterns suggest cumulative effects, where early misinterpretations and incomplete retrieval constrain later integration and expression. Implications include diagnostic feedback and targeted scaffolds for integration, justification, and explanation (interpreting/evaluating).

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TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE FOR TEACHING LENGTH MEASUREMENT AND ESTIMATION

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Teachers' pedagogical content knowledge (PCK) for involves skills of transforming content knowledge (CK) of particular topics into teaching practices. Such knowledge enables teachers to take proper actions in their teaching practices and contributes to their students' achievement. A cross-country comparison revealed that Taiwanese and German primary school students developed different profiles in their length estimation skills (Hoth et al., 2023). German students were found to have superiority in the situation where the to-be-estimated objects [TBEOs] were not small and not touchable, whereas Taiwanese students had better skills in the situations where the TBEOs were small as well as not small and touchable. It is assumed that teachers hold different culture-specific pedagogical views on classroom practices, shaped by their educational contexts (e.g., curricula, teacher training), which lead to variances in teaching practices and cause the different estimation skill profiles.

Sixty in-service primary school teachers (30 Taiwanese [T-Group]; 30 German [G-Group]) completed the PCK questionnaire on a) CK focusing on length measurement and estimation (LME) concepts, b) instructional actions for estimation activities, and c) approaches for teaching estimation strategies centring on levels of tendencies of using benchmarks and operation-and-touching TBEOs. The findings show significant differences between the two groups in all three sub-sections with medium to large effect sizes. The T-group had better CK of LME than the G-group. The result is consistent with those of Tatto et al. (2012). Although both groups preferred to choose TBEOs from classrooms and hall-ways, the G-group provided students more objects selected from school sport fields. Moreover, the G-group placed higher value on constructing benchmarks by memorizing the lengths of daily objects used frequently than the T-Group did. The differences in teachers' instructional actions and teaching strategies for estimation may help explain the results of Hoth et al.'s (2023) study.

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TEACHER PRACTICES ASSOCIATED WITH DIVERSE STUDENTS THRIVING IN MATHEMATICS

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Developing equitable education systems in relation to mathematics is an international goal. Research studies have shown that teacher practices influence student mathematical outcomes (Lynch et al., 2017) and there is a relationship between general student wellbeing and mathematical achievement (Estipona & Delos Santos, 2025). This highlights the potential for the classroom environment and teacher practices to influence equitable outcomes in mathematics. In this presentation, we examine the influence of teacher practices on mathematical outcomes for diverse students. We ask: *What pedagogical practices are evident in classrooms where diverse students are thriving in mathematics?* An ongoing professional learning and development (PLD) research project, Developing Mathematical Inquiry Communities (DMIC), focuses on supporting teachers to shift mathematics teaching to better serve diverse students in New Zealand. Drawing on data from a larger longitudinal study including student questionnaires, achievement data, and classroom observations, we present an exemplary case study of one primary teacher of Year 5 students and her practices following the DMIC PLD. Students in this classroom had substantially higher levels of self-reported mathematical wellbeing than other Year 5 students and strong levels of mathematical achievement. We examine the pedagogical practices used by the teacher that supported her class of Māori and Pacific students to thrive mathematically.

Our findings show that the teacher consistently drew on five pedagogical practices in classroom observations for which she showed high-quality enactment throughout entire lessons. This differed from the observation results in other classrooms. These practices included the use of challenging mathematical tasks, displaying high expectations, supporting substantive mathematical conversations, facilitating mathematical practices, and developing social norms. We argue that given evidence of diverse students thriving in this classroom, examining the pedagogical practices of the teacher offers important insights for equity.

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IMPROVING LOW MULTIPLICATION AND DIVISION FLUENCY THROUGH A DIGITAL GAME

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Some children continue to struggle with basic arithmetic skills, such as multiplication and division fluency, long after these topics have been taught in school (Psyridou et al., 2023). Because multiplication and division are foundational tools in many mathematical contexts, these children may benefit from targeted basic-skills training even when the curriculum moves on. Digital learning games offer one way to provide such training, though more research is needed on their effectiveness for children with learning difficulties (Alam et al., 2025). The aim of this study is to examine learning effects of the digital mathematics game REI-O for children with low multiplication and division fluency. REI-O game was designed to include different kinds of practice in adventurous gaming world. For example, it includes adaptive repeated practice with immediate feedback, conceptual instruction in form of a dialogue and practice, and minigames that require flexible consideration of different solution strategies.

Among voluntary participants in grades 4–6 (10–12 year-olds), children with dysfluency in multiplication and division skills were selected for the game intervention group to play REI-O for 12–15 minutes three times a week. In case of 4th graders, there was an enhanced control group receiving paper-and-pencil training. Multiplication and division fluency measure was implemented in two pre-tests and a post-test. After excluding incomplete cases, the game intervention group contained 85 and the control group 38 children. The data was analysed with repeated-measures ANOVA.

The increase in multiplication and division fluency during the intervention was statistically significant, with medium effect size. There was no difference between the game and paper-and-pencil groups for the 4th graders. The findings suggest that REI-O game can effectively support fluency development for children needing additional practice. Furthermore, teachers can choose between REI-O game and paper-and-pencil materials based on other considerations.

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WORDS FOR =: HOW DO THE WORDS WE USE RELATE TO THE MEANINGS WE HAVE

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The equals sign is used and interpreted in different ways in mathematics classrooms, intended to convey particular relationships associated with equality and equivalence. An understanding of the meanings associated with =, and flexibility in the uses of these meanings, are widely considered to be fundamental to algebraic thinking (Kieran, 2022). Distinctions are often made between an operational view, a relational view and a substitutive view (Jones & Pratt, 2012). Teaching students to recognise this polysemous nature of the equals sign is widely explored in existing research, but more recent work on using meaning-related language reveals how the words we use may support, or not, this teaching (Ward-Penny et al., 2026; Zentgraf & Prediger, 2024). In this study, we explore the relationship between the verbal representation of = and the meanings that mathematics teachers associate with these verbal representations.

An online survey was designed as a pedagogic activity for use during professional development workshops. It was based on the different examples of how the = sign is used in school mathematics, and the words used by teachers during their lessons, taken from a video study of mathematics teaching reported at PME 48. Where consent was given, the responses to the survey, were analysed using simple descriptive frequencies to examine relationships between the representations and the meanings. The results show that teachers associate particular words with specific relationships in many instances, whilst also using more ambiguous choices that could describe several relationships. Furthermore, some meanings of = were rarely referred to or distinguished from others. These choices have important implications for how we support students in understanding complex concepts such as equivalence and in making connections between words and symbols.

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TEACHERS' NOTICING GUIDED BY LEARNER OBSERVATION SHEET IN LESSON STUDY: AFRICAN EDUCATORS LEARNING JAPANESE PROBLEM-SOLVING LESSON

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This study investigates how a learner observation sheet influences the development of teacher noticing—particularly attending to and interpreting students' mathematical thinking—among African educators participating in a Lesson Study–based professional development program in Japan. Grounded in the theoretical framework of teacher noticing, the study conceptualizes professional learning as a transformation in how teachers perceive and make sense of students' thinking during instruction. Eight educators from African countries engaged in a nine-day training program in Feb 2025, where they observed and conducted problem-solving–oriented mathematics lessons using a learner observation sheet adapted from the TRU framework to systematically document students' thinking processes, engagement, and collaborative activities. The analysis revealed a clear developmental progression in participants' noticing across the training phases. In the early stage, teachers primarily attended to observable student behaviors and lesson outcomes, with limited attention to underlying cognitive processes. Gradually, their focus shifted toward students' reasoning, problem-solving strategies, and peer interactions, accompanied by emerging interpretations of students' mathematical thinking. In the later stage, participants demonstrated a more sophisticated and analytical perspective, capturing the full learning process—from individual work to collaboration, integration, and reflection—and recognizing causal relationships between students' actions, instructional design, and the learning environment. As emphasized by Sherin et al. (2011), teacher noticing involves not only observing classroom events but interpreting their meaning within instructional contexts. The findings indicate that the learner observation sheet functioned as a powerful mediating tool that guided teachers' attention toward student-centered aspects of learning and supported a shift from descriptive observation to analytical interpretation. By structuring observation through shared criteria, the tool also facilitated deeper reflection and common understanding during post-lesson discussions. Overall, this study suggests that the use of learner observation sheets can enhance teacher noticing and promote a deeper understanding of problem-solving–oriented mathematics instruction, particularly in cross-cultural professional development contexts. Limitations include the small sample size and the short duration of the program, indicating the need for further longitudinal and comparative research.

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AN ANALYSIS OF STUDENTS' IDEAS ON MULTIPLICATION PROBLEM SITUATIONS THROUGH REFLECTIVE THINKING

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Understanding students' ideas requires interpreting the reasoning underlying their responses rather than merely evaluating their correctness. Within reflective thinking, introspective thinking (IT) examines ideas that align with prior anticipation, while contemplative thinking (CT) reconstructs reasoning when students' responses diverge from what was anticipated (Emori, 2026). The transformative lesson study incorporating open approach (TLSOA) structures professional reflection through three interconnected processes: collaborative plan, do, and see (Inprasitha, 2022). However, how IT and CT function analytically in examining students' ideas in multiplication problem situations remains underexplored. Therefore, this study aims to analyze students' ideas through reflective thinking within the TLSOA model. This study purposively selected nine Grade 2 elementary students to solve three-lesson multiplication problem situations. One example of a problem situation that required 19 students to be arranged into groups of 2 or 3 students without leaving anyone alone. The data comprised the lesson plan, student worksheets, and classroom field notes. During the collaborative plan, the teachers anticipated possible ideas through the IT. During collaborative do, they observed the students' ideas in the classroom, and during collaborative see, they analyzed their written responses using IT and CT. Qualitative content analysis was employed to interpret the reasoning structures underlying students' solutions. The results revealed that most students produced valid groupings, such as $3\times5 + 2\times2$, coordinating with equal groups and members. These responses aligned with prior anticipation and were interpreted through IT. Some responses required CT: $19\times1=19$ ignored the grouping condition, whereas $(5\times2) + (9\times1)$ reached 19 but included groups of one member. These responses revealed structural miscoordination between the numerical calculation and contextual constraints. Although limited to three lessons, distinguishing between IT and CT provides a focused lens for examining students' ideas in multiplication problem situations.

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EXAMINING THE RELATIONSHIP BETWEEN MATHEMATICAL WELLBEING AND STEM CAREER INTEREST IN JAPANESE HIGH SCHOOL STUDENTS

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Across school subjects, mathematics is widely viewed as a core foundation for other STEM domains (Goos et al., 2023). Students' experiences in learning mathematics can shape their long-term sense of competence and influence educational and career choices. However, mathematics is also often experienced as a source of pressure (Dowker et al., 2016). Students' affective responses to mathematics, their sense of classroom relationships, and their value judgments regarding mathematical learning are therefore important conditions that may shape their STEM interest and career intentions. Although prior research has highlighted the significance of mathematical wellbeing which represents an emerging, multidimensional construct capturing students' subjective experiences of feelings and functioning in mathematics learning (Campbell, 2025), systematic evidence linking it to students' STEM career interest and intentions remains limited.

The present study surveyed 309 senior high school students from two schools in Japan using a well-established, internationally validated questionnaire. We computed composite scores, including a core mathematical wellbeing index and an indicator of mathematical tension/helplessness. Students' STEM career interest was positively correlated with their core mathematical wellbeing ($r = .314$, $p < .001$) and actual tension/helplessness ($r = .280$, $p < .001$). A linear regression controlling for grade and gender indicated that both predictors remained significant, with model fit $R^2 = .150$ (adjusted $R^2 = .139$). Grade and gender were not significant predictors ($p > 0.05$). This suggests a complex relationship: while students' stronger STEM orientation co-occurs with more positive core mathematical wellbeing, it is simultaneously associated with greater tension and helplessness in mathematics.

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VARIATIONS IN MATHEMATICAL COMPETENCIES AMONG HIGH-ACHIEVING STUDENTS

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High-achieving students in mathematics are often treated as a homogeneous group. However, using quantitative analysis, Prast et al. (2025) identified distinct subgroups among high-achieving students differing in cognitive, motivational, and achievement-related characteristics. Although their results show substantial heterogeneity, the identified differences primary concerns cognitive and motivational aspects rather than mathematical competence itself. Thus, there is a need to explore qualitative differences in how mathematical competencies are enacted by individual high-achieving students. This ongoing study investigates similarities and differences in individual mathematical competence profiles among high-achieving students. The study draws on the Mathematical Competency Research Framework (Lithner et al., 2010), which conceptualises mathematical knowledge as a set of competencies. To capture different aspects within each competency, three competency-related activities (CRA) are defined: Interpret, Do and Use, and Judge. The five competencies included in the Swedish mathematics curriculum: *conceptual*, *procedural*, *problem solving*, *reasoning*, and *communication*; are operationalised through these CRA. This enables a fine-grained, qualitative analysis of students' mathematical competence beyond grades or number of correct answers.

Eight high-achieving lower secondary students (age 13-15) individually solved up to 10 tasks, selected to allow demonstration of the different competencies. Written and oral solutions were analysed to create competence profiles, visualised as matrices across competencies and CRAs. Preliminary results show both homogeneity and heterogeneity among the students. Most students show strong performance in the productive activity "Do and Use", particularly in procedural competency. Clear variation emerges in reasoning and communication, especially in the analytical activity "Judge": some students engage in meta-reflection, while others show limited evaluation of methods or arguments. Our findings indicate that high achievement does not imply uniform development of mathematical competencies (cf. Prast et al., 2025). We contribute an empirically grounded mathematics-specific perspective on heterogeneity among high-achieving students, with implications for task design and differentiated instruction.

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UNPACKING COMPUTATIONAL THINKING THROUGH PROGRAMMING TASKS IN KOREAN INFORMATICS TEXTBOOKS

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Computational thinking (CT) has increasingly been integrated into mathematics education, particularly as students develop solution procedures, represent mathematical relationships, and systematically verify their results (Brennan & Resnick, 2012). Textbooks play a central role in shaping classroom practice, yet how CT is realized in textbook tasks remains underexplored. Programming tasks in informatics textbooks frequently involve mathematical ideas and representations. Therefore, examining these tasks enables the identification of practical approaches to integrating CT into mathematics learning. We conducted a textbook analysis of 23 Korean informatics textbooks (middle school: 12, high school: 11). We extracted 197 tasks in which mathematics was explicitly embedded in the coding process or was required to complete the task. Each task was coded along two dimensions: (a) mathematics concepts and (b) CT concepts, drawing on the CT components synthesized by Ye et al. (2023). CT concepts were identified as required programming features (variables, conditionals, loops, sequences, events, subroutines, and operators). When coding CT concepts, we examined whether they were represented as individual concepts or as bundled sets of multiple concepts, with a focus on elucidating the modes of thinking required by these representational forms. The analysis identified two main areas of focus. First, the mathematics concepts of programming tasks were concentrated in the domains of algebra and geometry, reflecting frequent use of numerical calculations and geometric constructions or transformations. Second, CT concepts formed procedural patterns centered on variable manipulation and the sequential execution of mathematical procedures. These patterns structured how tasks stored and updated quantities while guiding repeated calculations or geometric constructions. This analysis provides insights into how CT can be enacted through programming tasks and offers implications for the design of tasks and instructional practices related to CT in mathematics education.

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DESIGNING AUTOMATIC DIAGNOSIS OF STUDENTS' RATIONAL NUMBER STRATEGIES

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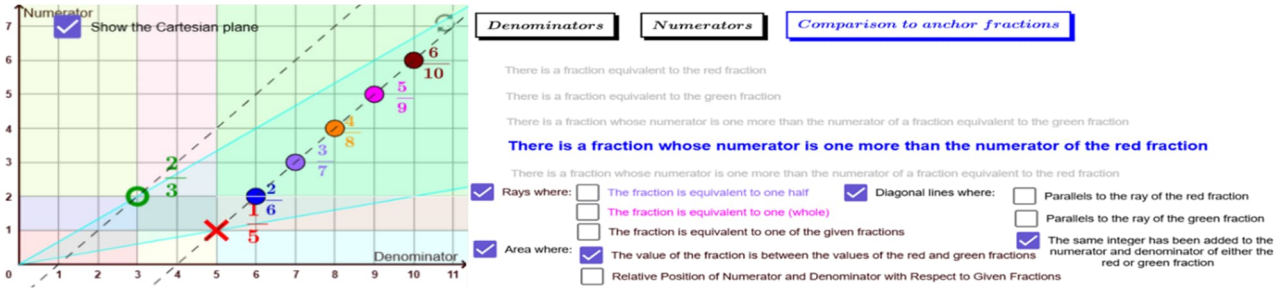
Open-ended fraction tasks provide access to students' reasoning by eliciting multiple responses, potentially making patterns visible (Watson & Mason, 2005; Yerushalmy, 2020). However, they create an interpretive challenge for teachers who must make sense of diverse responses. With AI support, we address this challenge with the goal of developing a tool to support teachers' interpretation of student-generated data.

We analyzed responses from 47 elementary students to seven example-eliciting tasks involving finding fractions between given bounds; their responses were represented on a Cartesian plane, making patterns in their work visible. AI helped us organize these patterns and design an interface for teachers. Figure 1 illustrates how this interface supports recognition of, for example, a strategy of adding 1 to both the numerator and denominator.

Figure 1: One dimension of feedback to teachers about a student strategy



The diagnostic framework, now embedded in software, organizes feedback on student reasoning along three dimensions: denominators, numerators, and comparison to anchor fractions (see Figure 1 for the comparison dimension). By grounding interpretation in observable patterns, the framework supports automated, interpretable reporting of reasoning. This work contributes a design perspective on how visual representations support the development of diagnostic tools for formative assessment.

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SPATIAL REASONING THROUGH COLLECTIVE ENACTMENT ACROSS SPACE SCALES

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Recent mathematics education research has moved beyond treating spatial reasoning as an individual, measurable skill and has instead highlighted its emergent, contextually organized character. This study examines how students' spatial reasoning was reorganized through shifts between collective enactment in a gym and video-mediated analysis in a classroom. Building on Davis et al. (2015)'s spatial reasoning model and an embodiment perspective on figural formation (Benally et al., 2022), this study focuses on how different space scales make distinct aspects of collective movement perceptually available and analytically negotiable. The study involved 36 ninth-grade students across three lessons. Students were asked to transform a cross-shaped collective formation into a circle while moving counterclockwise in a gym. Their activity was video-recorded, including an overhead view from the second floor. In subsequent classroom work, students analyzed the recorded movement through discussion and individual worksheets while observing the video. Data for analysis included a gym video, transcripts of classroom discussions, student-drawn diagrams, blackboard records, and observers' field notes. The analysis shows that some students attempted to form a circle by using the student in front of them as a local reference and trying to maintain a constant distance from that student. However, from their floor-level perspective, they could not determine whether the overall formation was actually circular. In the classroom, however, the overhead video observation made aspects of the collective movement more available for discussion. In one episode, students distinguished between outer and inner movements of the cross-shaped formation. They argued that, although the outers could trace circular paths, the inners had to move in a stair-step or straight-line manner relative to the rotating formation. Thus, movement initially treated as "making a circle" was reinterpreted as a set of rule-governed local motions that collectively generated circularity. This shift suggests that spatial reasoning in collective, embodied geometry cannot necessarily be reduced to the formal geometric construction of a figure. Rather, shifts across space scales affected what students could notice, discuss, and coordinate.

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ALGORITHMIC THINKING THROUGH COLORING ACTIVITIES IN PRIMARY MATHEMATICS CLASSROOMS

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Algorithmic thinking (AT) is closely related to mathematical thinking, as both involve interpreting conditions, analyzing relationships, and constructing possible solutions. In primary mathematics classrooms, unplugged coloring activities can provide opportunities for students to make these reasoning processes visible (Changsri et al., 2024). This study explores how coloring activities support students' AT in primary mathematics classrooms under the Transformative Lesson Study incorporating Open Approach (TLSOA): collaborative plan, collaborative do with four steps of Open Approach, and collaborative see (Inprasitha, 2022). This classroom-based qualitative study was conducted in two phases. First, six sets of coloring activities were analyzed to examine how task sequences could support students' AT. Second, the activities were implemented with nine Grade 6 students over six consecutive lessons, each lasting approximately 80–90 minutes. Data were collected from students' worksheets, classroom protocols, and field notes, and analyzed using Araya's (2021) framework of AT: Searching, Decomposing Problems, Linking Commands, and Recursion. The findings showed that students demonstrated three components of algorithmic thinking: (1) Searching was shown through identifying and selecting objects that met given conditions by examining attributes such as color, quantity, and position. Students compared possible choices and determined which objects satisfied task requirements. (2) Decomposing Problems was demonstrated by breaking coloring tasks into smaller conditions and considering each requirement step by step before completing the overall solution. (3) Linking Commands was evident in students' ability to organize coloring actions in a logical sequence to satisfy multiple conditions within the same task. These findings suggest that well-sequenced coloring activities can support students' algorithmic thinking in primary mathematics classrooms.

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THE STUDY ON GENDER DIFFERENCES IN MATHEMATICAL ANXIETY AND EMOTIONAL REGULATION AMONG MIDDLE SCHOOL STUDENTS IN TAIWAN AND SINGAPORE

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Emotions are increasingly recognized as central to mathematics learning. Control-Value Theory (Pekrun et al., 2002) suggests that students' ability to regulate emotions supports achievement, while mathematics anxiety can hinder cognitive processing and problem solving (Beilock & Willingham, 2014). Previous studies have reported gender differences in mathematics-related emotions, but findings vary across cultural contexts. This study examines emotional control and mathematics anxiety as predictors of mathematics achievement and explores gender differences in two high-performing education systems. The research questions are:

- Do gender differences exist in emotional control and mathematics anxiety?
- How do emotional control and mathematics anxiety predict mathematics achievement?
- Do these relationships differ between Singapore and Taiwan?

Data were drawn from PISA 2022, including Singapore (n=6,606) and Taiwan (n=5,857). Emotional control and mathematics anxiety were measured using validated student questionnaire scales, and mathematics achievement was assessed through standardized mathematics literacy scores. A cross-sectional multilevel modeling approach was used to account for student- and school-level variation, and Bayesian estimation was applied to improve parameter estimation and address missing data. Results indicated that higher emotional control significantly predicted higher mathematics achievement, while higher mathematics anxiety significantly predicted lower achievement in both countries. Female students demonstrated significantly higher emotional control in both contexts. Overall, male and female students in Taiwan has higher mathematics anxiety than in Singapore. These findings highlight the importance of emotional regulation in mathematics learning and suggest that gender differences in emotional experiences are similar and need to explore culture context in the future. The results support the development of culturally responsive instructional and emotional support strategies to improve mathematics learning outcomes.

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DIDACTICAL TRANSFORMATION OF A FRACTAL STRUCTURE THROUGH LESSON STUDY: DESIGNING A SIERPIŃSKI-BASED MATHEMATICAL GAME

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Fractals are mathematical structures that preserve their formal similarity under magnification and consist of recursively embedded substructures. However, such abstract structures cannot be directly transferred into instructional settings; they require pedagogical reconstruction. The framework of didactical transposition conceptualizes the process through which scholarly knowledge is transformed into an object of teaching (Chevallard, 1985). This study presents a qualitative case study examining the transformation of the Sierpiński fractal from a scholarly mathematical structure into an instructional object within a lesson study process. The participants were four prospective elementary mathematics teachers. Over a seven-week lesson study cycle, the participants designed, implemented, observed, and revised a mathematical game based on the Sierpiński fractal. Data sources included a game design bulletin, successive versions of the game, classroom implementation–observation notes, semi-structured interviews, and individual reflection reports. Data were analyzed within the framework of didactical transposition (Chevallard, 1985), tracing how fractal knowledge was reconstructed from its scholarly context into an instructional setting. Findings indicate that in the initial design, the fractal remained at the level of a visual representation. Following classroom implementation and observations, the scoring system was restructured to reflect the hierarchical and layered growth structure of the fractal. As a result of the revision process, the fractal evolved from a represented content element into the mathematical organizing principle of the game. Furthermore, the didactical transformation was found to be collectively constructed through mathematical reasoning within the lesson study process rather than individually enacted. These findings suggest that lesson study can function as a collective space of didactical transformation, enabling abstract mathematical structures to be reconstructed for instructional purposes while preserving their structural integrity rather than reducing them to superficial representations. Lesson study thus serves as a mediating space for the structural reorganization of mathematical knowledge.

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SELF-EFFICACY, ANXIETY, AND LEARNING STRATEGIES ACROSS MATHEMATICS ITEMS OF VARYING DIFFICULTY

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Students' self-efficacy and learning strategies are important for mathematical development and predictive of achievement (Bandura, 1997). Social cognitive theory (SCT) proposes that students with higher self-efficacy employ more efficient learning strategies and exert greater persistence with challenging tasks. Furthermore, SCT suggests a negative relation between self-efficacy and anxiety, such that stronger self-efficacy beliefs are associated with lower anxiety and fewer negative emotions (Bandura, 1997). However, mathematics achievement is not a homogeneous outcome: test items vary in difficulty, and relations between self-related perceptions (e.g., self-efficacy), learning strategies, and achievement may differ across items of varying difficulty. The aim of this study is to examine how self-efficacy, anxiety, and students' use of rehearsal and self-regulation strategies are related to mathematics achievement across items of different difficulty. A total of 1009 upper secondary students in Sweden (age 16–17) completed a questionnaire measuring self-efficacy, anxiety, and learning strategies (rehearsal and self-regulation). Questionnaire data were linked to students' results on a test at the end of the mathematics course. Partial least squares structural equation modeling showed that higher anxiety was associated with greater use of rehearsal strategies ($\beta = 0.41$), whereas higher self-efficacy was associated with greater use of self-regulation strategies ($\beta = 0.46$) and overall achievement ($\beta = 0.49$). Item-level logistic regression analyses further revealed that anxiety was associated with lower odds ratios for success, especially for difficult items. Self-efficacy was positively associated with success across all item types, and the odds ratios increased with item difficulty. Rehearsal strategies showed decreasing odds ratios as item difficulty increased (mean 1.03 for easy items and 0.78 for difficult items), whereas self-regulation showed increasing odds ratios with increasing item difficulty (easy: 1.27, difficult: 1.38). Odds ratios were estimated for each item using standardized predictors. Items were ordered by success rate in the sample, and grouped into difficulty terciles. By distinguishing achievement outcomes by item difficulty, this study provides a more fine-grained account of how self-related perceptions and learning strategies are associated with mathematical achievement. The findings suggest that self-efficacy and self-regulation may be particularly important for success on more demanding items, whereas anxiety and rehearsal are associated with lower odds as difficulty increases.

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MATHEMATICS TEACHER EDUCATORS AS DESIGNERS

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We are relational constructivist mathematics teacher educators (Steffe & D'Ambrosio, 1995) whose practices include curriculum design (Ackermann, 2012) for various goals in mathematics methods courses. Drawing from this perspective, we seek to improve our design practice through relationships with preservice teachers (PTs) alongside iterations of cycles of instructional activity design and researching those activities through PTs' responses to our designed activities (Zaslavsky & Sullivan, 2011). Drawing from Ackermann's designer dilemma we assume that PTs are autonomous and will interpret designed situations in unanticipated ways. Research on mathematics teacher educators (MTEs) as curriculum designers has identified general processes, categories for design elements, and influential factors. Less reported are insider perspectives of MTE development as curriculum designers. Our study of MTE design practice addressed the research question: How do MTEs design tasks to support PTs' development of curricular decision-making?

We use self-study, a practitioner focused methodology, and methods including dialogic analysis and reflections to develop awareness and attention to improve our design practice in the context of supporting preservice teachers' development as curriculum decision makers. Primary data sources from 2023-2025 included designed activities, reflections, and analytic dialogues. Secondary data sources included PTs' work samples, conference proposals, and proceedings encapsulating our design processes. Findings from dialogic analysis were supported by coded transcripts and reflections. Through dialogic analysis we developed awareness of constraints as a component of MTEs' curricular design. Although constraints initially were used to bound PTs' activity and resulting work samples, constructing models of PTs' curricular decisions resulted in our awareness of constraints as tools in activity design. Resulting intentional use of constraints provided PTs with situations that provoked discussion of curricular constraints. These discussions provided us with insight into PTs' rationales for curricular decisions in response to such constraints. Findings illustrate MTEs' awarenesses and attention to constraints as a component in MTE curricular design that supports PTs' development of curricular decision-making.

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EQUALITY UNDER UNCERTAINTY: STUDENTS' REASONING ABOUT SMALL MEASUREMENT DISCREPANCIES

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Recent research highlights an epistemic tension between school mathematics, centred on exact equality, and real-world mathematics, where results are approximate because they rely on measurement. In STEM activities, students must coordinate theoretical equality with empirical agreement based on measured data, even when small discrepancies appear. Klein's precision–approximation distinction remains useful for task design and mathematical literacy (Marco & Palatnik, 2024). Yet students often struggle to interpret minor numerical differences and decide whether they reflect meaningful effects or acceptable error (Wan, 2023). This study examines how students interpret small weight discrepancies when geometrical reasoning suggests equality.

We analysed 68 written submissions from 9th–11th-grade students in the Two Jewel Activity (Kazakevich & Marco, 2023), spanning four stages that required estimating and interpreting a difference in gold quantity. We identified four response categories; the two most salient were: (i) students who treated the weight discrepancy as negligible or explainable without undermining intended equal gold quantity (n = 29, ~43%), citing measurement imprecision, aging/dirt/material loss, or physical factors such as thickness, density, impurities, or voids; and (ii) students who treated the difference as critical (n = 15, ~22%), arguing that equal areas do not imply equal gold quantities and that the discrepancy undermines the geometrical proof.

Overall, the activity makes visible a productive epistemic tension between mathematical precision and empirical approximation. We argue that the design of the activity intentionally creates this tension and can support students' STEM reasoning by promoting coordination between theoretical proof and measurement-based interpretation. Such tasks may broaden students' conception of equality beyond strict numerical identity and strengthen their capacity to reason under uncertainty.

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MATHEMATICAL IMAGINATION WITHIN THE CLASSROOM

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Despite the relevance of imagination for mathematics learning (Kotsopoulos & Cordy, 2009), research on this topic is still scarce (e.g. Lajos et al., 2025). Imagination is a complex process; aspects central to it are its dynamic nature, temporality, the abstract-concrete tension, the interaction of action and perception, the interplay of actual and possible (Gfeller & Zittoun, 2021; Rucińska, 2024). Imagination evokes visual perception without being something that is actually depicted or seen, so it eludes linear accounts or contours. We see imagination as emerging in utterances and embodied interactions that may be situated and limited in space and time, but that may unfold covertly in a wider time span and may be dispersed across collectives rather than pertaining to the individual. We take imagination to be materially enmeshed with any learning assemblage and we suggest that, along these lines, characterizing mathematical imagination would be more like knowing the texture of a piece of cloth by smelling it, touching it, and/or hearing it move, than looking at its label.

We attempt to grasp the complex nature of mathematical imagination by analysing a classroom episode engaging grade 3 children with a pattern activity that was designed in the context of a larger intervention study aimed to develop early algebraic thinking. Using video data, we focus on how new mathematical relationships and regularities come to matter, making the imaginative process increasingly mathematical, as well as increasingly collective. We also highlight how different imaginative paths might have been elicited by different choices inside the classroom, speaking directly to the material, embodied, enactive and social dimension of mathematical imagination.

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ANALYSIS OF STUDENTS' COMMUNICATION IN LESSONS ON SIMILARITY USING DYNAMIC GEOMETRY SOFTWARE

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Similarity instruction tends to emphasize algorithms, which may result in student misconceptions. This study investigates mathematical communication facilitated by the DGS AlgeoMath's manipulability. We categorize learning into: similarity ratios, triangle similarity conditions, and the congruence-similarity comparison to derive pedagogical implications. Grounded in the communicative perspective of Sfard (2008) and theoretical framework established by Yu and Song (2013), 27 eighth-graders performed similarity tasks using the DGS's dilation, rotation, and reflection. Lesson data (transcripts, screen captures, worksheets) were analyzed on the basis of Table 1.

Table 1: Mathematics Communication Analysis Framework

Sub-domain	Communication Types (Codes)
Understanding Expression	Understanding of Term (U1), Misuse (U2)
Judging Others' Strategies	Questioning (J1), Counter-questioning (J2), Doubt (J3), Affirmation (J4), Comprehension (J5)
Reflection on Thinking	Revision (R1), Reflection (R2), Justification (R3), Clarification (R4), Generalization (R5), Inquiry (R6)

The students initially confused similarity ratios with area ratios. However, AlgeoMath's visual feedback led them to doubt (J3) these hypotheses and reconstruct meanings by reflectively re-examining (R2) expansion ratios based on side lengths. They comprehended (J5) why the two overlapped triangles were congruent and generalized (R5) AA similarity from ASA congruence. Furthermore, they refined their discourse by comparing the concepts that similarity involves overlapping after dilation/rotation (U1), whereas congruence requires the same size (R4). According to the above, AlgeoMath lessons expanded mathematical discourse, fostering student growth. These outcomes are significant when DGS features are combined with teacher questioning and noticing that stimulate student thinking.

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NEGOTIATING AUTONOMY AND PRESCRIPTION: ENACTING THE TRR CURRICULUM IN SWEDISH CLASSROOMS

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Recent efforts to introduce more structured approaches to mathematics instruction have renewed interest in how teachers interpret and enact prescriptive curricula. In Sweden, where curriculum materials have traditionally been optional and descriptive rather than mandatory, the introduction of the TRR (Thinking, Reasoning, and Reckoning) curriculum represents a marked shift. TRR consists solely of a teacher's guide and promotes activity-based learning through detailed lesson sequences. This shift creates a tension between prescriptive instructional design and a system characterised by strong teacher autonomy, reflecting a broader international challenge of balancing standardisation with professional judgement. Despite international research on teacher agency and curriculum enactment, we still know little about how Swedish teachers work with highly structured activity sequences in practice. This pilot study addresses this gap by examining one teacher's work with TRR over a six-month period. Drawing on nine video-recorded lessons and two interviews, the study explores the research question: How does a teacher balance professional autonomy with the prescriptive elements of the TRR curriculum in classroom practice? The analysis combines Priestley et al.'s (2012) ecological model of teacher agency with the Consolidated Framework for Implementation Research (CFIR) to clarify how intervention characteristics, contextual conditions, and teacher judgement interact during enactment. Preliminary findings suggest that the teacher actively negotiates the degree of adherence to TRR, alternating between following the prescribed activity sequences and adapting them to meet student needs, time constraints, and existing routines. Rather than simply complying or resisting, the teacher engages in situated, moment-to-moment decision-making that reflects both the affordances and constraints of TRR. The structured nature of TRR generates tensions—given Sweden's weak tradition of prescriptive materials—while also supporting lesson organisation and mathematical focus. These findings illustrate how teacher agency operates through moment-to-moment instructional decisions and how these decisions reshape the TRR model in practice.

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SENSING THE PROCEDURE, MISSING THE SENSE: EPISTEMOLOGICAL OBSTACLES IN PRE-SERVICE TEACHERS' UNDERSTANDING OF RADICAL EXPRESSIONS

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Several mathematical concepts have historically emerged from moments of epistemic crisis, one of which is radical numbers. Originating from attempts to resolve mathematical inconsistencies and conceptual tensions, these concepts carry traces of their historical struggles into contemporary classrooms. In mathematics education, such historically rooted difficulties have been described as epistemological obstacles. Research consistently indicates that students experience persistent challenges in learning and making sense of radical expressions (Fischbein et al., 1995; Sirotic & Zazkis, 2007). The present study aims to examine whether pre-service mathematics teachers are able to meaningfully interpret and justify the procedures they perform when working with radical expressions, thereby exploring the extent to which procedural fluency aligns with conceptual understanding. This qualitative case study involved 40 third-year pre-service mathematics teachers. A two-dimensional open-ended test designed to elicit both procedural and conceptual reasoning was administered, followed by semi-structured interviews with five participants to gain deeper insight into their reasoning. The collected data were analysed using descriptive and content analysis techniques. The findings revealed a clear discrepancy between procedural success and conceptual understanding. Although approximately 90% of participants correctly solved operation-based questions involving radical expressions, the majority struggled to explain why the procedures were mathematically valid or what underlying meanings justified them. From a procedural and conceptual learning perspective, the results indicate that correct procedures were not supported by strong conceptual understanding. Moreover, the parallel between the historical emergence of radical numbers and the conceptual struggles observed among future teachers suggests that epistemological obstacles may continue to shape contemporary learning. Given that these pre-service teachers will soon assume responsibility for teaching such topics, the study highlights the importance of fostering deeper conceptual engagement with mathematically and historically complex concepts that inherently embody epistemological tensions.

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FROM UNDERSTANDING TO ENACTMENT: HOW PRE-SERVICE PRIMARY TEACHERS IMPLEMENT FORMATIVE ASSESSMENT IN ELEMENTARY MATHEMATICS LESSONS

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Formative assessment (FA) is widely recognized as a powerful approach to improving pupils' learning and fostering deeper mathematical understanding. However, in mathematics, teachers may struggle to move from procedural evaluation to feedback that addresses pupils' specific misconceptions. This longitudinal study traces how five pre-service primary teachers develop their understanding and classroom enactment of FA in mathematics , drawing on Wiliam and Leahy's (2015) framework. The study addresses two research questions: (1) How does the understanding of FA in mathematics develop over time? and (2) Which FA tools do they enact in mathematics lessons to address pupils' thinking?

The study adopts a qualitative longitudinal multiple-case design following five participants over two years, from initial training to their first year as in-service teachers. Data were collected across four phases and included video-recorded mathematics lessons (15 lessons in total), semi-structured interviews with post-lesson reflections, and an online questionnaire. To analyze classroom practice (RQ2), a deductive coding scheme based on Wiliam and Leahy's (2015) five strategies and 29 subcategories was used as an observation protocol.

Regarding RQ1, analysis revealed a conceptual shift. Initially, participants viewed FA mainly as "kinder verbal feedback" or a descriptive alternative to marks. Over time, their accounts shifted toward a process-oriented view linking explicit learning goals, success criteria, and pupils' responsibility. Regarding RQ2, lesson analyses showed a gradual but uneven expansion of FA repertoires. After the university course, teachers introduced tools like mini-whiteboards and random-selection techniques to make mathematical thinking visible. However, explicit work with success criteria remained weak. For instance, when addressing a recurring error in written subtraction (52–48), a teacher successfully used mini-whiteboards to surface the specific misconception (forgetting to decrease the tens column after regrouping). Yet, the formative cycle remained incomplete, suggesting that participants were more successful in eliciting pupils' mathematical thinking than in turning such evidence into stable formative routines.

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USING STUDENT SOLUTIONS TO PROMOTE DISCUSSION OF MATHEMATICS TEACHING AND LEARNING

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Teachers who strive to understand how students approach a task and base their teaching decisions on this knowledge create a positive learning environment. One way teachers can gain insight into students' thinking is by analysing their solutions. In this paper, we present research aimed at encouraging productive discussions about student thinking among in-service mathematics teachers (ISTs). Some of them are Interpreters and some of them are Evaluators (Gehrtz, Hagman, & Barron, 2024). To promote such discussion, we created a research tool consisting of a geometric task from a mathematics competition for 14-year-olds and seven student solutions to this task.

The research took place at a summer school workshop designed to prepare students for a maths competition, with 26 teachers participating. The workshop consisted of three phases. First, the ISTs solved the problem individually. Second, they had to rank seven student solutions from best to worst. Third, they engaged in a collective discussion to evaluate the individual student solutions. The seven students employed five different strategies: two used an inductive strategy, while five used four different deductive strategies. These strategies were based on various properties of triangles and exhibited differing levels of argumentation. Two students reached the expected result.

Our findings indicate that the research tool effectively promotes discussion about mathematics teaching and learning during the third phase of the workshop. The ISTs tried to interpret how students approached the task and to consider possible reasons for errors or shortcomings in their solutions (Interpreters). This positively influenced those who focused mostly on the correctness of the result (Evaluators). Furthermore, some ISTs pointed out inconsistencies in the assessments. For example, some rated a solution containing an uncalculated numerical expression more highly than one with an incorrect calculation, and a solution without a visual representation more highly than one with an inaccurate sketch. Therefore, the discussion mobilised mathematics teacher's specialised knowledge of the participating ISTs (Carrillo et al., 2018).

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PROBLEM POSING AND SELF-EFFICACY IN MATH: DO BASIC NEED EXPERIENCES MEDIATE INTERVENTION EFFECTS?

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Self-efficacy—students' beliefs in their capabilities to organize and execute actions required to attain learning goals—is a central construct in motivation research and a strong predictor of engagement and achievement (Bandura, 1997). According to Self-Determination Theory (SDT), experiencing autonomy, competence, and relatedness supports motivational beliefs such as self-efficacy. From this perspective, self-efficacy is shaped not only by performance outcomes but also by the quality of motivational experiences in instructional contexts. Problem posing constitutes a promising instructional practice in this regard (Cai & Leikin, 2020). By allowing learners to generate their own mathematical problems, problem posing enables meaningful choice, ownership of learning content, and social interaction, which may support experiences of autonomy, competence, and relatedness, which in turn improve self-efficacy. Although previous studies have reported positive effects of problem posing on self-efficacy, little is known about the mechanisms through which these effects emerge. In the present study, we tested a model that connects problem posing as an autonomy-supportive intervention, experiences of basic needs during the intervention, and self-efficacy in mathematics. We conducted an experimental study with N = 194 tenth-grade students. We compared a four-lesson intervention in which students either posed and solved their own mathematical problems or solved given problems. After each lesson, students reported their experiences of autonomy, competence, and relatedness. Self-efficacy in mathematics was assessed before and after the intervention. Results showed no total effect of posing and solving own problems on self-efficacy in mathematics. However, posing and solving own problems increased students' experiences of autonomy and relatedness (but not competence). Experiences of autonomy, competence, and relatedness were positively associated with self-efficacy, and autonomy and relatedness mediated the effect of problem posing on self-efficacy. Prior self-efficacy did not moderate intervention effects on basic needs. The findings refine motivational accounts of problem posing by highlighting autonomy-supportive mechanisms that can indirectly foster self-efficacy.

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PARTICIPATION WITHOUT MATHEMATICAL AGENCY

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Undergraduate mathematics instruction has shifted in recent decades from lecture-based practices toward more frequent use of active, inquiry-based strategies. While these shifts may increase student agency, without explicit attention to classroom power structures and positionalities, these contexts may inadvertently perpetuate traditional participation patterns (Hagman, 2021). Applying positioning theory (Drageset & Ell, 2024) and Esmonde's framework for Opportunities to Learn (OtL) (2009) we investigate how participation, identity, and communication shape mathematical authority for students in small groups in an undergraduate precalculus classroom. We examined data consisting of conversational-turn transcripts from three consecutive class sessions, focusing on one focal student, a peer, and their instructor. Discourse was analysed at the turn-of-talk level, distinguishing between inquiry-oriented contributions (e.g., explanation, justification) and validation-oriented moves (e.g., seeking or confirming correctness) to understand the focal student's positionality within the mathematical community. Our findings show that the nature of the focal student's communications differed dramatically from those of their peer. While the focal student contributed over half the conversational turns (56%), these focused largely on seeking validation. The peer student had fewer conversational turns (44%), but those contributions consisted primarily of inquiry and providing validation. These patterns suggest a stable expert–novice dynamic, potentially limiting the focal student's opportunities to move toward more agentic participation. Our findings indicate that without explicit interventions in negotiating mathematical authority, students' roles can remain fixed, resulting in inquiry-based classrooms where social inclusion co-exists with mathematical marginalization.

Acknowledgements

This work was partially supported by the National Science Foundation (NSF) under Grant No. 2400902, No. 2400903. Any opinions and conclusions expressed are those of the authors.

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IMPLEMENTING THINKING CLASSROOM PRACTICES IN A UNIVERSITY CALCULUS REMEDIAL COURSE

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Previous research indicates that university STEM students are more likely to succeed when teachers use active teaching methods (Freeman et al., 2014), yet most university teaching still follows teacher-led instruction (Børte et al., 2023). This was also the case in Calculus I at the University of Iceland in fall 2024. Of 340 enrolled students, 257 signed up for the exam and 142 passed the course. The 115 students who failed were invited to participate in a remedial course in spring 2025. The remedial course was designed using thinking classroom practices (Liljedahl, 2020) to support students' active participation, shared thinking and individual sense-making. Students attended two weekly sessions for ten weeks (minimum 80% attendance) working in visibly random groups of three on vertical whiteboards, engaging with curricular tasks, and receiving formative feedback through viewing other groups' boards as well as through questioning, hints, and whole-class consolidation led by the two instructors. Weekly interactive check-your-understanding worksheets with immediate feedback and links to further exercise problems supported independent practice. In this case study, we looked for potential changes in students' perceived value of attending exercise classes and academic success. Of the 86 students who completed the remedial course, 69% passed the Calculus I equivalent examination. A post-course survey (37% participation) indicated increased study engagement and strong student support for the thinking classroom approach. Qualitative responses emphasized collaborative problem solving, peer explanation, and increased willingness to express uncertainty as key learning affordances. As a result of these outcomes, Calculus I exercise classes in fall 2025 were redesigned and tutors received training to adopt thinking classroom teaching practices. This talk reports further findings from the case study and discusses implications for course design, remediation structures, and the use of active learning pedagogies in first-year university mathematics.

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FACILITATORS' STATEMENTS ON DIFFERENTIATED TASKS IN MENTAL CALCULATION STRATEGIES OF SUBTRACTION

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Due to the heterogeneity of students in mathematics classrooms, student focus and instructions providing tasks for different competency levels are important. Students at the basic level with fragile conceptual understanding must be a focus, as gaps can hinder further learning. Therefore, teachers must use tasks aligned with students' levels. These tasks can be categorised as supporting, fostering, or challenging (Prediger et al., 2023; Vygotsky, 1978). Supporting compensates for short-term missing prerequisites by reducing cognitive load. Fostering targets the long-term development of basic concepts, whereas challenging involves tasks slightly above current abilities. Future professional development (PD) research should optimise approaches to overcome the prevalence of supporting practices and strengthen fostering aimed at learning progress (Prediger et al., 2023). Within the nationwide German QuaMath program, a teacher PD is designed to foster differentiated, student focused secondary mathematics instruction. Facilitators play a key role in mediating this content and must make informed statements about tasks for different learning levels. Against this background, this study investigates how facilitators evaluate task suitability for supporting, fostering, or challenging a basic-level student in mental calculation strategies of subtraction. To this end, 15 facilitators assessed three tasks for a Year 5 student. Data were analysed using inductive qualitative content analysis. Two patterns emerged: half of the facilitators identified the fostering task as most appropriate for the student's conceptual development, while others prioritized supporting or challenging tasks. The fostering task was rejected due to cognitive overload from additional work on the concrete level, a perceived mismatch between demands and learning level, and unuseful material use. In contrast, the supporting task was justified by its immediate useful function, and the challenging task by its role in promoting reflection or motivation, sometimes with reduced solution expectations. Findings indicate that some facilitators associate fostering with increased task difficulty rather than the development of conceptual understanding. This suggests a need for facilitator PDs to explicitly address the distinctions between supporting, fostering, and challenging.

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TRANSLANGUAGING AND AI IN EXPLANATORY VIDEO PRODUCTION

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This study investigates how multilingual primary students use artificial intelligence (AI) during explanatory video production in mathematics. In linguistically diverse classrooms, meaning is negotiated across languages, representations, and digital tools. While translanguaging supports meaning-making (García & Wei, 2014), little is known about how learners engage with AI-generated output. We conceptualize explanation as an interactional, multimodal process (Barwell, 2018). AI is understood as a mediating artifact that generates probabilistic linguistic proposals which must be interpreted in context. From this perspective, learner agency (Noren, 2015) becomes visible in how students accept, modify, or reject AI output. The study follows an interpretative qualitative design with multilingual students (grades 3–4) working on "Reach 1000" and producing explanatory videos; data include recordings, transcripts, and chat logs.

In one episode, a student uses AI to translate an explanation from Arabic into German. The output introduces a semantic shift ("minutes" instead of numbers) that does not fit the mathematical context and is rejected. After prompt reformulation, the output improves, yet shared understanding is not achieved, showing that correct translation does not guarantee mathematical understanding. Meaning emerges through negotiation. Learners display agency by critically evaluating AI output, while interaction supports reflection on language through prompt refinement.

These findings suggest that AI-supported translanguaging is an interpretative accomplishment rather than a transfer process. AI introduces an additional layer of mediation that can shift meanings in mathematically problematic ways, making learners' evaluative work central. Explanatory video production provides a productive setting to observe these dynamics (Oechsler & Borba, 2020).

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DIALOGISM OF MATHEMATICS: RETHINKING THE MATHEMATICAL EXPERIENCE

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Mathematics Education has extendedly theorized and studied the activity of doing mathematics: what it means to do mathematics, what it yields in terms of outcomes for the person and what the broader experience of mathematical activity itself is. While prior research emphasizes learning through doing mathematics, less attention has been given to the real time situated "happeningness" of mathematics. This study mobilizes a didactical perspective to theorize the process of meaning-making in mathematical practices. More precisely, it explores a multi-faceted perspective constituted of three main constructs: debugging/*bricolage* (Papert, 1980), ambiguity (Byers, 2007) and dialogic (Bakhtin, 1975/1981). Rather than attempting a comprehensive synthesis, this study adopts a focused articulation of these realities, aiming to capture mathematics as lived practices in which every mathematical act can be seen as opening a space between perspectives and making meaning from differences. Yet, the lived interaction of these dimensions in real-time practice remains to be examined. The research questions are: *How is the interplay of debugging, ambiguity and dialogism lived in mathematical practices? How does this dialogical reality shape mathematical meaning-making?*

Mobilizing a task-based interview protocol on the theme of spherical geometry, preliminary results of a pilot study reveal how dialogical structures of meaning-making emerge in mathematical activity. In particular, iterative debugging processes and situations of mathematical ambiguity appear to function as productive local sites of meaning negotiation. For instance, tasks pertaining to spherical triangles have led participants to live different versions of the mathematical notion of line straightness on spheres: both as locally straight from a phenomenological/embodied standpoint of someone walking *and* as globally curved from external geometric perspective. Here, the meaning of straight lines appeared to be less lived as converging toward a single stabilized understanding, and more as a situated alignment of differing perspectives that may remain, at least partially, irreducible to one another. Such observations lead to a view of mathematical activity as one of continual rearticulation in which meaning is constituted through the dynamic maintenance of tensions rather than their removal.

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IMPLEMENTATION OF THE RESPONSE TO INTERVENTION MODEL (RTI): THE EFFECTS OF TIER-2 INTERVENTION IN MATHEMATICS FOR AT-RISK FIRST GRADE STUDENTS

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Longitudinal data show that difficulties in mathematics identified at school entry tend to persist throughout schooling and into adulthood (Morgan et al., 2009). However, early and targeted interventions in mathematics, particularly RTI Tier 2, can significantly improve performance and reduce the proportion of students experiencing difficulties (Jitendra et al. 2021). This research therefore aims to test the effectiveness of implementing Tier 2 intervention in first-grade classes in Switzerland. These models are still relatively uncommon in Europe compared to North America. The research questions are as follows: a. How effective is the Tier 2 intervention program ? b. What is the perceived social validity of the intervention?

This randomized controlled trial (RCT) study was conducted with 24 classes from 9 schools with 356 students (age of 83.3 months SD=4.13) in French speaking part of Switzerland. All student were tested in mathematics and pupils with scores <0.75 standard deviation below the mean were considered at risk. In the study, 43 students were randomly assigned to the experimental group and 39 to the control group. Fifty-six scripted lessons were developed for the Tier-2 program and conducted by teachers to small groups of 4 to 6 students, 3-4 times a week for 14 weeks. Finally, the teachers completed a questionnaire in order to mesure social validity. The results showed that in the post-test, the difference in scores between the experimental group and the control group was significant with a moderate effect size $F(1.79)=6.16$, $p=.02$, $d=0.29$. In addition, none of the children in the experimental group were in the pathological range at the end of the intervention, compared to 13% in the control group. Secondly on a 6-point scale, teachers report that Tier 2 program is useful ($M=5.06$, $SD=0.65$), usable ($M=5.25$, $SD=0.86$), and acceptable ($M=5.33$, $SD=0.64$). Thus, the positive effects observed on learning, combined with strong social validity, indicate that structured, explicit, and scripted interventions can be meaningfully integrated into teachers' regular practices to support at-risk students.

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VARIATION IN SIXTH GRADERS' MULTIPLICATION ACROSS STRUCTURES: THE INTEGER–FRACTION TRANSITION

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This study explores how sixth graders' multiplication performance varies as they move from integer to fraction multiplication across multiplicative structures. While students often develop procedural fluency with whole-number multiplication, this knowledge does not readily transfer to fractions, which require coordination of quantities (Siegler et al., 2011). In addition, multiplication problems differ in their underlying structures, which impose distinct conceptual demands on learners (Greer, 1992), while task characteristics further shape students' problem solving (Vessonen et al., 2026). However, little research has examined how this transition varies systematically across multiplicative structures.

To address this gap, the present study examines differences in performance between integer and fraction multiplication across four structures: equal groups, multiplicative comparison, rectangular area, and Cartesian product. Thirty-five sixth-grade students from three rural primary schools in Taiwan completed a researcher-developed paper-and-pencil test including integer and fraction items across the four structures, with parallel tasks allowing comparison across number domains. Accuracy rates were compared across structures and number domains to identify structure-related patterns.

Results showed that students performed better on integer than on fraction multiplication across all structures. However, the magnitude of this difference varied by structure. The smallest gap was observed in equal group problems, while the largest gap occurred in rectangular area problems. Multiplicative comparison and Cartesian product showed intermediate patterns. These findings suggest that the difficulty of fraction multiplication is not uniform but depends on the underlying multiplicative structure. Structures involving continuous quantities (e.g., area) appear to impose greater conceptual demands than discrete structures such as equal groups.

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THE AFFECTIVE EXPERIENCES OF UPPER ELEMENTARY STUDENTS WHEN PLAYING CARD AND DICE GAMES INTENDED TO DEVELOP BASIC FACT FLUENCY

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Mathematics can be challenging for many students, and negative emotional experiences may hinder achievement. From Immordino-Yang's (2016) cognitive-affective perspective, emotion and cognition are inseparable, so students' feelings about mathematics shape how they engage with and learn mathematics. Likewise, Plass et al. (2015) describe game-based learning as the interaction of cognitive, motivational, affective, and social processes. Together, these perspectives suggest that mathematics games can become meaningful experiences that build fluency, confidence, and understanding.

- What are the experiences of upper elementary students when playing Sweet Sixteen and Tick Tock Roll a Clock to develop basic fact fluency?

This qualitative study examined how 28 Grade 4 students experienced these card-and-dice games while practicing fact fluency. Using a constructivist approach, data from videography, interviews, student drawings, and field notes were analyzed thematically to explore how gameplay influenced students' confidence, engagement, and mathematical understanding across individual and collaborative settings.

The findings suggest that mathematics games can create a positive affective learning environment by promoting enjoyment, confidence, and sustained engagement, although students' experiences remained diverse. Many students described the games as more motivating than traditional worksheets, with the social nature of gameplay reducing pressure and encouraging persistence. Collaboration, strategic choice, and manageable challenge appeared to strengthen students' sense of agency while transforming moments of stress into productive struggle.

A key contribution of the study is its focus on the diversity of students' affective experiences over a two-month period. These insights may help inform future game design and reinforce the importance of attending to affect in creating more supportive mathematics learning environments.

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TOOL-BASED FRAMEWORK FOR SENSE AND SENSING IN EMBODIED MATHEMATICS TEACHER EDUCATION

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Embodied cognition research (Lakoff & Núñez, 2000; Abrahamson, 2014) suggests that meaningful mathematical understanding emerges when students' sensorimotor experiences are intentionally connected to abstract reasoning. However, task design in mathematics education often separates procedural mastery ("sense") from conceptual understanding ("sensing"), which can hinder deeper learning. The Embodied Mathematical Imagination and Cognition (EMIC) framework addresses this gap by supporting pre-service teachers (PSTs) in designing tasks that purposefully integrate both dimensions. This study explores whether structured design tools can help PSTs bridge the sense-sensing gap in their mathematics lessons. Two research questions guide the inquiry: (1) How do design tools shape PSTs' pedagogical reasoning and collaborative task design? (2) How do tool-mediated tasks support students' movement from procedural activity to conceptual justification? A design-based study was conducted with four PSTs and mentor teachers in three Hong Kong schools over two months. PSTs were introduced to four principled design tools: Anchor-Explain-Generalize Template, Mode-Shift Planner Matrix, Controlled Dissonance Prompt, and Concreteness Fading Blueprint. Working in teams, they designed and implemented 2–3 lesson sequences. Data sources included completed design artefacts, lesson videos, and teacher reflections. Qualitative analysis focused on how the tools influenced design discussions and classroom interactions, particularly moments when students transitioned from perceptual anchors to verbal or gestured explanations, or when perceptual dissonance prompted justification. It is anticipated that the toolkit will help PSTs shift from using ready-made activities to engineering principled, embodied tasks. Tasks that employ perceptual dissonance are expected to stimulate student inquiry and support transitions from sensorimotor engagement toward formal mathematical reasoning. This research shows that well-designed tools, grounded in embodied cognition, can operationalize EMIC for PSTs. The project provides a replicable toolkit for teacher education, highlights perceptual dissonance as a key design principle, and offers classroom exemplars. Ultimately, it aims to empower future teachers to integrate embodied learning systematically into mathematics instruction.

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EXAMINING MATHEMATICS TEACHERS' CONCEPTIONS OF EQUITY THROUGH SPECULATIVE FICTION

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Internationally, there has been increasing recognition of the need to focus on equity in mathematics education (Vithal et al., 2024). Teachers play a key role in attending to and developing equitable mathematical learning contexts through their pedagogical practices (Gutiérrez, 2011). However, it appears that there are limited studies that focus on in-service teacher conceptions of equity. Vithal et al. (2024) contend that the concept of equity in mathematics education is difficult to define and there are both theoretical and methodological differences in conceptualising equity. We ask, how do teachers contextualise and frame the notion of equity in mathematics teaching and learning?

This presentation draws on data from four New Zealand teachers' conceptions and framings of equity in mathematics education using a methodological tool of speculative fiction. Speculative fiction is an umbrella term for stories that imagine worlds beyond our current reality, exploring possibilities that have not yet occurred, often focusing on a taken-for-granted norm or structure and considering what if it were *not* the case (Helliwell & Ng, 2022). Participating teachers were invited to write a story (a piece of speculative fiction) in response to a prompt, imagining a mathematics classroom of the future (in 10, 20, or even 50 years). It is these stories that form the basis of our analysis.

Our findings illustrate how the methodology drew out dimensions of equity, shifting teachers away from gap-gazing to imagining future worlds where all children embodied powerful mathematical identities. We contend that utilising speculative fiction as methodology enabled a group of teachers to voice what equity in mathematics education could be, going beyond those of dominant discourses. While each teacher was situated within unique and individual contexts, what emerged were distinct, yet overlapping articulations of a future world where equity flourished.

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“I FEEL WORRIED”: SENSING MATHEMATICS AS A THREAT: YOUNG CHILDREN’S EMODIED EXPERIENCES OF MATHEMATICS ANXIETY

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Mathematical anxiety (MA) involves cognitive worry, negative emotions, and physical responses like sweating, rapid heartbeat, and muscle tension (Luttenberger, Wimmer & Paechter, 2018). From an embodied cognition view, these physical reactions are integral to the mathematical experience—they reflect the body's sensing of mathematics as a threat even before cognitive understanding occurs (Varela, Thompson & Rosch, 2017). Despite the importance of early bodily sensing, the lived experiences of South African Grades 1-3 learners regarding MA are largely understudied, and children rarely serve as authoritative voices on their own mathematical-emotional experiences. This study explores these experiences from the children's perspective, with the main question: How do children perceive and express mathematics anxiety? It also investigates what insights children's voices might offer for developing suitable interventions. An interpretivist, qualitative multiple-case study was conducted at two after-school centres, involving 19 Foundation Phase learners. Data collection included the Maths Anxiety Scale, semi-structured interviews, focus groups, drawings, and individual reflection sessions. Analysis was conducted thematically, using the constant comparative method and inductive coding. The learners described MA through embodied language — helplessness, shame, and physiological reactions like sweating and freezing — showing awareness of their somatic responses to math threat. Child-informed coping strategies were shared, framing learners as active agents rather than passive recipients. The body's perception of math as a threat disrupted early sense-making. Recognizing children as experts in their own anxious experiences and using multimodal, child-centred methods — including drawing as embodied data — improves methodological and substantive understanding of early MA. Creating Foundation Phase maths environments where the body feels safe rather than threatened may be essential for mathematical sense-making.

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SENSING AND MAKING SENSE OF MATHEMATATICS THROUGH MATHEMATICAL HABITS OF MIND: A CROSS-CULTURAL STUDY OF GRADE 3 CLASSROOMS

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This initial study employs the construct of mathematical habits of mind to reconceptualize mathematical understanding as students' participation in coherent, epistemically grounded reasoning. Despite its theoretical significance, limited empirical research has examined how these habits are enacted and supported in everyday classroom practice, particularly in cross-cultural contexts. This study addresses that gap by investigating how mathematical habits of minds are cultivate in primary classrooms, thereby contributing empirical insight into how students develop the capacity to make sense of mathematics as a coherent system of ideas.

This study presents a comparative investigation of Grade 3 mathematics classrooms in Japan and Thailand, focusing on the development of mathematical habits of mind. The participants group were students in Grade 3 at the Attached School of Naruto University of Education in Japan and at the Demonstration School of Khon Kaen University in Thailand. Data were collected through video recordings, photographic documentation, and field notes. The data were analysed using a descriptive approach, which were based on Cuoco, Goldenberg and Mark (1996)'s framework.

The findings reveal different forms of mathematical engagement across the two Grade 3 classrooms. In Japan, students interpreted problem situations, identified multiplicative structures through distributive representations, and explained their reasoning to the class. In Thailand, students connected mathematical ideas to everyday contexts and articulated the meaning of a right angle conceptually. These contrasts illustrate how classroom practices shape students' opportunities for mathematical sense-making.

Grounded in mathematical habits of mind, this study compares Grade 3 classrooms in Japan and Thailand to examine how instructional structures shape students' opportunities for mathematical sense-making. The findings offer cross-cultural insight into how primary mathematics can be organized as a disciplined form of reasoning.

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COGNITIVE ACTIVATION AS PERCEIVED BY STUDENTS AND TEACHERS, AND ITS RELATION TO STUDENTS' MOTIVATION

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Cognitive activation is one dimension of instructional quality, and it refers to teachers' practices, which promote students higher-order thinking; for example, providing challenging tasks and asking students to discuss and explain their answers (Lipowsky et al., 2009). There is evidence that it is related to students' mathematics related educational outcomes like achievement and motivation, but evidence varies depending, for example, on measurement approaches (Charalambous et al., 2025).

In this study, we examine how cognitive activation in classroom is related to Finnish 8th graders' liking of learning mathematics and confidence in mathematics at student- and class level. We utilize the student and teacher questionnaires from TIMSS 2023 (Trends in Mathematics and Science Study) to measure cognitive activation as perceived by students and teachers.

In our sample, there were 5 808 students and 392 classrooms. Two-level structural equation models revealed a strong relationship between student reported cognitive activation and their liking of learning mathematics both at student ($\beta = 0.366$, $p < 0.001$) and class level ($\beta = 0.662$, $p < 0.001$). Similar results were found for students' confidence in mathematics at student ($\beta = 0.311$, $p < 0.001$) and class level ($\beta = 0.502$, $p < 0.001$). However, teacher reported cognitive activation was not related to students' motivational outcomes at class level.

These results highlight the importance of cognitive activation in fostering students' motivation but also leads to question about how cognitive activation should be measured (Kunter & Baumert, 2006). In the presentation the results are presented in more detail and varying results with different measures of cognitive activation are discussed.

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TALENT DEVELOPMENT IN INCLUSIVE MATHEMATICS CLASSROOMS: PD FACILITATORS' PERSPECTIVES

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The discourse on inclusion has reoriented giftedness research from a focus on the "identified gifted few" to talent development for the "talented many" in inclusive settings (Maker & Bahar, 2024). In mathematics education, this requires developing mathematical talents within regular classrooms. Addressing this challenge, our teacher professional development (PD) programme qualifies PD facilitators who lead networks of primary mathematics teachers. Building on principles for talent development in mathematics classrooms (e. g., Prediger & Schnell, 2017), our programme focuses on (1) providing individualised support for students' learning and developmental processes and (2) designing adaptive learning environments. The study asks: What characteristics of successful talent development in mathematics classrooms do PD facilitators describe, and to what extent are these descriptions mathematics-specific? *N* = 11 primary-level mathematics PD facilitators were interviewed via Zoom in nine interviews (two conducted in tandems; 33–55 min) at the end of their first qualification year. Data were analysed using content-structuring qualitative content analysis with a deductive category system derived from the two principles, yielding seven subcategories. With regard to the first principle (individualised support), facilitators emphasised social participation (mentioned in 8 of 9 interviews), variety of tasks (3/9), and process orientation (2/9). For the second principle (adaptive learning environments), qualitative differentiation (7/9) and openness (7/9) were most prominent, followed by interaction (6/9) and collaboration (4/9). Across subcategories, mathematics-specific concretisation remained largely implicit. Targeted use of materials and representations, substantial tasks that initiate cognitively demanding mathematical processes, and orientation towards mathematical structures were rarely articulated. Overall, PD facilitators' alignment with both principles tends to remain at a general inclusive-pedagogical level, with mathematics-specific concretisations articulated only selectively. This risks reducing "talent development for all" to surface features and points to the need for more explicit support in developing mathematics-specific interpretations of both principles within PD facilitator qualification.

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MATHEMATICS TEACHERS' AI LITERACY: A SYSTEMATIC REVIEW

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With the rapid evolution of artificial intelligence (AI), mathematics teachers face growing challenges: they must understand AI technologies and critically evaluate the mathematical accuracy of their outputs (NCTM, 2024). Yet, AI literacy remains largely absent from formal teacher education (Sperling et al., 2024), and research focusing specifically on mathematics teachers is still limited. Given that mathematics is grounded in logic, reasoning, and precision, a discipline-specific conception of AI literacy is essential. Since the release of ChatGPT in late 2022, AI has significantly influenced mathematics classrooms, but a systematic understanding of how teachers develop and apply AI literacy remains scarce. This study, therefore, asks: What are the recent developments in research on mathematics teachers' AI literacy since 2023?

We conducted a systematic literature search across ScienceDirect, SpringerLink, ERTC and Google Scholar for studies published between January 2023 and January 2026. Using the search string ("Mathematics" OR "Maths") AND ("Teachers") AND ("AI" OR "Artificial Intelligence") AND ("Literacy" OR "Competency"), we identified 228 records, from which 39 met the inclusion criteria. Two researchers independently coded the 39 studies through content analysis with reliability checks.

The review reveals that current research conceptualizes AI literacy as multidimensional but lacks mathematics-specific depth. Pre-service and in-service exhibit theoretical confidence yet limited practical competence, while in-service teachers often remain cautious observers. AI can act as a "creative partner" for lesson design, yet teachers frequently over-trust their outputs and overlook mathematical inaccuracies. Existing training programs encourage reflective engagement but inadequately address mathematical verification, emphasizing technical fluency over disciplinary precision. Overall, cultivating mathematics teachers' AI literacy faces multiple challenges. Future professional development should prioritize mathematically grounded, responsible AI integration.

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MATHEMATICS SELF-EFFICACY, ANXIETY, AND LITERACY: A MULTILEVEL STUDY OF TAIWANESE PISA 2022 STUDENTS

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Previous research has suggested that self-efficacy and learning anxiety are associated with students' academic literacy (Weissgerber et al., 2022). However, little is known about how these psychological factors jointly relate to mathematical literacy across student- and school-level contexts. Drawing on PISA 2022 data, this study investigated the relationships among mathematics self-efficacy, mathematics anxiety, and mathematical literacy at both levels by applying hierarchical linear modeling (HLM). At the student level, indices of socioeconomic status, gender, mathematics self-efficacy, and mathematics anxiety were included as predictors; at the school level, aggregated socioeconomic status and mathematics self-efficacy and anxiety were considered. Missing data were treated using Bayesian estimation. As for the results, the intraclass correlation coefficient (ICC) from the null model was .425, indicating substantial between-school variation and justifying multilevel modeling. At the student level, mathematics self-efficacy was positively associated with mathematical literacy and weakened the association between socioeconomic status and literacy, whereas mathematics anxiety was negatively related, even after controlling for gender. This pattern suggests that strong self-efficacy beliefs may buffer the disadvantages associated with lower socioeconomic status in mathematics learning across student groups. At the school level, average mathematics self-efficacy further reduced the influence of school-level socioeconomic status on mathematical literacy. Interestingly, when mathematics anxiety was included alongside self-efficacy at the school level, the previously negative association between anxiety and literacy became positive, implying that, in school contexts characterized by high self-efficacy, a certain level of anxiety may coexist with or even signal higher achievement. These findings highlight the complex interplay between self-efficacy, anxiety, gender, and socioeconomic factors across levels. By simultaneously modeling student- and school-level processes, this study extends prior work by clarifying how psychological factors and structural inequalities intertwine to shape mathematical literacy outcomes. The results suggest that, in addition to enhancing students' mathematics self-efficacy, educators may benefit from incorporating appropriately challenging tasks that elicit manageable levels of anxiety and from fostering school climates that support confidence in mathematics, thereby promoting the development of mathematical literacy.

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EXPLORING PRE-SERVICE ELEMENTARY TEACHERS' CLASSROOM LANGUAGE ACROSS BACKGROUNDS AND TEACHING PRACTICE: A TEXT MINING ANALYSIS

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Traditional evaluations of elementary mathematics teachers' classroom discourse are often subjective and difficult to scale (Tseng et al., 2018). To address this limitation, this study applies natural language processing and text mining to objectively analyze the lexical features of teachers' discourse across different instructional performance levels. The findings aim to verify the feasibility of automated text analysis for objective teaching evaluation and provide an empirical basis for future AI-assisted assessment. We analyze 335 ten-minute mathematics teaching demonstration videos conducted by pre-service teachers, simulating the formal teacher screening model in Taiwan. The study investigates how gender, college affiliation (Science, Humanities, Education), mathematics background, and teaching practice influence instructional language. Based on expert evaluations, participants were categorized into three instructional performance groups: Outstanding (***O***; n=55), Standard (***S***; n=257), and Needs Improvement (***NI***; n=23). To identify distinctive classroom discourse patterns, video transcripts were processed via text mining pipeline encompassing transcript extraction, CKIP-based Chinese word segmentation, keyword extraction, TF and TF-IDF analysis. The results reveal lexical differences across teacher backgrounds and performance levels. Regarding gender, female group more often used logical scaffolding terms such as "所以[so] " and "可以[can] ", whereas male group more often used "然後[then]" and addressing terms "老師[teacher]", "同學[classmate]". Education college group used "can" more often and de-emphasized teacher authority by ranking "teacher" lower, while Science college group more frequently used "teacher" and "多少[how much]". Regarding academic background, mathematics majors used concrete nouns "披薩[pizza]", "蛋糕[cake]", while non-majors relied on procedural terms "等於[equals]", "答案[answer]". Regarding performance, the ***NI***-group used "then" exceptionally often (double the TF-IDF of others) and pronouns like "我們 [we] " and "你們 [you, plural]" the least. Lexical data for the ***O***-group and ***S***-group were difficult to distinguish by terminology alone. Overall, these findings demonstrate that text mining provides objective, quantifiable insights into instructional discourse and teaching quality.

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HETEROPERCEPTION IN COLLABORATIVE 3D GEOMETRY: THE CASE OF BODY-SCALE POLYHEDRON CONSTRUCTION

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Group activities in mathematics education are often treated as advantageous contexts for conceptual learning, particularly when participants share resources and coordinate action to create common products (Ma, 2017). Yet a persistent pedagogical dilemma remains: groups may collaborate effectively and complete demanding tasks, while individual participants develop divergent understandings that remain invisible to instructors and even to themselves. In body-scale construction tasks, this issue becomes especially salient, as participants may simultaneously manipulate the same material elements and appear to share attentional focus. But does coordinated action imply shared perception?

We propose the construct of *heteroperception* to characterize cooperating agents bearing different perceptual organizations of a shared domain, even while jointly acting on the same objects (Abrahamson, et al., in press). Drawing on interactional perspectives on cognition, we consider how perception and action are co-constituted in collaborative mathematical activity, and how divergence in perceptual organization may play a productive role in conceptual development.

Four adult participants co-assembled a body-scale icosahedron using wooden dowels and links, with minimal structural guidance. The session was audio–video recorded and was followed by individual stimulated recall interviews (SRI).

Results indicate that participants coordinated effectively while attending to different structural features. For example, one participant monitored the number of edges meeting at a vertex, while another organized the same configuration as a “roof” composed of triangular faces. These divergent perceptual constructions did not surface during the interaction and did not hinder task completion. Yet, during the SRIs, some participants detected these differences and enriched their insights.

Joint action need not require shared perception. Divergence may remain implicit in action, suggesting that collaborative geometry tasks should engage reflection.

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HOW MANY FEET DO FOUR CHICKENS HAVE? MAKING SENSE OF KINDERGARTNERS' FUNCTIONAL THINKING

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Functional thinking is central to early algebra. Identifying patterns in particular cases (whether recursive or functional) builds toward generalizing relationships (Blanton et al., 2015). To study how young learners coordinate quantities in particular cases, we asked: *what responses do kindergartners give to a prompt about the number of feet for four chickens, and how can we make sense of their responses?* We analyzed the video data of 742 semi-clinical interviews. Interviewers presented a two-column table showing chickens (1, 2, 3) and feet (2, 4, 6) and asked: "How many feet would four chickens have?" We tallied numerical answers and took a grounded approach to analyze focal interview videos, interpreting how students constructed the answers. Four answers were most frequent: eight (n=191, 25.7%), five (n=136, 18.3%), four (n=92, 12.4%), and six (n=80, 10.8%). The following ways of reasoning corresponded to the four most frequent answers. Functional: Coordinated across columns (e.g., 4+4=8). Recursive: Skip-counted the feet column to eight. Next-number: Read the chicken column as a counting sequence continuing to five. Input-matching: Noticed repeated 2s and assumed input four resulted in output four. Add-two: Applied the "up by two" rule from the feet column (2, 4, 6) to the four chickens (4+2=6). We observed that what children noticed in the table and the prior knowledge they brought to the task worked together to shape their answer (Levin, 2018). Ultimately, our contribution frames the ideas children bring to a task as initial, legitimate ways of functional thinking.

Acknowledgements: This research is supported by the National Science Foundation's DRK-12 Award #2404984. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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PRE-SERVICE TEACHERS' USE OF GESTURE IN THEIR BILINGUAL TEACHING

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Multimodal communication can support teaching and learning mathematics in bilingual settings. It includes verbal explanation, written and drawn representations and gesturing, to help learners access meaning by drawing on their multiple semiotic resources. Pre-service teachers (PSTs) are not necessarily prepared for taking full advantage of these multiple ways of communicating (Prediger, 2019), which is especially useful in contexts where learners' home language is not the language of learning and teaching (LoLT). We therefore aim to explore, 'How do PSTs use gesture in their bilingual teaching of the Jump Strategy?'. This study is framed by Vygotsky's social cultural theory (1978), acknowledging that learning takes place in social contexts through language, and recognising the value of mother tongue in learning. The qualitative case study took place in a South African University, with first-year PSTs. The mathematics teacher educator (MTE) demonstrated the Jump Strategy on the empty number line (ENL) by following scripted lessons in a teacher guide with supporting demonstration videos. PSTs then worked in pairs to video-record their own pretend teaching of the strategy to an imagined learner whose home language is not the LoLT (English). Here we focus on three PSTs who opted to pretend teach a subtraction calculation bilingually in English and isiXhosa, a previously marginalised language in South Africa. Videos were transcribed, translated into English, and broken down into steps to code, including the demonstration video as a baseline. We coded gesture as including writing (W), drawing (D), pointing (P) and showing jump direction and size (J). The PSTs' videos aligned closely to the demonstration video with implementing the writing (calculations and numbers), drawing (ENL and jumps), and pointing; however, the gesturing of the jump direction and size in the PSTs' videos was less prevalent than in the demonstration video. This suggests a missed opportunity to convey meaning. Implications include the value of MTEs and PSTs carefully analysing their pretend teaching in the form of videos (and comparing to a baseline). This enables critical reflection on their implementation fidelity, making explicit the pedagogical value of each multimodal component for enhancing access to meaning for their bilingual learners. The presentation will explore the PSTs' videos and transcripts in more detail.

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PROBLEM POSING FOR INTEGRATED STEM: A STUDY OF PROSPECTIVE ELEMENTARY SCHOOL TEACHERS

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Advocates for integrated STEM education argue that integrated STEM experiences help students develop an understanding of the characteristics of STEM disciplines as forms of human knowledge and inquiry. Such integration increases students' awareness of how STEM fields shape human environments and empowers them to take ownership of their learning (English, 2016; Roberts et al., 2022). Furthermore, proponents recognize that meaningful STEM integration relies on teachers who can successfully navigate the unique instructional challenges inherent in this approach.

This study examined opportunities for integrated STEM learning embedded in problems posed by 22 prospective elementary school teachers. Drawing on a conceptual framework that characterizes integrated STEM as encompassing authentic contexts, engineering design thinking, STEM content integration, STEM practices, 21st-century competencies, and exposure to STEM careers (Roehrig et al., 2021), the analysis provides a nuanced portrait of how prospective elementary school teachers enact STEM integration through problem posing. The results document that many prospective teachers successfully framed their problems around human needs, constraints, and system-level considerations, generating opportunities for engagement in engineering design, STEM practices, development of 21st-century skills, and exposure to STEM careers.

The findings also reveal specific patterns of STEM disciplinary connections within the posed problems, with mathematics and science being the most frequent pairing. These patterns reflect the dominant positioning of mathematics and science within elementary curricula and teacher preparation, as documented in previous research (English, 2016). Ultimately, these results underscore concerns raised in the STEM education literature and call for a greater emphasis on technology and engineering in teacher training (Roehrig et al., 2021).

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PROSPECTIVE SECONDARY TEACHERS' MATHEMATICAL KNOWLEDGE: EVIDENCE FROM ENTRY-LEVEL ASSESSMENTS IN A MASTER'S PROGRAMME

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Research on mathematics teacher education highlights that effective teaching requires solid mathematical knowledge closely linked to teaching practice (Carrillo-Yañez et al., 2018); however, students entering the Master's Programme for Secondary Teacher Education (MUFPS) show persistent difficulties in problem solving, reasoning, and conceptual understanding, particularly in specialised and horizon knowledge, even after completing their practicum experiences (Hine & Thai, 2018). This study aims to show that many students entering the MUFPS in Spain do not possess the mathematical knowledge required to fully benefit from the programme's didactic training, as well as to identify these gaps and propose supplementary support measures. The study analyses the initial mathematical knowledge of 48 students entering the MUFPS through a one-hour diagnostic test consisting of four pre-university-level problems (geometry, calculus, matrix algebra, and linear programming). The results indicate low performance, with mean scores below 4 out of 10 in three of the four problems (3.60; 3.91; 2.84), while only linear programming achieved a mean score above the pass mark (5.14). Indicator-based analysis reveals that, compared to acceptable results in basic content (means between 6.25 and 7.08), tasks involving differentiation, solving irrational equations, and classifying extrema obtained mean scores below 2, pointing to significant shortcomings in analytical procedures essential for teaching at secondary and upper-secondary levels. The findings show that many students enter the MUFPS without the expected level of mathematical knowledge, particularly in calculus and algebra. These shortcomings call into question the adequacy of the current teacher education model and support the need for entry-level diagnostic assessments and reinforcement measures.

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CONNECTING TEACHER AND STUDENT PERSPECTIVES IN AN UNDERGRADUATE MATHEMATICS COURSE

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Despite ongoing efforts to improve university mathematics instruction, challenges remain in aligning teaching approaches with student learning needs. In particular, in the prevalent lecture-based teaching, there is often a disconnect between what lecturers and students perceive as meaningful teaching and learning moments (e.g., Lew et al., 2016). Our research investigates this gap by examining how lecturers' views of their teaching align (or misalign) with students' learning experiences, and how insights from this relationship might inform undergraduate mathematics teaching. Theoretically, we draw on two constructs that offer insight into meaningful classroom episodes. From a teacher perspective, critical events refer to moments when students' thinking becomes apparent to teachers, offering opportunities to deepen mathematical understanding through teachers' ability to identify, interpret, and respond to these moments (Rotem & Ayalon, 2022). From a student perspective, memorable events are conceptualised as learning moments that hold significance and meaning for the student who experienced them, and are typically accompanied by strong emotions (Marmur, 2019). Taken together, these offer complementary lenses, capturing the cognitive and affective significance of classroom events from both teacher and student viewpoints. The study is part of an ongoing wider research project. Here, we report on data from a large undergraduate mathematics course (approximately 120 students). Data sources include four lecture observations, pre- and post-lesson interviews with the lecturer, and questionnaires completed by students. Preliminary analysis reveals both similarities and tensions between learning moments identified as meaningful by the lecturer and by the students, and the ways in which they were characterised. In the presentation, we will share selected moments from the lecturer's and students' reflections, and use these to examine how lecturer planning and noticing may shape students' mathematical and emotional experiences in undergraduate mathematics classrooms.

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PROBLEM POSING AS A PROCESS: A CASE STUDY

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This study examines how pre-service teachers move through the problem-posing process when working on structured tasks in a technology-rich environment. Our analysis is guided by Cai and Rott's (2024) model, which distinguishes four phases: orientation, connection, generation and reflection, and makes it possible to trace how participants' activity develops across the process. The study addresses the following question: How do pre-service teachers move through the problem-posing process in a technology-rich environment?

A qualitative case-study approach was adopted. The participants were a pair of mathematics undergraduates enrolled in a teacher education course who engaged in two problem-posing tasks explicitly organized around the four phases of the process. Data consisted of the pair's written productions, records of their work during the tasks, GeoGebra constructions, and interactions with AI chatbots. The analysis focused on how they interpreted the task, organized ideas, formulated candidate problems, and revised their proposals, while also considering the role of GeoGebra and AI chatbots during both problem solving and problem posing. In orientation, they tried to interpret the situation, prompt, and purpose of the task, although they initially approached it from a problem-solving perspective. In connection, they identified relationships, concepts, and possible directions for problem posing, building on ideas that emerged while solving the original problem and through GeoGebra exploration; this made the phase central for later work. During connection and generation, AI supported the examination and validation of emerging ideas. In generation, they proposed, compared, and selected candidate problems; in reflection, they evaluated them and considered revisions. Throughout the process, digital tools supported the pair's work, but the main interest of the study lies in how they negotiated what to do, which ideas to pursue, and how to pose a meaningful problem. The study shows how a phase-based analysis can make visible how participants' activity develops throughout problem posing. For teacher education, the findings point to the need to support future teachers in recognizing problem posing as a mathematical activity in its own right.

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This research was partially funded by the research project PID2022-139007NBI00.

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RECONSTRUCTING AN INTEGRATED LEARNING TRAJECTORY OF ANGLE CONCEPTS: DEVELOPMENTAL COORDINATION IN GRADE 3–4 STUDENTS

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Angle is a fundamental geometric concept involving both static (e.g., corners and openings) and dynamic (e.g., turning and rotation) aspects and is introduced in Grades 3 and 4 in Japan. Despite its importance, prior research shows that students struggle to develop a unified understanding of angle, often treating "corners," "openings," and "turns" as separate categories and experiencing difficulties in inclination and rotation contexts (Mitchelmore & White, 2000; Huang et al., 2018). This study examines how students coordinate geometric and quantitative interpretations of angle across contexts and how this coordination develops along a learning trajectory. Clements and Sarama (2021) proposed separate learning trajectories that conceptualize angle as a measurable quantity and part of geometric development. This study integrates these trajectories into a unified framework of eight stages, enabling contextual difficulties to be interpreted not as isolated recognition problems but as indicators of incomplete integration. A questionnaire survey was conducted with Grade 3–4 students to examine their positions along this trajectory.

Results showed that students more readily recognized angles when two lines were explicitly visible, whereas selection rates were lower in inclination and rotation contexts. Beyond contextual difficulty, students struggled to coordinate geometric interpretations with inclination and rotation as dynamic–quantitative interpretations, which require constructing an implicit reference line and interpreting angle as the measurable amount of turning. These findings suggest that the central challenge lies not merely in recognizing angles in specific contexts, as previously reported, but in coordinating geometric structure with dynamic–quantitative meaning within a unified concept of angle, a coordination essential for later mathematical development.

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EXPLORING THE VALUES OF A LEARNING COMMUNITY USING PHENOMENOLOGICAL METHODS

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In mathematics learning, values are defined as characteristics of specific sociocultural contexts rather than personal beliefs, and as the meaning-making mechanisms underlying discourse and interactional norms in mathematics learning (Seah & Wong, 2012). However, the process by which values are formed remains unclear. This research aims to explore the process of value formation in mathematics learning and to provide an example of a method for such exploration.

This study aims to clarify the process of value formation using phenomenological methods. Phenomenology is a philosophical approach that emphasizes discerning the essence behind individual phenomena from a subjective perspective (Brinkmann & Friesen, 2018). In 2022, the author conducted continuous observation and analysis using phenomenology for one hour each week in a class of fourth-grade elementary school students in Japan. Observations revealed 12 episode descriptions related to students' values regarding mathematics learning. These episodes consist of Ep.1 through Ep.12.

Each of the 12 episode descriptions is embedded with words, actions, and emotions related to Teacher A's support and the students' values. Examining this series of episode descriptions reveals a shift in the learning community's values from student-teacher collaboration to student-to-student collaboration. In Episode 1, the value of Teacher A supporting students and facilitating the entire learning process to deepen their understanding becomes clear. However, in Episode 9, the students object to including Teacher A in their discussion, reflecting their value system of wanting to build new knowledge independently. This suggests that as the students continued to learn under Teacher A, they came to realize the value of building mathematical knowledge on their own. This study partially clarified the process of value formation using phenomenological methods.

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This work was supported by JSPS KAKENHI Grant Number JP22K13708.

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POSSIBILITIES FOR QUALITY MATHEMATICS TEACHING: FOCUS ON FUNCTIONS

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This paper extends Prediger et al.'s (2022) framework by proposing explicit recognition rules for analyzing high-quality mathematics teaching. Using a Grade 10 lesson episode on hyperbolic functions, I illustrate the coding process and examines enacted pedagogy.

Around the world, there is increasing acknowledgement that high-quality mathematics teaching must be a central educational priority, considering its role in enabling students' epistemological access to powerful mathematical knowledge and, by extension, meaningful participation in academic, civic, and economic spaces (Prediger et al., 2022). Ensuring such access requires more than curriculum coverage; it demands teaching practices that deliberately support learners in engaging with, making sense of, and productively using mathematical ideas. One crucial strategy for advancing this agenda lies in the development of comprehensive theoretical frameworks that make it possible to examine, with analytic depth, the quality of teaching that unfolds in classrooms. These frameworks not only allow researchers to methodically characterize pedagogical practices, but also provide practical guidance for teachers, helping them adapt their teaching in ways that foster richer learning opportunities. One such framework is that developed by Prediger et al. (2022), which consists of five interconnected principles: conceptual focus, cognitive demand, longitudinal coherence, student focus and adaptability, and enhanced communication. Prediger et al.'s framework outlines principles for high quality mathematics teaching but offers limited guidance on recognizing and grading them in practice. This paper proposes detailed recognition rules, illustrates their use through a Grade 10 lesson on functions in South Africa, and examines teaching quality patterns and the analytic value of the extended framework.

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COGNITIVE DEMAND OF TASKS IN PRIMARY MATHEMATICS TEXTBOOKS: AN ANALYSIS OF MEASURES AND SHAPE AND SPACE IN IRELAND

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Findings from the Trends in International Mathematics and Science Study (TIMSS) have consistently indicated relative weaknesses among Irish students in the domains of Measurement and Geometry. This study focuses on one component of a doctoral project examining alignment across curriculum, instruction, and assessment by investigating the content focus and cognitive emphasis of tasks in primary mathematics textbooks. Specifically, it analyses tasks from three Fourth Class (Fourth Grade) mathematics textbooks within the Measures and Shape and Space strands.

This study draws on two frameworks: the revised Bloom's taxonomy of educational objectives (Krathwohl, 2002), used to classify cognitive emphasis, and Porter's Survey of Enacted Curriculum (Porter & Smithson, 2002), used to examine alignment across content and cognitive dimensions. The cognitive levels in the adapted Bloom's framework include remember, understand, apply, analyse, evaluate, and create.

Using the above frameworks, all tasks within the selected textbooks were classified and the resulting data were transformed into matrices representing frequencies and proportional distributions. Visual representations, including tables and heatmaps, were generated to illustrate patterns across the textbooks. Porter's SEC was then used to calculate the alignment between each of the textbooks.

Across all three textbooks, a clear concentration of tasks at the apply level is evident, particularly within the Measures strand. Higher-order tasks involving evaluation, or creativity remain scarce. Alignment scores indicate moderate consistency between the three textbooks in terms of content focus and cognitive demand.

These findings suggest that the textbooks emphasise procedural practice over higher-order mathematical thinking. This raises important questions about the extent to which students are provided with opportunities to engage in the kinds of reasoning and problem solving reflected in large-scale assessments such as TIMSS.

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CONTRASTING TEXTBOOK CONCEPTIONS AND STUDENTS' RESPONSES TO PIVOTS QUESTIONS

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The PROTEUS project (Mesa et al., 2023) developed interactive open questions for open-source calculus and linear algebra textbooks housed in the Runestone Academy learning analytics platform. The responses to these questions, designed to be answered prior to formal instruction, allow us to map how students reason about specific mathematical topics. Using Balacheff's cK¢ model (Balacheff & Gaudin, 2009), we contrasted the control structures that can be inferred from the textbook with the control structures present in students' responses to textbook questions about pivots. The data come from 38 students from three different teachers in the U.S.

We found three conceptions in the textbook related to the question of determining how many solutions a consistent linear system has. The control structures supported by the textbook were: 1) Pivots on the columns (students could check whether every column of the coefficient matrix contains a pivot leads to a unique solution), 2) Free variables, (identifying at least one free variable supports the conclusion of infinitely many solutions), and 3) Parametric description (expressing variables in terms of basic variables leads to the conclusion of infinitely many solutions). In the student responses, we identified two additional control structures, Degree of freedom and Relationship between number of pivots and variables. Most responses (n=17) used Free variables or a combination of two control structures (n=12).

The emergence of control structures not explicitly supported by the textbook suggests that students' mathematical validation processes cannot be fully inferred from curricular propositions alone. In addition, we can observe that, despite students developing control structures beyond what is explicitly presented, there is a noticeable influence of the textbook content in their responses, reflecting how curricular materials might shape the strategies students employ.

Acknowledgment. Funding for this work was provided by the National Science Foundation. Any opinions, findings, and conclusions or recommendations are those of the author(s) and do not necessarily reflect the views of the NSF.

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FEELING FIT: NEGOTIATING MODELLING PRACTICES

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The prominence of data has intensified calls to integrate data science into K–12 mathematics (NCTM, 2024). Integrating mathematical and data perspectives can support students in taking up and extending disciplinary ways of knowing, but also raises pedagogical challenges, as domain-specific practices involve distinct forms of reasoning and ways of structuring and evaluating knowledge (Lehrer & Schauble, 2021). Drawing from a larger design-based research project studying teachers' integration of data and mathematics, this analysis focuses on a lesson in which an experienced Algebra 1 teacher, whom we call Veronica, worked to coordinate mathematical ideas of rate of change and nonlinear relationships with data modelling. We explore how mathematical and data-oriented modelling practices interactionally emerged and were coordinated or brought into tension to interpret nonlinear data. Drawing on the construct of disciplining perception (Stevens & Hall, 1998)—a deliberate practice of seeing in ways that are socially organized and relevant to a particular discipline—we used interaction and discourse analysis on multi-perspective classroom video from the focal lesson to understand how language and gesture were used to evaluate models across mathematics and data science. In the lesson, Veronica asked students to construct linear models of nonlinear data that "have the most dots on the line," introducing a provisional criterion for fit that supported models that might work locally but not across the entire distribution. Through affective evaluations (e.g., "that sucks"), she destabilized this criterion, positioning aesthetic judgment as central to evaluating models without emphasizing disciplinary notions of fit. Different ways of seeing emerged and were taken up, including going through points, representing rate of change, and balancing error. Students felt when linear models were insufficient, motivating the need for piecewise models. However, this ambiguity was not productive for all students. In moments of breakdown, these ways of seeing came into conflict, resulting in Veronica prioritizing the mathematical structure of rate of change. These findings highlight the complexity of fostering epistemic diversity, as efforts to integrate and sustain multiple ways of seeing can shift toward privileging specific disciplinary criteria.

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PUPILS' CRITERIA FOR EVALUATING MATHEMATICAL ARGUMENTS

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Evaluating arguments is a key objective in mathematics education, yet pupils often struggle in this area. Although there are theoretical frameworks for proof competence, these tend to focus on methodological knowledge rather than the specific criteria pupils use in practice (e.g., Heinze & Reiss, 2003). This study takes an entirely empirical approach to identifying the criteria applied by pupils when evaluating mathematical arguments. A qualitative study was conducted with 37 eighth-grade pupils (aged 13–14) to investigate these criteria. The participants were asked to evaluate 15 different arguments. These arguments varied in terms of their representation (verbal, symbolic or iconic), as well as in terms of the completeness and correctness of their structural elements (Toulmin, 1969). The pupils assigned a grade to each argument and provided a justification for each grade given. The data were analysed using an interpretative approach based on hermeneutics to reconstruct the underlying criteria and corresponding categories.

The analysis revealed a comprehensive list of 15 distinct criteria used by the pupils, covering everything from content to recipient orientation. Some examples of the categories of criteria are: correctness and logic (pupils focused on the correctness of the result and argument, as well as the conclusiveness and internal logic of the argument), communication and structure (criteria such as understandability, linguistic expression, and the quality of visualisation or representation), metacognitive aspects (pupils distinguished between their perceived understanding of the author and the traceability of the explanation provided), formal and structural elements (completeness, mathematical formality, and the generality of the argument). The findings demonstrate that pupils possess multidimensional perspectives and are capable of evaluating arguments from various perspectives including pedagogical and subject-specific ones. The identified criteria provide a valuable foundation for discussing what constitutes a "good" argument in mathematics education for pupils. The findings also provide a starting point for classroom discussion about which criteria are useful for evaluating (mathematical) arguments and how these criteria can be applied when constructing one's own arguments.

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CONCEPT CARTOONS IN THE TEACHING OF MATHEMATICS EDUCATION RESEARCH FINDINGS ON GEOMETRY

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Understanding how prospective mathematics teachers (PMTs) interpret research findings is a key challenge in Mathematics Education. While mathematics teaching generates extensive research, the pedagogy of teaching these findings to PMTs remains underexplored, with teacher educators often relying on a "show and tell" approach. Drawing on Ochoviet & Parodi (2024), this paper presents preliminary findings from an ongoing study that employs concept cartoons (Naylor & Keogh, 2013) to engage PMTs with Wilson's (1990) research on inconsistent ideas about geometric definitions.

The study adopted a qualitative exploratory case study design, involving seven PMTs enrolled in a final-year teacher education course. Data generation centred on a sequence of three concept cartoons depicting PMTs discussing authentic student productions on geometrical definitions. Participants' written responses to the CCs and transcriptions of the audio-recorded plenary discussions were analysed following a thematic approach: open coding was first employed to identify emergent patterns, after which axial coding enabled the grouping of these patterns into coherent categories, which were subsequently interpreted in light of Wilson's (1990) framework.

Analysis of data reveal how PMTs progressively identified five key difficulties: transition between definition and representation; set inclusion; deeply-rooted conceptual prototypes; construction of definitions; and testing of definitions. These closely mirror Wilson's (1990) sources of student inconsistency: implications in definitions, imprecise language, strong prototypes, and logical reasoning difficulties, yet emerged spontaneously from PMTs' interpretations. The findings suggest that CCs constitute a productive context for eliciting pedagogical content knowledge, offering a low-stakes alternative to traditional practices that deepens PMTs' sensitivity to student cognition.

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GENDERED VISUO-SPATIAL REASONING IN FIFTH GRADE: THE ROLE OF THE DIDACTIC CONTRACT

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Spatial reasoning is a central component of mathematical sense-making in geometry, although literature documents a persistent gender gap in visuo-spatial tasks, with males often outperforming females as early as primary school (Lauer et al., 2019). In this context the study is framed through the notion of the didactic contract, understood as the system of implicit expectations that defines legitimate mathematical activity (Brousseau, 1986). These expectations can privilege explicit and verbally justifiable strategies, potentially constraining intuitive spatial reasoning, which has been shown to develop differently across genders during the primary school years (Lauer et al., 2019). The study adopts a mixed-methods design combining large-scale assessment data with an exploratory qualitative analysis of students' written justifications on a visuo-spatial item involving non-standard orientation. Preliminary quantitative results indicate a gender effect favouring males, while qualitative analyses provide initial interpretative insight into this pattern. Female students' responses show recurring tendencies such as reliance on school-familiar but geometrically irrelevant properties, visually unstable reasoning and hesitation in mentally manipulating spatial configurations. In some cases, initially correct intuitions appear to be abandoned when they cannot be aligned with expected forms of justification. In contrast, such patterns are not systematically observed in male students' responses, which more often rely directly on spatial configurations and tend to preserve initial intuitions. Overall, these findings suggest that gender differences in visuo-spatial performance may be related to the interaction between spatial sense-making and classroom norms. The didactic contract operates privileging compliance with rules and verbal control over intuitive exploration. For students, the pressure for formal justification and procedural correctness can lead to a suspension of agency in tasks that require generative and autonomous visualization. Recognizing these socio-didactic constraints is essential for designing practices that validate visual intuition and foster more equitable conditions for mathematical sense-making. This study contributes to ongoing research by providing empirical insight into the interplay between gender differences in visuo-spatial reasoning and the didactic contract and opens the way to further work aimed at strengthening and refining this relationship through a broader mixed-methods design.

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TAKING ADVANTAGE OF SCHOLARLY CONTENT FOR PROMOTING MATHEMATICAL COMPETENCE

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In recent years, there has been a proliferation of international curriculum proposals promoting teaching practices that make sense of school mathematical content (e.g., Battista, 2017). In this way, teaching of mathematics is intended to develop mathematical competencies such as modelling, generalisation, connections, among others. However, teaching mathematics this way at preuniversity levels is not without difficulties. This is due to teachers' low confidence in their own ability to work directly on mathematical skills in the classroom, as well as other curricular and contextual constraints. Educational initiatives, such as Math4U (Vondráková et al., 2019) may help to address this shortcoming, because it integrates real world problems as part of resources aimed at teaching curricular content. These apparently provide opportunities to develop mathematical skills, which motivates the question addressed in this research: To what extent problems based on scholarly content are useful for promoting modelling competencies in pre-university students?

An exploratory and descriptive study based on qualitative data is developed to address this question. The sample is composed of a group of 20 high school students, who solved individually the task 'The Parabolic Calculator,' obtained from the Math4U platform and devoted to exploring numerical properties of the parabola from a geometric construction. The students' written productions are analysed according to the strategy proposed by Montejo-Gámez et al. (2021) for school mathematical models.

The results of the study show the reasoning used by the students to analyse the geometric construction, interpret, validate, and generalise the model through its algebraic treatment. The discussion reflects on the potential of this type of task to foster students' mathematical competence and to facilitate the transition from mathematics at the end of secondary education to the formalism of mathematics in the first years of university.

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MAKING SENSE OF MATHEMATICS THROUGH A BOUNDARY ARTEFACT IN VOCATIONAL NURSING EDUCATION

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In vocational mathematics education, many students are demotivated to learn mathematics, which to them looks detached from their desired professions (FitzSimons & Boistrup, 2017). To address this disconnect, we investigated how students in nursing education can make sense of mathematics through interaction with a blood-pressure measurement device. Underlying our research is (1) the Theory of Semiotic Mediation and its concept of didactical *artefact* that evokes mathematical meaning (Bartolini Bussi & Mariotti, 2008), and (2) the concept of *boundary object* that preserves its identity across disciplines (Akkerman & Bakker, 2011). We combined these concepts into a new concept, *Boundary Artefact* (BA), which is a mediating tool for meaning making in both the vocation and in mathematics. The classroom activities, in which students measured their blood pressure using the BA, were co-orchestrated by a mathematics and a vocation teacher. The video-recorded data were analysed on how students' interaction with the BA gave rise to personal signs, such as numerical values being considered as "normal" or "problematic" and gestures indicating increase or decrease. In the ensuing classroom discussion, these evolved into shared meanings from the mathematical and vocational domains. Students' meanings related to measurement, variability, and thresholds, which remained closely connected to professional judgement and institutional norms. Our study reveals that a BA, such as a blood-pressure measurement device, can enable students in vocational education to engage with mathematical meanings emerging from professional practices. In this semiotic mediation, the BA functions as a generator of signs and then progressively mediates mathematical and vocational meanings simultaneously. Our study contributes to PME research by offering a theoretically grounded analysis of mathematical sense-making in vocational contexts, where mathematics does not announce itself (FitzSimons & Boistrup, 2017).

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MATHEMATICAL MODELLING KNOWLEDGE AND ATTITUDES OF FIRST-YEAR SOCIAL SCIENCE STUDENTS

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Research in mathematics education has established mathematical modelling (MM) as a key practice for developing conceptual understanding and connecting mathematics to real-world contexts (Marchisio & Roman, 2021). To date, MM research has largely focused on science, technology, engineering, and mathematics (STEM) disciplines with limited attention to social science and art students, particularly at the transition to university contexts (Marchisio & Roman, 2021). Addressing this gap, this study investigates the MM knowledge and attitudes of first-year social science students at a South African university thereby establishing a necessary foundation for the informed and responsive research on mathematical modelling in undergraduate social science contexts. The study is guided by the research question: *What are the levels of mathematical modelling knowledge and attitudes among first-year social science students* Using a descriptive, quantitative cross-sectional design, survey data were collected from 156 first-year psychology students at a South African university. Adapted versions of the Mathematical Modelling Knowledge Scale and the Mathematical Modelling Attitude Scale (Asempapa, 2016) were administered, and descriptive statistics and regression analyses were used to analyse the data. Findings indicate moderate levels of modelling knowledge, with students demonstrating strong recognition of modelling as a translation process and the use of multiple representations, but weaker understanding of the distinction between modelling and problem-solving and the role of technology in modelling. Attitudes toward modelling were generally positive, particularly with respect to real-world relevance, constructivist learning beliefs, and motivation. However, students reported lower confidence in their own understanding of modelling. While broadly aligning with earlier studies that used the same instruments, the present findings extend this work by demonstrating that these patterns also hold for first-year social science students, highlighting the relevance of mathematical modelling beyond traditionally studied STEM populations.

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MATHEMATICAL MODELLING AND EARLY ALGEBRA: CHANGES IN TEACHERS' SENSE-MAKING IN A CONTINUING PROFESSIONAL DEVELOPMENT PROCESS

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Mathematical modelling (MM) and early algebra have a prominent place in primary education and curricular guidelines. However, within mathematics education research, empirical evidence on MM in primary education—particularly its potential to foster algebraic thinking at early ages—remains limited (Anhalt, 2021). Given that implementation largely depends on teachers' interpretations and decisions, this study analyses the senses primary education teachers construct about MM to enhance early algebraic work in a continuing professional development course in Chile. We conducted a qualitative case study with primary education mathematics teachers in this course, focused on early algebraic work and the use of MM to promote it. To explore changes in the senses attributed to MM, we examined initial narratives and final group interviews through the lens of Esteley and Cruz (2021). The results show changes in the senses attributed to MM for enhancing algebraic thinking. Initially, teachers mainly linked algebra to equations and MM to contextualised single-solution problems. They later recognised MM's potential to promote Blanton's (2008) algebraic practices: generalisation, representation, justification, and reasoning. They also value the open nature of the tasks, particularly the selection and discussion of variables. For example, they note that pupils "discover new variables within a problematic situation and go beyond a simple result". By the end, they attribute a more open role to MM, highlight its richness for early algebraic work, and design and implement classroom proposals aligned with this orientation.

Acknowledgements: This study is funded by the research project PID2024-157106NB-I00; financed by MCIN/ AEI/10.13039/501100011033, and FEDER, UE.

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THE NEW NEEDS FRIENDS

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What we call ‘new’ in math education is often linked with how students construct relevant math identities (Gee, 2000; Gutiérrez , 2013). These identities, further, are studied as sociopolitical constructs, with a focus on issues like ‘agency’ and ‘power’. However, the student also emerges at a more fundamental and ontological level: at the level of Being or bare subjectivity (see Brown, 1996). At this level, we assume that what (or who) emerges within ‘proper’ math classrooms must be a math student, since that is our curricular intent or the aims of (math) education. In other words, we justify ‘the new’ because *it is new*, thus fetishizing ‘the new’. To ontologically justify the new, we must resist the idea that math classrooms always construct math students and trace how *mathematical subjectivities* emerge.

This emergence is a temporal problem. In this study, I thus ask: “In what time do mathematical subjectivities emerge?”, or “When does the new appear?” Conventional answers might suggest that the new appears only in futures intended by the curriculum. Instead, within assignment submissions from 24 pre-service teachers enrolled in a course on math anxiety that I taught, I find evidence that my students also constructed themselves (as math subjects) in the past and the present, as well as in the future.

From their assignments, I recognized three distinct ways that the ‘new’ emerges and the their three corresponding modes of mathematical engagement. The ‘new of the past’ appears when the student considers herself already constructed as a math student and *refuses* to be re-constructed during the current lesson. The ‘new of the present’ appears when the student *intuits* her own construction as a math student as it happens. The ‘new of the future’ happens when the student *speculates* her construction as a math student and uses empty gestures to represent a future non-representable self. I read their temporal emergence through various continental philosophers and advocate a different role for the teacher. Instead of a guide, mediator or facilitator, I argue for a teacher-friend: one who travels into various ‘times’ to gather her students.

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EXPLORING RELATIONAL THINKING IN TEACHERS' WORKING WITH CONNECTED ADDITION TASKS

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This study draws on research on relational thinking, defined as "using fundamental properties of number and operations to transform mathematical expressions rather than simply calculating an answer following a prescribed sequence of procedures" (Carpenter et al., 2005, p.54). Connected tasks afford such thinking by enabling prior results to be adapted when solving subsequent problems. However, South African research highlights gaps in instruction that promote sense-making and relational thinking for efficiency (Askew, 2019). We therefore examined how eight primary teachers, participating in task-based interviews preceding a WhatsApp-based professional development programme linked to their teaching after-school Maths Clubs in low-attainment contexts, responded to two connected addition tasks (A: 53 + 20; B: 53 + 19), in terms of solving for themselves and for supporting learners. Our analysis involved coding teachers' interview response video clips for the extent of relational thinking in their solving the two tasks. Relational thinking was identified when teachers used the structure or result of the first task to transform the second. For instance, seeing 53 + 19 as 53 + 20 − 1, or as one less than 73, rather than solving it as an independent calculation.

Across the two tasks, a clear pattern emerged in teachers' strategy use. While Task A was mostly solved mentally using jump strategies, place-value decomposition, or column algorithm, Task B was largely treated as a more complex and independent problem, which was solved using written column algorithms or Dienes' block representations for both self and learners. No instances of relational thinking were observed in teachers' own solutions, as teachers did not use the result of Task A to transform Task B, even when invited to consider the tasks collectively. Only one teacher indicated the possibility of using relational thinking in her teaching, using rounding and adjusting from 53 + 20. Teachers favoured written column algorithms and Dienes' blocks for their perceived accessibility to a wide range of learners. Overall, teachers' responses suggested that Task B was almost always recomputed as an independent task. This indicates limited uptake of the relational affordances intentionally built into our connected task pair.

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TASK DESIGN PRINCIPLES FOR SUDENTS' DEFINING

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In mathematics, definition and defining are essential, as with proof and proving. Some frameworks that capture students' defining have been proposed (Kobiela & Lehrer, 2015; etc.), and revealed diverse processes when students construct or refine definitions of mathematical concepts. Based on the accumulated research, the present study aims to develop classroom interventions especially focusing on mathematical tasks that students can engage in constructing and refining definitions. I am tackling the research question: How can we design tasks to promote students' defining? The present study has been conducted using the methodology of topic-specific design research. I constructed a conceptual framework to promote activities where students themselves construct their tentative definitions and refine them into mathematically appropriate definitions. This framework consists of two phases: (a) purposefully selecting a definition of a familiar concept or giving a name of a new concept, (b) refining tentative definitions into appropriate ones. Given these aspects in the framework, I established learning the goal to engage defining in the context of mathematical inquiry and designed the task for university students (preservice teachers) that is the extending inclusion relations of special quadrilaterals. This task involves students constructing definitions by examining examples and properties of quadrilaterals they know the names (e.g., trapezoids, kites) and refining their definitions into more correct and more sophisticated ones. Using this task, classroom interventions that consist of three sessions were implemented with 35 students. The lessons were transcribed and analyzed qualitatively with the help of a research assistant. Through the analysis, I identified several scenes in the classroom lesson episodes where students selected their definitions of special quadrilaterals and refined them by discovering or non-examples of the concepts. Students were able to not only define and classify squares, rectangles, rhombuses, parallelograms, and trapezoids, but also extend the inclusion relation of them to include kites, isosceles trapezoids, and other quadrilaterals that have special properties. The results suggest that the designed tasks are effective for promoting students' defining. I concluded that the following two task design principles for defining are formulated: (1) To present questions about an incomplete system that generates students' doubts, (2) To suggest a consideration about the relationship between multiple concepts in order to construct their definitions.

Acknowledgements

This work was supported by the JSPS, Grant Numbers 23K12778 and 24K00410.

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PRE-SERVICE MATHEMATICS TEACHERS' NOTICING OF CRITICAL EVENTS ASSOCIATED WITH ARGUMENTATION

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Argumentation is increasingly recognized as essential in mathematics classrooms, yet teachers often struggle to capitalize on opportunities for its development. Given the multifaceted nature of argumentation, its successful facilitation in the classroom depends on teachers' ability to recognize moments that can foster the constriction, critique, and refinement of mathematical ideas. We refer to such moments as Critical Events Associated with Argumentation (CEAs) – instances identified by teachers as opportunities for argumentation to develop and to use them as a lens for analyzing teacher noticing (Ayalon & Nama, 2025).

This study focuses on preservice mathematics teachers (PSTs) and examines the CEAs they identify while observing classrooms during their teacher education program. Focusing on self-selected events offers insight into what PSTs perceive as potential for argumentation at early stages of professional learning. The research question is: What characterizes the CEAs identified by PSTs during classroom observations?

The study sample comprised nine secondary PSTs. During the program, they learned about argumentation in mathematics classrooms and conducted classroom observations to identify CEAs. Events were coded as structural or dialogic, with the structural aspect focusing on argument construction and justification, and the dialogic aspect on peer interactions. Participants identified 21 events: 11 structural, 10 dialogic, with two involving both aspects. Most structural events reflected challenges in argument construction, including constructing invalid arguments (6/11) and using inadequate justifications (4/11). Dialogic events frequently highlighted difficulties in participation, such as non-participation or withdrawal from discourse (4/10) and disrespectful responses to peers' arguments (3/10). The same number of events (3/10) focused on productive discourse that led to mathematical progress.

Taken together, the events identified by PSTs highlight multiple ways in which argumentation is recognized in classroom practice, encompassing both reasoning processes and interactional dynamics. While based on a small sample, these findings suggest that examining the events PSTs select can offer insight into their emerging understandings of what counts as an opportunity for argumentation.

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WHAT KINDS OF STUDENTS DOES MATHEMATICS ASSESSMENT PRODUCE?

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Mathematics assessment is often conceptualised as an objective practice of measurement (Nieminen et al., 2023). This dominant paradigm has been challenged by views of assessment as a socio-historical practice that shapes how students are 'known' in mathematics education (Yolcu & Popkewitz, 2019). Namely, mathematics assessment creates and reifies the 'constructs' it purports to measure (Wiliam et al., 2004, p. 45). What kinds of students are mobilised by mathematics assessment, then?

In this study, this objective is explored by analysing curriculum and policy documents regarding mathematics assessment in Finland. The analysis relies on Popkewitz' policy analysis tools that call us to focus on those practices of mathematics education that make it possible to generate truth statements about who children are, who they are becoming, and how they differ from each other (Yolcu & Popkewitz, 2019). Assessment produces precisely such truth statements about who students 'are'. Through a socio-politically oriented document analysis of Finnish mathematics assessment, this study contributes to assessment research by making visible the 'ideal' learners of mathematics that are mobilised by these policy frameworks.

The policy analysis shows that the desired student subject is mobilised by assessment as individualistic and objectified. To summarise, mathematics assessment (and summative assessment in particular) aims to target the individual student's mathematical abilities and in doing so creates a particular historical idea of an individual knower and doer of mathematics (cf. Nieminen et al., 2023). On the other hand, through assessment, (and formative assessment in particular), students are made into objects to be managed, controlled, and 'put into shape' (cf. Yolcu & Popkewitz, 2019). This study calls for further attention to the (un)desired ways of being a student that are made possible by mathematics assessment.

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PROSPECTIVE TEACHERS' USE OF REPRESENTATIONS IN A LEARNING STUDY ABOUT THE COMPLEMENT PRINCIPLE

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This study investigates what prospective teachers (PTs) learned when teaching the relationship between addition and subtraction in Grade 1 during a learning study. The aim is to identify how the use of representations changed from the first to the second lesson in the iterative process. While representations can enhance mathematical learning (Schoenherr et al., 2024), their effectiveness depends on how they are aligned with the object of learning. The study was conducted within a mathematics course in primary teacher education at two universities, where 100 PTs participated in a learning study (Pang & Runesson Kempe, 2019). A case from one group working with the complement principle (if a+b=c, then c−b=a) is presented. The theoretical framework of the study is variation theory (Marton, 2015). Data include video recordings from Lesson 1 (L1) and Lesson 2 (L2), in which PTs used variation theory to design and plan the lessons and identified critical aspects for pupils' learning. One identified critical aspect was the same part-whole relation between the numbers for addition and subtraction. In L1, the prospective teachers used the bar model to illustrate the part-whole relation. A torn A4 sheet was used to help pupils experience a part-whole structure, but the structural relationship was not explicitly foregrounded. After revision, a more fine-grained use of the bar model was present, highlighting new critical aspects of the complement principle. The bar model in L2 was coordinated with symbolic expressions, strengthening connections between visual and numerical representations to increase understanding of the complement principle. In L1, the paper bar model was used as a general model, while in L2 it was used to illustrate a specific relation (5/4/1), accompanied by the question: "How can we make this a subtraction?" The bar model was used to find the missing parts (L2) instead of the whole (L1). In L2, the bar model was used to work with subtractions $5-4=1$ and $5-1=4$. PTs visualised the invariance of the whole, keeping 5 constant while varying the missing part, explicitly discerning how the same part–whole structure underpins both operations. The findings illustrate how participation in a learning study can support PTs in developing a theoretically grounded and purposeful use of visual representations in teaching relationships between addition and subtraction in early arithmetic.

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MATHEMATICAL CONCEPTS AS FUNCTIONAL PROPERTIES

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Learning mathematics within an enactive-ecological radical embodiment (EERE) framework is characterized as 'living' in the relationships and interaction of the learner and the world around them. Meanwhile, traditional cognitivist frameworks understand conceptual understanding and concept formation as part of the cognitive structure of a learner, merely being influenced by this interaction. Designing learning opportunities for conceptual understanding within EERE now needs to clarify and explicate what we are actually aiming for in terms of learning outcomes, problematizing what we understand as a mathematical concept within radical embodiment to take into account the entanglement of cognized and cognizer (Baggs & Chemero, 2021).

This theoretical contribution explores this question by discussing the nature of concepts on the basis of a central tenet of radical embodiment—that of *affordances.* Affordances are possibilities for action that arise as relational properties between an individual and features of an environment. Engaged in loops of perception and perceptually guided action, individuals become aware of certain affordances, which allow for coordinated interaction. Inspired by Thompson (2023), we propose a concept of a mathematical concept with the selective focus on affordances and the accompanying possibility of exclusion at its core: Assuming that changes in the layout of affordances can be perceived (Chemero, 2003), a certain affordance A1 can be distinguished from non-A1. The affordance A1 serves as a template, leading to a dichotomy of A1 and non-A1 through the process of exclusion, not based on similar properties of the objects being engaged with, but on similar sensorimotor responses in the course of goal-oriented tasks. In this sense, concepts are not 'some-things' but highlighting aspects of dynamic processes that "are a functional part of situations and have a participatory role in structuring them" (Thompson, 2023, p.147). From an educational standpoint, it is hypothesized that learning opportunities are designed so equivalence classes of sensorimotor response can be built by learners. Our idea of a mathematical concept will be discussed in the context of empirical illustrations.

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INFORMAL AND FORMAL MODES OF PROOF-RELATED REASONING IN EUCLIDEAN GEOMETRY TASKS IN SOUTH AFRICAN TEXTBOOKS

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Proof is a central practice in mathematics, particularly in Euclidean geometry, where students are expected to organise knowledge using a system of definitions, axioms, and theorems. However, Tall (1998) argues that students are often introduced to proof as a finished product characterised by formal, symbolic language beyond their conceptual grasp. Consequently, they tend to memorise procedures rather than make sense of the logical relationships that justify each step. In response, contemporary research (e.g., Zhang & Qi, 2019) conceptualises proof within a framework of proof-related reasoning (PrR) that includes informal and formal modes. Proof is therefore understood not only as a formal deductive endpoint but as part of a continuum linking informal reasoning to structured deduction. Following Zhang and Qi's (2019) framing, informal modes, such as *identifying patterns*, *conjectures*, *non-proof arguments*, and *evaluating claims*, function as precursors to formal proofs. Formal proof (*demonstrate, generic examples, counterexamples*) requires theorem-based justification (Zhang & Qi, 2019). In this study, we adopted this framework to examine how modes of PrR are embodied in expository and exercise tasks and to determine their pedagogical orientation toward proof. We conducted a qualitative content analysis of nine purposively selected mathematics textbooks for Grades 10-12 that are widely used in public schools. A total of 1993 tasks were analysed. In South Africa, textbooks are among the primary resources for teaching and learning, particularly in rural and under-resourced schools. They therefore shape how mathematics is learnt. The results indicate that while textbooks provide substantial engagement with PrR (67.93%, 1354 of 1993 tasks), they emphasise formal proofs. Minor opportunities (11%) for informal reasoning are evident in Grade 10 but diminish progressively in Grades 11 and 12, where a strong formal pedagogical orientation prevails. As a result, the continuum from informal exploration to formal deduction is not consistently presented across the grades. This suggests a misalignment between theoretical models of proof development and their integration in the sampled textbooks. To facilitate sense-making, it is suggested that the sampled textbooks integrate both formal and informal modes of PrR across the grades.

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INVESTIGATION OF GIFTED STUDENTS' COLLABORATIVE PROBLEM-SOLVING SKILLS IN ONLINE ENVIRONMENT

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Today, many problems require teamwork; therefore, collaborative problem-solving (CPS) is regarded as a critical 21st-century skill (Graesser et al., 2018; OECD, 2017). OECD (2017) defines CPS as the capacity of two or more individuals/agents to build a common understanding by integrating their knowledge, skills, and efforts to produce solutions to the problems they encounter. With the development of technology in recent years, the ability to collaborate in online environments has come to the fore and is considered among the critical skills of the future (OECD, 2017). The presence of collaboration skills is of great importance in gifted students, who are expected to be at the forefront of the development of countries and the solution of global problems (Renzulli, 2012).

In this study, it was aimed to examine the CPS skills exhibited by gifted students in an online environment. Three gifted middle school students continuing their education at a Science and Art Center in the province of Aydın, Turkey, participated in the study. During the data collection process, six open-ended mathematics problems in an eco-mathematics context (Chesky & Milgram, 2021) were used. These problems require the use of mathematical relations and operations in the solution of complex problems such as global warming and energy issues. The PISA 2015 CPS Matrix was utilized in the analysis of the data. It was observed that the participants most frequently exhibited skills related to establishing a common understanding, while they least exhibited skills aimed at establishing team organization during the problem-solving process. It was determined that the participants did not exhibit skills A1, A2, and A3 belonging to the 'Exploring and Understanding' step throughout the process. In this study, it is concluded that the participants exhibited at least one skill from the three collaborative competencies and were able to conduct the CPS process in an online environment.

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FROM PERCENTAGES TO SOCIAL JUSTICE: MEANINGS CONSTRUCTED BY MATHEMATICS TEACHERS WHEN INTEGRATING ECONOMIC AND FINANCIAL EDUCATION

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The integration of Economic and Financial Education (EFE) in Mathematics Education is often justified in terms of everyday usefulness. Here, we examine it from a social justice perspective, focusing on how such integration redistributes, recognizes, and represents knowledge in the classroom. The study addresses the following question: What meanings do mathematics teachers construct, when reflecting on the integration of EFE, to promote mathematical activity (MA) from a social justice perspective? The study articulates three theoretical strands: (i) The conception of social justice (redistribution, recognition and representation) of Fraser (2008), used to interpret access to learning opportunities, validation of students' experiences without moralization, and the definition of curricular priorities; (ii) the Reflection and Action model (RyA) of Parada y Fiallo (2022), which characterizes teaching practices as situated reflection leading to action taken, allowing decisions about tasks, resources, interaction, and assessment to be interpreted as expressions of professional meaning-making; and (iii) EFE understood as a school field that mobilizes mathematical notions as well as positions regarding consumption, saving, credit, and indebtedness. A descriptive–exploratory study was conducted through a questionnaire administered to mathematics teachers in public schools. The instrument included closed items and an open question about financial situations addressed in class. The analysis combined synthesis across categories of practice with thematic categorization of reported examples to reconstruct meanings associated with mathematical activity and social justice. Teachers' reports suggest that EFE enters the classroom unevenly and without sustained development: it gains presence when teachers design tasks based on familiar situations but weakens when institutional conditions are required. Interpreted through RyA, four meanings of mathematical activity with EFE emerge: using mathematics to calculate and interpret; comparing options to make decisions; viewing EFE as a right linked to unequal opportunities; and opening a critical perspective in development, particularly when arguing and discussing credit and indebtedness.

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INTERPRETING STUDENTS' MATHEMATICAL THINKING INVOLVING NATURAL NUMBER BIAS

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Natural number bias (NNB) has been widely documented as a persistent difficulty in learning rational numbers. However, less is known about how prospective mathematics teachers (PMTs) interpret student responses, in which this bias becomes evident when working with fractions and decimals. This paper examines how PMTs reconstruct the reasoning underlying such responses by asking: What characterizes PMTs' interpretations of student productions involving manifestations of NNB?

The study draws on two complementary frameworks. First, professional noticing (PN), focusing on interpretation as inferring students' mathematical understanding from the details of their productions and from knowledge of mathematical thinking (Jacobs et al., 2010). Second, conceptual change research provides the basis for understanding students' NNB as a persistent reasoning framework arising from the overgeneralization of natural number properties to rational numbers (Van Dooren et al., 2015).

Nine PMTs participated in a qualitative exploratory case study within a teacher education course. The study was based on a sequence of two teacher-role concept cartoons (TRCC) whose characters are PMTs discussing authentic student productions involving NNB. Data consisted of audio-recorded plenary discussions. Segments containing explicit inferences about students' thinking were analysed and coded into three levels (Jacobs et al., 2010): lack, limited, and robust evidence (LaE, LiE, RoE).

Analysis of the data showed the levels to be characterized as follows. At LaE, errors were reduced to procedural shortcomings; NNB was explicitly ruled out. At LiE, errors were attributed to the inappropriate transfer of learned rules or analogies, with NNB implicitly suggested but not conceptualized as an underlying framework. At RoE, errors were interpreted as coherent results of reasoning with rational numbers as if they were natural numbers, reconstructing NNB as an overgeneralization of natural number properties. These levels capture qualitative differences in interpretive depth rather than developmental stages, positioning TRCC as a resource for inquiring into PMTs' noticing of NNB in students' mathematical thinking.

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REPRESENTATIONS AND GENERALIZATION PROCESS IN FIFTH-GRADE

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This study addresses generalization in the functional context of algebraic thinking. Torres et al. (2021) proposed a model of the generalization process that comprises three successive stages: abduction, induction, and generalization. The existing research on this process has primarily focused on the early years of elementary education. Our interest here is the generalization process in fifth-grade students by observing the tabular and graphical representations they use in their progress toward generalization. Additionally, there is limited research on the effects of utilising diverse representations on this model of generalization process (Torres et al., 2021), especially with regard to the use of Cartesian graphs. In this study, the aim was defined through three specific goals: (a) to identify the different stages occurring in the student's generalization process, (b) to identify the representations used to document the structure in each stage of the process, and (c) to determine whether the representations utilised during the process facilitated the student's generalization.

Through a case study, we interviewed a student to provide an in-depth analysis of the representations used during the generalization process. We analyzed a semi-structured interview with the student in which they worked on a task that included various representations, in particular Cartesian graph. We selected this student because their interview was representative of an intermediate level, reflecting the typical performance of the fifth-grade class; and because of their active participation.

The research findings encompassed the student progressed through the three phases of the model (Torres et al., 2021). On the other hand, verbal, primarily, and gestural representations were the most prevalent throughout the process. The novelty of this study is that in this instance, the representation which most facilitated progression towards generalization was Cartesian graph, in conjunction with the gestural one, primarily due to its visual characteristics and the procedural manner of its construction.

Acknowledgment

This study was conducted with the support of Projects PID2020-113601GB-I00 and PID2024-157106NB-I00 financed by MCIN/AEI/10.13039/501100011033 and Fondos FEDER. Grant PRE2021-097391 financed by MICIU/AEI/10.13039/501100011033.

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ENACTING THE "WE": COLLECTIVE IDENTITY IN UNIVERSITY ROLE-BASED COLLABORATIVE PROBLEM SOLVING

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Within university mathematics education, undergraduate students' identity dynamics play a crucial role in shaping their academic experience. Against this backdrop, collaborative problem solving has proven to be a space in which students can explore mathematicians' problem-solving practices, thus providing a valuable setting for investigating these dynamics. Building on previous research examining university students' identity dynamics during role-based collaborative problem solving and focusing on the interplay between students' identities and the institutional discourse surrounding mathematical activity (Picariello et al., in press), this study aims to extend this line of research by investigating the configuration of collective identities in university collaborative problem solving. This study draws on data from a role-based collaborative problem-solving activity conducted in a second-year General Topology course within a mathematics degree programme in southern Italy, involving eight groups of students. The activity was designed to place students in a space where they could embody different ways of engaging in mathematical activity, allowing them to align with or question institutionally shaped norms of doing mathematics. Drawing on Judith Butler's understanding of collective identity (2015), the analysis sought to trace the plural actions around which collectivities, represented by groups engaged in role-based collaborative problem solving, are constituted.

Findings indicate actions enacted in and through the activity itself configuring a collectivity around shared theory of success for university mathematics and implicit norms that frame mathematical activity fundamentally articulated around rigour, in which the affective component is deemed as less necessary. The study highlights how narratives, implicit institutional norms surrounding mathematics, and individual identity are consolidated, shared, and enacted at the group level, thus adding a new layer to our understanding of the identity dynamics in which undergraduate mathematics students are involved.

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VISUAL MEDIATORS IN LECTURES: BRIDGING COMOGNITIVE CONFLICT

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Research on instructors' practices in the university mathematics lecture has expanded over the past decade, but research on how students understand the lecture has received less attention (Melhuish et al., 2022). However, evidence suggests that communication breaches exist that may hinder students' access to the meaning conveyed by the instructor (Lew et al., 2016). The commognitive framework (Sfard, 2008), which views mathematics as a discourse, suggests that these breaches may be due to incompatible mathematical discourses (commognitive conflict). One proposed aspect that could help bridge this conflict is the use of visual mediators. Having access to visible objects on which to conduct operations can potentially lead to agreement on the objects of discourse, enabling more effective communication in situations where there are discursive mismatches.

This contribution explores the role of visual mediators in shaping students' commognitive engagement with a first-semester Real Analysis lecture. We analysed data from two distinct lectures where the definition of limit was introduced, along with stimulated-recall interviews with two or three student pairs per instructor. Our analyses focussed on the visual mediators used by the two instructors during their presentations and those invoked by students during their discussion of the definition episode.

Preliminary results indicate that instructors employ visual mediators differently in their lectures. Moreover, when discussing the definition excerpt, students rely on both the visual mediators introduced by the instructor and additional mediators they produce themselves. This suggests that comprehension of the lectures may depend on students' ability to interpret the visual mediators used by their instructors, potentially shaped by the additional mediators spontaneously available to them. We explore the implications of these findings for the effectiveness of university mathematics lectures.

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BAR MODELS IN EMBODIED MATHEMATICAL REASONING

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This study is grounded in an embodied and semiotic perspective on mathematical cognition, which conceptualizes reasoning as emerging through the coordination of perception, action, and representation. From this perspective, thinking is understood as a multimodal and embodied process, mediated through gestures, language, and artifacts (Luis Radford, 2009). Within this framework, bar models are understood as visual-spatial representations that mediate embodied reasoning, enabling learners to structure and interpret mathematical relationships. The study aims to investigate how bar models support embodied mathematical reasoning in students' problem-solving processes. It focuses on how learners coordinate gesture and visual representation to attend to, structure, and communicate mathematical relationships. A single-case study was conducted focusing on a Primary 3 student from Singapore with prior experience using bar models for the four basic operations (addition, subtraction, multiplication, and division) but no formal instruction in fractions. As part of the research design, an instructional sequence was developed around a word problem task completed on a tablet device. The tablet interface captured the student's incorrect solution. This solution was then placed alongside a valid solution provided by the teacher-researcher, creating a side-by-side comparison intentionally designed to stimulate reflection and explanation, using comparison between solutions as a guiding reference for the student to articulate their reasoning verbally and through gesture. Video recordings of teacher–student interactions served as the primary data source for analysis.

The analysis of the embodied episodes shows that bodily actions played two complementary roles. First, they directed attention to salient elements of the bar model, including structural similarities. Second, they externalised structural relationships including sequences, part-whole connections or comparisons. Building on these episodes, evidence of embodied visual-symbolic cognition appears to emerge toward the end of the observation, as the student increasingly coordinates visual structuring in the bar model with symbolic calculation, mediated through gesture. This suggests a developmental progression from earlier embodied activity to symbolic reasoning. The findings offer preliminary evidence that teachers can be supported to attend to embodied strategies as indicators of reasoning not yet verbalised, recognising how students' gestures may reveal structural insights or signal underlying misconceptions.

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SEMIOTIC TOOLS FOR RELATIONAL THINKING IN EARLY GRADES: PRELIMINARY DATA FROM A LONGITUDINAL STUDY

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The interest in early relational thinking as a precursor for algebraic thinking is growing (e.g., Cabral et al., 2022). Based on Davydov's idea of quantitative relationships (Davydov, 2008), we define relational thinking as the noticing, communication, and use of quantitative relationships to solve contextual problems (e.g., word problems in primary school). In our longitudinal study, we aim to understand and describe the possible pathways of relational thinking development in students from grade 1 up to grade 4. Our participants are 33 students from two schools in Canada. At the end of grade 1, we conducted individual interviews where the participants solved 2 word problems: one presenting an additive composition relationship and the other presenting an additive comparison relationship. Based on Radford's objectivation theory (2016) and our previously developed bidimensional frame (Polotskaia et al., 2024), we analyzed the students' use of language, drawings and gesture in 63 video-enregistered cases. In 30 cases, students produced some drawings, which we categorized into several types. Only 7 of these drawings present a relational thinking character. In 17 cases, students produced some oral arguments which can be associated with relational thinking. In 5 cases, we identified gestures associated with relational thinking. We aim to discuss the variety of semiotic tools available for learners solving word problems. We also plan to present the Grade 2 data that will be available in May 2026 and discuss the progress.

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STUDYING 11TH-GRADE STUDENTS' UTILIZATION OF PSEUDO-OBJECTS IN PROOF BY CONTRADICTION

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The role of Dynamic Geometry Environments (DGEs) in proof by contradiction has been studied mainly through the notion of the pseudo-object. Pseudo-objects are geometrical figures that display conflicting properties and may degenerate due to conflicts between direct invariants—imposed by construction commands—and indirect invariants—emerging as Euclidean consequences of the construction. Experiencing a pseudo-object can support students in visualizing and engaging with proof by contradiction (Baccaglini-Frank et al., 2018). However, it remains unclear under what conditions pseudo-objects are utilized productively and how the coordination of figural and conceptual elements, central to geometric reasoning, operates in this context. To address this, we draw on the theory of figural concepts (Fischbein, 1993), which views geometric objects as integrating these elements, and investigate how students reinterpret the dynamic experience of a pseudo-object by exercising theoretical control—imposing and manipulating theoretical elements onto a figure. The research question is: How does the pseudo-object contribute to upper secondary students' productive geometric reasoning in proof by contradiction? Three interventions were conducted in an 11th-grade geometry classroom. Students worked on non-constructability tasks designed to promote the emergence of pseudo-objects, using both a DGE and paper-and-pencil environment. The analysis focuses on how students integrate figural and conceptual aspects of a figure in their argumentation. Our results suggest that when invariant conflicts remain purely figural, pseudo-objects fail to support students' reasoning. This contrasts with the pseudo-object framework, which posits that perceiving such conflicts suffices for pseudo-object actualization. Conversely, pseudo-objects proved productive when students interpreted these conflicts within Euclidean theory through theoretical control. This coordination involved attributing conceptual elements to figural ones and was mediated by transitions between dynamic and paper-and-pencil environments and peer interaction. The intertwined contribution of these settings—so far unexplored—emerged as central to the productive use of pseudo-objects, underscoring the need for further investigation.

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EXPLORING RELATIONAL UNDERSTANDING IN PRIMARY ALGEBRA LEARNING

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In primary algebra learning, relational thinking is recognised as an essential component of algebraic reasoning. To explore the manifestation of students' relational thinking in algebraic activities, this study is guided by two research questions: RQ1: How do students move between quantitative relationships and algebraic expressions during algebraic problem-solving? RQ2: How does relational thinking manifest when students evaluate and justify equivalence in mathematically incoherent situations (MIS)? MIS are context-based tasks that present structurally similar expressions for different quantitative relationships (e.g., b = a - c and b = a + c) without an immediate question to answer (Savard & Polotskaia, 2017). To address the research questions, nine Grade 6 students from a primary school in China participated in a first interview, a one-week learning session using a digital rod tool to support relational thinking in algebra, and a second interview involving parallel tasks. Students' written work from the interviews was also collected. The first and second interviews were designed to examine how students moved between quantitative relationships and algebraic expressions across the learning activities. Based on a sociocultural perspective, the analysis draws on Radford's (2021) theory of objectification, conceptualising relational thinking as a collective objectification process mediated by multimodal semiotic resources (e.g., explanations, written inscriptions, tool actions). The study found that students' movement between quantitative relationships and algebraic expressions was coordinated in their speech, written inscription, and actions. This coordination became stable and consistent across the two interviews, and in the second interview, relational thinking was expressed in a more settled symbolic form. Further, in MIS, relational thinking seems to help students view an algebraic expression as a quantitative relationship, enabling them to identify a relational mismatch and justify the correct expression. These initial findings show how relational thinking can become visible in algebra activities. The next stage of this study will further explore how relational thinking unfolds through students' speech, gestures, and actions in algebra classroom practices.

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FROM SCAFFOLDING TO SUBSTITUTION: STUDENTS' SENSE-MAKING OF STATISTICAL CONCEPTS USING GENERATIVE AI

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The growing integration of Generative Artificial Intelligence (AI) in mathematics education presents both opportunities and challenges, as evidenced by an expanding body of literature (e.g., Rahmatika et al., 2025). In teaching statistics, this integration has been associated with both enhanced student performance and concerns about diminished critical thinking. These contrasting possibilities invite us to understand learning as the dynamic formation of connections within networks of human and non-human nodes, such as digital tools and intelligent systems. From this perspective, this study aims to explore how students use generative AI to make sense of statistical concepts, with a focus on measures of central tendency and variability. Using a qualitative descriptive design, 23 high school students were asked to complete a set of statistical tasks in a 90-minute session, during which they were allowed to use any generative AI tool to help them solve the problems. Thematic analysis was used to examine the nuances in how the students prompted AI and transformed the AI feedback into their handwritten solutions.

Three sense-making behaviors characterized students' use of AI in mathematics learning. First, some students used AI as a recall tool, prompting it to revisit previously learned concepts before attempting to solve problems independently based on the responses. Second, other students engaged in a more guided approach by uploading images of tasks and asking AI to teach them how to solve similar problems through alternative examples. Third, a number of students relied on AI for direct solutions by submitting task images and prompting the tool to solve them. Across these behaviors, AI functioned as a conceptual scaffold and cognitive clarifier, supporting understanding by breaking down ideas and processes. In some cases, students demonstrated procedural delegation with partial sense-making, allowing AI to handle computations while they attempted to interpret results. However, instances of full substitution were also evident, where students depended entirely on AI outputs with little critical engagement. These patterns highlight how AI can become embedded in students' learning networks, emphasizing the need for pedagogical strategies that maintain productive engagement with generative AI tools.

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PROMPTED, NOT TOLD: UNLOCKING CONCEPTUAL FRACTION COMPARISON WITH A CHATBOT

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Fraction comparison can be a challenging topic in primary school because students often apply whole-number reasoning to numerators and denominators rather than coordinating them in relation to the same referent whole (González-Forte et al., 2023). To address this difficulty, we designed and implemented *FractionsCoach*, a pedagogically controlled conversational assistant with intentionally constrained AI. This chatbot does not provide full solutions, but delivers graduated formative feedback (brief prompts/targeted explanations), requiring students to state an explicit comparison criterion and supporting evidence. Its adaptive pathway is organised into performance-gated phases that support differentiated pacing, from representation work to targeted comparison types, and then to mixed comparisons and justification. The research reported here aims to examine FractionsCoach's contribution to fraction comparison teaching and learning. Specifically, we investigate how the tool shapes mathematical justification (stating criteria, using representations as evidence) and whether the students' achievement is affected after using it. Our study employed an experimental pre-test/post-test control group design. It was conducted in two typical Grade-4 Greek classes (N=33; aged 9-10 years; intervention group n=17; control group n=16) and data were collected from tests, system logs, audio-recordings, group interviews and field notes. The intervention group improved significantly (M=56.82 to 92.12, $p<.001$); ANCOVA on post-test scores (pre-test as covariate). The results showed no statistically significant group effect ($p=.135$), though adjusted means favoured the intervention (92.96 vs 85.30). Log data indicate high accuracy on first attempts and greater difficulty in same-numerator than same-denominator comparisons (92.9% vs 96.5%), while qualitative evidence suggests that the tool strengthened criterion-based justification and collaborative regulation without providing full solutions. Overall, the findings reveal not only measurable achievement gains but also a qualitative improvement of the students' mathematical justification skills. As a result, this study contributes to the literature by introducing a pedagogically controlled, constrained-AI assistant which promotes mathematical argumentation and reasoning rather than simply aiding task completion.

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EYE MOVEMENTS OF 5- AND 6-YEAR-OLDS IN SPONTANEOUS FOCUSING ON NUMEROSITY WITH NATURALISTIC STIMULI

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Spontaneous FOcusing on Numerosity, SFON, is a process in which a child spontaneously focuses their attention on the exact number of items and incidents and exploits the knowledge in action (Hannula & Lehtinen, 2005). Although a lot is known about SFON and how it supports learning, there is limited knowledge on the cognitive and attentional basis of SFON. Eye tracking has proven to be a valuable tool to study children's enumeration processes (Schindler et al., 2020). However, previous research has only used explicitly mathematical tasks. In this study, we investigated the variation in children's average number of fixations when they spontaneously focused on numerosity and enumerated sets of objects in naturalistic pictures versus when they did not and, thus, sought indications of the link between visual attention and SFON. We looked at set sizes two and four to distinct sets in subitizing and counting range. Total of 62 children aged 5 to 6 years completed a Memory card game task. The child looked at eight pictures on a screen for five seconds each while their eye movements were tracked with screen-mounted Tobii Pro Spectrum eye tracker. After each picture, the child was asked to describe what they saw as carefully as they could remember, and we registered for which targets they stated numerosity verbally or non-verbally. The pictures included sets of 1-6 familiar objects in varying arrangements. A 2 (SFON / no SFON) x 2 (set size: 2 / 4) repeated measures ANOVA revealed significant main effect of SFON on the number of fixations on the object sets, $F(1, 19) = 9.97$, $p = .005$, $\eta_p^2 = .34$, but no effect of set size or interaction of SFON and set size. Paired samples *t*-tests confirmed that there were significantly more fixations on average when the child spontaneously focused on numerosity compared to no SFON with sets of two objects $t(31) = -4.51$, $p < .001$, $d = -.80$, but not with four objects. The results indicate differences in eye movements when the child focuses on numerosity and when they do not. Since some children only enumerated one of the two set sizes, the samples in *t*-tests were not identical. Other background variables, such as enumeration skills, may explain why SFON related to the number of fixations only for the sets of two objects.

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WHY DO SOME STUDENTS PROGRESS AND OTHER DO NOT? SELF-REGULATED LEARNING IN STATISTICAL REASONING

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Statistical reasoning skills are important for students facing the challenges of a data-driven society. However, many students have difficulty understanding and interpreting data from graphical displays to make appropriate decisions due to a lack of confidence and dependence on teacher examples. Statistical reasoning in this study refers to students' ability to interpret, analyze, and draw conclusions from data based on Garfield & Ben-Zvi (2008). Self-regulated learning is conceptualized as students' ability to plan, monitor, and evaluate their learning process based on Zimmerman (2002). This study addresses the following question: *How does self-regulated learning relate to students' statistical reasoning development before and after instruction in a sociocultural context?* This study was conducted over 12 classroom meetings with 152 high school students in Nias Island, Indonesia. The students took an initial test to measure their skill level, and then they received statistics lesson in classroom context. Afterward, data were collected again for comparative analysis of statistical reasoning ability and self-regulated learning using the Stacking method via Rasch analysis. The findings showed that an increase in self-regulated learning led to an increase in statistical reasoning ability. Students with high levels of self-regulation were able to advance their statistical reasoning level from level 2 (verbal) to level 5 (integrated process). However, some students remained at level 2 (verbal) due to moderate self-regulation, highlighting that low self-regulation can hinder improvement in statistical reasoning. Interviews with these students revealed that, although they attempted to solve problems independently, they failed to reflect on or evaluate the difficulties they encountered. This indicates the importance of providing support for self-regulated learning to improve statistical reasoning skills. These findings provide insight into how self-regulated learning supports students' progression in statistical reasoning within sociocultural-based learning contexts and giving the individual center analysis perspective using Rasch analysis.

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ADDRESSING MATHEMATICAL PROMISE IN TEACHER EDUCATION: PRE-SERVICE TEACHERS' EXPERIENCES

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Although pre-service teachers (PSTs) are expected to address diverse learners, mathematically promising students (MPS) risk marginalisation in contexts where inclusion is primarily framed in remedial terms. Mathematical promise is here understood as a dynamic and developable potential shaped by cognitive, affective, and social factors and nurtured through opportunity, challenge and support (Sheffield, 2003). Whereas prior research has focused on PSTs' individual conceptions of MPS. In contrast, this study adopts a systemic perspective, analysing how tensions within teacher education shape how MPS are understood and addressed, as manifested in PSTs' experiences following a targeted seminar and practicum. Drawing on cultural-historical activity theory (CHAT) (Engeström, 1987), teacher education is conceptualised as a culturally mediated activity system in which PSTs act as subjects. The object concerns how MPS are understood and addressed in mathematics teaching, mediated by tools, rules, division of labour, and community. Semi-structured interviews were conducted with eight voluntary PSTs (grades 4–6) after participation in a targeted three-hour seminar on mathematical promise and a subsequent five-week practicum in mathematics. The analysis was focused on identifying contradictions within the activity system as expressed in PSTs' experiences. Four interrelated contradictions were identified across the data. A primary contradiction emerged within the object itself, as remediation-oriented interpretations of student diversity clashed with the development-oriented perspective on MPS introduced in the seminar. Secondary contradictions arose between pedagogical tools and rules concerning MPS presented in teacher education and their enactment during practicum. A quaternary contradiction became visible between the activity systems of teacher education and school practice, as seminar-informed approaches to nurturing MPS conflicted with established school routines and priorities. Rather than indicating failure, these contradictions signal a potential reorientation of the object, teaching MPS, making previously implicit tensions visible. The study thus shows how systemic contradictions can function as developmental resources and illustrates the dynamic nature of the object in an activity system, which may be temporarily reoriented through deliberate changes in its mediating components.

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THE ROLE OF MOTIVATION IN SELF-REGULATED LEARNING OF MATHEMATICS

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Self-regulated learning (SRL), defined as learners' active regulation of motivation, cognition, and behaviour, is a key predictor of academic achievement and is closely connected with motivation (Zimmerman, 2008). This study examines motivational and self-regulatory processes during the instructional format of *FSA* (German "Freie Stillarbeit", *free silent work*). FSA provides students (during grades 5 to 7 in secondary school) with autonomy regarding learning content, timing, and pace, while instructional support is offered only upon request. This emphasizes self-regulation processes such as help-seeking, goal-setting, and time management. FSA differs from interventions like those in previous studies in terms of its duration. In the school year 2025/26, 50 fifth-grade, 40 sixth-grade, and 37 seventh-grade students at a German secondary school took part. Data collection comprised a paper-pencil pre-test assessing prior mathematical knowledge, questionnaires assessing intrinsic motivation, perceived instructional support (Deci & Ryan, 2003; Reinhold et al., 2021), and strategy use, as well as a post-test at the end of the school year evaluating the learning gain. The post-test is accompanied by a questionnaire about learning experience, time management, and satisfaction with the provided material. In this paper, the findings of the pre-tests and the first questionnaires are included. The performed ANOVA shows a significant difference in motivation during the FSA between grades (p = .023). The Bonferroni post-hoc test reveals a significantly lower motivation during FSA in grade 7 than in grade 5. Whereas in grades 5 and 7, there is no significant difference between regular lessons and FSA regarding motivation, there is one in grade 6 (p = .012). Those findings imply that there could be a specific grade where SRL is particularly effective in fulfilling learners' needs of perceiving autonomy and competence.

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INTERWEAVING ETHICAL AND MATHEMATICAL REASONING IN MATHEMATICS TASKS

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Critical mathematics educators and researchers (e.g. Skovsmose, 1994) seek to teach mathematics while also developing students' critical consciousness. In the study reported here, we situate critical consciousness within the broader field of ethics, meaning the moral principles that govern behavior, because as Atweh & Brady (2009) posit, ethics includes and transcends considerations of social justice and by situating social justice within the larger idea of ethics, justice becomes not something done out of charity but rather a moral obligation to do what is right. Critical mathematics curriculum designers frequently struggle to integrate mathematical and critical reasoning effectively within instructional design, often grounding their social justice components in established theory, but rarely incorporating instructional design theories to develop mathematical ideas. To address this imbalance, we integrate Realistic Mathematics Education (RME), a theory that anchors mathematical learning in realistic contexts and supports the gradual development of abstract reasoning (Gravemeijer, 1994) in our instructional design. We collaborated with two teachers and an administrator to develop an instructional sequence on ratio, proportion, and percent change within the context of critical and ethical reasoning about teen nicotine vaping. After a post-hoc qualitative document analysis of the instructional sequence, we set out to revise the sequence, with the primary goals of (a) integrating aspects of ethical reasoning beyond critical consciousness earlier on in the sequence and (b) developing tasks that simultaneously engage students' proportional, critical, and broader ethical reasoning. In this oral communication, we will highlight excerpts of the revised sequence aimed at simultaneously engaging mathematical, critical and ethical reasoning and share results of piloting these tasks. We hypothesize that such tasks are important because they provide opportunities for future mathematical professionals to develop their ability in mathematics to reason ethically and critically about the potential impact of their decisions.

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PROPORTIONAL STRUCTURE IN INTEREST CALCULATION: THE INTEREST BAR AS A COHERENT REPRESENTATION

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Interest calculation is often introduced in mathematics classrooms as an application of percentages, yet it is central to financial education. Mathematically, interest calculation can be described as a proportional structure in which capital (or credit), interest rate, and interest are linked by multiplicative relationships. These must be explicitly coordinated to ensure conceptual understanding (Lamon, 2007). Instruction frequently emphasizes formula-based procedures, leaving the proportional structure implicit and the quantities treated as isolated elements. This pattern was also confirmed in a diagnostic study conducted by the authors involving 61 students: interest problems were predominantly solved using memorized formulas, while the underlying proportional structure was rarely considered. Mismatches and unstable calculations indicate that the underlying relational dependencies are not systematically coordinated. These connections within the proportional structure can be supported by a representation. Specifically, the use of an interest bar in a learning environment developed in design-based research (Bakker, 2018) will be investigated to answer the research question: How does the interest bar, as a coherent representation, support the explicit coordination of proportional thinking and the use of formulas in interest calculation? Based on the principles of conceptual focus and longitudinal coherence (Prediger et al., 2022), the interest bar was selected as the central representation in the learning environment, as it visually structures the proportional relationships between capital (or credit), interest rate, and interest. The learning environment is iteratively implemented and refined. Ultimately, the aim is to use the interest bar to integrate procedural processes with relational understanding and reduce decontextualized routines.

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PRE-SERVICE TEACHERS' ASSESSMENT LITERACY IN MATHEMATICS: NEGOTIATING FORMATIVE IDEALS AND DISCIPLINARY DEMANDS

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Assessment is increasingly framed as integral to learning rather than merely certificative. In mathematics, this implies attending to students' reasoning and supporting instructional regulation. Yet persistent tensions remain between formative ideals and institutional routines. How pre-service teachers negotiate these tensions in mathematics during practicum remains underexplored. This study examines how pre-service elementary teachers enact assessment literacy in mathematics and how institutional, disciplinary, and equity-related constraints shape this enactment. Drawing on DeLuca et al.'s (2019) model of assessment literacy and research on integrating assessment and mathematics teaching (Suurtamm, 2018), assessment literacy is conceptualized as situated professional judgment.

A qualitative interpretive design was used. Eleven pre-service teachers were interviewed at the end of their practicum. Data were coded inductively and organized across four dimensions: functions, processes, equity, and validity. Four patterns emerged. Participants endorsed formative assessment, yet summative practices remained predominantly paper-based, revealing tension between discourse and institutional norms. Equity and differentiation were valued but constrained by time and accountability pressures. Concerns about validity—especially criteria and curriculum alignment—were intensified by the epistemic complexity of mathematics. Practicum functioned as a mediating space where assessment knowledge was reconfigured, often through mentor influence and in tension with coursework. Assessment literacy in mathematics thus appears negotiated at the intersection of formative ideals, disciplinary demands, and institutional constraints. The study positions practicum as a critical site where evaluative judgment becomes embedded in the epistemic structure of mathematics teaching.

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OPPORTUNITIES AND CHALLENGES IN GENERATIVE AI-BASED INSTRUCTION FOR FORMULATING STATISTICAL INVESTIGATIVE QUESTIONS

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The 2022 Revised Mathematics Curriculum in South Korea emphasizes "formulating Statistical Investigative Questions (SIQ)" for statistical literacy. While this is a critical starting point, students often face challenges in refining their inquiries into structured questions with specific variables and populations. Generative AI (GenAI) serves as a cognitive tool addressing these limitations (Lee & Jho, 2024). This study explores the quality of SIQ formulated with GenAI, students' conceptual understanding of these questions, and their prompting strategies. Questions were assessed on a 3-level scale using three criteria refined from Arnold (2013): clarity of variables and population, and requirement for variability. Student-AI interactions were examined via seven prompting strategies (e.g., "Specify clear instructions", "Split complex tasks") based on prompt engineering principles (Bozkurt & Sharma, 2023).

Participants were 20 middle school students in five groups using "Wrtn" to formulate SIQs and data collection plans. Data included GenAI dialogue logs, student worksheets, and video/audio recordings of class and group activities. Each group's final SIQ achieved Level 3 in variable and population clarity, significantly supported by GenAI. However, some students confused "population" with "sample" during discussions, despite the clear SIQs formulated by GenAI. Students naturally used "Specify clear instructions" but faced a breakdown in interaction due to their limited understanding of data types, failing to refine prompts effectively. While GenAI scaffolds structuring students' inquiries, its output does not guarantee their deep conceptual understanding. Teachers must bridge this conceptual understanding gap by fostering "prompt-based communication competence" and providing targeted feedback on statistical concepts.

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WHAT SHOULD TEACHERS NOTICE IN COLLABORATIVE PROBLEM-SOLVING SITUATIONS?

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Teacher noticing is widely conceptualized as the capacity to perceive, interpret, and decision making in respond to students' thinking (Jacobs et al., 2010). Empirical studies of teacher noticing have mainly examined how teachers attend to the reasoning of individual students. Yet collaborative problem-solving (CPS) situations challenge this framing. In CPS situations, what teachers must notice is more than individual cognition. They must attend to the emergent, distributed reasoning of the group, the interactional and social processes that shape understanding, and the impact of their own interventions on collective reasoning. This paper as part of a research project focused on understanding teacher noticing in problem solving situations invites a reconsideration of the question: What do teachers consider important to notice in CPS contexts? Drawing on teachers' written responses to a specific classroom scenario portraying CPS episode with teacher intervention, the analysis aims to interrogate how they conceptualize the scope and boundaries of noticing in collective settings. Based on the preliminary exploratory analysis of the responses, two conceptual tensions emerge. First, teachers' responses suggest a shift from noticing individual cognition to attending to the collectively negotiated understanding of the problem, concepts, and strategies that emerges through group interaction and discussion. The participants describe noticing as skills to perceive how ideas circulate, transform, and stabilize across participants. Second, teachers' responses point to a reflexive noticing, in terms of noticing not just to students' cognition, but also to the teacher's own influence or role on the group. Participants emphasized the importance of noticing changes in the task's cognitive demand, for example, whether a question or hint makes the problem too easy or maintains productive challenge. Noticing, in this sense, includes attention to how instructional actions reorganize collaborative system itself. As short, in problem solving when collaboration is central, the unit of analysis shifts: from student thinking to collective reasoning involves attending to collective mathematical activity as well as reflecting on the teacher's role in shaping this collaborative system. These findings point to the importance of both conceptual and methodological development: conceptually, in understanding how noticing operates at problem solving situations, and methodologically, in exploring ways to study and capture such distributed and interaction-mediated noticing in the group level.

Acknowledgment: This research supported by FONDECYT Project No. 1250933.

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EMBODIED MENTAL MODELS IN UNDERGRADUATE STUDENTS' TOPOLOGICAL REASONING

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Topology is among the most abstract branches of mathematics, yet learning it is deeply sensory: students draw diagrams, mentally deform objects, and use spatial language like "close enough" and "around it" to anchor formal definitions in bodily experience. Despite this, research on how students construct mental models for topological concepts remains scarce. Embodied cognition holds that abstract mathematical concepts arise through metaphorical extensions of sensory-motor schemas such as containment, proximity, and motion (Lakoff & Núñez, 2000). Topology is particularly salient here, since concepts like open set, neighbourhood, and ball inherently invoke spatial notions of "being inside" and "being near." Drawing on this perspective with Johnson-Laird's (1983) mental model theory, this study investigates how embodied spatial intuition and formal reasoning interact in the mental models that mathematics majors construct for topological concepts (Şahin, 2024).

Six cognitive dimensions—topological reasoning, set-theoretic explanation, symbolic logic, visual representation, geometric reasoning, and dimensional comprehension—were examined across seven topological concepts. Nine third-year mathematics students participated in two rounds of video-recorded, think-aloud interviews; data were analysed following Miles and Huberman's qualitative procedure.

Six mental models emerged on a gradient from formal abstraction to embodied grounding: a *Topological Mental Model* integrating all dimensions; primitive *Set-Theoretic* and *Symbolic* models; a *Geometric-Topological* model; and perceptually grounded *Euclidean-Geometric* and *Spatial* models. The formal model was rarely attained; most students relied on diagrams, Euclidean dimensional intuitions, and spatial metaphors that simultaneously *enabled* sense-making and *constrained* abstraction, generating misconceptions such as equating topological with metric spaces. These findings show that sensory resources are constitutive of topological meaning-making; instruction should explicitly engage and extend embodied intuitions to support the transition from spatial sensing to formal sense-making.

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MAPPING PRESERVICE TEACHERS' INTEGRATION OF CULTURALLY RELEVANT MATHEMATICS TEACHING (CRMT) THROUGH LESSON PLANNING

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Culturally Relevant Mathematics Teaching (CRMT) (Zaval & Aguirre, 2024) has gained traction among U.S. mathematics teacher educators as research increasingly highlights its effectiveness for supporting culturally and linguistically diverse learners and addressing persistent achievement disparities. National organizations in the U.S (e.g., NCTM) reinforced this momentum by urging teachers to integrate students' cultural identities and experiences into mathematics instruction. This study examines how lesson plan assignments utilized as approximations of practice in a mathematics methods course, support PSTs in developing CRMT approaches. Using Epistemic Network Analysis (ENA), we analysed 10 preservice teachers' (PSTs') lesson plans. ENA modelled the complex connections among the three dimensions of the CRMT rubric: Knowledge & Identities, Rigor & Support, and Power & Participation. The research questions: a) How do PSTs integrate dimensions of the CRMT rubric in their plans for mathematics instruction? b) Where do PSTs' lesson plans provide space and opportunities for culturally responsive mathematics teaching elements?

Our analysis and continued reflection on our practices highlighted the pressing challenge of supporting PSTs in integrating CRMT practices, such as Affirming Multilingualism, and in addressing Power and Participation. The lesson plan assignment provided the greatest space for PSTs to include CRMT practices, showing significant connections across all three CRMT dimensions. However, this configuration revealed a disconnect in how PSTs bridged their mathematical knowledge for teaching with effective instructional strategies. The study noted limited inclusion of CRMT practices related to the Power and Participation dimension, suggesting a lack of focus on teaching mathematics with a social-justice orientation. Additionally, the sub-dimension of Affirming Multilingualism was largely absent, pointing to a gap in instruction on meeting the needs of multilingual students. These findings resonate with international research showing that CRMT emerges differently across global contexts due to variations in cultural norms, teacher preparation, and curriculum structures. The study holds implications for MTEs to situate problems in community collaboration, multimodal representation, language use, and iterative module design.

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THE DEVELOPMENT OF PRESERVICE PRIMARY SCHOOL TEACHERS' RATIONAL NUMBER KNOWLEDGE DURING A MATHEMATICS DIDACTICS COURSE

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Oliveira and Hannula (2008) state that aspiring mathematics teachers aim to learn mathematics and to learn how to teach it. Difficulties in mathematical content knowledge are especially pronounced in rational numbers, with misconceptions persisting even into adulthood (Vamvakoussi et al., 2012). Effective instruction therefore requires teachers to have a solid understanding of rational numbers themselves. Earlier studies on preservice teachers' rational number knowledge have been rather cross-sectional in nature or focused on a limited number of aspects of this knowledge. To address this gap, the present study examines six different aspects (adaptive rational number knowledge, conceptual knowledge of density and number system, arithmetic procedural knowledge, and procedural and conceptual knowledge of representations) during a mathematics didactics course.

The longitudinal quantitative study examines preservice teachers' (N=102) rational number knowledge at the beginning of the course and its development during the course. Differences at course entry were examined across six aspects of rational number knowledge, and development was analysed focusing on the four aspects most likely to change.

A cluster analysis identified three distinct profiles of rational number knowledge — low, medium, and high — which differed both in performance and in how they developed during the course. Although the mathematics didactics course primarily focused on pedagogical knowledge, preservice teachers' mathematical content knowledge improved over time. These findings suggest that mathematics didactics courses should place greater emphasis on supporting preservice teachers' mathematical content knowledge, particularly in rational numbers, while maintaining a strong integration of pedagogical and mathematical content knowledge.

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COMPARING PRE-SERVICE MATHEMATICS TEACHERS' SELF-EFFICACY ACCORDING TO TEACHING PRACTICES

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Teacher self-efficacy is linked to teaching practices in different ways. On one hand, teacher self-efficacy impacts students' motivation and outcomes as well as teachers' performance, commitment, and interactions with students. On the other hand, field experiences influence preservice teacher self-efficacy. There are few studies in Latin America focused on the relationship preservice mathematics teacher training and self-efficacy. However, studying such a relationship may provide insights into how initial teacher training can be improved. Our research question was: Are there differences in preservice mathematics teachers' self-efficacy across different pedagogical practices?

Understanding teacher self-efficacy as teachers' beliefs in their ability to influence students' learning, we studied preservice mathematics teachers' self-efficacy in a program characterized by a predominant focus on teaching practices. The program has six educational criteria: non-segregating ethos, mathematical activity, problematization and research, proactive attitude, critical attitude, and collaborative work. The program's curriculum includes a core of practices. Progressively, student teachers transition through different pedagogical practices starting with research practices, moving through supervised teaching practices, and finishing with mentored teaching practices. We adapted and applied a teacher self-efficacy scale (Avilés-Canché & Marbán, 2024) that best fit the program's educational criteria. We applied the instrument to 84 preservice mathematics teachers. 20 were in the research practices, 31 in the supervised, and 33 in the mentored practices. We run an ANOVA after verifying normality and homoscedasticity. Assuming practice types as the factor, we found significant differences between groups in five educational criteria. Then, teacher self-efficacy varies depending on the type of practice. For instance, there are significant differences in teacher self-efficacy for including all the students in their class (ETHOS) between student teachers in research practices, supervised and mentored teaching practices ($F(2,81) = 8.38$, $p < .001$). This and other results allow us to conclude that different types of field experiences as part of a mathematics teacher training program increase the level of teacher self-efficacy in terms of mathematical activity, inclusive and questioning practices, a proactive attitude, and collaborative work.

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OPERATIONALIZING MATHEMATICS TEACHERS' RESPONSIVENESS IN AI SIMULATIONS

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Mathematically Responsive Teaching (MRT)—grounding instruction in students' mathematical thinking—remains insufficiently theorized. Our AI-based teaching simulations, in which teachers interact with an agent-student, provide a controlled yet interactive setting for examining such responsiveness. We developed an AI-based teaching simulation platform, that includes a scenario featuring a middle school student who incorrectly adds probabilities instead of multiplying them in a two-event independence task. Whereas prior work has conceptualized responsiveness primarily as a set of instructional moves (e.g., Jacobs & Empson, 2016), we conceptualize MRT as a multi-dimensional construct. In this context, we ask: What can teacher–bot interactions in AI-based teaching simulation reveal about prospective teachers' MRT? Building on prior research (Schwarts, 2025), we developed a holistic framework that distinguishes commonly conflated MRT dimensions, comprising: (D1) identifying error, (D2) eliciting student perspectives, (D3) leveraging error as a resource, (D4) building on student perspectives, (D5) interpersonal responsiveness, and (D6) promoting mathematical reasoning. We applied the framework using a 0–3 scale to code seven prospective teachers' simulation logs ($M \approx 12$ teacher turns per simulation).

Both Eliciting (D2) and Using Error as a Resource (D3) had the lowest means (0.86), indicating these dimensions were most challenging for teachers. D5 was high ($M = 2.71$), reflecting strong interpersonal engagement in the simulation. Our analysis shows that the MRT framework distinguishes eliciting (D2) from building on student perspectives (D4; $M = 2$): two participants scored 0 on eliciting, yet closely followed the student's logic (coded 3 on D4) and did not introduce a tree representation. This underscores a key contribution of the framework: MRT is multifaceted and not reducible to talk counts; eliciting and building are analytically distinct.

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FACILITATORS´ DISCUSSION-LEADING AND ADAPTATION PRACTICES DURING IMPLEMENTING A PD COURSE

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When implementing a professional development (PD) course for mathematics teachers, facilitators have multiple responsibilities, such as *adapting PD materials* and *leading discussions* (Prediger et al., 2022). To ascertain how facilitators manage these tasks, six interviews were conducted with each of two facilitator tandems, reflecting on *adapting PD materials* and *leading discussions*. Each interview took place after one of the three sessions of a PD course had been delivered by the respective tandem as part of the QuaMath Project, which aims to improve the mathematics teaching quality across a range of topics. In order to conceptualize the discussion leading practices employed by the facilitators, we adapted the framework for leading mathematics discussions by Shaughnessy et al. (2021) from the classroom to the PD level. Four discussion-leading practices could be defined: *making contributions*, *eliciting teacher thinking*, *probing teacher thinking* and *orienting teachers towards the thinking of others* (Shaughnessy et al., 2021). Initially, both tandems focused solely on *making contributions*, emphasising the highlighting of relevant teacher statements and their link to the PD course content. During the subsequent PD sessions and follow-up interviews, the focus remained on *making contributions*, but *eliciting* and *probing teacher thinking* became a more prominent theme in the facilitators' reflections on their discussion-leading practices. However, *orienting the teachers towards the thinking of others* did not occur in either tandem's discussion-leading practices as the facilitators found this aspect particularly challenging. Regarding the task of *adapting PD materials*, the facilitators' practices remained consistent throughout the three sessions: One tandem kept the provided materials largely in their original state, while the other tandem omitted some of the provided conversation starters and activities for the teachers, resulting in a constrained discussion culture. The facilitators mostly justified their discussion-leading and adaptation practices by referring to the limited time available. As many PD courses have a constrained timeframe, a careful planning in terms of adaptation and discussion-leading, with a focus on best supporting teacher learning, should form a more intensive part of facilitator preparation.

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MIDDLE SCHOOL STUDENTS' USE OF GENERATIVE ARTIFICIAL INTELLIGENCE IN PROVING PROCESSES

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This study aims to explore the use of generative artificial intelligence (genAI) in middle school students' proving processes related to argumentation in geometry. At the middle school level, proving primarily focuses on constructing arguments about geometric properties, which students often find difficult due to the demands of logical reasoning and formal justification. Yoon et al. (2024) proposed a framework illustrating the processes of genAI use in proof construction within an undergraduate context. Building on this work, this study seeks to refine understanding of how genAI is used in school-level settings, particularly in relation to argumentation in geometry. Therefore, this study investigates the ways in which students use genAI when engaging in proving tasks involving argumentation in geometry at the middle school level. An exploratory approach was adopted in this study, and clinical interviews were conducted with eight Korean eighth-grade students who had learned geometric properties. Seven problems related to argumentative tasks in geometry at the middle school level were presented, and students were asked to select two problems to solve freely. Considering the difficulties students encounter in proving, they were instructed to explain their solutions using dynamic geometry software. They then constructed proofs while autonomously using genAI through verbal interaction. Data collected included students' interaction logs with dynamic geometry software and genAI, written proofs, and verbal utterances. Based on these data, students' use of genAI during proving was analyzed using an open coding approach. Students used dynamic geometry software to confirm conclusions at an intuitive level and to justify those conclusions using geometric properties, thereby laying the groundwork for recognizing key components of proving. Students primarily used genAI to obtain or verify key ideas for constructing their proofs. When constructing written proofs, students focused on key ideas generated by genAI rather than copying detailed explanations verbatim or examining all provided content. During this process, students demonstrated varied strategies, such as setting constraints in prompting and varying the timing of genAI use. Findings suggest that genAI functions as a mediational resource in students' proving by supporting the generation of key ideas, shaping prompting strategies, and influencing the timing of proof construction.

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EXPLORING PERCEIVED CURRICULM OF AN EXPERIENCED TEACHER THRUOGH A RESEARCH-LESSON DESIGN

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The current study aims to examine the alignment of different aspects of school mathematics curriculum from the eyes of experienced teachers. The study has a particular focus on the relationship between the "intended" and the "implemented" curriculum in the process of implementing the new national curriculum standards. A theoretical framework of curriculum alignment in the earlier TIMSS curriculum analysis (Valverde et al., 2002) was used to conceptualize the role of teachers across three different aspects of curriculum. By taking an ecosystem approach to curriculum (OECD, 2020), we address the research questions: How an experienced teacher interprets the "intended" curriculum? To what extents a research lesson design in school-based lesson study helps researchers to understand the "perceived" curriculum?

Data were collected at a public elementary school in an urban area in Japan. An experienced teacher was invited to conduct a research lesson in school-based lesson study in 5th grade classroom in the unit of "geometrical figure and angles" that is treated differently between the previous and the current curriculum standards. The research lesson and the post-lesson discussion were video-taped and then analysed qualitatively. Data sources included a detailed lesson plan, verbatim transcriptions of video-recorded lesson and the post-lesson discussion, and the fieldnotes of researchers.

The results of analysis reveal that the teacher interpreted the curriculum standards in the way that his own perspectives can be manifested on the values of school mathematics, in general, and on the development of mathematical reasoning, in particular. It was also found that he intended to change the sequence of the appearance of figure types different from those in the textbook to "customize" the instructional materials to fit with his students. Through the analysis the importance of textbooks was discussed both as a "potentially implemented curriculum" and as a platform for the teacher to exemplify what is officially intended in his classroom.

Acknowledgement: This research was supported by JSPS KAKENHI 24K00408.

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A CROSS-GRADE ANALYSIS OF STUDENTS' STRATEGIES IN DIRECT, INVERSE, AND PSEUDO-PROPORTIONAL PROBLEMS

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Proportional reasoning is understood as the ability to identify and coordinate multiplicative relationships between quantities, distinguish proportional from other relations, and use different representations while recognising invariant structures, rather on mere use of rules (Silvestre, 2012). Research on proportional reasoning shows that students often succeed procedurally without demonstrating conceptual understanding (Lamon, 2007). Difficulties persist across schooling, raising questions about how students' reasoning, strategies, and justifications evolve over time. In Portugal, direct proportion is taught in 6th and 7th grades, while inverse in 9th grade. This study analyses 6th- and 9th-grade students' solutions to one pseudo-proportional and three proportional problems (direct and inverse) to examine whether their reasoning reflects genuine proportional understanding or mainly procedural approaches. It involved 74 6th-grade and 78 9th-grade students from four schools whose teachers agreed to participate. Both correct and incorrect responses were analysed. Correct responses were code by strategy type, distinguishing additive and multiplicative reasoning, including relational thinking between and within measures and algorithmic rules, while incorrect ones were code by error type. In the direct proportion missing value problem, 6th graders tend to use a scalar strategy, while 9th graders mainly rely on the rule of three. In a pseudo-proportional problem, 6th graders show a higher success rate, although both groups reveal misuse of direct proportionality. In the inverse proportionality problem, a slightly better performance is observed in 9th grade, but half of the 6th graders are already able to solve it before formal instruction using a scalar strategy based on inverse covariation. For the ratio comparison problem, both groups performed similarly, with a predominant use of ratio comparison, whereas additive reasoning was the most frequent error. Performance did not increase with schooling and seams linked to understanding multiplicative relationships, as shown by 6th graders' success with inverse proportionality. Difficulties reflect additive reasoning, linearity assumptions, and indiscriminate use of the rule of three, evident when intentionally juxtaposing problem types.

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TEACHING FOR LINGUISTIC AND CONCEPTUAL TRANSFER: CROSSLINGUISTIC TRANSLANGUAGING IN MULTILINGUAL PRIMARY MATHEMATICS CLASSROOMS IN ENGLAND

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Translanguaging is a pedagogical approach, which facilitates flexible use of learners' full linguistic repertoires and has gained increasing interest from mathematics education researchers in recent years (Planas & Chronaki, 2021). Cummins'(2021) crosslinguistic translanguaging theory (CTT) provides a theoretical framework, which advocates teaching for the transfer of concepts, skills and learning strategies across languages. From this perspective, languages are viewed as separate but connected, dynamic and fluid, rather than unitary and undifferentiated (Cummins, 2021). CTT is garnering increasing interest within mathematics education research, such as Farrugia et al. (2026), for connecting languages and mathematical concepts. In this discussion, I present examples of mathematical teaching and learning which draws from two studies based in Oxfordshire, UK. The first study, part of a PhD, analysed examples of translanguaging in maths lessons in a French-English bilingual classroom. In this setting, the co-teachers working with 8-9-year-old learners created specific opportunities for crosslinguistic and cross-conceptual transfer between the languages. The findings identified how CTT could be used to support the development of conceptual understanding through exploration of mathematical structure, representation and variation. The second study explored how teachers from English-medium schools trialled similar strategies with their learners, some of whom were multilingual. The preliminary findings from this study reveal that CTT can afford opportunities for pupils to utilise their linguistic repertoires, even where teachers do not share the learners' home languages. Furthermore, the findings indicate the possibilities for translanguaging across multiple languages within the same classroom, using representation, structure and variation as the core foci for teaching and learning.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' CRITERIA FOR EVALUATING SUBJECT-SPECIFIC VS. INTERDISCIPLINARY TEACHING MATERIALS

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Interdisciplinary teaching offers the potential to motivate learners and connect and apply mathematical content across different subjects, but also relies on careful lesson planning. For that process, teachers often draw on existing or AI-generated teaching materials, which must be carefully evaluated to ensure their justified selection. Evaluating teaching materials should, for example, be based on objective quality criteria which mirror established criteria for teaching quality, such as instructional quality (e.g. Praetorius et al., 2018). While evaluating teaching materials is challenging for prospective secondary teachers (Cevikbas et al., 2024), there is little research on the evaluation of materials in interdisciplinary teaching contexts and it is unclear whether it differs from that in subject-specific contexts. To gain insights into prospective mathematics teachers' knowledge and skills regarding the criteria-based evaluation of teaching materials, our study addresses the research question: Which quality criteria do prospective secondary mathematics teachers a) mention and b) use when evaluating subject-specific and interdisciplinary teaching materials?

Using a within-subject design, 54 prospective secondary mathematics teachers in Germany were asked to provide criteria to evaluate (i) subject-specific teaching materials on basic concepts of functions and (ii) interdisciplinary teaching materials on basic concepts of functions and photosynthesis in randomized order. Subsequently, they received concrete teaching materials and were asked to evaluate these. Data were analysed using a qualitative content analysis.

Overall results show only few differences in the frequency of mentioned criteria between subject-specific and interdisciplinary teaching materials. However, the frequency of some criteria differed significantly between teaching materials, for example use of representations. The ongoing analyses focus on participants' use of criteria when evaluating subject-specific and interdisciplinary teaching materials and compare them with the criteria they previously mentioned.

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CHILDREN'S USE OF DURATIONAL QUANTITATIVE PROXIES

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Durational reasoning, or operating on and with the abstract quantity of "time", extends beyond the clock reading and elapsed time calculations of early schooling. Young children begin conceptualizing duration as a quantity before such schooling through their attention to other measurable attributes of experience, such as the amount of Lego pieces or the distance traveled to grandma's house (Smith, 2021). These other attributes serve as quantitative proxies for the invisible quantity of "time" and allow young children to begin reasoning about the durations of their lived experiences. We interviewed 4-, 5-, and 6-year-olds from across the United States (n=32) about their durational experiences and how they worked to mathematize this intangible, invisible quantity. Each child was interviewed twice, once focusing on reflections on past experiences, the second quantifying common experiences through tangible means. Using constant comparison analysis (Glaser & Strauss, 1967), we identified moments when participants worked to quantify durational aspects of their experiences. We found that young children conceptualized duration indirectly, treating other measurable attributes of experience, such as distance travelled or gross quantity, as quantitative proxies for time. Consistent with Piaget's (1969) notion of intuitive time, our participants seemed to reason about duration through these covarying, perceptible attributes, coordinating them in measurable ways (e.g., the time of vacuuming 3 rooms versus completing a puzzle with 20 pieces). In doing so, their activity reflects Thompson's (2011) definition of quantification, conceptualizing an attribute with a unit of measure and operating on the assumption of a proportional relationship—here, between their experience and a durational proxy, such as distance or accumulation.

Additional information

This work was supported by a Spencer Research Grant on Education: Small.

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TEACHER LEADERS' MOVES TO FOSTER TEACHERS' NOTICING OF ARGUMENTATION IN MATHEMATICS CLASSROOMS

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Several countries have incorporated argumentation into mathematics curricula to promote a culture in which knowledge construction is viewed as a situated, critical, and reflective activity involving group participation. Despite its curricular relevance, teachers often experience difficulties fostering argumentation in classroom, are essential to develop teachers' competences to support argumentation in mathematics classrooms with a focus on teachers' noticing of argumentation (Ayalon, 2024). While Professional Development Programs (PDPs) in teachers' noticing have increased (Amador et al., 2024), less attention has been paid to how Teacher Leaders (TLs) enact moves in PDPs. This gap is particularly relevant with respect TLs' moves enacted in PDPs aimed at teacher noticing to foster argumentation in mathematics classrooms. Accordingly, this study explores the moves enacted by TLs during a PDP designed to developing teachers' noticing to foster argumentation in mathematics classrooms.

The study adopts a qualitative descriptive approach based on the analysis of nine video-recorded coaching sessions led by two experienced TLs. The analysis identified two main types of TL moves: those aimed at challenging teachers' practices and those aimed at eliciting teachers' noticing of argumentation-related classroom events. Important differences emerged between the two TLs. One TL tended to orient teachers toward predetermined distinctions, resulting in largely procedural participation. In contrast, the other TL promoted collective interpretation by inviting teachers to revisit their initial analyses in light of classroom evidence and conceptual resources. The analysis further suggests that collective, evidence-based elicitation may play a particularly relevant role in supporting teachers' noticing of argumentation. Although based on a limited number of cases, these preliminary findings point to the need for further research on how different TLs' enactments may influence teachers' professional learning in PDPs focused on argumentation.

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MATHEMATICAL STYLE: AN ANALYTICAL FRAMEWORK

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While learners' intellectual output is a primary focus of mathematics' education research, their mathematical insight is always expressed through a communicative manner. A conjecture can be stated with a smile, a grimace, an apology, a gesture that is magnanimous or hesitant. At times, these mannerisms of doing mathematics become patterned and therefore sufficiently distinctive that they could be considered as a mathematical style. In sociolinguistics, style refers to "a *system of distinction* in which a style contrasts with other possible styles that can produce functionally similar social outcomes (Irvine, 2002, p. 22, original emphasis). Styles use multimodal resources to signal affiliation or disaffliation with others and frequently carry a sense of value, beauty, or distaste (Machin & van Leeuwen, 2005). In this communication, we consider what a "system of distinction" might mean for mathematical activity. We propose a framework for analyzing distinct, styled manners of doing mathematics through a two-axis system. One axis identifies whether a style is invoked by naming the style or by performing it. The other axis tracks whether co-participants, including researchers, identify the distinctiveness of a style as locally or socially scoped. We layer sociolinguistic analysis upon Kollosche's model of styles of reasoning to ensure that we attend to the mathematical nature of sociolinguistic style moments (2021). We share examples of mathematical style among teacher candidates during their methods course, and describe ways in which analyzing moments of mathematical style can deepen our understanding of their developing pedagogical practice. We suggest that scholarship on sociolinguistic style can restore the richness of students' mathematical experiences to our research record, and more broadly, contribute to understanding how expressive practices shape mathematical engagement.

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REFINING A TOOL FOR EVALUATING STUDENTS' MATHEMATICAL PROBLEM POSING

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Problem posing (PP) is increasingly recognized as a core component of mathematical proficiency, fostering creativity and higher-order cognitive functions. In this generative process, students construct personal interpretations of concrete situations to formulate meaningful problems (Stoyanova & Ellerton, 1996). Despite its benefits, PP remains marginalized in curricula as teachers often hesitate to integrate it due to a lack of familiarity, time, or practical frameworks (Cai & Hwang, 2023). To provide teacher support and facilitate classroom implementation, we examined Leavy and Hourigan's (2022) eight-dimensional Framework for Posing Elementary Mathematics Problems (F-PosE). Our study employed mixed methods to evaluate the applicability of F-PosE in a secondary education context with 16 high-achieving 10th-grade students. Over two weeks, groups developed problem sets covering algebra and geometry. The resulting 20 tasks were independently scored by two raters using F-PosE indicators. To ensure objectivity, a specialized form of inter-rater reliability (IRR) was applied to compare the scores and identify discrepancies in the framework's interpretation. The IRR analysis revealed mixed results regarding objectivity. While the raters were statistically consistent in scoring certain dimensions, other factors proved highly subjective, leading to differing interpretations. Based on these findings, we conducted a thorough investigation into the causes of these discrepancies. Consequently, we propose a refined F-PosE framework that retains the consistent indicators while broadening the definitions of others to include both contextual and intrinsic mathematical motivation. The goal is to provide a more teacher-independent feedback tool that bridges the gap between PP theory and formative assessment in the classroom.

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EXAMINING FIDELITY OF IMPLEMENTATION IN A KINDERGARTEN EARLY ALGEBRA STUDY

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Mathematics educators argue for an early introduction of algebra in school curriculum (NCTM 2000). However, further research is necessary to understand how this shift affects students' later mathematical development. To explore this topic, we implemented a large-scale, cluster-randomized study to test the effectiveness of a Grades K-2 early algebra intervention. We focus here on the fidelity of implementation (FOI) and examine how variation in FOI is related to student performance. We use data from Year 1 of the intervention.

Our study took place in three school districts, involving 40 schools, 189 classrooms and 2144 students. Classrooms were randomly assigned to treatment or control groups. We randomly selected 40 classrooms for FOI analyses and collected digital lesson logs, classroom video-recordings of specific lessons and teacher interviews. Students received a pre- and post-implementation digital test on algebra.

Lesson log and interview data suggested that, overall, teachers followed the curriculum as intended. Video coding and analysis revealed a subtly different picture: our coding aligned with teachers self-reporting regarding the logistical aspects of lessons (e.g., use of manipulatives, completing tasks, getting students to work in groups), but deviated on the prevalence of teachers' use of algebraic practices. After controlling for pre-test performance, Hierarchical Linear Modelling (HLM) showed that Logistical Curriculum Implementation was not a significant predictor of post-test scores ($p = .79$), whereas Algebraic Instructional Practice showed a trend toward significance ($p = .08$).

Overall, we noticed that teachers need support, not only to implement the new curriculum, but also to understand and appreciate the algebraic ideas at hand. As this was only the first year of the implementation, we expected teachers to focus first on the logistical aspects, particularly in Kindergarten, and on the bigger algebraic ideas.

Acknowledgements

This research is supported by the National Science Foundation's DRK-12 Award #2404984. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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THE GREAT LEARNING & THE CONTEMPORARY RELEVANCE FOR MATHEMATICS LEARNING

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This theoretical paper seeks to clarify Confucian methods of learning through a textual analysis of Chu Hsi's commentaries on the *Great Learning* and other works. To illustrate the contemporary relevance, *the Learning Compass for Mathematics* proposed by the OECD (2023) is used as an example. I argue that Confucian learning is fundamentally experiential – *the unity of knowledge and action*.

> When you know something but don't act on it, your knowledge of it is still superficial. After you've personally experienced it, your knowledge of it will be much clearer, and its significance will be different from what it used to be (Chu, 1990/1270, p.116).

In the process of learning, knowledge is acquired through *investigation of things* and *extension* to perfection. "Action is more significant" (*ibid*) because the objective of learning is human functioning and flourishing to ensure a healthy society and collective well-being. Confucian's approach to learning mathematics, I would argue, is to investigate the principles in those things that learners encounter, such as identifying patterns in data and solving real-world problems. To take responsible actions thereby to have a deeper understanding. To apply mathematical knowledge in different contexts for an extensive period to develop genuine knowledge naturally. *Great Learning* is not rote memorization but by experiencing the practical implications of knowledge personally. Learners study for themselves and not for the sake of examinations (*ibid*). Confucian epistemology is comparable to the OECD's (2023) epistemic knowledge, which emphasises knowing how to think and act like a partitioner. Chu sees knowledge as tentative and subject to change over time; therefore, an intellectual virtue of open-mindedness is valued in learning (1990/1270). In the context of school mathematics, some ideas have to be rejected as scholarly knowledge is constantly evolving. Confucian iterative learning process also resembles the OECD's approach (2023) whereby learners continuously think critically and reflect upon their actions and those of others. Confucianism may have been misinterpreted in the past as the pedagogy and principles, such as investigation, experiential learning, and open-mindedness, are often neglected in the cultural influences on mathematics learning.

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PRE-SERVICE MATHEMATICS TEACHERS' EXPERIENCES IN A GAMIFIED UNIVERSITY COURSE: A THEMATIC ANALYSIS

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A growing body of research demonstrates the potential of integrating thoughtfully designed gamified systems into education. While gamification is increasingly recognised as a legitimate pedagogical tool, empirical studies continue to report mixed or even negative outcomes – underscoring that its effectiveness is highly contingent on design, context, and implementation (Zainuddin et al., 2020). This highlights a critical gap: we know whether gamification can work, but we know far less about how students experience it, particularly in high-stakes, emotionally charged contexts like the transition to university mathematics. Understanding the learner's perspective is therefore essential to move beyond performance metrics and design systems that are not only effective but also meaningful and sustainable. This study explores pre-service mathematics teachers' experiences in a point-based gamified Number Theory course during the sudden shift to online education in 2020. Students gained points through weekly quizzes, weekly assignments and active participation in lectures and practices. In addition to the point system, other game elements were implemented to make the course engaging. While quantitative results of the experiment (Muzsnay et al., 2025) showed significant gains in achievement and decreased dropout rates, this study reveals how students perceived the course design. The thematic analysis of 20 reflections followed Braun and Clarke's (2006) approach, using inductive, semantic coding. Students reported that they felt "seen", as their effort was recognised and mistakes were framed as developmental, not punitive. The course design helped them "see where they stood," enhancing certainty and reducing anxiety. Though some noted issues with pacing or clarity, the overall experience was empowering, and many plan to adopt similar systems in their future teaching. Our analysis reveals that gamification can help transform mathematics learning from a source of fear into an experience of agency.

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POETIC DIAGRAMMING: AN EXPLORATORY METHOD IN TRACING EMBODIED MATHEMATICS

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Mathematical embodiment posits that thinking and doing are entwined, and mathematical meaning emerges through the coordination of bodily actions with material forms (de Freitas & Sinclair, 2012). Following this view, we argue that acts of gesturing and diagramming constitute the doing of mathematics, and not mere representations of mathematics ideas. Consequently, these processes require a different way of transcribing and analysing, to better attend to how mathematical meaning and the distribution of agency emerge from them. Specifically, we propose that the approach of poetic diagramming can address this need. While poetic inquiry in mathematics education has mostly focused on the repetition and parallelism present in words, phrases, syntax and sounds (Staats, 2008), we see a potential to extend the poetic function in linguistic data to non-linguistic data by attending to repetitive visual patterns and moments of mobility in the making of the diagrams. Thus, in this paper, we focused on students' gestures and diagrams and explored how they might reflect poetic structures that communicate students' sense-making of mathematics and the nature of mathematical agency. Methodologically, we also experimented with different ways to "transcribe" the diagrams and gestures to illuminate these poetic expressions.

The data consists of video captures from a university course focused on problem solving where students worked on vertical whiteboards with non-permanent markers to solve problems collaboratively. Using the students' actual diagramming process in the video data as the basis, we created poetic diagrams to illuminate the poetic form and function of their diagramming and gesturing motions over time. These poetic diagrams allow us to see how the solution and students' mathematical understanding of a specific problem are constructed through the entwinement of rhythmic diagramming and gesturing. It also helps us to attend to the emergence of distributed agency across numerical relations. Overall, by attending to the visual and temporal structures, the poetic diagrams make mobility visible in ways that linear and verbal transcripts cannot. In the presentation, examples and findings will be discussed.

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STUDENTS' DESCRIPTIONS OF ARITHMETIC SYMBOLS

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Learning to use and understand symbols is crucial for learning mathematics. Using symbols can include how to perform calculations, for example, 5+3=8. Understanding symbols can include the ability to see "through" symbols (Alibali et al., 2014) to their referents (i.e., what the symbols stand for), for example, by linking the symbol "5" to a set of five objects. When linking symbols to referents, language can play a central role, for example, when "five" is used to describe both the numeral and the quantity. However, Powell and Fluhler (2020) found that it is more difficult for students to describe the meaning of arithmetic symbols than to use the symbols, but we do not know if their descriptions develop as they become more familiar with the symbols. Our study investigates the following research question: How do students' descriptions of referents to symbols vary depending on how familiar the symbols are to them? We asked 66 students (grades 3–9) how they would explain different symbols to a friend. Students' level of familiarity with the symbols was based on when they are introduced in school. The symbols, here sorted in decreasing level of familiarity, were, 3, =, +, %, π, and √. All symbols were presented within mathematical statements. We identified what basic parts of speech (verbs, nouns, and adjectives) students used when describing the meaning of each symbol. Signs of a more advanced mathematical discourse can be the use of more nouns than verbs, due to objectifications (Pimm & Wagner, 2003) and more adjectives, to focus also on properties (Davies et al., 2022). There was no correlation between familiarity with the symbols and types of words used, suggesting that the discourse about symbols is established early and maintained through grades 3–9. Further research that simultaneously analyses the discourse in teaching and the discourse of students could perhaps explain this lack of progression.

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A COMPARATIVE CASE STUDY ON FORMATIVE FEEDBACK IN PROBLEM-BASED TEACHING

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Problem solving is a key part of primary school maths lessons. Previous research shows that problem-based teaching can be divided into different phases, based on Pólya's ideas (Rott, 2020), with the *Look back* phase being particularly important. At the same time, feedback practices vary depending on the timing or phase of a lesson (Smit et al., 2024). However, the relationship between the structure of the phases and feedback practices throughout a lesson remains unclear.

To this end, we conducted a comparative case study on teacher behaviour in problem-based teaching combining two different analyses and their respective instruments: Teachers *closely managing* their students' ideas by highlighting correct solutions vs. teachers *emphasizing* their students' ideas and strategies (Rott, 2020), combined with a detailed coding of teachers' feedback behaviour, classifying every teacher-student interaction (Theile & Rott, 2026). We assume that the structure of the *Look back* phase relates to the type of feedback provided in preceding phases. To investigate this, we analyse one lesson from each of two primary classes where students worked on a non-routine problem. The lessons were selected based on the design of the *Look back* phase, with one being closely managed and the other emphasizing students' strategies.

In the lesson featuring a strategically oriented *Look back* phase, responsive feedback predominates, focusing primarily on students' understanding of the task's structure, their own reflection, and various approaches to solving the problem. In the lesson featuring a closely managed *Look back* phase, invasive feedback occurs more frequently, serving primarily to ensure the correct result is achieved at an early stage. It also emerges that the same categories of feedback (e.g., offering partial solutions) serve different functions depending on the lesson design. The results suggest a link between the structure of the *Look back* phase and the teacher's feedback behaviour in the earlier phases of the lesson, even though the direction of causality remains unclear.

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GEOMETRIC ABILITIES OF FIRST GRADE STUDENTS IN PROBLEM SOLVING APPROACH

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Geometry represents a fundamental domain of mathematical knowledge that individuals encounter and apply consistently in their daily lives (Clements et al., 2018). Consequently, it is essential to introduce geometric concepts to students at an early stage of their academic development, particularly during the lower primary years.

This study was designed to investigate the geometric reasoning abilities of first-grade students through implementation of a problem-solving instructional approach. The research utilized qualitative methodology, employing video documentation and analytical description as primary data collection and analysis techniques. The participant group consisted of 36 first-grade students enrolled at Khon Kaen University Demonstration Elementary School during the first semester of the 2025 academic year. The students received instruction based on a problem-solving pedagogical framework across six sequential lessons. Data analysis was conducted using the theoretical framework developed by Clements et al. (2018).

The research findings demonstrated that first-grade students exhibited four core geometric abilities, (1) ability in shape recognition by correctly identifying circles, triangles, squares, and rectangles, (2) ability in shape reproduction, successfully recreating geometric shapes from memory, (3) ability in spatial orientations, engaging in discussions about whether shapes were positioned vertically or horizontally, and (4) ability in pattern, organizing figures according to identified patterns.

The study revealed two noteworthy findings beyond the initial framework. First-grade students exhibited the ability to recognize and articulate relationships between different geometric shapes. Moreover, they demonstrated capacity for modelling real-world objects through geometric representation, providing reasonable explanations and justifications for their reasoning. These findings suggest that first-grade students possess more advanced geometric understanding than conventionally anticipated, particularly in their ability to connect abstract geometric concepts to concrete real-world applications.

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PERSPECTIVES ON LEARNING TO TEACH MATHEMATICS FROM A VISUALLY IMPAIRED PRESERVICE TEACHER

Helen Thouless, René Hartmann, Zahara Hussain

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The theme of this conference is sensing and making sense of mathematics, but what happens when one sense is impaired? In this study we examined the following research questions: What aspects of our BA (Hons) Primary Education Core Mathematics sessions that were accessible to a preservice teacher (PST) with a visual impairment (VI)? What could have made the module more inclusive? What did the PST wish her lecturers knew about teaching and learning mathematics with a VI?

This study took a participatory research framework, where the PST with the disability was involved with the creation, production, and dissemination of the research (Zarb, 1992). The PST was an active contributor throughout the research process, discussing the ethical implications of the research, developing the interview questions, engaging in a semi-structured interview with her mathematics education lecturers, and participating in the dissemination process. After the interview did a membership check about the accuracy of the transcript and analysed the data to find themes.

This PST found the supportiveness of her mathematics lecturers an important aspect of the accessibility of the course. They were supportive by pre-teaching both the concepts and the use of manipulatives, by ensuring that the PST did not feel singled out because all the PSTs had access to the same manipulatives, and by adapting the materials for the PST. The use of an online mathematics audit was one of the least inclusive portions of the course, with multiple visual images taken from the Key Stage 2 National Curriculum Test and a small font size. The PST wished that her lecturers understood that students who learn mathematics through Braille are having to function in multiple languages and so giving them the same content, but fewer questions would be fairer. She also noted that consistent lighting is important, but being sat at the front or having the font size increased was not necessarily helpful. Students with VI are a heterogeneous group with regards to both the type and degree of their VI and their achievement in mathematics (Klingenberg et al., 2019), therefore these findings cannot be generalised to all PST's with VI.

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TOWARDS SIMULATION COMPETENCE IN STEM-EDUCATION: VALIDATING THE NATURE OF SIMULATION

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Using and understanding digital simulations, and fostering these competencies in STEM education, is becoming increasingly important, as many societal decisions are informed by economic, ecological, and scientific simulations. Preliminary research based on expert interviews suggests that the Nature of Simulation (NoSim) can be described as a three-dimensional construct including understanding models, modelling processes, and system thinking (Bleymehl, 2025). Building on Bleymehl's study, variability and covariational reasoning are added here as theoretically relevant dimensions, as both may support students in understanding stochastic and parameter-based simulations. The goal of the SIMBA-MINT project is an initial validation of the NoSim construct as a foundation for developing effective simulation tools and learning environments. In a first step, the following research questions are addressed:

- How many core dimensions underlie the NoSim construct?
- How is NoSim related to covariates such as digital experience and attitudes, cognitive abilities, understanding of the Nature of Science (NoS), and computational thinking?

To address these questions, a pilot online survey (approximately 30 minutes) is being conducted in April and May 2026 with N = 100 students aged 14–15 in southern Germany. The questionnaire combines multiple-choice competence tasks and a small number of self-report items across the five proposed NoSim dimensions and several covariates. It further includes an additional scale capturing students' epistemic understanding of simulations, for example their conditional character, limits, and the role of repeated runs in stochastic simulations. This initial validation step explores whether NoSim can be identified as a five-dimensional construct in an exploratory factor analysis and whether its dimensions show meaningful but distinct relations to the covariates. A better understanding of simulation competence can help to develop tools and learning environments that are better aligned with students' ways of thinking and working with simulations. In this way, students may come to see simulations as valuable tools for investigating complex and important real-world phenomena.

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GIVING A VOICE TO THE SILENT MATHEMATICAL SYMBOLS

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With an increasing emphasis for communication as an important aspect in mathematics learning, research has looked into how students could be "seen as producers of mathematical language rather than merely as consumers" (Morgan, 1998). However, the notion of communication in the mathematics classroom is often interpreted as talking to learn mathematics, while the role of writing as a mode of communication to learn mathematics has not been as widely discussed, practised and researched. Specifically, writing in the mathematics classroom is mostly associated with the writing of mathematical equations, notations and symbols as the focus is primarily placed on the presentation of final solutions rather than the process of getting to the solutions. Consequently, this form of writing is usually depersonalised and decontextualized, and often leads to student work that is universally similar. Yet, the process of writing with characteristics (*integrative, connective* and *self-rhythmed*) which are deemed akin to effective learning strategies can potentially empower students in the learning process as they engage and coordinate their different senses to make sense of mathematics (Emig, 1977; Morgan, 1998). Hence, we are keen to explore how writing in the mathematics classroom could better focus on students' (different) processes or experiences of doing mathematics. Based on the data collected from an exploratory teaching experiment conducted in the grade 11 mathematics classroom, we examined how the process of writing mathematics, through the structure of a double-entry journal – with examples recorded on the left, supported by interpretations and explanations on the right – could help students attend to both the mathematics and their sense-making processes. Students' writings over a period of three days were analysed for the purpose of this study. In particular, we noticed how these students were intentionally engaging different senses (e.g., visual, temporal) to organise their examples, steps and explanations/reasoning. We could also see and hear students' voices emerging through their examples and style of writing as the depersonalised and decontextualized mathematical notations are now embedded with the meanings students afforded them. We will discuss more about the teaching experiment and the findings during the presentation.

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MAKING SENSE OF INTERCULTURAL MATHEMATICS PRACTICES OF THE TIKMŨ'ŨN BRAZILIAN INDIGENOUS

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In our research, we investigated how Tikmũ'ũn Brazilian Indigenous, named Maxakali, teach, understand, and use mathematics in environment restauration practices of their territories. To Maxakali (2025), tikmũ'ũn school education should be guided by their ancestral educational practices and the flow of their spiritual entities (*yãmĩyxop*). However, *tikmũ'ũn* schools have prioritized Western mathematical concepts and procedures that can be expressed in Maxakali language: the use of money, numbers, and operations algorithms, only as a visual representation (Tomaz & David, 2018).

Since 2023, we have been focused on the environmental restoration practices and planting of agroforest systems in their villages, with the participation of 30 agroforest agents, 16 nurseywomen, and 32 teachers and undergraduates. This project became central in Tikmũ'ũn's lives as their territory has less than 12% forest and the water courses are polluted and stilted up, leading to extremely poverty.

We selected from a two-year period the empirical material composed of videos of the collective practices and participants' field diaries. We identified numeracy practices (Fonseca, 2020), theoretically and methodologically grounded in the tikmũ'ũn knowledge system, transmitted by chants, drawing, see-listen-do with the whole body (Maxakali, 2025) and agroforest knowledge from non-indigenous. The findings show the appropriation of intercultural numeracy practices by the indigenous, for instance, maps for the reforestation area, use of de measures and proportionality when mixing soil and fertilizer, which support Indigenous well-being, including languages and mathematics are rooted in relationships to land and intergenerational cultural practices (Cajete,1994). These practices have been introduced into schools, strengthening their connection with the culture and redefining mathematics for the Tikmũ'ũn people.

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DIFFERENCES IN TURKISH ELEMENTARY AND MIDDLE SCHOOL STUDENTS' LENGTH ESTIMATION SKILLS

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Length estimation skill is a multidimensional construct influenced by object size and accessibility (Hoth et al., 2023). There is indication that estimation skills vary across distinct dimensions: estimating the length of small objects, not small and touchable objects, and not small untouchable objects. While the 2024 Turkish Education Model emphasizes length estimation skills as a relevant learning goal of mathematics education, research on how these skills develop across grade levels remains sparse. In this regard, the primary goal of this study is to examine how Turkish elementary and middle school students' length estimation skills differ.

A total of 601 students (53,1% boys, 21 classes, four different schools) solved the adapted length estimation test developed by Hoth et al. (2023). The sample of elementary school students comprises $n = 334$ students from Grade 4, while the sample of middle school students holds $n = 267$ students from Grades 5 and 6. Students' estimates were scored with regard to their deviation from the objects' actual length. The scale's reliability was acceptable ($\alpha = .79$). For all children, the mean score was determined for the three dimensions of estimation skills.

Results show that the mean scores of elementary and middle school students differ in all three dimensions of length estimation in favour of the middle school students. However, differences were only significant for the Small Objects dimension as well as the not small and touchable dimension. The results suggest that developmental gains in length estimation are dimension-specific and may reflect curriculum emphasis on concrete manipulation.

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CO-DEVELOPMENT OF MATHEMATICAL AND COMPUTATIONAL THINKING VIA VARIABLES IN SCRATCH-SUPPORTED GEOMETRY AND MEASUREMENT TASKS

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The integration of Computational Thinking (CT) into mathematics instruction has become increasingly important in recent years. However, the simultaneous development of CT and mathematical thinking (MT) in problem-solving processes has not yet been sufficiently explored (Ye et al., 2023). This study aims to examine the co-development of *mathematical variables* and *variables* as a CT concept, as 5th-grade students solved geometry and measurement problems in the Scratch environment. Accordingly, the research question is: How did lower-secondary school students develop the concept of variables in mathematics and CT contexts while solving geometry and measurement problems in the Scratch environment? A class of 30 fifth-grade students engaged in Scratch-supported geometry and measurement problem-solving tasks. Two of them were selected based on their knowledge of variables. The data sources consisted of students' Scratch projects, worksheets, and interview transcripts. We qualitatively analyzed students' transitions between tables of values, pseudocode, and Scratch scripts. The findings show that Scratch tasks provided an environment that enabled students to reinterpret mathematical variables during algorithmic operations. Students established relationships among mathematical variables using tables of values and translated these relationships into pseudocode, enabling them to approach mathematical operations in a more step-by-step manner. As these relationships became clearer, the process of creating algorithms became easier, supporting the mutual development of MT and CT. However, students experienced significant difficulties when converting their pseudocode into scripts in the Scratch environment, which were related to how they used *mathematical variables* and *variables* as a CT concept. Testing and debugging in Scratch enabled students to reflect on their solution steps. During this process, students questioned not only the algorithm but also the mathematical operations, variables, and generalizations they used, revealing the co-development of mathematical and algorithmic representations. These findings indicate that the co-development of mathematical and CT variables took place through the processes of transposition (converting pseudocode into scripts in the Scratch environment) and debugging.

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TEACHERS' PRACTICES IN SUPPORTING STUDENTS' MATHEMATICS MOTIVATION IN PRIMARY SCHOOL

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Motivation plays a significant role in learning mathematics, and identifying instructional practices that improve students' motivation is therefore important (Schukajlow et al., 2023). The purpose of this study is to deepen understanding of how students' motivation in mathematics can be supported in primary education. Specifically, we aim to explore the practices teachers use to foster motivation.

We examine the phenomenom through data-driven content analysis of semi-structured interviews conducted with 15 primary school teachers as part of an international longitudinal MATHMot project. Teachers were selected for the interviews based on the high or increased motivation levels observed in their classrooms in grades 3-5. The theoretical framework is grounded in two motivational theories: SEVT (Eccles & Wigfield, 2020) and SDT (Ryan & Deci, 2020).

According to the interviews, teachers seek to enhance student's motivation in mathematics by supporting learning with differentiated and diverse instructional methods, providing constructive and positive feedback, creating opportunities for success, encouraging positive interactions, connecting mathematical content to everyday life, making learning mathematics enjoyable, and promoting persistence. In the presentation, results will be discussed in detail with examples and connections to motivational theories.

The results show that supporting students' motivation in mathematics is a continuous and multifaceted process. Future research is needed to investigate instructional strategies that foster motivation from the students' perspective.

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LEARNING OPPORTUNITIES FOR UNDERSTANDING THE PLACE VALUE SYSTEM IN TEXTBOOKS FOR EARLY MATHEMATICS EDUCATION

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Place value understanding is fundamental for early arithmetic and predicts later mathematical performance (Moeller et al., 2011). Jensen et al. (2024) describe regrouping and place value as the two core principles underlying the base-ten system. Regrouping involves combining and decomposing units in tens. The place value principle assigns meaning to digits based on their position and value. Place value instruction is, among other factors, also influenced by mathematics textbooks (Pepin et al., 2001). This study examines place value related learning opportunities in textbooks, focusing on task types, number ranges, and concepts of place value and regrouping. We aim to address the following research questions: (1) To what extent do textbooks offer place value learning opportunities? (2) Which types of learning opportunities address place value concepts? (3) How frequent are core constructs of the place-value system addressed? Five German Grade 1 and 2 textbook series were analyzed using quantitative content analysis (Mayring, 2015). All learning opportunities instructing the place value understanding were coded by type, number range, and principle. Frequencies and percentages were compared across textbooks and grade levels. The Kappa coefficient ranged from .92 to 1. Place value understanding was instructed in 3.74 % (182) of the 4,863 identified learning opportunities. Most of them were exercises (75.27 %) and two-digit numbers (73.63 %). 54.95 % of the learning opportunities focused on place value alone. 37.36 % instructed place value and regrouping combined. The study is limited to a textbook analysis and does not capture classroom implementation.

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DOES STATISTICS HAVE A MEANINGFUL STATISTICAL SIGNIFICANCE FOR YOU?: VIEWS ON STATISTICS EDUCATION AND STATISTICAL LITERACY

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Increasing data intensity worldwide and the rise of biased statistical interpretations have made it necessary to include statistics in educational curricula. Within this context, statistical literacy has been identified as a necessary skill for becoming active and informed citizens (Franklin et al., 2007). Schoen et al. (2025) likewise emphasized that statistics has been incorporated into mathematics curricula relatively recently and that its distinction from mathematics is still not fully understood. This study investigates, from the views of teachers and pre-service teachers, how statistics education and statistical literacy are defined, and the importance attributed to statistics education. In this context, the research question is "What are the views of mathematics teachers and pre-service mathematics teachers about statistics education and statistical literacy? A qualitative research design was employed, and semi-structured interviews were conducted. The interview questions were developed by researchers and subsequently reviewed by two experts in mathematics education. The participants consisted of nine high school mathematics teachers whose professional experience ranged from 4 to 30 years ($\bar{X} = 18.56, SD = 8.79$), and eight pre-service mathematics teachers, four of whom were in their third year and four in their final year of study. Teachers and pre-service teachers primarily focused on the concepts of analysis, interpretation, and decision-making when describing statistics. Regarding statistics education, participants generally highlighted the teaching of statistical concepts and procedures. While teachers and pre-service teachers shared a common understanding of statistical literacy as the ability to read, analyse, and interpret data, graphs, and tables, one of the most striking findings of the study was the divergence in their views regarding the definition and perception of statistical literacy. While teachers emphasized drawing meaningful inferences from concrete data, pre-service teachers foregrounded reasoning under uncertainty. This divergence suggests that teacher education programs and curriculum designs should explicitly address both views to support a more comprehensive and nuanced understanding of statistical literacy.

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ALGORITHM DESIGN IN SOLVING PERMUTATION PROBLEMS: AN ANALYSIS OF PRE-SERVICE TEACHERS' REASONING

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Computational Thinking (CT) is increasingly recognized as a key educational competency, involving processes such as abstraction, decomposition, and algorithmic structuring (Shute et al., 2017). Within this perspective, Algorithmic Thinking (AT) refers to organizing complex goals into logically ordered and operationally coherent steps. In mathematics education, this activity has been described as algorithmizing mathematical situations—transforming mathematical relationships into executable structures (Lehmann, 2024). Permutation problems offer a productive context for examining such processes, as they require coordinating constraints, distinguishing cases, and systematically enumerating possibilities which are the areas where students often rely on formula-based reasoning (Lamanna et al., 2022). Addressing the limited research on algorithmic approaches in specific mathematical domains, this qualitative study investigates pre-service teachers' algorithm design processes in solving permutation problems. Sixteen pre-service mathematics teachers enrolled in a Computational Thinking course participated in the study. Data included pseudocode drafts, algorithms developed in Flowgorithm, and semi-structured interviews with five participants. Findings show that participants produced more accurate algorithmic designs when constraints were explicitly stated. Performance declined when they had to identify multiple conditions. Difficulties were especially evident in distinguishing permutation from combination structures and in operationalizing implicit constraints within conditional and iterative constructs. The process of algorithm design provided a context to see participants' conceptual gaps. Therefore, programming-based algorithm design may function primarily as a formative assessment lens, providing insight into students' reasoning. Successful execution of an algorithm design related to a particular mathematical concept have potential to provide students to come up with a clear and robust conceptual understanding.

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UNLOCKING UNDERSTANDING: DOES FRACTION MAGNITUDE DEVELOPMENT SHAPE STUDENTS' FRACTION SENSE-MAKING?

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Despite the crucial importance of a good fraction magnitude understanding, students' understanding is often marked by misconceptions (e.g., Gabriel et al., 2023; Reinhold et al., 2023). While these misconceptions have been well documented, little is known about their development. Moreover, it is unclear whether development on a frequently used fraction magnitude task (ordering fractions) truly reflects a deeper fraction magnitude conceptual understanding, or if it results from rule-based strategies. This study followed 182 sixth and seventh graders over one school year and found that most retained their fraction magnitude misconceptions, with only a few students progressing toward correct understanding. Next, we tested whether improvement in fraction ordering was related to performance on a fraction sense-making task. Students estimated 12/13 + 7/8 by choosing 1, 2, 19, 21, or "I don't know." With good fraction magnitude understanding, the answer is easily seen as near 2 because both fractions are close to 1. Fraction sense-making task performance did not differ between improvers and non-improvers on the fraction ordering task, suggesting the ordering gains were more procedural than conceptual. Our results confirm previous research showing students' struggle with the conceptual understanding of fractions, which may be partly due to instructional practices that emphasize procedural skills rather than conceptual comprehension. Therefore, fraction instruction should not only focus on teaching how fractions work, but also put emphasis on conceptual understanding of fractions. Next, the methodological implication is that the use of fraction ordering tasks might need to be reconsidered as measure of students' conceptual understanding of the magnitude of fractions.

Acknowledgements

The study is part of the EDUCA RCF Flagship (#358924, #358947).

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SPATIAL ABILITIES AND MATHEMATICS TEACHER EDUCATION

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As spatial abilities gain prominence in mathematics and science curricula, the complexity of spatial teaching has become increasingly evident (Bufasi et al., 2024). It is therefore necessary to reexamine the questions originally posed by Bishop (1980) regarding mathematics teachers' spatial abilities: "Should experimental teaching methods in this area take into account the spatial abilities of the teacher? If so, how? […] How much responsibility should mathematics teachers take for the training and teaching of spatial abilities?" (pp. 266–267). To address these questions, a systematic literature review is being conducted using multiple databases, focusing on preservice and in-service mathematics teachers. The search terms include spatial abilities, literacy, thinking, reasoning, visualization, and related constructs. The selected studies are classified according to key perspectives that contribute to understanding visuospatial reasoning *(cognitive science, mathematics education, sociocultural)*; their attention to diagrams, gestures, and embodiment (Sinclair et al., 2016); and their approaches to *training* and *teaching* (Bishop, 1980). Additionally, connections between these perspectives are examined to address the gap in integrating *cognitive science* and *educational research* (Bufasi et al., 2024).

Preliminary findings suggest an increasing interest in mathematics teachers' spatial abilities, with particular emphasis on *cognitive science* research. Studies also indicate connections between *cognitive science* and *mathematics education,* often linked to pedagogical knowledge and pedagogical content knowledge. Moreover, mathematics teachers' responsibility appears to be increasing, driven mainly by policy agendas focused on identifying and training future scientific talent, rather than promoting spatial literacy for citizenship or addressing social inequalities.

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INVESTIGATING MATHEMATICAL PROBLEM POSING BY PROSPECTIVE PRIMARY TEACHERS: GAPS AND STRENGTHS

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Recent studies have shown that many prospective primary teachers (PPTs) have difficulties posing a "good" problem. This is generally defined as non-routine (NRP), solvable, with a motivating context and involving a variety of solution strategies (Leavy & Hourigan, 2022). In fact, many PPTs tend to choose closed, routine problems that have one right answer that requires low cognitive demand (Leavy & Hourigan, 2022). However, to understand the quality of the problems posed, it is not enough to examine them in terms of these "general-mathematical elements" ("general") (Ulusoy, 2025). Some problems may be solvable and non-routine, but they may not relate to the mathematical concept in the problem-posing task. We therefore need to examine the problems on the basis of "content-specific elements" ("content") (Ulusoy, 2025). This study's objective is to analyse the characteristics of problems posed by PPTs, considering in the light of both their pedagogical knowledge regarding "good" problems ("general") and their ability to identify the mathematical purpose of those problems ("content").

We administered a multiple-choice questionnaire (MCQ) to 108 PPTs to investigate PPTs' knowledge in these two respects. In addition, 24 PPTs participated in semi-directed interviews. They were asked to pose what they considered to be a "good" problem and to justify their choice. From the "general" point of view, our results showed a more nuanced reality than what is usually described in the research literature. In the MCQ, a large majority of PPTs stated that a "good" problem must be complex, develop reflection in students and involve multiple solving strategies. However, when they posed a problem, half of them posed NRP, but all their problems were solvable. Regarding the "content", many PPTs failed in the MCQ to recognise the mathematical purpose of a Grade 4 problem and in the interviews, half of them were unable to identify the mathematical objective of the problem they themselves posed.

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HOW COMPUTATIONAL THINKING MAY SUPPORT MATHEMATICS LEARNING IN PRIMARY EDUCATION: ON SUPPORTIVE RELATIONSHIPS IN A LITERATURE REVIEW

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Computational thinking (CT) is increasingly discussed in primary mathematics education, where prior reviews highlight its potential to support mathematical competencies and raise questions about how this support is realised (Nordby et al., 2022). To systematically analyse how relationships between CT and mathematics are described in the literature, a systematic literature review following PRISMA guidelines (Page et al., 2021) identified *N = 82* documents in Scopus, of which *n = 32* studies were included. The review reconstructed multiple types of relationships between CT and mathematics. Focusing on supportive relationships, this contribution examines how CT facets are described to support mathematics learning in primary education. Of the 32 included studies, 22 describe CT as supporting mathematics; of these, 20 include supportive effects without reference to specific facets, and in 10 of these no specific facets are identified. Where specified, abstraction and decomposition are among the few recurring facets, illustrating how support is described. Therein, decomposition is described to structure problems into subproblems, while abstraction supports the construction of mathematical representations (e.g., patterns, variables). These descriptions are often situated in programming-based tasks, suggesting that support may emerge through programming as a mediating activity providing context and purpose for applying abstraction and decomposition. This raises the question whether some support attributed to CT is, at least in part, tied to programming as the activity enacted, which itself is described as a supportive CT facet in certain studies. The small number of facet-specific studies and conceptual heterogeneity limit comparability and obscure potential mechanisms. These findings indicate a need for clearer articulation of CT facets and further research on how such supportive relationships are realised across programming and non-programming contexts.

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5-7-YEAR-OLDS DESCRIPTION OF PATTERNS IN A FUNCTIONAL THINKING TASK

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Functional thinking is a core strand of early algebra and has continued to gain increasing attention within mathematics education research. It well recognised that many young students can express variational patterns and corresponding relationships in more sophisticated ways than previously perceived. However, there is a concern that current learning trajectories may be holding the research field back from recognising the algebraic potential in young children's experiences, due to varying interpretations of what counts as algebra (Walkoe & Levin, 2020).

This oral communication focuses on data drawn from a task-based assessment gathered as part of a larger ongoing project within New Zealand. The task was based on an AAB repeating pattern, and required students to predict further positions, as well as create a table of data using the units of repeat. However, we focus on the open response question “what patterns do you notice” which was the final item in the task. In total, we examined n = 735 responses of 5- 7-year-olds. Using both deductive (Stephens et al., 2017; Warren & Miller, 2010) and inductive approaches, we found that students responses fell under three overarching pathways. First, a variational/correspondence pathway which aligned with the levels of sophistication framework (Stephens et al., 2017). Second, a unit of repeat pathway, consisting of eight unique codes. Third, a foundational pathway, consisting of six unique codes such as “context statement maker” and “number and answer noticer”.

Overall, we identified that whilst young students were able to describe patterns in ways which aligned to functional thinking or unit of repeat pathways, many also described patterns in ways which are not reflected in current algebraic trajectories. In the oral communication we will present examples and further detail on the three over-arching pathways, as well as discuss how codes in the foundational pathway might be harnessed in the development of algebraic thinking.

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TEACHERS' EXPECTATIONS REGARDING THE USE OF LEARNING PROGRESS ASSESSMENT

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There have been repeated calls for evidence-based teaching and adaptive, formative student support using digital assessment tools. Learning progress assessment (LPA) enables data-driven instruction and holds great potential, as stated by Fuchs et al. (2024), using short, (today mostly) digital, and standardized tests. Based on the provided information, teachers can track students' performance to identify learning difficulties early, respond accordingly, and monitor subsequent changes to assess the effects of the support provided (Nelson et al., 2023). Despite this potential, LPA remains underutilized in German elementary schools. To facilitate the implementation and utilization of LPA and to inform its design, it is imperative to consider elementary school teachers' expectations regarding LPA. These will be crucial for the acceptance of LPA as a tool for data-based teaching and individual support, particularly prior to first-time implementation, when expectations are not yet shaped by experience. These considerations led to the following research questions: *RQ1 What are elementary school teachers' expectations regarding the use of LPA in mathematics lessons? RQ2 Do these expectations differ based on individual and systemic factors?* To answer these questions, this study examines the expectations of 60 elementary school teachers regarding the use of LPA, as well as the influence of individual factors (e.g., assessment skills, ICT literacy) and systemic factors (e.g., technical equipment). Prior to using the intended LPA, data were collected via a questionnaire. The results of the study indicate that teachers primarily have functional expectations, particularly regarding automated, in-depth data interpretation. Expectations regarding easy, time-efficient use of LPA, as well as support and professional development, were mentioned less often. Expectations were largely independent of individual and systemic factors, yet assessment skills were significantly negatively associated with a need for support. Subsequent studies will compare teachers' expectations before and after using LPA to examine how initial expectations change through experience.

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DISPOSITIONAL STRUCTURE OF THINKING SKILLS IN PROPORTIONAL REASONING

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Proportional reasoning is a foundation for algebraic and advanced mathematical thinking. It requires the coordinated use of thinking skills—critical, creative, reflective, and systems thinking. Critical and creative thinking enable students to question assumptions and generate alternative strategies, whereas reflective and systems thinking support the evaluation of reasoning and the coordination of quantitative relationships and structures for effective problem solving (Wang, 2025; Yang et al., 2026). Together, these skills underpin meaningful engagement in proportional reasoning. Grounded in Bandura's Social Cognitive Theory (SCT) (Bandura, 2001), this study conceptualizes students' dispositions toward these skills through three constructs: behavioral intention, self-efficacy, and value. In SCT, engagement is shaped by students' intention to act, their confidence in their capability, and the value they attribute to the activity. Twelve items were developed by integrating the three dispositional dimensions across the four thinking skills. Two competing hypotheses were tested: H1, that the four skill constructs would better represent students' dispositions; and H2, that the three dispositional constructs would provide a superior representation. Using Partial Least Squares Structural Equation Modeling with a valid sample of 122 seventh graders, the findings supported H2. The three dispositional constructs provided a better structural representation of the 12 items, indicating that, at the dispositional level, the four thinking skills are more coherently organized under broader motivational dimensions than as distinct skill-based constructs.Theoretically, this study extends SCT to the domain of thinking skills in mathematics by clarifying the structural nature of related dispositions. Practically, it suggests that instructional efforts may be more impactful when designed to strengthen these motivational dispositions in ways that generalize across thinking skills.

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STUDENTS' CHALLENGES IN GRAPHING LINEAR AND QUADRATIC FUNCTIONS

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From covariational reasoning perspective, graphing helps students to reason about covariation by treating points as paired values of two quantities and the graph as a trace of how changes in one quantity correspond to changes in the other. Covariational reasoning was defined to be *the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other* (Carlson et al., 2002, p.354). In graphing, students need to sustain covariational reasoning throughout the process, so breakdowns are likely to occur at transitions between adjacent graphing steps. This study investigates into the discontinuity between adjacent graphing steps and examines the grade differences.

Data was collected from 335 students in Grades 7-10 from one school in Xinjiang, China. Eight booklets were developed from three guided graphing tasks related to linear and quadratic functions, each having two versions (Cronbach's α ranged from 0.86 to 0.91), involving. Each student needed to complete one booklet that was randomly assigned. Students' responses were coded using a five-step graphing framework with a three-level scoring 0-1-2: (1) making a table, (2) constructing axes with appropriate labels, (3) labeling scales, (4) plotting points, and (5) constructing the graph. For each adjacent transition (step k to step $k+1$), conditional success rates were calculated to quantify stepwise attrition.

The results revealed that the lowest conditional success rate occurred at the transition from step 1 to step 2, which is around 0.49-0.53 across all participants. This pattern was replicated at each grade level. This indicates that the greatest discontinuity happened at the representational shift from a correspondence display to the construction of a coordinate system that couples two quantities. Meanwhile, the success rates became higher and higher as the grade levels grew.

The results of the current study suggested that the transition from making a table to constructing the coordinate system is not only the starting step of graphing, but also the most possible difficult transition. Suggestions for instructional improvement include the use of dynamic software.

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WHAT KNOWLEDGE ARE PROSPECTIVE TEACHERS EXPECTED TO BRING FROM UNIVERSITY? A SURVEY STUDY

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In many countries, secondary mathematics teacher education includes both academic and practical education. Subject-specific education during the academic phase is often organized in university courses for academic mathematical content knowledge (CK) and pedagogical content knowledge (PCK) with a focus on the former. There is a consensus in recent research in mathematics education that mathematics teachers at secondary schools do not only need CK and PCK but also a profession-specific knowledge about connections between school mathematics and academic mathematics (e.g., Dreher et al., 2018; Wasserman et al., 2023). We refer to this knowledge as *school-related content knowledge* (SRCK). There is an ongoing discussion about the extent to which and how SRCK should be integrated into curricula for teacher education. The aim of our study is to examine the views of teacher educators from the practical phase on this topic, as they – alongside teacher educators from the academic phase – are deeply involved in teacher education and therefore can be seen as experts. We present first results of an ongoing online survey in 14 of 16 federal states in Germany with a sample of 124 teacher educators. Results reveal that a majority of our sample (81.4%) viewed SRCK as requiring targeted learning opportunities beyond those for CK. While 7% of these participants identified the practical phase as the ideal setting for such learning opportunities, 93% favored the academic phase. Subsequently, most participants (74.8%) suggested adjustments in university curricula, specifically calling for a greater emphasis on SRCK at the expense of CK. In sum, our findings support the theoretical argument that (1) SRCK represents a relevant aspect of teachers' professional knowledge, the acquisition of which requires specific learning opportunities that go beyond mere CK training, and (2) that SRCK should be addressed in the academic phase of teacher education. The surveyed views of teacher educators complement existing initiatives that aim to strengthen academic mathematics courses and support SRCK. A next step should be to gain further insight into effective learning opportunities for SRCK and their ideal implementation in teacher education.

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UNPACKING THE RELATIONSHIP BETWEEN MATHEMATICS SUBJECT MATTER KNOWLEDGE AND TEACHER NOTICING

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Teacher noticing (TN), which comprises attending, interpreting and responding, is a core competence for understanding students' mathematically thinking and responsive teaching (Jacobs et al., 2010). While scholars hypothesize a shared theoretical space where mathematical knowledge for teaching intersects with professional noticing (Thomas et al., 2017), empirical evidence detailing this exact relationship remains scarce. Specifically, it is unclear how sub-dimensions of subject matter knowledge (SMK) (Ball et al., 2008), namely common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK), predict the specific components of TN. This study investigates the extent to which CCK, SCK, and HCK predict pre-service teachers' abilities to attend, interpret, and respond to real classroom. The study involved 160 mathematics pre-service teachers (PSTs) from China who completed two validated instruments: a written SMK test and a video-based TN assessment. Given the ordinal nature of the dependent variable, three ordinal logistic regression models were constructed to determine the predictive power of SMK dimensions on each TN skill. Results indicate that HCK emerged as the sole positive predictor for attending (β = 0.227, p < .001), suggesting an understanding of the broader mathematical landscape enables teachers to identify critical classroom events. For interpreting, both CCK (β = 0.297, p = .009) and HCK (β = 0.165, p = .013) showed significant positive effects. Finally, only CCK significantly predicted responding (β = 0.361, p = .003). The dominance of CCK in the interpreting and responding phases implies that PSTs rely heavily on basic mathematical correctness rather than leveraging specialized knowledge for highly adaptive instruction. Notably, SCK did not predict any noticing skill, suggesting that for novice teachers, this specialized knowledge may remain inert and not yet functionally connected to practice. These findings empirically validate the intersection of knowledge and noticing, clarifying how specific SMK domains are differentially activated during the different phases of the noticing process.

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MATHEMATICS STUDENTS' CHATBOT ADOPTION

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Large language model (LLM) chatbots can explain concepts and provide step-by-step solutions in mathematics, yet evidence on learning gains is mixed and adoption drivers remain unclear (Cheng et al., 2024). We examined which learning-related factors predict mathematics students' chatbot adoption. We conducted a cross-sectional online survey of upper-secondary mathematics students in Austria (grades 9–13; ages 14–19; N=2,027). Chatbot adoption served as the dependent variable. Predictors included educational-psychology constructs, i.e. student well-being and cognitive engagement, and student characteristics: age, parental education, mathematics performance, and interest in mathematics (school-track proxy). Scale reliabilities were satisfactory (e.g., cognitive engagement, ω=.878; chatbot adoption, ω=.946). We estimated a structural equation model in R/*lavaan* (Rosseel, 2012) using robust maximum likelihood. Model fit was acceptable: CFI = .944, TLI = .936; RMSEA = .043, SRMR = .048. Three direct effects were significant. Lower mathematics performance predicted higher chatbot adoption (β = .132, p = .002). Higher cognitive engagement predicted lower adoption (β = −.116, p = .014). Older age predicted slightly lower adoption (β = −.069, p = .029). Effects of well-being, parental education, and mathematics interest were not significant. Adoption appears driven primarily by performance- and process-related factors rather than affective well-being, socio-economic background, or broad subject interest. Consistent with prior indications that chatbots do not uniformly raise performance (Asare et al., 2023; Cheng et al., 2024), lower-performing students use them more, and students with higher cognitive engagement use them less. Practical implications are twofold: i) Position chatbots as adaptive scaffolds for weaker learners, emphasizing step decomposition, worked-example support, metacognitive prompts, and planned fading of assistance. ii) Implement age-sensitive strategies: more motivational, dialogic support in lower grades and critical evaluation/quality-control practices in higher grades. Limitations include the cross-sectional design, self-reports, and operationalization of *adoption* as frequency rather than quality of interaction.

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SENSING MATHEMATICS BEYOND THE CLASSROOM: STUDENTS USING PHOTOGRAPHS TO EXPLORE MATHEMATICS IN EVERYDAY CONTEXT

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Mathematics education research emphasizes that mathematics plays a significant role in societal, political, and environmental contexts and cannot be understood as an isolated school discipline (e.g., Skovsmose, 2023), yet students often struggle to identify such connections in practice. This difficulty is linked to their perception of mathematics, which is shaped by school, peers, and social experiences (De Corte et al., 2002) and is often reduced to procedural calculations and formal algorithms (Reiss & Hammer, 2021). This study employs an autophotography approach to explore what (societal) topics students focus on when photographing aspects of their environment that interest them, and how they describe or apply mathematical concepts in these contexts. A total of 140 students from grades 5 to 12 in Austria submitted photographs accompanied by written explanations, resulting in 303 analyzable contributions.

The findings indicate that only about one-third (110) of all submissions explicitly addressed societal topics. Most of them focused on physics (69), some on biology-related topics (including human body, plants, environment, and sustainability; 19), and finance (8). Students either focused on the photographed objects and mathematized them or used them as a starting point for broader themes and questions, sometimes combining both approaches. The ways in which students used mathematics in their questions varied, with nearly half relying on simple, previously learned procedures such as direct quantification and single-step calculations. The findings align with research showing that students view mathematics as a primary goal and focus on procedural calculation. Even when addressing societal contexts, many mathematized objects and focused on basic procedures. As only one third addressed societal issues, this raises questions about their understanding of "societal" and the limits of the autophotographic approach – two issues we would like to discuss further.

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NAVIGATING BOUNDARIES AND INCLUSION: A NARRATIVE INQUIRY INTO KOREAN MATHEMATICS-RELATED MAJORS' CAREER TRAJECTORIES

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This study explores the career trajectories of students in mathematics-related majors through narrative inquiry. In Korea, demographic shifts, advances in emerging technologies, and changes in educational policy have reshaped recruitment across education and industry, contributing to increased career uncertainty among these students (e.g., Ministry of Education, 2023). Across national contexts, students in mathematics-related majors increasingly find themselves navigating hybrid professional identities spanning education, industry, and research. This study examines how students narratively construct and interpret their career trajectories. Six undergraduate students majoring in mathematics or mathematics education participated in semi-structured interviews focusing on motivations for choosing their major, experiences in mathematics coursework, and post-graduation plans. The data were analyzed through narrative analysis, informed by Polkinghorne (1995), to understand how career trajectories and their meanings are constructed over time. The narrative analysis yielded two major themes. First, students perceived boundaries across multiple career domains, including mathematics or mathematics education as disciplinary pathways, pursuing graduate studies, entering the teaching profession, and seeking employment in industry, and they experienced tensions while navigating these boundaries. Students moved between certainty and uncertainty regarding particular career paths, which led to internal conflicts. Second, experiences in mathematics coursework were closely associated with students' boundary perceptions. Students selected courses with particular career directions in mind, while in other cases, coursework experiences led them to reconsider and reconstruct their career trajectories. Thus, course taking functioned not merely as an academic requirement but as a site for developing a sense of inclusion and direction within specific career domains. These findings highlight that mathematics coursework plays a mediating role in shaping career trajectories and suggest the need for curriculum design that connects course structures with opportunities for career exploration within mathematics-related higher education contexts.

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EARLY CHILDHOOD TEACHERS' BELIEFS ABOUT THE NATURE OF MATHEMATICS AND TEACHING PRACTICE

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Beliefs are key determinants in teachers' mathematics practices. This relationship is well-established, with teacher-centred practices commonly associated with more traditional, transmission-oriented beliefs about mathematics and student-centred practice typically associated with more connectionist or discovery-based understandings (e.g., Swan, 2006). In early childhood education, teachers reportedly prefer student-centred practice but evidence of benefit is still contested (Lerkkanen et al., 2012; Lindner & Simpson, 2018). This raises an important question: to what extent are these preferred pedagogic approaches reflected in actual practice, and how do they relate to early childhood teachers' disciplinary beliefs about mathematics? Whilst previous research has explored early childhood educators' beliefs about the importance of teaching mathematics, little research has explored the relationship between their practice and beliefs about the nature of mathematics. Therefore, we surveyed 257 Reception teachers (4-5 years) in England about their mathematical teaching practices and beliefs. This age group is comparatively under-researched yet particularly interesting in England because it is the first year of statutory schooling. We found that Reception teachers reported adopting student-centred teaching approaches more frequently than teacher-centred approaches. This strongly aligned with their beliefs: the vast majority agreed with statements reflecting discovery-based or connectionist conceptions of mathematics, with more disagreement across traditionalist statements. Teachers working in areas with the highest and lowest levels of socio-economic disadvantage tended to espouse more traditionalist practices, and choice of schemes/resources were also associated with variation in practice. Together, these findings raise questions about how school context and perceptions of learners' needs may shape early childhood teachers' practice and the beliefs that underpin them.

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PROBLEM-POSING PATTERNS ACROSS DATA-BASED TASKS WITHIN A 2×2 FRAMEWORK

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With the growing prominence of authentic data (OECD, 2023), incorporating real-world datasets into problem-posing has become increasingly important. Yet prior research has emphasized product quality in simplified contexts (Cai et al., 2023), leaving students' framing processes underexplored. This study examines students' problem-posing patterns across two data-based tasks adapted from a released PISA item, focusing on their framing orientations. Participants were 149 Taiwanese 8th graders (randomly selected from four schools). Two open-ended tasks based on the same dataset context were administered: Task 1 (problem posing of questions) required students to construct mathematical questions from a given dataset, and Task 2 (problem posing of justification) required them to generate evidence and explanations for a trend from the same dataset. (score 0-3; inter-rater Kappas > .89). Students were classified as low (0-1) or high (2-3) in each task, forming a 2×2 framework (LL, LH, HL, HH).

The distribution across the 2×2 framework was as follows: LL (38.3%), LH (28.2%), HL (24.2%), and HH (9.4%) revealed that over 52% of students showed inconsistent originality across tasks, supporting its task-dependent nature. Qualitative analysis identified two distinct problem posing patterns: (1) Structurally Oriented Tendency Reasoning (SO-TR), characterized by structural re-representation and systematic reorganization of numerical relations. (2) Trend-Oriented Interpretive Reasoning (TO-IR), characterized by reclassification, holistic interpretation, and trend-oriented reasoning. Notably, even among the high-performing HH group, students demonstrated diverse orientations, with some favoring SO-TR's structural depth while others excelled in TO-IR's contextual inference. Taken together, these findings highlight the importance of examining originality across tasks rather than treating it as a fixed individual trait. Detailed distributions and illustrative examples will be presented during the conference session.

Acknowledgements

The authors gratefully acknowledge the support of the National Science and Technology Council (NSTC), Taiwan (Grant No. MOST 110-2511-H-003-007-MY3).

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AN ERP STUDY ON COGNITIVE FLEXIBILITY AND UPDATING IN FRACTION EQUIVALENCE ACROSS REPRESENTATIONS

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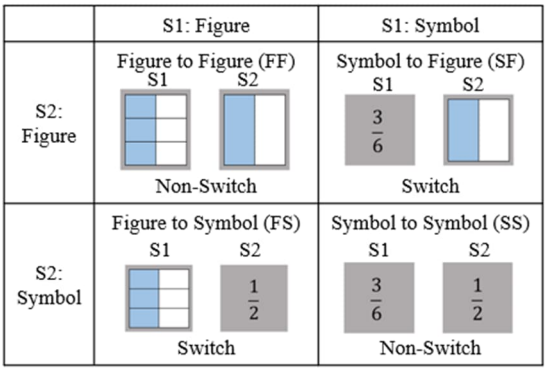
Understanding fraction equivalence is fundamental to mathematical learning, requiring learners to flexibly coordinate and update information across different representation systems. This study investigates the behavioral and neural dynamics of fraction equivalence judgments to disentangle the costs of rule switching from representation-based updating. Grounded in Dual Coding Theory (Paivio, 1990), we examine how processing within a single representational system (symbolic-symbolic or figural-figural) versus transitioning across systems (symbolic-figural or figural-symbolic) differentially shapes cognitive load.



Thirty right-handed adults performed a fraction comparison task while EEG data were recorded. The task required participants to judge whether two sequentially presented fractions were equivalent using two formats: symbolic (Arabic numerals) and figural (area models). This generated four transformation conditions: Figure-to-Figure (FF), Symbol-to-Symbol (SS), Figure-to-Symbol (FS), and Symbol-to-Figure (SF). Behavioral performance (accuracy and reaction time) and Event-Related Potentials (N2 and P3 components) were analyzed to evaluate cognitive flexibility and working memory updating demands. Contrary to classical task-switching theories, behavioral results revealed better performance in switch conditions (FS, SF) compared to non-switch conditions (FF, SS). Crucially, processing equivalence within the figural system (FF) imposed the highest cognitive load, resulting in lower accuracy and longer reaction times. Conversely, the symbolic system (SS) was faster and more accurate. Neural data corroborated that SS elicited larger N2 and P3 amplitudes—reflecting efficient cognitive control—poorer FF performance suggests figural representations impose a high cognitive load that may consume a lot of working memory resources. These findings challenge the assumption that visual models universally simplify learning. Cognitive load in fraction equivalence is driven by representational updating rather than rule switching. Evaluating equivalence in area models (FF) requires simultaneously maintaining visual features and actively extracting underlying relational structures. Therefore, identifying a figural fraction and performing equivalence operations upon it are cognitively distinct.

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IN-SERVICE TEACHERS' LEVELS AND BELIEFS ABOUT STRATEGIC FLEXIBILITY IN NON-ROUTINE PROBLEM SOLVING

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This study aims to explore in-service teachers' strategic flexibility level and their underlying beliefs in non-routine problem solving. Strategic flexibility involves generating and selecting appropriate strategies depending on context (Star & Newton, 2009). Non-routine problems provide a key setting for observing this competence (Schoenfeld, 1985). In this context, a qualitative case study design was employed with nine teachers from different school levels (Yin, 2018). Data were collected through semi-structured interviews consisting of two parts. In the first part, teachers solved five non-routine problems and were asked to explain their thinking aloud and produce alternative strategies. In the second part, open-ended questions explored teachers' beliefs about strategic flexibility, including its meaning, classroom use, and instructional conditions. Written solutions and audio recordings were collected for analysis. Strategic flexibility was assessed using a rubric based on multiple strategies and appropriate strategy use. Beliefs were analyzed through content analysis by identifying themes across interview responses.

Results showed most teachers demonstrated high flexibility despite limited prior familiarity. They were strongest in selecting appropriate strategies, while generating multiple strategies within a problem was less common. Teachers emphasized the importance of flexibility for learning, teaching, and real life, but noted students' limited flexibility. Confidence and persistence were identified as important factors. Supportive conditions included safe classroom environments, error tolerance, discussion, and open-ended tasks. However, time and curriculum pressures were key constraints. Teachers consistently linked non-routine problems with flexibility development. These findings highlight the need for professional development focused on evaluating and comparing strategies.

Acknowledgement

Supported by The Scientific and Technological Research Council of Türkiye (TÜBİTAK)

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EXPLORING TRANSITION DYNAMICS BETWEEN ALGORITHMIC THINKING COMPONENTS IN FLOWGORITHM-SUPPORTED MATHEMATICAL TASKS

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Algorithmic thinking (AT) encompasses components such as *decomposition* (breaking down a problem into manageable parts), *abstraction* (distinguishing unnecessary information), *algorithm design* (structuring a problem by creating sequential steps), *pattern recognition* (identifying recurring relationships), and *debugging* (identifying and correcting errors) (Lehmann, 2024). Research on how programming tools shape the dynamic relationships among AT components in the context of mathematical tasks is quite limited. Therefore, this study aims to investigate how the components of AT are activated and interconnected as a pre-service mathematics teacher (PMT) engaged in Flowgorithm-supported mathematical tasks. The research question is: What patterns characterize the interconnections among the components of AT when a PMT was engaged in Flowgorithm-supported mathematical tasks?

We designed an embedded single-case study. A task-based interview containing four open-ended mathematical tasks was conducted with a PMT. The interview transcripts were analyzed based on Lehmann's (2024) *AT competency model* adapted to the study context. Interview data were coded based on the AT components and their subskills. The transition orders were determined, and patterns characterizing the interconnections among the components were illustrated using diagrams.

The findings revealed nonlinear, cyclical, and multidirectional transition patterns among AT components. In particular, repetitive transitions were observed between the *algorithm design* and *debugging* components. Although the PMT mostly performed debugging by returning to the *algorithm design* and *decomposition* after detecting the error, consistent bidirectional transitions were observed between the *algorithm design* and *debugging* components. Therefore, although the PMT returned to dividing the algorithm into sub-goals to correct errors, she did not return to *debugging* without first reorganizing the *algorithmic design* or moving to the *abstraction* component. Transition patterns and frequency varied significantly depending on the nature of the mathematical task. Based on the findings regarding the dynamics of AT components, implications for classroom practice regarding the implementation process of Flowgorithm-supported mathematical tasks will be presented.

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NEGOTIATING THE LABEL: HIGH SCHOOL MATH TEACHERS' POSITIONING OF EMERGENT BILINGUALS

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Supporting emergent bilingual (EB) students in mathematics is critical, as they often face academic challenges rooted in language-based deficit positioning (Yi et al., 2025). This study examines how high school teachers position EBs in mathematics classrooms after participating in collaborative, situated professional development focused on EB learners. Research highlights the importance of engaging EBs in rigorous mathematical tasks while leveraging their cultural and linguistic assets (DiNapoli & Morales, 2021). We implemented a 5-Act Task Framework curriculum designed to support challenging mathematics in culturally and linguistically relevant contexts. Using positioning theory, which examines how roles and responsibilities are constructed through interactions and storylines, we analyze teachers' interactive positioning of EBs in relation to mathematical learning. Few studies examine how teachers' positioning of EBs develops over time; this study addresses that gap. The study took place in an urban district where 21.9% of students are EBs speaking over 100 languages. Over three years, we worked with seven mathematics teachers teaching 100% EB Algebra and Pre-Algebra classes. Data included video recordings of teaching, planning, reflection, and interviews. Three coders analyzed the data in three phases: (1) open coding of positioning acts, (2) development of a coding manual through consensus, and (3) construction of individual and collective storylines. All teaching transcripts were coded and analyzed for themes. This presentation focuses on Phase 2 findings related to teachers' interactive positioning of EBs. Findings show both deficit and asset-based positioning, with asset-based views occurring more frequently but not predominating. Monolingual veteran teachers most often used the "Learner with Difficulties" code, whereas bilingual and early-career teachers more frequently positioned students as "Mathematically Capable." Deficit and asset framings coexisted, with students sometimes positioned simultaneously as capable contributors and as struggling learners. Notably, students were more often positioned as mathematical contributors within collaborative structures. Further implications will be discussed in the presentation.

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NOT ALL CORRECT ANSWERS ARE EQUAL: EXAMINING THE ROLE OF STRATEGY IN REPEATING PATTERN ABSTRACTION

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Patterns are a central concept in mathematics education and support later achievement and early algebraic thinking (e.g., Rittle-Johnson et al., 2017). Repeating patterns, a foundational pattern type in early childhood, have been widely studied; however, research has largely emphasized accuracy rather than strategy use. This study investigates whether abstraction strategy use predicts performance in Repeating Pattern Abstraction Task (RPAT) beyond the effects of age and Visuospatial Working Memory (WM) in 4- to 6-year-old children. A convergent parallel mixed method design was employed. 56 preschool children participated in one-to-one interviews. The researcher-developed RPAT (12 items; α = .88) and the Corsi span task were administered to assess visuospatial WM. Responses were scored dichotomously, and children's explanations were coded using a hybrid thematic analysis based on Lüken and Sauzet's (2021) strategy codes. A binomial generalized mixed-effects regression tested the effect of strategy use on RPAT accuracy, while a second regression examined the contributions of strategy use, controlling age and visuospatial WM to RPAT accuracy. Binomial generalized mixed-effects analyses indicated that only sequence awareness and unit identification strategies significantly predicted accuracy. Age 6, compared to age 5, was also a significant predictor, whereas no differences were found between 4- and 5-year-olds. Regression analyses further revealed a quadratic relationship between abstraction strategy use and performance. The final model, including age and visuospatial working memory, explained 76% of the variance in abstraction scores. The current study extends the existing literature about repeating patterning by highlighting the importance of children's explanations and strategy use, beyond the effects of age and visuospatial working memory. In the presentation, further results and implications will be discussed in detail.

Acknowledgements: This study was supported by TUBITAK under the Grant Number 124K343 and BIDEB 2210-A National Graduate Scholarship Program.

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MIDDLE SCHOOL STUDENTS' USE OF GENERATIVE ARTIFICIAL INTELLIGENCE IN PROBLEM SOLVING AND PROBLEM POSING

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This study aims to explore how middle school students use generative artificial intelligence (genAI) in problem solving and problem posing. Problem solving and problem posing provide students with opportunities to experience productive challenges in mathematics learning and to develop their mathematical thinking (Cai, 2022). As genAI has become a tool capable of generating mathematical problems, it has begun to be used within the domain of problem posing (e.g., Walkington et al., 2025). However, there remains a need for a systematic approach to integrating genAI into problem solving and problem posing processes. This study analyzes students' use of genAI in such processes to propose practical directions for its implementation. Seven Korean eighth-grade students individually participated in end-of-semester problem solving and problem posing activities. Given the timing of the study, the activities focused on counting and probability topics included in the curriculum for that grade level. Students used three genAI models (ChatGPT, Gemini, and Copilot) to generate problems and solved one problem from each model. Students then posed their own problems and solved them. Data collected included students' genAI interaction logs, written work, and verbal utterances. Adopting an exploratory approach, Students' use of genAI in problem solving and problem posing was analyzed using an open coding approach. Students engaged with and solved genAI-generated counting or probability problems that varied across models in formatting, contexts, and levels of complexity. After solving these problems, students' posed problems often reflected the situations and contexts of the genAI-generated tasks. Rather than merely modifying numbers or changing conditions, students constructed new problems by drawing on the situations and contexts that structured the original tasks. Students flexibly set conditions when posing problems and solved them using counting or probability concepts. These findings suggest that students can proactively leverage genAI to support both their problem solving and problem posing processes.

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ANTICIPATORY TEACHER NOTICING IN THE PRE-ACTIVITY PROCESS OF MATHEMATICAL MODELLING ACTIVITIES WITH GIFTED STUDENTS

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Teacher noticing is defined as the process of paying attention to students' mathematical thinking, interpreting this thinking, and relating it to pedagogical decisions (Jacobs et al., 2010). This process has mostly been examined through classroom interactions; relatively little research has been conducted on how teachers anticipate students' possible ways of thinking prior to implementation. Teachers' prior cognitive constructions play a decisive role, particularly in the planning of open-ended tasks. This study aims to examine the anticipatory noticing structures that emerge in teachers' planning process prior to a modelling activity to be conducted with gifted students. The research was conducted using a qualitative multiple case study design. Two teachers who taught mathematics to gifted students were asked to plan a modelling task, and semi-structured interviews were conducted based on these plans. In the interviews, teachers were asked to anticipate possible solutions, cognitive risks, and intervention strategies. The data were analysed through deductive thematic analysis informed by Jacobs et al.'s (2010) framework of professional teacher noticing—anticipatory attending, anticipatory interpreting, and anticipatory deciding.

The findings indicate that teachers engage in structured mental simulation during the planning process. They anticipate the variables and representations that students may focus on (predictive attention), focusing particularly on the tendency towards rapid abstraction and early generalisation. Assessing that these tendencies may superficialise the modelling cycle (predictive interpretation), they plan interventions that will reopen the neglected stages of the process (predictive decision). The gifted context qualitatively differentiates anticipatory noticing: teachers develop a planning organisation aimed at slowing down the process, preventing early closure, and preserving epistemic depth rather than supportive intervention. These results demonstrate that anticipatory noticing serves a strategic function in modelling pedagogy by preserving epistemic integrity. The study extends teacher noticing beyond the moment of application, making visible the attention–interpretation–decision organisation in the planning process and offering a cognitive contribution to modelling instruction in the context of gifted education.

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IMPROVING MATHEMATICAL THINKING BY PLAYING BOARD GAMES IN MATHS LESSONS – A SECOND TAKE

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Playing board games stimulates the prefrontal and parietal areas essential for mathematical cognition (Newman et al., 2016). This suggests possible advantages to their educational use. While benefits in school settings for students aged 6–13 are increasingly documented (e.g., Vita-Barrull et al., 2024), research in secondary education remains sparse. To address this, our first project phase (Szörényi, Muzsnay, Szabó & Zámbó, 2026) involved 10th graders (N=170) from six schools. For nine weeks, one mathematics lesson per week was replaced by gameplay using non-digital logical or geometrical board games. We examined the effect of the intervention on students' geometrical and logical skills via pre- and post-tests. Results obtained using Anova indicated equivalent performance in logic skills, but significant improvement in geometrical skills in the experimental group (n=79) versus controls (n=91).

In this current proposal, we present the extension of the intervention to a second phase, where the original control group was switched to experimental – playing board games for 1 lesson/week for 9 weeks – and vice versa in one school (N=62 in total), culminating in a post-test measuring logic and geometry skills. The research goal is to examine the medium-term effects on the initial experimental group and within-subject gains, using the input and output test results, supplemented by a qualitative review of test responses for thinking processes. We hypothesize an improvement in the geometry skills of the newly experimental group and maintained performance of the original experimental group. The expected outcomes support integrating board games into mathematics lessons to enhance thinking skills and suggest the need for longitudinal, multi-game replications of the study for broader implementation.

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RETHINKING MATHEMATICAL KNOWLEDGE AND PRACTICE IN THE AGE OF AI: PERSPECTIVES FROM STEM FACULTY REGARDING UNDERGRADUATE MATHEMATICS

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The rapid integration of artificial intelligence (AI) across STEM disciplines is reshaping expectations of what it means to be mathematically prepared for undergraduate study and professional practice (Awang et al., 2025). While gaps in tertiary mathematics readiness are well-documented, how these expectations are being reconfigured in the age of AI remains underexplored. Grounded in the Anthropological Theory of the Didactic (ATD) (Chevallard & Bosch, 2020), this study conceptualizes university STEM faculty as the "noosphere"—the sphere of actors influencing the didactic transposition of mathematical knowledge. Accordingly, this study addresses two research questions: (1) What specific mathematical content do STEM academics perceive as essential for STEM undergraduates in the age of AI? and (2) What AI interaction strategies and associated concerns within mathematics learning and problem-solving do they expect from undergraduates? An open-ended qualitative survey was administered to 35 university STEM academics in Hong Kong, a context undergoing rapid AI adoption across industries and education. Data underwent deductive and inductive thematic analysis to uncover faculty's views. The results are organised into three interrelated domains: (1) valued mathematical content, (2) forms of AI use in learning and problem-solving, and (3) concerns surrounding AI integration. The results indicate an expanded and interdisciplinary conception of foundational mathematics (e.g., algebra and calculus), accompanied by an emphasis on data-driven areas (e.g., statistics, probability and mathematical modelling). Furthermore, findings revealed a complex duality in AI use—simultaneously positioned as a tutor to provide individual scaffolding and an assistant to enhance computational efficiency, yet cautioned against as a potential source of over-reliance that may undermine foundational understanding and critical thinking. These insights provide an empirical account of how the academics redefine mathematical knowledge and practice in AI-rich STEM education, ultimately extending the ATD framework to AI-integrated contexts.

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EMERGENT BILINGUAL CHILDREN USING SEMIOTIC RESOURCES TO SUPPORT THEIR MATHEMATICAL ACTIVITY

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Research on the mathematical learning of bilingual students shows that children draw on multiple communicative resources, not only language, to support their learning (Moschkovich, 2002). For example, research on semiotic resources in mathematics shows that gestures, speech, and inscriptions help make meaning visible (Arzarello et al., 2009). However, research on semiotic resources involving bilingual learners has primarily focused on older students. More needs to be known about how young emergent bilingual children use semiotic resources to support their learning. This qualitative case study examined the following question: How do emergent bilingual kindergarteners use semiotic resources to support their mathematical activity as they work on challenging mathematical tasks? Participants included four five-year-old Chinese-English emergent bilingual children from a pre-K private Chinese immersion school in the northeastern United States. Data included videos of three 30-minute sessions of the students working on two combinatorial tasks. Analysis using a video methodology (Powell et al., 2003) and drawing on semiotic research resulted in six semiotic episodes showing that students used semiotic resources to support their mathematical activity in mainly two ways. First, the children used semiotic resources to represent mathematical ideas, making them visible and concrete to others. For example, they used gestures and manipulatives to display solutions and explain strategies. Second, the children used semiotic resources to support sense-making. For example, they used semiotic resources to interpret what counted as a valid solution, negotiate task premises, and modify tasks. The results show that semiotic resources do not only serve communicative purposes but also shape participation in mathematical activity. The children used semiotic resources to make sense of tasks and decide how to proceed. The study extends semiotic research to early childhood bilingual learners and shows that semiotic resources enhance meaningful participation in learning and should be encouraged in mathematics classrooms.

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PREREQUISITES FOR TEACHING MATHEMATICS TO STUDENTS WITH VISUAL IMPAIRMENTS IN INCLUSIVE CLASSROOMS: A SYSTEMATIC REVIEW OF BEST PRACTICE RECOMMENDATIONS

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Introduction: The learning process of students with visual impairment (VI) requires specific adaptations, particularly in visually demanding subjects such as mathematics (Hayes & Proulx, 2023). In inclusive settings, teachers need operational and structured guidance to effectively support these learners. However, existing recommendations remains fragmented and unevenly organised.

Aim: This study aimed to identify and systematically structure good practice recommendations for teaching students with VI using an instructional-phase learning.

Method: A systematic review was conducted across major databases covering the period 1990–2025. Sixteen guidelines meeting inclusion criteria were selected and appraised using the AGREE II instrument. Extracted recommendations were analysed and organised according to Hattie's (2017) model: preparation, opening, delivery (learning and feedback), and conclusion.

Results: A total of 192 specific recommendations were identified and showed an uneven distribution across lesson phases. Most recommendations focused on lesson preparation (e.g., adapting the environment and instructional materials), followed by the opening and delivery (learning) phases (e.g., establishing a supportive climate and using multisensory instruction). Only minimal guidance addressed feedback, and no recommendations explicitly targeted the lesson conclusion phase.

Conclusion and implication: The findings confirm the pivotal role of teachers in ensuring accessibility for students with VI and highlighted essential guidance to support instructional decision-making across all phases of a mathematics lesson.

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SENSING PROPORTION: AN EMBODIED LEARNING DESIGN FOR INCLUSIVE MATHEMATICS EDUCATION

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In mathematics, students are often able to compare numbers but fail to understand the relationships between quantities. In the context of ratios, they tend to use additive comparisons rather than proportional relationships. This difficulty is more pronounced in students with mild learning difficulties, who process quantities separately and struggle to coordinate them simultaneously. From a cognitive perspective, this relates to limitations in working memory, while embodied cognition emphasizes the coordination of action and perception in building understanding (Abrahamson, 2014). However, how students move from comparing numbers to understanding relationships has rarely been empirically explained. This study aims to explain how students with mild learning disabilities develop proportional reasoning through embodied and contextual learning. Using a design research approach, we developed Local Instructional Theory based on RME-Embodied Learning (Gravemeijer et al., 2006; Zulkardi, Putri & Wijaya, 2020) in the context of designing tour packages. Learning activities included planning travel routes, comparing distances between destinations, selecting modes of transportation, and calculating and comparing tour package costs based on the number of participants. Students were also asked to interpret information from maps, price tables, and itineraries to make efficient decisions. The results showed a consistent reasoning trajectory, from comparing numbers to coordinating quantities, and then establishing multiplicative relationships through representations such as ribbon diagrams and double number lines. These findings confirm that embodied activities are effective when they support relational coordination rather than simply object manipulation.

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SENSING PROBABILITY: REFLECTIONS OF SIXTH GRADE STUDENTS' EXPERIENCES

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Probability is one of the conceptually most challenging areas of school mathematics for students. Students often struggle to establish the relationship between theoretical and experimental probability and to understand why repeated trials can produce different results each time. These difficulties largely stem from the abstract and conceptually demanding nature of probabilistic reasoning. The embodied learning approach proposes that abstract mathematical concepts can be made meaningful through bodily experience. Embodied learning in mathematics education emphasizes representation through bodily actions (e.g., gesture and movement), which are inherently connected to mathematical concepts and processes. In the context of probability, practices such as taking physical positions and using perception-based designs (Abrahamson, 2014) may support students in constructing abstract concepts through more concrete experiences. This study was conducted to reflect how 6th graders experienced the concept of experimental probability through an embodied learning approach. In this classroom-based study, 24 students participated in three activities. In the first activity, students determined the likelihood of their own verbally constructed statements by positioning themselves on a probability spectrum drawn on the classroom floor and explained why they chose that particular position. In the second activity, students were individually asked to conduct a coin-toss experiment 20 times, represent the outcomes using two different colors, and visualize their results by coloring boxes on a sheet of paper. They were then asked to represent the relative frequencies of heads and tails using their bodies. Finally, the experimental probabilities of heads and tails were discussed based on the total of 480 trials conducted. In the third activity, a spinner divided into four equal sections in different colors was drawn on the classroom floor. Before each spin of the spinner, students predicted which section would occur and, according to predetermined bodily codes based on whether their prediction was correct or not, they observed the distribution of the class. This process created an environment for conceptualizing experimental probability. Observational evidence from the classroom suggested that these activities designed through embodied learning may have supported students' emerging conceptualization of experimental probability.

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VALIDITY AND RELIABILITY STUDY OF COGNITIVE LOAD SCALE TEST FORM: TURKISH VERSION

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Various tools have been developed to assess students' cognitive load (CL) during instruction. One of the most widely used scales is Paas's (1992) single-item scale, which measures the mental effort during learning. In recent years, instead of this single-factor structure, multifactorial structures focusing on the dimensions of CL have been developed and CL has been measured accordingly (Eysink et al., 2009). Furthermore, in addition to measuring students' CL related to the effectiveness of the instruction, it is also important to assess their CL regarding a specific mathematics topic both before and after the implementation. From this point of view, we aimed to develop a Turkish CL scale test form (CLST) that originated from Leppink et al. (2013), which was designed to measure instructional effectiveness and assess three types of CL, whereas CLST aims to assess two types of CL related to a specific mathematics topic, namely algebra. The data were obtained from 7th grade students through a convenient sampling technique. The CLST was presented to 7th graders after an algebra test. Factor analysis was used to assess validity, and reliability was evaluated using Cronbach's alpha coefficients.

The CLST was conducted with the data obtained from 223 seventh-graders to perform EFA. According to EFA, the CLST consisted of two factors: intrinsic load (IL) and extraneous load (EL), which together explained 63.97% of the total variance. CFA was conducted with the data obtained from 354 seventh-graders. The first-order CFA provided a good fit to the data, χ^2 (8) = 36.67, p<.001, RMSEA= .07, NFI=.97, CFI=.98. The reliability statistics for both IL and EL showed acceptable levels of internal consistency that were .85 and .81, respectively.

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ABSTRACTING FROM COMPUTATION: ASSESSING PRE-SERVICE MATHEMATICS TEACHERS' ALGEBRAIC THINKING

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Algebraic thinking extends beyond computing answers to reasoning about structures. For pre-service mathematics teachers (PSMTs), formatively assessing this is challenging. While the Algebraic Habits of Mind framework (Driscoll, 1999) conceptualizes this reasoning, its formative application remains limited. Magiera et al (2017) state that PSMTs need tools to decode reasoning hidden behind correct computations. Addressing this, we operationalized Abstracting from Computation into a diagnostic rubric. The research question is: How can this rubric reveal qualitatively distinct profiles of PSMTs' algebraic thinking masked by procedural correctness? This qualitative study involved 20 third-year PSMTs in an Algebra Teaching course. Data emerged from the "Reversals task," requiring participants to reverse a three-digit number, subtract the smaller, and justify the pattern. This task deliberately distinguishes between empirical trial-and-error and structural abstraction. Written responses were analysed using a three-level rubric (proficient, emerging, no evidence) evaluating sub-habits like generalizing and symbolic justification. Two researchers independently coded all data, resolving discrepancies through discussion to ensure interrater reliability. To illustrate the rubric's diagnostic power within space limits, we contrast two representative cases highlighting the abstraction spectrum among the 20 participants. Both achieved the correct invariant outcome (1089), yet their reasoning differed entirely. PSMT#5 showed no evidence of abstraction; their reasoning was empirical, relying solely on numerical trials (e.g., 541-145=396) to "prove" the pattern. Conversely, PSMT#12 exhibited emerging abstraction. They successfully used symbolic representation (100a+10b+c) to find the structure 99(a-c). However, lacking deductive justification, they reverted to plugging in numbers to verify their own formula. This research highlights that correct answers do not guarantee algebraic thinking. By pinpointing transitional gaps—like PSMT#12's shift from symbolic manipulation to deductive proof—the rubric serves as a crucial formative tool. It offers teacher educators a principled method to move PSMTs from arithmetic verification to structural reasoning.

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THE EMERGENCE OF COMPUTATIONAL THINKING IN PRE-SERVICE TEACHERS' GEOGEBRA-BASED ACTIVITY DESIGN

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Computational Thinking (CT) enables individuals to tackle various challenges systematically (Wing, 2011) through decomposition, pattern recognition, abstraction, algorithm design, debugging, and evaluation. Since the intellectual processes underlying CT closely parallel mathematical reasoning, supporting pre-service teachers (PSTs) in integrating these skills is crucial for effectively designing technology-enhanced mathematics environments. However, existing research focuses heavily on programming and robotics-based practices, while studies using dynamic software largely target K-12 students. Consequently, how CT emerges beyond coding, particularly utilizing underutilized dynamic tools like GeoGebra's (GGB) spreadsheet view, remains underexplored. Thus, we ask: "How do GGB's dynamic tools—specifically the spreadsheet view, checkboxes, and command writing—support the emergence of CT components during PSTs' activity design processes?"

This qualitative study investigates two PSTs designing an 8th-grade linear relationships activity using GGB. Methodological triangulation was achieved by cross-checking interviews with final design products. Findings report on one representative case (Emir) for an in-depth micro-analysis. Ensuring reliability, two researchers independently conducted thematic analysis based on the CT framework, resolving disagreements through consensus.

Emir demonstrated *decomposition* by breaking the topic into modules explicitly using GGB's checkboxes. For *pattern recognition*, he utilized the spreadsheet view alongside graphical representations to dynamically model the coefficient-slope relationship. His *abstraction* practices involved restricting negative slopes to manage students' cognitive load. He engaged in *algorithm design* through command writing, applying conditional logic ("if-then") and coordinates to link variables. Encountering technical errors, he demonstrated *debugging* by developing pragmatic limitations. Finally, *evaluation* emerged as he critically reviewed the spreadsheet view's pedagogical efficiency.

In conclusion, GGB's dynamic tools directly support the simultaneous emergence of CT components. Findings suggest CT functions not merely as a technical proficiency, but as a multidimensional cognitive tool enhancing PSTs' analytical decision-making and pedagogical practices in non-programming contexts.

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PME 49
Helsinki, Finland
2026

POSTER PRESENTATIONS

CONTEXTUALISING MATHEMATICS THROUGH WATER: DISCOVERING RIVERS AND ARISING ENVIRONMENTAL RESPONSIBILITY

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The research explores how numeracy can be applied in a context of citizen science developing the Sustainable Development Goals 6 and 15, addressing water resource management and biodiversity conservation (Andrà et al., 2024). Citizen science serves a dual purpose, providing both valuable ecological data and fostering environmental learning and civic engagement (O'Donoghue, Taylor & Venter, 2018). Numeracy, defined as the practical application of mathematical knowledge in various real-world contexts, plays a crucial role in citizens' understanding of river-related issues. Following Andrà et al. (2024), the study aims to unravel citizens' perceptions and knowledge regarding rivers and the life inside them, within an interdisciplinary perspective. The research questions are: How does numeracy manifest when individuals are asked to answering questions about rivers? How is it related to knowledge about rivers, even outside mathematics? The research methodology involved administering multiple-choice questions to 90 high school Italian students. The aim of the questions is to uncover students' conceptions about quantities, ratios, and percentages concerning river-related phenomena, and the correspondence with real river data (Roccatello et al., 2025). The development of the questionnaire is discussed, as well as its potential for measuring interdisciplinary knowledge in mathematics, biology and ecology to understand rivers and their biodiversity. The data revealed insights into students' perceptions of river water availability, water quality, and the impact of human activities on river ecosystems. The students were also able to identify invasive species and natural phenomena that concern rivers. Possible relations among different forms of knowledge are explored.

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STUDENTS' COLLECTIVE REASONING WHILE WORKING ON VERTICAL WHITEBOARDS

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Problem solving and reasoning are fundamental components of mathematical competence. Although research on the role of collaboration in problem solving and reasoning is already a well-established field, it has recently gained renewed attention through studies of the teaching approach *thinking classrooms*. In a thinking classroom, students are assigned to random groups of 2-4 students and work on vertical whiteboards with problem-solving tasks that are intended to promote mathematical thinking (Liljedahl, 2016). This poster reports from a video study investigating student reasoning in an upper secondary mathematics classroom. The sample consists of three groups of three students (aged 15–16). We collected video data of the students during their group work. Reasoning can be understood as justifying the validity of statements and arguing to convince oneself or others (Lithner, 2008), using identifying, predicative, verifying, and evaluative arguments. In our study, we analysed the students' reasoning in three stages, identifying (1) what types of arguments they made, (2) how they used the vertical board during the argumentation, and (3) what type of agency they had when constructing the arguments. We distinguished between shared and unshared agency, depending on whether the students were mutually dependent on one another in the reasoning process. All the groups we observed made use of all four types of arguments. They constructed a large proportion of predicative and verifying arguments, while the proportion of identifying and evaluative arguments was smaller. We observed that the students shared individual ideas with each other by writing them on the vertical board and referred to their writings during argumentation, indicating that the boards functioned as support for the collective reasoning. Within the groups, shared and unshared agency appeared in approximately equal amounts during the construction of arguments. The purpose of the poster is to present illustrative examples of how students engage in collective reasoning while working in groups on vertical whiteboards. Our results will be displayed as three doughnut charts, presenting a profile of each group in terms of arguments made, use of the vertical board, and agency.

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BENCHMARKING INSTITUTIONAL SUPPORT FOR UNIVERSAL DESIGN FOR LEARNING IN HIGHER EDUCATION MATHEMATICS

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The 2024 United States (U.S.) Department of Justice update to the Americans with Disabilities Act (ADA) Title II requires public universities to ensure digital accessibility of instructional materials (U.S. Department of Justice, 2024). Universal Design for Learning (UDL) frames accessibility as proactive design rather than reactive accommodation (CAST, 2018). In mathematics, UDL presents distinctive challenges due to symbolic notation, equation editors, and assistive technology compatibility. Accessible participation requires structural support for MathML rendering, equation authoring workflows, and integrated learning management systems. This study benchmarks how land-grant universities are structurally supporting UDL principles in higher education mathematics under the DOJ mandate. The research questions are: (1) How are institutions operationalizing UDL-aligned accessibility in mathematics? (2) What structural supports are provided to faculty? and (3) What patterns of variation emerge? A cross-case benchmarking design was employed. Peer land-grant universities subject to DOJ requirements were selected for comparison. Data sources included publicly available accessibility policies, mathematics-specific guidance, IT accessibility resources, and documented tool recommendations. A structured protocol ensured comparability. A heuristic instrument assessed institutional support across documentation clarity, mathematics-specific workflows, accessible tooling integration, and service infrastructure. Institutions were categorized along a four-level scale from Reactive (ad-hoc compliance) to Managed/Integrated (systematized, proactive support). Cross-case analysis identified implementation patterns. Preliminary heuristic evaluation results indicate substantial variation in institutional alignment with UDL principles. While most institutions publish general accessibility policies, fewer provide mathematics-specific guidance or proactive workflow support. Compliance mandates alone do not ensure equitable mathematical participation; institutional design capacity significantly shapes accessibility enactment.

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EXAMINING SEVENTH-GRADE STUDENTS' PROPORTIONAL REASONING SKILLS USING SCENARIO-BASED DIGITAL GAMES

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Proportional reasoning is a significant and complex domain in school mathematics and plays a critical role in students' mathematical development (Lobato et al., 2010). It permeates a major part of students' mathematical understanding and is related to learning many concepts, such as similarity, scale factor, probability, and instantaneous rate of change (Lobato et al., 2010). Moreover, proportional reasoning extends beyond mathematics, supporting learning across other disciplines and guiding many aspects of everyday life. Despite their importance, students often struggle to understand ratios and proportions, and teachers find teaching these concepts challenging (Lamon, 2007).

Given the potential benefits of digital games for mathematics education, we developed scenario-based digital games (SBDGs) to investigate their effectiveness in supporting ratio and proportion instruction in seventh grade. The study employs a mixed-methods design and includes two groups of seventh-grade students (the experimental and control groups). In the experimental group, instruction is supported by SBDGs, which students play in pairs on tablets for five of the 20 lesson hours, whereas the control group receives only traditional textbook-based instruction. To assess students' learning, we developed a 30-item proportional reasoning test. Additionally, we are collecting qualitative data through screen recordings of students' gameplay on tablets to better understand how SBDGs support the development of students' proportional reasoning, as well as affective, social, and metacognitive skills.

This is an ongoing study. The data collection process began in February 2026 and will conclude in March 2026. The collected quantitative and qualitative data will be analysed, and the results will be presented in detail at the conference.

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AN EXAMINATION OF THE SPECIALISED CONTENT KNOWLEDGE THAT PROSPECTIVE MATHEMATICS TEACHERS BRING FROM HIGH SCHOOL TO THE UNDERGRADUATE LEVEL

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The knowledge and skills required of teachers have been widely conceptualised in the literature (Shulman, 1986; Ball et al., 2008). Ball et al. (2008) emphasised the critical role of specialised content knowledge (SCK) in effective mathematics teaching. SCK refers to mathematical knowledge specific to teaching and does not consist solely of pedagogical knowledge. It involves not only using mathematical concepts correctly but also analysing students' thinking and explaining misconceptions on mathematical grounds. However, mathematical knowledge acquired at the high school level often focuses primarily on procedural performance, rather than on the development of teaching-specific mathematical understanding. This situation necessitates the deliberate and systematic development of specialised content knowledge during the undergraduate education process of prospective teachers.

This study was conducted within the scope of the "Foundations of Mathematics" course in the elementary mathematics teacher education undergraduate program. In line with a pedagogy of teaching, the researcher first aimed to determine the current level of prospective teachers' content knowledge. In this process, the argumentation technique was used. The study aims to examine the specialised content knowledge that prospective mathematics teachers carry from high school to undergraduate education. The researcher assumed the dual role of course instructor and practitioner. The study employed a qualitative research design. Participants consisted of 30 first-year prospective mathematics teachers. Data were collected through classroom video recordings, transcripts, and field notes and analysed using descriptive methods. The findings of the study indicate that prospective teachers possess incomplete and, in some cases, incorrect knowledge in several mathematical content areas they are expected to teach. For example, most participants were unable to explain the mathematical meaning of the process commonly referred to as "bringing down a zero" when extending division with remainders to decimals

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ENHANCING (PRE-SERVICE) TEACHERS' PROFESSIONAL VISION OF MATHEMATICS INSTRUCTIONAL QUALITY

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Professional vision, referring to *noticing* and *knowledge-based reasoning*, plays an important role in (pre-service) teachers' classroom behaviour, and ultimately affects student learning. Specifically, teachers' professional vision of mathematics instructional quality needs to include subject-specific characteristics (e.g., cognitive activation, content selection and presentation, Praetorius & Charalambous, 2018). Developing teachers' professional vision of mathematics instructional quality requires theory-based observation frameworks and their effective use in practice (e.g., Santagata et al., 2021). Beyond the lack of a coherent observation framework, it is also unclear how teachers apply such frameworks when observing classroom instruction.

Therefore, we developed a model of mathematics-specific instructional quality based on $n = 5$ research-based frameworks, resulting in $n = 6$ criteria for *cognitive activation* and $n = 9$ criteria for *content selection and presentation*. Each criterion comprises four specific and concrete level descriptions to provide a criterial reference for evaluating classroom instruction. Exemplary classroom situations from the existing frameworks further illustrate each level. We report on the development process of the model and present its structure and exemplary criteria. Based on a qualitative Think Aloud study with $n = 8$ pre-service teachers, we describe strategies when applying the criteria, as well as the affordances and challenges introduced by the criteria descriptions.

The think-aloud interview results show that the explicit criteria descriptions support pre-service teachers' knowledge-based reasoning, enabling them to go beyond solely describing the observed situations. However, pre-service teachers mostly focused on teachers' actions in their explanations, with limited attention to the mathematical content or to the effects of these actions on learning processes. While the results speak to the general feasibility of the approach for generating a model of mathematics instructional quality, future work needs to explore which teacher characteristics, such as content-related knowledge and external supports, can facilitate its application.

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INVESTIGATING THE EFFICACY OF PROFESSIONAL DEVELOPMENT FOR MATHEMATICS GRADUATE STUDENTS

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Through a longitudinal study of mathematics graduate students' (MGS) development as teachers during their graduate programs (Beisiegel et al., 2019), we found that MGSs had little time or support to reflect on and understand teaching practices and they most often replicated the lecture-based instruction they had experienced frequently in their undergrade education. Additionally, we found that after two to three years of experience as a teaching assistant, they were ready to think more deeply about engaging students in active learning and to consider teaching practices that are proven to improve students' learning outcomes and sense of belonging in mathematics. To accommodate the learning-about-teaching stages we observed in the MGSs, we designed a three-year Engaged Learning and Innovative Teaching Experiences (ELITE) PD program that offered weekly graduate seminars during which MGSs were presented with increasingly complex evidence-based teaching practices (e.g., from using exit cards at the end of class to gauge student understanding of a newly-learned concept to having students give class presentations about their understanding of a concept). Our research questions include: (1) What impact does the MGSs' adoption of ELITE PD practices have on student outcomes (e.g., experiences, belonging)? (2) What identifiable aspects of individual and department cultures inhibit or support sustainable change? We are investigating individual change through Azjen's (1991) Theory of Planned Behaviour and institutional change through Bolman and Deal's (2008) Four Frames Model. To study the efficacy of the ELITE PD program, we collected multiple sources of data: (1) interviews with MGSs and faculty leadership, (2) observations of MGSs' teaching and post-observation debriefs, (3) surveys from the MGSs' students, (4) artifacts from the PD program. In our poster, we aim to convey the interwoven nature of our findings across individual and institutional contexts.

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GEOMETRY TEACHING IN A RURAL SCHOOL: TENSIONS BETWEEN POPULAR AND ACADEMIC MATHEMATICS

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Mathematics teaching in rural contexts often reveals disconnects between students' lived knowledge and formal schooling. Knijnik (2006) distinguishes "popular mathematics" mobilized by rural workers from "academic mathematics" transmitted through formal education. Skovsmose (2001) proposes "landscapes of investigation" engaging students' lived experiences through open exploration. This study examines how these tensions manifest in geometry instruction at a rural school in southern Brazil, asking: *How do tensions between popular and academic mathematics manifest in geometry instruction?*

Through an ethnographic study (February-July 2023) involving 66 secondary students and eight teachers, data from field diaries, interviews, and photographs were analyzed using content analysis. Outdoor activities created landscapes of investigation where students explored geometric concepts through measurement: sketching circles with varying radii to investigate area relationships and estimating arc length using diverse strategies. These activities showcased students' advanced spatial reasoning, grounded in agricultural practices. One student discussed the spacing requirements for cassava (1.5 meters to support root growth), while another explained how diagonal soybean harvesting helps protect combine equipment. However, when moving indoors, instruction reverted to academic mathematics, emphasizing standard formulas and leaving students unable to bridge their practical reasoning with formal representations. Gravemeijer and Kindt (2001) observe that this tension exists between the mathematising practices from everyday, culturally rooted activities and the formal mathematics established and endorsed by school curricula. The findings suggest that pedagogical approaches that explicitly articulate relationships among these knowledge forms can legitimize students' existing understanding while providing access to conventional mathematical tools. Future research should explore micro-to-macro curricular approaches that support both knowledge systems.

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FOUNDATIONS FOR DISCERNING NUMBERS' PART-WHOLE RELATIONS IN A PRESCHOOL INTERVENTION STUDY

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Four Swedish preschool groups participated in an 8 months long intervention, implementing numerical activities where numbers were presented and handled in ways that make units discernible and emphasising numbers' part-whole relationships in arithmetic problem solving (see Venkat et al., 2019). This approach has in previous research shown positive impact on children's developing number sense and emerging arithmetic skills. However, some children do not benefit from taking part in the interventions. There are likely more foundational aspects, for learning to discern the part-whole relationship of numbers. The current study specifies the necessary baseline, in terms of discerned aspects of numbers, for the intervention to have expected outcome. The specific research question is: What aspects are critical as a baseline for learning to discern numbers' part-whole relationships?

61 children participated in task-based interviews pre and post intervention (2:9–5:8 years at first interview). Responses were coded qualitatively based on identified (un)discerned aspects (cardinality, ordinality, part-whole relations) in the children's verbal expressions and finger use. Results are presented in a crosstabulation, comparing (un)discerned aspects of numbers expressed pre and post intervention.

Two conclusions are drawn from the analysis: If ordinality and cardinality are discerned as differentiated, children will more often benefit from the intervention, and, children who have not discerned ordinality or cardinality are less likely to take advantage of the teaching intervention. Across tasks, 56-60% of the children demonstrated partial numerical discernment in the pre-test (cardinality and/or ordinality). Of these, 35-60% were coordinating cardinality and ordinality in the post-test, expressing an advanced structural approach to the tasks. 52% of the children did not receive the tasks or answered non-numerically in the pre-test; 80-93% of them were able to engage with the task after the intervention, however mostly discerning ordinality and/or cardinality as differentiated. The results indicate that the intervention broadened participation and the opportunity to experience tasks numerically, providing the field of early mathematics with knowledge how to design teaching interventions that benefit *all* children.

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A COMPLEX GENDERED PHENOMENON IN GRADE 1: MATHEMATICAL AFFECT AND ASSESSMENT

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In France, since their introduction in 2018, national assessments reveal (NA) an early mathematics performance gap disadvantaging girls within the first four months of compulsory schooling. However, existing studies mainly focus on performance or/and the threat of gender stereotypes and rarely consider pupils' activity in assessment situations (AS) together with their mathematical affect (MA).

This study is part of a broader empirical research grounded in a didactical analysis structured through an approach centred on MA. Two previous case studies were conducted within the problem-solving domain: one on pupils' writing strategies in NA items, and one on learning regulations in informal AS just after NA period. The analysis results showed differentiation effects according to gender due to implicit mechanisms in the didactical assessment contract. The research question is: are these differentiation effects perceptible in Grade 1 pupils' verbalizations of mathematical affect ?

The data consist of semi-structured interviews conducted with 34 Grade 1 pupils (gender-balanced) across 7 classes, following informal AS. Pupils were selected by their teacher to represent a range of mathematical competence levels in their classes. The analysis relies on an analytical grid adapted from Genoud and Guillod (2014), articulated with DeBellis and Goldin's (2006) model of MA. It combines binary and qualitative coding with exploratory quantitative analyses.

The analysis results show no substantial gender differences, except a tendency for boys to associate success with speed ($r = .01$, Cramér's $V = .61$). However, more refined analyses reveal effects related to teacher practices, strong co-occurrences between affective variables and differentiated patterns of verbalization. Gender bias don't appear as a primary structuring factor of MA at this stage; it is mainly shaped by assessment practices and individual factors. However, an early gender differentiation in interpretation of performance norms, may contribute over time to the construction of gendered relationships to mathematical activities in AS.

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STUDENTS' STRATEGIES WITH UNKNOWNS IN DIGITAL BALANCE TASKS: INSIGHTS FROM CONTRASTING CASES

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This study investigates primary school students' problem-solving processes involving unknowns in balance-scale tasks in a digital environment. Early algebra research highlights relational understanding of equality as an extension of operational thinking when dealing with unknowns (Knuth et al., 2006). While balance models can support such thinking (e.g., Warren & Cooper, 2005), less is known about maintaining equivalence with multiple unknowns in digital environments. Although such reasoning underpins early algebraic thinking, it is rarely foregrounded in primary mathematics. The present study addresses the following research question: *How do primary school students maintain equivalence in digital balance-scale tasks with unknowns, and what strategies can be identified in their solution processes?* Students' interactions, strategies, and reasoning (N = 22, Grade 4) were analysed using qualitative content analysis of screen recordings and task-based interviews. The tasks featured two given balances representing equivalence relations with unknown quantities and a third to complete equivalently (Bräuer, 2022). The digital environment allowed students to manipulate the balance without automated feedback. Two contrasting cases show different approaches to reasoning about unknowns and equality in the same task: Eva transforms both sides and substitutes, enabling relational reasoning aligned with previous research on the balance model; in contrast, Claire mainly displays possible results and estimates isolated elements, reflecting operational thinking. These cases illustrate that successful problem solving is not determined by the frequency of digital interaction, but by coordinating actions with equivalence strategies to support relational reasoning. Findings reveal (a) how maintaining equivalence becomes visible in students' interactions, and (b) identification of and diverse strategies for dealing with unknowns in digital balance tasks beyond final answers.

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EVALUATING A DOMAIN-SPECIFIC AI MICROCREDENTIAL IN MATHEMATICS TEACHER EDUCATION: A PRE–POST STUDY ON BELIEF CHANGE

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The integration of generative AI into mathematics education requires a profession-specific AI literacy that combines epistemic awareness, didactic judgment, and ethical reflection. Building on the distinction between generic, ethical, and domain-specific AI literacy (Knoth et al., 2024), we conceptualize AI competence for mathematics (pre-service) teachers as a subject-anchored professional capacity aligned with the discipline's norms of rigor, justification, and conceptual understanding. In response, the University of Hamburg implemented the ECTS-based microcredential "AI Navigator" in the teacher training programs, which embeds AI-related use cases into mathematics education modules, integrates structured ethical reflection, and concludes with a performance-based digital workpiece. The certificate thus represents a curricular intervention aimed at professionalizing AI use in mathematics instruction. The ongoing study examines whether participation leads to measurable changes in pre-service teachers' AI-related beliefs. Based on prior survey research on beliefs and self-efficacy in AI-supported in mathematics education (Buchholtz & Schorcht, 2025), we apply a validated 87-item instrument in a longitudinal pre–post design. Data are collected at entry (T1) and completion (T2) of the certificate; a matched comparison group is planned. We hypothesize that the certificate fosters not merely increased acceptance of AI, but more differentiated and professionally grounded belief structures, characterized by strengthened domain-specific self-efficacy, heightened awareness of pedagogical potentials, and sustained sensitivity to epistemic and ethical risks. The study contributes empirical evidence on the impact of a structurally anchored microcredential on the development of domain-specific AI literacy in mathematics teacher education.

Additional information

Funded as part of the ReAction (Relationships and Academic belonging in Higher Education) project by the German Foundation for Innovation in Higher Education.

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DEEPENING MATHEMATICAL DISCUSSIONS THROUGH SMALL-GROUP SCAFFOLDING AND NOTICING

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The poster presents a professional development program based on three interrelated theoretical frameworks that integrate (1) the Small-Group Scaffolding Tool (SGS Tool), (2) teacher noticing, and (3) classroom discussions around rich mathematical tasks. The SGS Tool was developed because teachers find guiding small groups challenging (Calor et al., 2022). The tool consists of six steps that help teachers determine timing and manner of providing support that meets the group's needs and purposely stimulates mathematical discussions (Calor et al., 2022). Teacher noticing comprises three core components: noticing students' mathematical thinking, interpreting this thinking, and deciding how to respond in ways that promote student learning (Rotem & Ayalon, 2022). Strategies for classroom discussions, focus on students' diverse solution strategies and thinking processes (Kooloos et al., 2023).

This research follows a design research approach that aims to adapt the SGS Tool from a theoretical and practical perspective. After introducing the theoretical frameworks and a short training how to apply the SGS Tool, teachers contribute classroom situations that are analyzed using the SGS Tool and the teacher noticing framework. Seven secondary school mathematics teachers (7^{th} grade-12^{th} grade), all familiar with the classroom discussions framework, participated in this study. Video recordings of guiding group discussions and teacher reflections were collected.

Preliminary results show that teachers find all steps of the SGS Tool necessary, that noticing should be part of all these steps and that an additional step could be added to the SGS Tool: step 7 'listening, noticing and leaving', which entails listening to the group discussion, noticing and deciding whether to stay or leave the group. We propose, also from a theoretical perspective, to add this seventh step to the SGS Tool.

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OBSTACLES AND SOLUTIONS IN SOLVING ARITHMETIC PROBLEMS THROUGH GRAPHIC MODELS IN THE PRIMARY EDUCATION

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Problem solving is a powerful and effective vehicle for learning (Van de Walle, 2011). Zlate (2006) emphasizes that problem solving ranks first in cognitive psychology, by involving various reasoning *processes* and *mechanisms.* The mental representation of the problem and the choice of the solving method are crucial for the successful development and completion of the process. Moving from the text of the problem to the mathematical model involves a modelling effort: building a system of representation that explains, "formalizes" various particular objects (Raynal &Rieunier, 2005), developing a formal representation, or even a support for understanding the mathematical structure of a problem and for justifying the arithmetic operations. The graphic model, among the first models studied in elementary school, operates with graphic language, on different degrees of abstraction. *The objective* of the research was to implement and evaluate a two-stage strategy for learning problem solving through the graphic models based on using straight line segments: 1. Familiarizing students with graphic language; 2. Using graphic language to solve problems through the graphic model.

The research method is mixed and consist of a didactic experiment alongside observation of the problem-solving behaviour of students. The experimental activity was performed on 2 classes of students (experimental class and control class), suming up 46 students aged 8-9. The pre-test and post-test comparative assessments (28% increase in the performance average of the experimental class) and the observations noted (significant decrease in the incidence of bottlenecks caused by graphic modeling in problem solving) confirm *the hypotesis of research*: separating the learning of graphic language from the problem-solving process enhances students' ability to solve mathematical problems through graphic modeling. With the enrichment of the instrumental repertoire for solving mathematical problems, students' problem-solving behavior also develops.

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INTEGRATING GENERATIVE AI AND DISCOURSE-BASED ASSESSMENT TO PROMOTE SELF-REGULATED LEARNING IN ALGEBRA

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Generative AI (GenAI) can personalize mathematics learning, yet current research confounds "media" with "method" (Clark, 1983). Studies treat the AI tool as an intervention without specifying pedagogy, leaving causal mechanisms unclear (Weidlich et al., 2025). In the absence of intentional instructional design, GenAI may lead to cognitive offloading and reduced metacognitive engagement. Thus, this study adopted Discourse-based Assessment Practice (DAP), a method of extended questioning and adaptive feedback (Chen et al., 2012), as the instructional guideline and GenAI as the delivery medium. We examined how a DAP-integrated GenAI application scaffolded seventh-graders' self-regulated learning (SRL) in algebra.

The application runs on a Large Language Model API with Retrieval-Augmented Generation (RAG) to reduce hallucinations. Following DAP principles, the AI identified algebraic misconceptions and provided graduated scaffolding, from metacognitive questioning to strategic hints, fading support as student mastery grew. A pilot study with students at different achievement levels assessed usability and initial effectiveness. We collected system logs, interaction records, and qualitative observations to refine the feedback cycle. Preliminary observations suggested that when DAP serves as the explicit method within the AI medium, cognitive offloading is reduced. Low-achieving students engaged more willingly in iterative problem-solving; high-achieving students used the system to verify solutions and detect errors. By clarifying the tool-strategy distinction, this study informs GenAI environment design. Future research will enlarge the sample and apply epistemic network analysis to trace student-AI interaction patterns.

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LLM-SUPPORTED FORMATIVE ASSESSMENT FOR WHOLE-CLASS DISCUSSION IN INQUIRY-BASED LESSONS

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Large language models (LLMs) are increasingly studied in mathematics education, for example, to generate feedback on students' work or to support teachers in designing tasks. Recent studies report that their outputs can be convincing but wrong or inconsistent, and may add to teachers' workload. These findings suggest that teachers need to validate LLM outputs and often adapt them before they can be used for formative assessment. Currently, little is known about how this validation and adaptation takes place when mathematics teachers work with students' live digital contributions (e.g., posts on Padlet, an online sharing platform) and prepare for whole-class discussion under time pressure in real lessons.

Our research question is: How do mathematics teachers check and adapt suggestions from an LLM-based assistant (LLMa) when real-time evaluating students' Padlet posts for orchestrating whole-class discussions? To frame teachers' checking and adaptation of LLMa outputs, this study draws on the Ask-Answer-Analyse-Adapt cycle for formative assessment with technology (Dalby & Swan, 2019) and the theory of instrumental orchestration, which concerns how teachers arrange digital tools in lessons (Drijvers et al., 2010). In a design-based study with 4-6 secondary mathematics teachers, we will use LLMa as a teacher-facing assistant built on a large language model. Teachers will use the LLMa to cluster Padlet posts and generate short summaries and suggestions for whole-class discussion. Data includes Padlet posts, LLMa outputs, lesson audio of whole-class discussion, and post-lesson interviews. Data will be thematically coded for teachers' LLMa-mediated orchestrations.

We expect that LLMa-supported analysis of Padlet posts will help teachers under time pressure to carry out formative assessment: to gain a quick overview of students' thinking and to identify a small set of posts for whole-class discussion. We also expect teachers to organise this feedback by evaluating the LLMa's suggestions in relation to their lesson goals and deciding what to keep, modify, or reject; the study will clarify how LLMa can support teachers' adaptive use of formative assessment evidence.

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EXPLORING PRIMARY PRE-SERVICE TEACHERS' EVALUATION OF GENAI-GENERATED MATHEMATICAL MODELLING TASKS

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The ethical and pedagogically sound use of generative artificial intelligence (GenAI) has become a central issue in teacher education. Pre-service teachers' (PSTs) ability to evaluate mathematical tasks is a key component of teacher preparation (Bardy et al., 2024); integrating GenAI meaningfully into this process is especially important. Recent studies have begun investigating GenAI's potential to evaluate and modify mathematical tasks (Schorcht et al., 2025). Yet, how PSTs interpret and validate GenAI-generated tasks specifically for mathematical modelling remains an underexplored dimension of teacher preparation. Accordingly, this research examines the question: How do primary pre-service teachers evaluate mathematical modelling tasks generated by GenAI? This action research adopted the task-selection processes (Bardy et al., 2024) and the features of mathematical modelling tasks Geiger et al. (2022). as the research framework. The study was conducted in a primary mathematics education course. PSTs participated in four lectures introducing mathematical modelling and task design. Forty-one written assignments, including screenshots of GenAI interactions and written evaluations, were collected for content analysis. The initial findings show that nearly half of the PSTs (22/41) provided independent evaluations. However, their reasoning was largely descriptive and focused on surface-level mathematical modelling task features. The remaining PSTs relied on GenAI's evaluation without offering their own reasoning. While GenAI supports task design, it risks shaping PSTs' judgment toward superficial reasoning. This underscores the need for teacher education to move PSTs from passive users to critical evaluators of GenAI-generated content.

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MONEY WITHOUT CASH? A MATHEMATICS EDUCATION PERSPECTIVE

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CHANGING MONETARY PRACTICES

In many countries, everyday monetary practices are undergoing rapid transformation due to the declining use of cash and the increasing prevalence of digital payment systems. For learners, money is no longer primarily encountered as a tangible medium but as symbolic, often invisible, numerical information on screens and cards. From a mathematics education perspective, this shift raises questions about how learners *sense* and *make sense of* monetary quantities, value, and transactions. While traditional school mathematics has long relied on cash-based contexts (coins, notes, making changes), these contexts may no longer adequately reflect students' lived experiences.

DIDACTICAL IMPLICATIONS

The reduced materiality of money changes the epistemic and cognitive conditions under which learners develop quantitative and financial understandings. Cash-based situations typically support embodied and perceptual access to magnitude and equivalence, whereas digital payments foreground symbolic representations and algorithmic processes. This development may both open and constrain learning opportunities. On the one hand, digital contexts can foster work with large datasets, functional dependencies, and modelling situations. On the other hand, the loss of sensory grounding may weaken intuitive understandings of value and proportionality. From a modelling perspective, money provides a rich domain for linking arithmetic, functional thinking, and economic reasoning.

QUESTIONS FOR MATHEMATICS EDUCATION

The poster discusses how mathematics education can respond to changing monetary practices. It argues for a reflective integration of cash and cashless contexts that supports both experiential grounding and formal reasoning. Particular attention is given to implications for modelling tasks and for fostering students' critical understanding of quantitative information in digital financial environments.

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HYBRID COLLABORATION DYNAMICS IN THE CO-DESIGN OF A MATHEMATICAL MODELLING ACTIVITY-SEQUENCE

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Interest in teacher collaboration and task design is growing internationally in mathematics education, yet hybrid environments remain understudied (Marshman & Goos, 2025). One barrier to effective teaching of mathematical modelling (MM) in compulsory education is the limited availability of teacher-usable tasks (Borromeo Ferri, 2018), making collaboration in the design and adaptation of MM proposals relevant. Within this framework, four Argentine teacher educators collaborated to design an MM activity-sequence for a hybrid initial teacher education group. The collaboration combined face-to-face and digitally mediated interactions. The study examines the dynamics of collaborative work during the design process. This qualitative case study analyses how hybrid interactions shaped participation, negotiation, and decision-making among the teacher educators (Wegerif, 2020).

Our results show that the collaborative work was configured as a dynamic process of positioning and repositioning around MM. What began as an apparent consensus, sustained by a shared repertoire regarding MM, became strained when specific institutional conditions became visible (for example, a format alternating face-to-face and virtual weeks). This situation reopened negotiation about what it means to design "authentic" tasks under constraints of time, curricular legitimacy, and limited interaction. This tension (authenticity–time–hybrid conditions) was not resolved; rather, it became a productive focus that restructured roles, reoriented decisions, and was materialised in the resulting MM activity-sequence, organised into five "moments" articulating reflection, extended MM work, and conceptualisation through a combination of face-to-face formats and digital mediations. Overall, the study contributes to research on collaboration among teacher educators by showing how hybrid environments configure collaborative dynamics in the design of an MM activity-sequence and by offering insights for future implementations of MM.

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MATH LANGUAGE WHEN SPEAKING A DIFFERENT HOME LANGUAGE: AN EXPLORATORY STUDY IN 2ND GRADERS

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Understanding mathematical texts, instruction and mathematical exercises requires both general language abilities and math language abilities (i.e., the content-specific language used in mathematics, such as the words *double*, *majority*; Bailey & Wilkinson, 2022; De Keersmaeker et al., 2025). Research shows that both general and math language explain individual differences in children's math performance, with math language being more influential (De Keersmaeker et al., 2025). Despite the growing linguistical diversity in western classrooms, research comparing children's math language in their home language (HL) and instructional language (IL) is scarce. Therefore, we conducted an exploratory study examining (1) differences in math language across IL and HL, and (2) relations with math. 28 second graders (10 boys, 18 girls, mean age: 7 years 11 months) participated in this exploratory study. They attended school in Flanders, Belgium (IL = Dutch) and spoke Turkish at home. Children completed a math language task (in IL and HL), general language task (in IL), and a math task (in IL).

A paired samples t-test and binominal GLMMs suggest that children's math language abilities are comparable across languages, as children showed similar performance in accuracy and error patterns. Next, children were grouped into three clusters (low, average, high) based on their math language scores in both languages. General language did not differ between groups, and math language was comparable across languages, within groups. Across groups, higher math language was related to higher math abilities, implying that a higher math language ability in any language (even not the instructional one) may act as a protective factor.

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THE REFLECTIVE TEACHER IN THE DIGITAL MATHEMATICS CLASSROOM: EMOTIONS, BELIEFS, AND SELF-EFFICACY IN PRIMARY EDUCATION

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Recent research increasingly recognizes that teacher affect and beliefs profoundly influence instructional practices (e.g. Di Martino, 2011), particularly when integrating digital technologies in mathematics education. While Leijen et al. (2023) demonstrate that teacher beliefs significantly mediate students' opportunities for agency in primary education, there remains limited understanding of how affect and beliefs on one side, and self-efficacy and reflective processes on the other interact to shape teachers' digital mathematics practices. Drawing from established frameworks including Bandura's self-efficacy theory, Pekrun's control-value theory of emotions, and the TPACK model, this study addresses this crucial gap by investigating the multidimensional nature of teacher orientations toward technology integration.

The research aims to identify digital-pedagogical profiles that emerge from these interactions, and explore how reflective processes mediate these orientations. We employ an explanatory mixed-methods design structured in two stages. Phase 1 involves administering a comprehensive questionnaire to 300 primary teachers measuring emotions, self-efficacy, epistemological beliefs, and orientations toward digital mathematics teaching. Phase 2 consists of explicitation interviews (Vermersch, 1994) with approximately 20 teachers selected based on emerging profiles from Phase 1 to explore teachers' experiential and reflective processes in greater depth. Data analysis combines quantitative cluster analysis with qualitative thematic coding to identify teacher profiles. Based on existing literature and preliminary theoretical triangulation, we anticipate the emergence of different profiles that we expect to reveal significant correlations between emotional regulation, reflective capacity, and effective technology orchestration. In the presentation, further results will be discussed in detail.

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STUDENTS' SCEPTICAL CONCEPTIONS OF GENERATIVE ARTIFICIAL INTELLIGENCE IN MATHEMATICS EDUCATION

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Mathematics education research has shown growing interest in the integration of generative artificial intelligence (GenAI) tools in teaching and learning contexts (Pepin et al., 2025). Blahopoulou and Ortiz-Bonnin (2025) focused on undergraduate students who choose not to use GenAI, showing how these students can shed light on possible risks and misuse of these tools. Building on this perspective, this study focuses on secondary school students who limit or avoid the use of GenAI in mathematics learning. The study addresses the following research question: What conceptions do secondary school students who are sceptical about GenAI have regarding its use in mathematics learning? The construct of conceptions is understood as encompassing beliefs, meanings, and mental representations (Philipp, 2007).

To that end, 36 students' short written essays on GenAI in mathematics learning were collected and analysed: 19 from non-users, 8 former users, and 9 rare users. The categorical analysis was then conducted to investigate students' conceptions. In particular, several recurrent conceptions emerge: (1) GenAI may reduce autonomy and mathematical reasoning; (2) GenAI can support learning and (3) may reduce motivation and interest in mathematics; (4) GenAI may lead to reduced understanding and (5) may reduce anxiety and increase accessibility of mathematics; (6) GenAI may be unnecessary or distracting for learning; and (7) GenAI may be unreliable and misaligned with school mathematics. These findings align with prior research on the potential of GenAI to support learning and on concerns about overreliance, reasoning, and reliability (Pepin et al., 2025; Blahopoulou & Ortiz-Bonnin, 2025). Moreover, the study highlights how, in mathematics learning and among non-users, students' conceptions also involve aspects such as motivation, anxiety, and accessibility.

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ATTITUDES AS AN OUTCOME OF BLENDED MODELLING INSTRUCTION IN PRIMARY TEACHER EDUCATION

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This poster reports on changes in primary mathematics student teachers' attitudes towards mathematical modelling following an independence-oriented modelling intervention implemented in a blended learning environment. It also visually presents the instructional design of a ten-lesson modelling unit (five face-to-face and five online lessons) on fractions, developed within the project *Studies in Teaching Approaches for Mathematical Modelling at Primary Level (STAMM@prim)*. Instructional design and affective factors are central to modelling learning. Carefully structured interventions that combine instructional guidance with opportunities for autonomous construction can foster learner independence and support substantial competency gains (e.g., Lindl et al., 2025). In modelling contexts, attitudes are especially salient because tasks are open-ended, cognitively demanding, and situated in real-world contexts. Such characteristics can both challenge and strengthen learners' motivational beliefs and emotional responses (Di Martino, 2019). Using a pre–post design, quantitative data were collected from 34 student teachers in 2025 with the adapted Survey of Attitudes Towards Mathematical Modelling (SATMM). The instrument comprises 36 items across six dimensions (*effort*, *interest*, *difficulty*, *value*, *cognitive competence*, and *affect*) measured on a seven-point Likert scale. Paired-samples *t*-tests revealed significant gains in *effort* ($t(33) = 5.26$, $p < .001$, $d_z = 0.90$, large) and in *interest*, *value*, *cognitive competence*, and *affect* ($t = 3.37$–3.60, $p \leq .002$, $d_z = 0.58$–0.63, medium). Perceived *difficulty* did not change significantly ($t(33) = 1.04$, $p = .308$), indicating that while tasks remained challenging, participants reported increased motivation, confidence, and positive emotional engagement. Overall, the findings suggest that a well-designed, independence-oriented modelling intervention in a blended environment can strengthen key affective and motivational dimensions without reducing cognitive demand. For PME research, the study highlights the importance of considering attitudes as an integral component of modelling competence and as a meaningful outcome of instructional design.

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UNDERGRADUATE STUDENTS' MATHEMATICAL-DIGITAL COMPETENCY IN SOLVING SOCIO-ECOLOGICAL CONTEXTUAL PROBLEMS: A CASE FROM INDONESIA

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Introduction and study goal. Learning mathematics involves applying mathematical knowledge to real-world problems. The use of digital tools to explore socio-ecological issues is increasing. Indeed, when students use digital tools to solve these contextual problems, they activate mathematical modeling processes to understand and change the world (Coles et al., 2024; Geraniou & Jankvist, 2019). However, many studies treat students' mathematics and digital skills as separate areas. Hence, this study examines how university students combine mathematical and digital competencies to solve a socio-ecological problem.

Methods. The study used a qualitative approach. We collected data from task-based activities with thirty-five second-year undergraduate students (ages 19-20), all of whom had taken a geometry course. The task focused on water scarcity to spark socio-ecological engagement in mathematics. Students completed the task in GeoGebra Classroom with a 30-minute time limit.

Preliminary results. The initial analysis shows that students' mathematical and digital competency in solving socio-ecological problems can be described in five stages. First, the socio-ecological conceptual and contextualization stage is about understanding the narrative and translating real-world constraints into an abstract mathematical statement. Second, the digital hypothesis stage involves forming a mathematical hypothesis and choosing a digital tool strategy. Third, digital formulation is building and executing the model in a digital environment. Fourth, reasoning and iteration use the digital tool's features to validate and adjust the mathematical output. Fifth, socio-ecological encoding and communication translate the validated mathematical results back into a meaningful, contextual conclusion.

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FROM PAPER TO DIGITAL: STUDENT PERFORMANCE IN THE KANGAROO OF MATHEMATICS COMPETITION

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The "Kangaroo of Mathematics" is the world's largest school mathematics competition, engaging over six million students annually with short, timed multiple-choice problems that emphasize inventive reasoning over routine classroom procedures. In recent years, the digital transformation of such competitions has been discussed both as an opportunity and a risk for a more inclusive and sustainable world. While digitalization is often associated with resource conservation, researchers like Shengjergji and colleagues (2024) highlight possible "hidden costs" and environmental impact of digitalisation, suggesting that the ecological balance of digital learning is complex. Furthermore, moving a competition from paper to screen is not a neutral technical change. While Kenderov (2022) demonstrates a positive example with the Bulgarian "Theme of the Month" contest—where students use dynamic geometry software like GeoGebra to explore mathematical ideas—the direct transfer of the Kangaroo contest reveals different results.

Based on the 2023–2025 Austrian Kangaroo cohort (94,750 students; 160,360 entries), we analyzed results from a digital replica of the paper format, which reached ~10% online participation by 2025. First findings indicate a significant "digital gap": students participating online achieved consistently lower scores compared to their peers working on paper. Specifically, the analysis shows a stable performance drop of 3.00 percentage points, which corresponds to a relative loss of nearly 8% in achievement. This disadvantage is educationally non-trivial, as it nullifies around 50% of the average annual learning progress typically observed between consecutive grade levels. These results suggest that unless tasks are specifically designed for digital exploration—as seen in the Bulgarian model—the mere digitalization of traditional formats may introduce systemic frictions that hinder mathematical performance.

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EXPANDING THE SCOPE OF AN INSTRUMENT TO SELECT SOCIAL MEDIA VIGNETTES FOR MATHEMATICS EDUCATION

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The ability to select (digital) learning resources represents a fundamental component of digital competence, which must be cultivated among pre-service mathematics teachers. Evaluating the effectiveness of its development within teacher education requires objective, reliable, and valid assessment instruments. Gonscherowski (2026) devised such an instrument and subsequently validated it using GeoGebra applets. The concise four-item instrument requires participants to describe the learning content of a GeoGebra applet (item 1), select the learner's age and specific educational needs (items 2-3), and provide a rationale for or against its selection in relation to the latter (item 4). The open- and closed-ended items are scored on a scale from 1 to 7. Given the widespread use of social media also for disseminating video vignettes for learning mathematics (Engelbrecht et al., 2025), the present study aims to determine whether the developed four-item instrument can also be used to assess the ability to select such vignettes. Additionally, the study examines how a sample of pre-service teachers (n=16), participating in a mathematics lecture during the winter of 2025/2026, reasoned for or against the selection of such vignettes. The instrument was utilized in a voluntary assessment that included a video vignette, a GeoGebra applet, and a non-digital teaching method for the Pythagorean theorem. The same coding procedures outlined by Gonscherowski (2026) were applied, which demonstrated that identical coding and scoring could be used for the video vignette. A notable result was that the predominant reason for selecting the video vignette was a single factor: the demonstration of the Pythagorean theorem's application in a real-world context. In contrast, reasoning related to the GeoGebra applet exhibited greater variability, encompassing factors such as reduced cognitive load and increased motivation. Convrsely, reasoning about the non-digital method for teaching the Pythagorean theorem was less specific or absent. These findings are constrained by the small sample size and the evaluation of digital resources for only one learning content. Nevertheless, they underscore that the sample of pre-service teachers placed significant importance on the real-world context of learning materials. This insight should be considered when developing the facet of digital competence and selecting digital learning resources.

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MATHEMATICAL FLEXIBILITY OF LITHUANIAN PUPILS

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Mathematical understanding is essential for personal and societal success, yet many Lithuanian pupils underperform in mathematics, raising concerns about teacher quality and preparation. In the age of digitalization and artificial intelligence, questions arise about whether the mathematics taught offers a solid foundation for developing essential mathematics-related skills. In the long term, this also affects the country's capacity for high-tech and post-industrial development. Mathematical flexibility supports problem-solving, reasoning, conceptual understanding, and adaptability (Star et al., 2022). Evaluating whether Lithuanian pupils possess these skills is crucial, as this has not been studied in the Lithuanian context. The research questions are: (1) how mathematical flexibility vary by mathematical achievements, (2) what flexibility profiles emerge from the data.

The sample consists of nearly 1,000 Lithuanian pupils, divided into two groups according to grade level: secondary and upper secondary education. The intervention followed procedures used in previous studies (e.g., Star et al., 2022; Xu et al., 2017) and was implemented in collaboration with the pupils' teachers. Pupils were given the same linear equation tasks on two occasions and were asked to solve them using different methods, then asked to evaluate with is more efficient.

Several distinct profiles emerged from the data, including one profile of students who experienced significant difficulties in solving linear equations, particularly more challenging ones, and a second profile of students who were able to solve the equations but demonstrated no flexibility, as they did not present any alternative strategies. These pupil profiles provide a more nuanced understanding of the Lithuanian student population.

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USING FAIRY TALES TO BREAK ALGORITHMIC FIXITY IN PROBLEM POSING

Gülnihal Güven, Çiğdem Arslan

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Mathematical problem posing is a fundamental cognitive activity that fosters creativity (Silver, 1994); however, pre-service teachers often default to routine patterns due to algorithmic fixity. This study investigates the potential of children's literature, particularly universal fairy tales, to overcome this constraint. Fairy tales provide a familiar yet flexible context, enabling participants to concretize abstract concepts beyond standard procedures (Zazkis & Liljedahl, 2009). The research analyzes how such narrative contexts shape pre-service teachers' cognitive integration and affective responses during the problem-posing process.

This study is a case study conducted with 25 pre-service mathematics teachers. Data were collected through an individual problem-posing task supported by visual cards depicting *Pinocchio*, *Cinderella*, *Snow White*, and *Rapunzel*, along with learning journals. Participants were asked to construct mathematical problems based on these universal narrative contexts. The 25 written problems were analyzed using qualitative content analysis to identify how fairy tale elements were connected to mathematical concepts.

The findings indicate that the participants transformed fairy tales into mathematical variables and structures. In the *Rapunzel* context, the desire to escape the tower was modeled through hair growth percentages and the Pythagorean theorem based on the tower's structure. In *Pinocchio*, the concepts of "lie" and "truth" were mathematized without detaching them from their ethical dimensions, through the Fibonacci sequence and "good deeds" designed to neutralize the effects of lying. In *Cinderella*, the pressure of midnight evolved into dynamic problems involving speed–time calculations and step combinations. In *Snow White*, the theme of sharing was represented through the division of gold among the seven dwarfs using prime relationships or the greatest common divisor, while the poisoned apple was incorporated into survival probability scenarios.

Endnote: This study is a part of the author's ongoing master's thesis.

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VIRTUES AS PREDICTORS OF INTRINSIC MATHEMATICS MOTIVATION IN YOUTH

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Virtues are positive qualities of character commonly studied in positive psychology. In the present work, character strengths and virtues are referred to collectively as virtues, reflecting the traditional meaning of virtue. The aim of the study was to analyze how virtues predict intrinsic motivation in mathematics in youth. Compassion, leadership, and patience were identified as core virtue dimensions in youth aged about 10–15 years and operationalized in an earlier stage of the study. Although there is much prior research about virtues and intrinsic motivation separately, research combining virtues with intrinsic mathematics motivation is limited.

In self-determination theory, intrinsic motivation means interest and enjoyment in doing something for the sake of the activity itself (Ryan & Deci, 2000). We operationalized intrinsic mathematics motivation as an adaptation of *Intrinsic Motivation Inventory* interest/enjoyment scale (IMI, 2020; Ryan & Deci, 2000).

The sample consisted of 264 pupils aged about 10–15 years. As virtues are considered rather stable character traits, we assigned virtues as independent variables and intrinsic mathematics motivation as a dependent variable for a multiple regression analysis. According to the results, self-reported skills in leadership and patience predicted intrinsic mathematics motivation about the same amount each ($N = 264$, $R^2 = .234$: leadership: $\beta = .313$, $p < .001$; patience: $\beta = .266$, $p < .001$; compassion: $\beta = –.002$, $p = .973$). Compassion did not predict intrinsic mathematics motivation.

The results indicate that the virtues of leadership and patience are positively associated with intrinsic motivation in mathematics. While there are theoretical arguments for assuming virtues to be a more stable construct than motivation, we cannot rule out a reciprocal relation. Nevertheless, the results indicate an interesting relationship between personal virtues and motivation in mathematics, warranting further study.

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TEACHERS' NOTICING RELATED TO MODELLING-SPECIFIC METACOGNITIVE SUPPORT – A VIGNETTE-BASED STUDY

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Modelling-related meta-knowledge, e.g., about working steps in the modelling cycle (e.g., Blomhøj & Jensen, 2003) can be very helpful when students have to monitor their own modelling process, for instance. Consequently, teachers' noticing (Sherin et al., 2011) of non-optimal modelling-specific metacognitive support in classroom situation contexts is hence a potentially impactful teacher competence aspect, because teachers should be able to foster modelling-specific meta-knowledge in their students. As research on this modelling-specific aspect of teacher noticing is scarce, this study explored the following research question: *To which extent do pre-service and in-service teachers notice non-optimal modelling-specific metacognitive support in classroom situations shown in classroom cartoon vignettes?*

For exploring this research question, more than 200 pre-service and 18 in-service teachers were asked to comment on two classroom cartoon vignettes which showed situations in which a retrospective consideration of the modelling process would have been necessary in order to make the students aware of their modelling process and to enable them to improve their work throughout the stages of this process – a learning opportunity missed by the vignette teacher. Hence, an opportunity of building up knowledge at the metacognitive level on the modelling process had been missed in the vignette situations. The participants' comments on the classroom situation as represented in the cartoon vignette were coded in a top-down criterion-based approach using content analysis (Mayring, 2015). The analysis yielded comparatively low frequencies of this aspect of noticing both for pre-service and in-service teachers. A qualitative look at the answers also suggests that there was little awareness of metacognitive support with respect to modelling. For mathematics teacher education, the findings thus call for a reinforced emphasis not only on professional knowledge related to modelling and the modelling cycle, but for learning opportunities focused on classroom situations in which modelling activities are implemented.

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ELEMENTARY TEACHERS' TAKEAWAYS ACROSS TWO PROFESSIONAL DEVELOPMENT PROJECTS IN RURAL APPALACHIA

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INTRODUCTION AND BACKGROUND

This poster presentation focuses on shared themes and uniqueness across two projects supporting professional learning in mathematics for elementary teachers. One project, Math Counts, is a three-year series of Summer Institutes in mathematics content and pedagogy. A second project, Computer Science for Appalachia (NSF 2219418), is a five-year project with mathematics embedded professional development in the context of learning on computational thinking and culturally responsive pedagogy. Research on learning has emphasized the importance of hybrid spaces that support the participation of multiple identities, perspectives, experiences, and ways of knowing (Guterrez et al., 1999). Through the prioritizing of stories and community connections, our work the past ten years has sought to support teachers' content understanding, engagement, and pedagogy. In addition to supporting teachers through open tasks, we draw on a decentering approach that positions teachers as experts of their students and communities. This approach prompts us to ask teachers two questions: How does a given task connect to the community? How will students experience this task? The overarching research question we have explored is: What are the tasks, experiences, and resources of professional development that support teacher learning and connections related to mathematics?

THEMES AS REFLECTED IN TEACHERS' WORK

Two themes appeared across teachers' work from both projects. First, community and individual stories provided an effective way for teachers to meaningfully connect with the mathematics content being presented. Second, there seemed to be some missed opportunities in leveraging stories to build content understanding that have implications for future design. One theme that reflected uniqueness of the two projects was how mathematics embedded in computer science professional development offered another point of connection for some teachers. These themes hold implications for mathematics teaching and learning and professional development.

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EXPLORING A CHATGPT-BASED MATHEMATICS TUTOR: FEASIBILITY, REINFORCEMENT, AND SOCRATIC GUIDANCE

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This study explores the application of generative artificial intelligence (ChatGPT) in mathematics education, assessing its problem-solving accuracy and its potential to serve as a Socratic math tutor for junior high school students. Using mathematics questions from Taiwan's Comprehensive Assessment Program (CAP), the research was conducted in three phases: preparation, training, and implementation. In the preparation phase, the untrained ChatGPT demonstrated limitations in visual and spatial reasoning, with over 70% of its errors stemming from textual and graphical misinterpretations. This echoes findings that AI struggles with interpreting graphical data in mathematical reasoning (Dao & Le, 2023).

During the training phase, a three-level reinforcement strategy was applied to address these limitations: modifying input formats to text/LaTeX, introducing Polya's (1945) four-step problem-solving framework, and providing hierarchical prompts. This reinforcement strategy effectively mitigated the AI's initial weaknesses, significantly improving its problem-solving accuracy to an average score of 95.77.

Finally, in the implementation phase, an AI-based math tutor was constructed utilizing Pólya's framework and Socratic questioning (Elder & Paul, 1998) to guide students through the problem-solving process. While the AI could systematically structure the problem-solving guidance, results showed that it still struggled with executing authentic Socratic dialogues—such as guiding students to reflect on blind spots rather than directly providing answers—and interpreting complex geometric diagrams. This study provides in-depth insights into the potential and limitations of utilizing generative AI for personalized mathematics learning.

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INTERNATIONAL COMPARATIVE ANALYSIS OF THE QUALITY OF ELEMENTARY MATHEMATICS INSTRUCTION IN EL SALVADOR, NICARAGUA, AND BOLIVIA

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The Mathematical Quality of Instruction (MQI) framework evaluates the quality of mathematical content in classroom instruction through structured observation (Hill et al., 2008). Previous research has demonstrated associations between teachers' mathematical knowledge and observed instructional quality using MQI (Hill et al., 2008). At the same time, methodological studies emphasize that MQI scores must be interpreted carefully, accounting for multiple sources of variation such as raters, lessons, and instructional contexts. Questions remain regarding how MQI functions across different national and cultural settings.

This study examines how MQI functions across different national contexts. The research questions are: (1) How does the mathematical quality of instruction differ across elementary mathematics classrooms in El Salvador, Nicaragua, and Bolivia? and (2) To what extent does the background of evaluators influence MQI scores?

Video recordings of Grade 5–6 mathematics lessons were collected in the three countries. Lessons were segmented into instructional episodes and coded using the MQI protocol. Coding was conducted by three mathematics education experts from different cultural backgrounds. Coders jointly analysed sample lessons prior to independent coding. Inter-rater reliability was examined using intraclass correlation coefficients, and variance components associated with raters and lessons were estimated using a generalizability approach.

MQI profiles are expected to reveal both shared characteristics and cross-national differences. Lesson-level variation is expected to exceed rater effects, suggesting that MQI can provide a stable comparative framework across cultural contexts.

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VISUAL AND BEHAVIOURAL SIGNATURES OF ARITHMETIC STRATEGIES IN CHILDREN: AN EYE-TRACKING STUDY

Samuel Juma, Ariane Tretow, Susannah Otieno-Leppänen, Tuire Koponen, Jarmo Hämäläinen, Paavo. H.T. Leppänen

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Arithmetic strategy use is central to numerical competence, since one cab draw from multiple co-existing strategies depending on task demands and individual differences (Hickendorff, 2022). Thus, understanding how children deploy different arithmetic strategy is critical for instruction. However, the cognitive signature of strategy use is difficult to identify from behavioural measures: accuracy does not indicate how a solution was produced, and reaction time offers only a coarse proxy of processing. We argue that eye-tracking offers a process-sensitive window into real-time interaction with symbolic number representations. We hypothesize that hidden strategy implementation leaves traces in gaze behaviour that behavioural measures alone may miss, thus complement outcome-based assessments. The research questions are:

- Which eye-movement features differentiate arithmetic strategies during single digit addition task?
- How do task characteristics (e.g., operand magnitude and order) influence strategy selection and corresponding gaze patterns?

The study included 133 first-grade students (mean age 8.2 years; 49% boys) from 25 public schools in Finland. Children solved computerized single digit addition questions while eye movements were recorded; reaction time and accuracy were logged, and strategy use was coded from children's post-task reports.

Results show that counting showed longer dwell times and frequent operand transitions, decomposition involved structured digit-level shifts, and retrieval showed sparse fixations and shorter viewing time. Small →large operand order elicited faster responses and fewer transitions than large →small, indicating that operand structure influences strategy selection.

Based on the results, we conclude that arithmetic strategy use is reflected in measurable gaze patterns during symbolic processing. Fixation duration, spatial allocation, and operand transitions function as observable markers of strategy implementation. Task features such as operand order and visual layout need to be considered in task design as they shape strategy use across learners.

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COLLABORATIVE PROBLEM-SOLVING PROCESSES IN AN ESCAPE GAME CONTEXT

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This poster presents part of a doctoral study examining collaborative problem-solving processes in an educational escape game context. Although educational escape games have attracted increasing attention in education, their contribution to mathematical problem-solving processes remains only partially documented. This study aims to explore how mathematical problem-solving processes are implemented by Grade 6 students in an educational escape game context.

The conceptual framework is grounded on research addressing the contributions and limitations of escape game context (Piñero, 2020) and on problem-solving models such as Poirier (2001). The study adopts a qualitative interpretative approach. A digital escape game integrating mathematical puzzles was designed for research purposes. The intervention was conducted with three Grade 6 teams composed of three students each. Data were collected through video recordings, classroom observations, field notes, and students' written problem-solving work. The analysis focused on observable manifestations of problem-solving processes.

The poster presentation highlights one pre-algebra episode in which students were asked to determine the mass of objects based on two relational constraints. Results show that problem-solving processes were dynamic, non-linear, and iterative. Consensus-building dialogue, trial-and-error reasoning, and digital feedback opportunities supported collaborative problem solving by transforming errors into exploratory learning steps. Interface-based validation functioned as a regulatory indicator of students' answers.

Overall, the study contributes to understanding how educational escape game can support collaborative problem-processes processes, highlighting their potential to foster iterative exploratory reasoning in pre-algebra tasks.

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MATHEMATICAL ARGUMENTS AND "FAKE NEWS": EXPLORING EVALUATIVE CRITERIA ACROSS CONTEXTS

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The critical evaluation of claims and arguments is a central goal of mathematics education, both within mathematical contexts and, more broadly in the sense of general education, in fostering a critical and reflective attitude towards arguments beyond it (Skovsmose, 2023). In this sense, it is particularly relevant given the increasing spread of (socio-ecological) misinformation. We conceptualise the evaluation of statements and arguments as the use of criteria to assess claims and their justification (Toulmin, 2003). This qualitative study investigates: *Which evaluative criteria do students use in mathematical and socio-ecological contexts, and to what extent are criteria used in one context taken up in the other?*

We conducted semi-structured pair interviews with 16 eighth-grade students (aged 13–14). They evaluated mathematical claims and arguments (true and false) as well as climate-related social media posts in a fixed sequence. Data were analysed interpretatively focusing on the evaluation criteria used. The analysis revealed a range of similar and different criteria. For instance: Concerning the initial mathematical claims and arguments, they focused primarily on correctness. When dealing with climate misinformation, they introduced additional criteria such as plausibility, source origin, and data presentation. For example, the credibility of a claim was questioned solely based on its visual and linguistic design. Remarkably, these criteria re-emerged during the valuation in the second mathematical context, where students paid attention not only to correctness but also to the structure and presentation of arguments. Our findings suggest that students' evaluative practices are context-sensitive and can expand when engaging with socio-ecological claims. The results indicate that criteria introduced in one context may also be applicable in other contexts. This underscores the potential of integrating socio-ecological contexts into mathematics education to provide students with a broader range of evaluation criteria for arguments, which previous studies have shown to be limited (Meyer & Schnell, 2020). Thus, the study highlights the potential of socio-ecological contexts to extend students' evaluative practices in mathematics education and to contribute to critical mathematical thinking.

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EARLY PROPORTIONAL REASONING BEFORE AND AFTER INSTRUCTION IN MULTIPLICATION AND INTRODUCTORY FRACTIONS: AN EXAMINATION OF SECOND GRADERS' USE OF PICTORIAL REPRESENTATIONS

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Proportional reasoning has been widely investigated due to its educational importance (e.g., Lobato et al., 2010). However, limited research has examined in detail how proportional reasoning is fostered through formal schooling. In this study, we focus on instruction in whole-number multiplication and introductory fraction concepts. Based on written survey data, we examine how this instruction relates to the development of early proportional reasoning (EPR) before and after the instructional sequence. In our previous study (Hino et al., 2024), we analyzed how children's EPR developed before and after learning multiplication, as well as how the children began to use multiplication in solving proportional reasoning problems. Building on this work, the present paper further refines the survey tasks and adds pictorial representations to the problem statements. We report the results of a paper-and-pencil survey conducted with 30 second-grade elementary school students, consisting of a pre-survey administered prior to instruction in multiplication, followed by instruction in multiplication and basic unit fractions, and a post-survey administered afterward. The results of the written survey indicate that the percentage of correct answers to Problem 7 ("If 8 eggs cost 60 yen, how much do 4 eggs cost?") increased from approximately 17% to 40%. In contrast, the percentage of correct answers to Problem 3 ("If 8 eggs cost 60 yen, how much do 2 eggs cost?") increased only from 0% to 13%. An analysis of children's written responses and strategies, interpreted from the perspectives of EPR and their learning experiences with multiplication and fractions, revealed that children engaged in exploratory attempts to coordinate the two quantities—the number of eggs and the price— by adding their own details to the pictorial representations provided in the problem. A remaining challenge is to examine in greater detail the diversity of these responses in relation to different patterns of children's learning trajectories.

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UNPLUGGED DATA SCIENCE LEARNING IN MATHEMATICS EDUCATION: A MODELLING PERSPECTIVE

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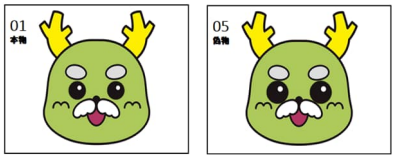
The advent of big data and AI has heightened interest in instructional approaches that ensure students' data science (DS) learning in mathematics education (Biehler et al., 2025). *Unplugged* DS tasks and activities that do not rely on computational tools and programmes are effective for elementary and middle school students (e.g., Podworney et al., 2025). However, designing such tasks and activities remains challenging. This study aims to develop unplugged DS tasks and activities based on *Models and Modelling Perspectives* (MMP; Lesh & Doerr, 2003) and verify them through a practice for seventh graders (aged 12–13 years). MMP focuses on real-world problem-solving and conceptual development through the modelling process. Through this process, students elicit their own thoughts within meaningful contexts, express them as mathematical models, and evaluate and revise these models to enhance their shareability and generalisability. Consequently, MMP aligns with task and activity design to develop student models and key ideas related to DS. Based on MMP, this study developed the *Unplugged Image Classification Task*, which requires students to develop convincing and sharable models for classifying a real school mascot character according to cards labelled as real or fake (Figure 1). In five lessons, the students repeated three cycles of generating a classification model using the cards and verifying and modifying it with new cards. Preliminary analysis of the students' models revealed various measures of classification, including the measurement of different attributes (e.g., eye size) and the simplification of characters using geometric figures. These measures were articulated through diverse representations, such as texts, tables, pictures, figures, and decision-making process diagrams, similar to decision trees. These results highlight the potential for a modelling approach to achieve open-ended and creative DS learning.

Figure 1: An example of cards (Left: real; Right: fake)



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THE SITUATED NATURE OF MATHEMATICS MOTIVATION: DIFFERENCES BETWEEN TOPIC-SPECIFIC AND GENERAL MATHEMATICS MOTIVATION

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Eccles and colleagues (1983) proposed a theory of achievement motivation, now referred to as the situated expectancy-value theory (Eccles & Wigfield, 2020). Therefore, to understand a students' motivation within a subject, statements regarding that subject instead of general motivation gives a clearer insight into students' experiences and differences. We argue, however, that not all situations are created equal and, thus, what constitutes a situated nature may require a more in-depth analysis than remaining subject-specific. Therefore, the current study assesses mathematics motivation comparing both general mathematics-related motivation, i.e. situating motivation in mathematics, with a topic-specific type of motivation, more specifically, situating motivation to the topic of functions.

The students in the study were Grade 7 Estonian students (N = 194, 48% female) with the approximate age of 13-14 years who participated in the study during spring 2024. They were asked to assess their self-efficacy and utility value through a questionnaire which was based on the situated expectancy-value theory. Four statements were provided for mathematics-related self-efficacy (ω = .88), function-related self-efficacy (ω = .87), mathematics-related utility value (ω = .92), and function-related utility value (ω = .89) with a total of 16 statements.

A paired samples Wilcoxon test showed that students rate mathematics-related self-efficacy (*Median* = 3.50, $W = 10484$, $p < .001$, $r_{rb} = .73$) higher than function-related self-efficacy (*Median* = 3.00), and mathematics-related utility (*Median* = 4.00, $W = 15338$, $p < .001$, $r_{rb} = .99$) higher than function-related utility (*Median* = 3.00).

Therefore, situating motivation in a mathematics-related context may not be enough to understand the students' day-to-day experiences and motivation with different topics.

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BUILDING STUDENT QUESTIONING THROUGH STUDENT-CREATED AI RESOURCES IN SECONDARY MATHEMATICS

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Recent technological developments in AI have introduced unique creative pathways for student success in secondary mathematics. Focusing on student development of advanced questioning, this study reports on student uses of AI to develop their own instructional and assessment materials as well as their evaluations of both materials as they develop their own confidence and abilities as math learners (Canonigo, 2024). Using a situated learning perspective (Lave & Wenger, 1991), we as researchers sought to understand how secondary math students refined ways to use AI to assist in their ability to question and produce authentic materials to better learn and assess their learning of high school mathematics. The research questions are:

- How have secondary students come to use AI to develop their own questioning about math?
- In what ways do secondary math students view AI as a resource for deeper mathematical understanding?

The sample comprises 37 secondary mathematics students in their last two years of high school in an independent school in an urban city in the southeastern United States. The intervention takes place over one semester of teaching, with students incorporating three iterations of AI questioning to determine how best to use AI as a resource they can use to improve mathematical understanding. The students answered qualitative questions about their perceptions and use of AI in secondary math classrooms after each incorporation of AI.

Qualitative responses show a strong positive preference towards the use of AI in secondary mathematics classrooms as a resource for question development and review material, citing that the AI is not as helpful in catching student error or cannot personalize learning in the way that a human teacher can. Further, results show that students more creatively learned how to word prompts for AI so that students gained more autonomy in their own learning and understanding of their own learning.

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DEVELOPING ETHNOSTEAM LESSON PLANS THROUGH DESIGN-BASED RESEARCH

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In this poster, we present findings from a design-based research (Bakker & Van Eerde, 2014) study examining how cultural artifacts (e.g., carpets, weaving practices, traditional games) can be meaningfully integrated into mathematics lesson plans through an interdisciplinary approach called EthnoSTEAM. The conceptual framework of this research integrates ethnomathematics and STEAM education. The study involved 11 middle school mathematics teachers from diverse regions of Türkiye. Using design-based research, the lesson plans unfolded across three iterative phases. In the preliminary phase, a needs analysis was conducted, followed by online training sessions introducing the EthnoSTEAM approach and the stages of the design thinking process (empathy, define, ideate, prototype, test) (Cook & Bush, 2018). In the prototyping phase, teachers collaboratively developed and refined eight EthnoSTEAM-based lesson plans addressing algebraic expressions; numbers and operations; rational numbers and operations with rational numbers; and square root expressions. In the evaluation phase, lesson plans were implemented in classrooms and revised through structured expert feedback loops. Data were collected through teacher diaries, lesson plans, and meeting records, and analyzed using descriptive and content analysis. Kendall's W coefficient was used to examine expert agreement across design cycles. Findings indicate that the EthnoSTEAM approach strengthened students' motivation, increased cultural awareness, and supported deeper mathematical connection-making. Teachers described the design process as professionally enriching, though time-intensive. Expert analysis revealed that while teachers demonstrated strong performance in the ideation phase, they encountered the greatest difficulty during the testing stage, particularly in assessing interdisciplinary student outputs. We will share the iterative design process, teacher experiences, and the development of a mathematics-centered EthnoSTEAM lesson plan template.

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ARITHMETIC COMPUTATIONAL THINKING TEST: DESIGN AND PILOT STUDY

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The term Computational Thinking (CT) describes the competencies for understanding and solving problems by drawing on core concepts of Computer Science. Numerous countries have implemented CT into their curricula even on a primary school level. The integration of CT poses new challenges such as a "lack of tools and resources for teaching" and the adequate "assessment of CT" (Bocconi et al., 2022, p. 39). In response to this problem, this study presents the development of the ACTt, a new test, assessing CT within the field of arithmetic.

Adapted from the CTt by Román-González (2015), the test focuses on the assessment of computational concepts such as sequences, loops and conditionals and partly includes computational practices such as abstraction and debugging. The test integrates CT into arithmetic, reflecting its common use in math instruction and its underrepresentation in existing CT assessments. While CT is considered a general, transferable problem-solving skill, we chose to test CT within a specific mathematical field to assess the basic concepts needed to develop higher level CT skills. The test is aimed at students in grades 4 and 5 (ages 9-11) to assess CT at the transition from primary to secondary school. The ACTt consists of 30 multiple-choice questions. In developing the test, we conducted an expert rating ($n = 14$) excluding items with a low content validity index $k^* < .75$. A pilot study ($n = 67$, grades 4 and 5) indicated medium item difficulty $p_{\emptyset} = .54$ ranging from $p_{min} = .15$ to $p_{max} = .85$. We also conducted cognitive interviews ($n = 13$, grade 4) to analyse the students approaches and sources of error. The items were revised based on the student's errors and feedback. Future studies will be conducted with bigger simple sizes and address criterion validity of the ACTt. Corresponding results will be presented at the conference.

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MAKING SENSE OF THE MODELING PROCESS: PEDAGOGICAL GAINS OF PRE-SERVICE MATHEMATICS TEACHERS

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In mathematics education, the Models and Modeling Perspectives (MMP) framework emphasizes that teacher knowledge is not static but evolves through the continuous recognition, interpretation, and refinement of classroom instructional situations (Doerr & Lesh, 2003). Consequently, Model-Eliciting Activities (MEAs) serve a crucial function in pre-service teachers' professional growth. Designing and implementing MEAs allows teachers to develop tools to elicit and deeply analyze students' mathematical thinking. Therefore, providing pre-service mathematics teachers with extensive opportunities to experience this process during their education is vital for their professional development (Doerr, 2007).

The purpose of this study is to reveal the pedagogical gains attained by pre-service mathematics teachers (PMTs) through the implementation of activities they developed during a mathematical modeling course. Fifty senior pre-service teachers enrolled at a state university participated in this study. A case study design, situated within qualitative research methodologies, was adopted. The PMTs were required to develop an authentic mathematical modeling activity during their modeling course and implement it with students. Subsequently, they were asked to report on this entire process. The gathered data were subjected to document analysis. The findings of the research indicate that PMTs found the experience of guiding students through the modeling process highly valuable. Furthermore, they stated that they had learned to refrain from providing evaluative feedback—such as direct approval or rejection—on students' questions, and they understood that the modeling process itself holds greater significance than a merely result-oriented approach.

The findings of the study indicate that mathematical modeling practices can foster process-oriented pedagogical gains for the professional development of pre-service teachers.

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DEVELOPING AN INSTRUMENT TO MEASURE MATHEMATICS TEACHERS' DIGITAL WELL-BEING

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Digital well-being has emerged as a critical dimension of teachers' professional functioning. Teacher well-being is conceptualized as a multidimensional construct (Viac & Fraser, 2020), while digital well-being research highlights risks such as digital overload and boundary management (Büchi, 2021). In addition, Artificial Intelligence (AI) has become increasingly integrated into teaching, creating new demands (Sperling et al., 2024). This study investigates (1) which dimensions characterize mathematics teachers' digital well-being in digitally mediated and AI-integrated teaching contexts, and (2) which factors should be incorporated into an instrument for measuring mathematics teachers' digital well-being. Drawing on the OECD Teachers' Well-being Framework (Viac & Fraser, 2020) and framed by Activity Theory (Blunden, 2015), digital well-being is conceptualized as emerging within a mediated activity system in which AI functions as a pedagogical tool. Semi-structured interviews were conducted with ten upper-secondary mathematics teachers from schools across Estonia and analysed using thematic analysis with particular attention to dimensions relevant for instrument construction. The analysis identified several core dimensions of mathematics teachers' digital well-being: cognitive load, emotional responses, perceived competence, teacher identity, and institutional support. Where institutional support structures were present (clear guidelines, IT assistance), using digital tools and integrating AI into teaching was associated with confidence and reduced workload. Where these structures were absent, teachers reported reduced cognitive well-being. The study provides an empirically grounded basis for constructing a multidimensional measurement instrument.

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‘DIDACTIC SUPERVISION ON MATHEMATICAL TUTORIALS’: DESIGN OF AN EDUCATIONAL COURSE ON INSTRUCTORS’ ATTITUDES TOWARDS TEACHING MATHEMATICS

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Strengthening teaching skills with the aim of fostering viable student attitudes towards mathematics is seen as a major task at tertiary educational institutions. The methodological lens is hereby often geared towards the students’ side. However, instructors’ attitudes towards teaching mathematics (ATTMs) play just as crucial of a role in this process. I therefore explore the impact of targeted educational interventions on ATTMs of instructors at university level in a design-based research project. Herein I aim at revealing their effectiveness, describing design principles and fostering the theoretical understanding of ATTMs. The theoretical framework of the three-dimensional model for attitudes (Di Martino & Zan, 2010) was adapted to the context of ATTMs to create explicit interventions for a new university course called *“Didactic supervision of mathematical tutorials”*. During the summer semester of 2026 six tutors of mathematic-related tutorials participate in this educational training course at the University of Graz. Since “attitude is produced by different evaluative processes” (Hannula, 2002, p.30), the implemented interventions focus highly on reflective practices. Various measurements are integrated into the course to target the different dimensions of ATTMs as seen necessary (ibid.): pre- and post-interviews for a holistic view, written learner’s diaries to capture the continuous development, qualitative surveys for associations regarding teaching mathematics as well as reflecting collegial observations from a more didactically developed perspective. Although this course only spans over one semester, I aspire to obtain data on the development of the participants‘ ATTMs. The poster displays results of the initial literature-based conjecture mapping (Sandoval, 2014), which show the need for an embodiment that incorporates reflection on practical application as well as educational resources and mediating processes that focus highly on educational justification to foster viable ATTMs. Furthermore, the interventions constructed according to the multidimensional character of ATTMs are illustrated.

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EARLY ALGEBRA MOMENTS AND WAYS OF LISTENING

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Many students struggle with algebra, and more research is needed on *how* teachers can make effective instructional decisions and act responsively in the classroom to support students' development of algebraic practices from the beginning of formal schooling. I explore how Grade 4 teachers can support students in justifying generalisations of patterns. My theoretical framework comprises four algebraic thinking practices—generalising, representing, justifying, and reasoning (Lee et al., 2023)—and the concept of early algebra moments (EAMs) (Lee et al., 2023). I investigate how three ways of listening—evaluative, interpretive, and hermeneutic (Davis, 1997)—can interact with EAMs, and how these ways of listening may help teachers act on EAMs to support students when they are justifying their generalisations. The participants in this study are a Grade 4 teacher and his students from a private school in Norway. The teacher and I selected tasks for 15 lessons that provided students with opportunities to engage in the four algebraic practices. We planned and discussed the lessons together. Taking an enactivist approach, my role was that of a participant- observer, understanding research as a process of learning together with the teacher. I video-recorded, transcribed, and analysed selected episodes, and discussed with the teacher his experiences in supporting students as they justified their generalisations. The poster will present a visualisation of how different ways of listening can interact with EAMs. I will show an example of each type of listening within an EAM in which a student had the opportunity to justify a generalisation. My findings show that interpretive and hermeneutic ways of listening can help the teacher act effectively in EAMs, and that the students' representations play an important role. The hermeneutic way challenges the teacher to relinquish control and to listen in a genuinely open manner. The evaluative way—checking the correctness of answers—does not support students in developing the practice of justifying. My next step is to explore how purposes (Brown & Coles, 2011) can help teachers listen and act in EAMs.

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CONSTRUCTION OF A CROSS-LAGGED MODEL OF MATHEMATICS GROWTH MINDSET

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Growth mindset serves as a critical catalyst for student achievement, particularly within the challenging domain of mathematics. Extant literature underscores that growth mindset exerts a profound influence on student performance; specifically, when students believe the belief that mathematical ability can be developed through effort, their academic achievement improves significantly (Boaler, 2016; Dweck, 2006). The primary objective of this study is to construct a cross-lagged model of mathematics growth mindset to track the development of mathematics growth mindset over time. In the initial stage, cross-sectional data were utilized to establish a model based on the "Context-Self-Action-Outcome" framework. In this model, teacher mathematics growth mindset served as the context variable, student mathematics growth mindset as the self-component, and three dimensions—cognitive, affective, and behavioral—as action variables (mathematics engagement), while mathematics academic performance constituted the outcome variable. The participants of this study are 754 seventh-grade students from public middle schools in five counties and cities in central Taiwan. Stratified cluster sampling ensured representativeness. The fit indices of structural equation modeling demonstrated that the observed data aligned well with the theoretical model (GFI, CFI, TLI > .90; RMSEA < .08), and all parameter estimates reached statistical significance. Findings revealed that teacher mindset significantly predicted student mindset and mathematics engagement, with the latter further predicting academic outcomes. Subsequently, the study extended this framework into a longitudinal design across two waves to construct cross-lagged models. The cross-lagged model for mathematics growth mindset and mathematics achievement yielded a $\chi^2 = 22.24$ (df =14, p=.07), with robust fit indices: GFI, CFI, TLI > .90; RMSEA < .08. A similar model fit was obtained for the relationship between mathematics growth mindset and mathematics engagement. However, preliminary analysis of two- waves indicated that while within-wave correlations were significant, the cross-lagged effects between waves did not reach statistical significance. This necessitates further analysis of the complete three-wave dataset to refine the cross-lagged model for mathematics growth mindset and its associated factors.

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PROSPECTIVE TEACHERS' VIEWS ON THE PRACTICE RELEVANCE OF MATHEMATICS EDUCATION – EMPIRICAL INSIGHTS FROM A VIGNETTE-BASED SURVEY

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Prospective teachers' (PTs') views on mathematics education (MEd) and on its practice relevance in particular can have a filtering role for their profession-related learning (e.g., Pajares, 1992), i.e., the development of their professional knowledge (Shulman, 1986). Despite the importance of such views resulting from this role, the empirical research base still needs to be broadened, in order to identify ways of optimally dealing with potentially detrimental views in teacher education. In particular, vignette-based methods have hardly been explored in this context so far. This study consequently uses vignettes to investigate PTs' views on the relevance of MEd and whether they see MEd as a scientific discipline which can inform classroom practice, according to the following research questions: *(1) Which views of prospective mathematics teachers on the relevance of MEd can be identified? (2) To which extent is MEd identified as a scientific discipline which can inform classroom practice?* 44 PTs from a German university of education (33 female, 11 male) at the beginning of their masters' degree were asked to comment on given vignette teacher statements. These were presented in three cartoon vignettes, which showed short dialogues of in-service mathematics teachers in the teachers' room. The PTs had to respond to the following prompt: "Please comment the views of the six teachers in the following three [teachers' room] dialogues by (a) describing what views the teachers (probably) hold and (b) stating how you judge on these individual views (please provide justifications for your judgements)." Content analysis (Mayring, 2015) was used to identify and reconstruct PTs' views according to the research questions; it yielded the following results: Very opposed PT views were identified in the sample, from not seeing MEd as relevant for classroom practice at all – and/or not seeing MEd as a scientific discipline – to reflected views of how the science of MEd can inform teachers and their practice, by teachers' critical reflection on research findings. The results call for reducing this disparity in PTs' views by enriching PTs' meta-knowledge on the status and relevance of MEd, as well as by strengthening PTs' understanding of scientific studies in MEd.

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EFFECTIVE PROFESSIONAL DEVELOPMENT FACILITATOR CHARACTERISTICS: ATTENTION TO MULTIPLE STAKEHOLDER PERSPECTIVES

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While the design of professional development (PD) programs has been widely studied, less attention has been given to the characteristics of effective PD facilitators. Subsequently, research has increasingly focused on the development of PD facilitators and characterized the knowledge and skills they need (Borko et al., 2014). This study adopts a holistic approach, drawing on the perspectives of all actors involved in a specific PD context to identify key characteristics of effective facilitators. We sought to answer the following research question:

- What are the effective PD facilitator characteristics from the perspectives of one particular PD's coordinator, facilitator, and participants?

This phenomenological study involved a PD coordinator, a PD facilitator, and five PD participants. Prolonged interviews (~2 hours) were conducted to capture the essence of each participant's experience with the phenomenon. During data analysis, open coding was used to capture more of what the interview transcriptions revealed. Our findings both align with and extend existing literature on effective PD facilitator characteristics. Consistent with prior studies, all participant groups emphasized the importance of facilitators' classroom teaching experience, experience in the same learning and teaching process, and subject-matter expertise. While high-quality feedback has been less frequently highlighted, our findings underscore its importance—particularly when it is grounded in authenticity and expertise and delivered in a non-directive manner that supports teacher reflection and adaptation. Additionally, accessibility and the creation of a safe, collaborative learning environment were identified as essential, reinforcing prior work on the pedagogical dimension of facilitation (Perry & Booth, 2021). Finally, our results highlight the need for greater attention to facilitators' prior facilitation experience, as novice status may shape participant perceptions and introduce potential tensions, even as facilitation itself remains an iterative learning process.

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ADAPTATIONS AND INSTRUCTIONAL STRATEGIES FOR TEACHING MATHEMATICS TO LEARNERS WITH VISUAL IMPAIRMENT: UN UMBRELLA REVIEW

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Mathematics education presents particular challenges for learners with visual impairment (VI) due to the discipline's reliance on visual representations, symbolic notation, and spatial organization. Curricula and textbooks frequently depend on diagrams, graphs, geometric figures and visually structured layouts, which may create significant barriers to equitable learning (Klingenberg et al., 2019). Although the inclusion of students with VI in mainstream classrooms is expanding, teachers reported a lack of knowledge and resources to adapt mathematics instruction effectively. Designing instruction to be accessible from the outset is therefore essential. While recent reviews have examined specific tools (e.g., assistive technologies, tactile graphics), no comprehensive synthesis covers all teaching phases. This umbrella review aims to identify and summarize the adaptations and instructional strategies to teach mathematics during the phases of learning, according to Hattie's model (2023).

A systematic literature review was conducted following PRISMA guidelines across six databases. Strategies were synthesized and organized according to instructional phases (preparation, opening, delivery, feedback, closure). Twelve reviews met inclusion criteria (mean methodological quality: 59%). Despite varied aims and methods, all reviews report convergent strategies that have proven effective for students with VI in mathematics. Results highlighted several key domains of effective practice: (1) diagnostic assessment aimed at understanding students' specific visual functioning; (2) anticipatory lesson preparation, including the provision of accessible materials such as Braille, Nemeth Code, and large-print formats; (3) adapted lesson delivery strategies (including assistive technologies, multimodal representations, tactile manipulatives, and explicit verbal descriptions). Providing concrete guidance for teachers without specialised training in VI appear essential to promote equitable access to mathematics for students with VI in inclusive classrooms.

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EVIDENCE FOR TEACHING MATHEMATICS TO STUDENTS WITH INTELLECTUAL DISABILITY: AN UMBRELLA REVIEW OF 50 YEARS OF RESEARCH

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The acquisition of mathematical skills is a major challenge for students with intellectual disabilities (ID), as it contributes to their independence in academic, daily, and professional activities (Bouck & Long, 2023). However, recent analyses show that their mathematical skills are often weak, mainly because of the difficulty teachers have in adapting teaching methods (Ruppar et al., 2023). Over the past 50 years, several authors have found that students with ID can acquire academic skills, even complex ones, using evidence-based practices (e.g. Bouck & Long, 2023). In order to summarize these evidence-based practices, this umbrella review provides an extensive overview of all systematic reviews and meta-analyses published up to 2024 on effective interventions in mathematics for students with ID aged 5 to 18. The research questions were as follows: 1. What patterns emerge related to (a) content area, (b) design, (c) school grade (d) origin of study, (e) person conducting the intervention, (f) type of intervention? 2. Which interventions can be considered evidence-based? Twenty-two systematic reviews or meta-analyses that included 514 studies from 1975 to 2023 with 3,678 participants were reviewed. After recoding 119 empirical studies met the inclusion criteria. The findings reveal that the areas of numbering (21%) and operations (28%) are the most studied. In contrast, measurement (7%), geometry (9%), and algebra (9%) remain underexplored. Most studies use a single-case design (83%) compared to group design (17%). Geographically, 89% of studies were carried out in the United States, followed by Europe (7%). Regarding instructional approaches, systematic instruction is the most tested (30%), followed by explicit instruction (19%) and concrete represented abstract (CRA) (9%). Effectiveness results show that CRA and strategy teaching are the most effective methods. Explicit instruction and systematic instruction also scored highly. This umbrella review offers many avenues for professionals in the field and helps identify gaps in current research, opening up research prospects for the coming years.

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A CURRICULAR MAPPING ANALYSIS OF MATHEMATICS AND ENGINEERING IN VOCATIONAL HIGH SCHOOLS

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In the era of Industry 4.0 and digital transformation, there is an increasing demand for technical talent with strong mathematical modelling and analytical skills. However, Technical and Vocational Education and Training (TVET) institutions often maintain rigid disciplinary boundaries, leading to a disconnect between general mathematics and vocational subjects. This study utilizes visual mapping analysis to reveal the structural gaps between the "Basic Electricity" curriculum and the national mathematics curriculum in vocational high schools. The aim is to identify the cognitive challenges posed by these disciplinary boundaries and provide evidence for curriculum reconstruction. This research is grounded in Boundary Crossing theory. According to Akkerman and Bakker (2011), boundaries between different fields (e.g., academic vs. vocational) serve as potential sites for learning and innovation. In TVET, students must navigate the boundary between abstract mathematical symbols and practical engineering applications. By using visual tools as boundary objects, this study identifies where knowledge translation and integration fail due to curricular misalignment. A mixed-methods design were employed included content analysis and visual mapping. The visual mapping analysis reveals a complex many-to-many relationship between engineering tasks and mathematical tools. Key findings include: The visual mapping analysis reveals a complex many-to-many relationship between engineering tasks and mathematical tools. While 10th-grade vocational units in the first semester demonstrate stable horizontal alignment by corresponding with junior high or early 10th-grade mathematics. The curriculum in the second semester exhibits a significant vertical temporal disconnect. Visual evidence confirms a cross-grade leap where 10th-grade electrical units demand 11th-grade mathematical foundations. This disparity generates a knowledge vacuum in engineering practice, characterized by a high density of connections in the Sankey diagrams pointing to advanced topics such as derivatives and integrals for units like transient circuit analysis. Ultimately, this structural vacuum compels students to perform complex physical modelling before they have mastered the required mathematical principles in their vocational education curriculum. The research findings highlight a severe disconnect between vocational demands and the sequencing of the mathematics curriculum. This knowledge vacuum compels students to rely on rote memorization rather than conceptual understanding, hindering their ability to construct mathematical meaning in professional contexts.

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TIMING MATTERS: STUDENTS' USE OF GENERATIVE ARTIFICIAL INTELLIGENCE IN PROVING MATHEMATICAL STATEMENTS

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This study explores the timing of generative artificial intelligence (genAI) use during mathematical proof construction. Building on Yoon et al. (2024), which examined how students use genAI in proving, the present study focuses on the sequencing of students' interaction with genAI. We aim to investigate how the point at which genAI enters the proving process influences students' proving activity.

The study involved 30 undergraduate students who engaged in two proving activities. We compared two approaches: Initial Independent Proving (IIP), in which students used genAI after an initial independent attempt, and Initial Dependent Proving (IDP), in which students used genAI before drafting a proof. Participants were assigned to IIP and IDP conditions through a crossover design and experienced both while completing the same proof tasks. Data included students' initial proofs, genAI interaction logs, revised proofs, a similar proof task completed without genAI, and reflection surveys. We analyzed students' proofs using Stylianides's (2007) three components of a mathematical argument.

Findings indicate that students in both IIP and IDP conditions tended to integrate genAI-generated proofs into their final work, although aspects of this integration differed. In the IIP condition, students showed a tendency to revise their initial proofs or shift strategies after interacting with genAI, suggesting that genAI functioned as a resource for review and conceptual refinement. In contrast, students in the IDP condition showed stronger alignment with genAI-provided proof structures from the outset, resulting in fewer revisions. Students' reflections highlighted a tension between the efficiency of early genAI use and the learning value of using genAI after independent attempts, underscoring the importance of instructional design that carefully considers the timing of genAI integration.

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CONCEPTUALIZING CRITICAL THINKING IN MATHEMATICS CLASSROOMS

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Critical thinking (CT) is widely emphasized as a central goal of mathematics education in curriculum and policy documents. Despite this prominence, CT remains theoretically diverse, with definitions originating from philosophy, psychology, and education. Frameworks such as Facione's (1990) Delphi study identify core cognitive skills including interpretation, analysis, evaluation, and inference. In mathematics education, CT is often intertwined with constructs such as reasoning, problem solving, and mathematical communication, resulting in conceptual ambiguity (Schoenfeld, 2016). Although policy documents promote CT, limited research has examined how mathematics teachers conceptualize this construct in classroom practice. This study explores how mathematics teachers conceptualize CT in their classrooms. The guiding research question is: How do mathematics teachers conceptualize critical thinking in mathematics teaching and learning?

An interpretivist qualitative design was employed. Semi-structured interviews were conducted with mathematics teachers within a single educational jurisdiction. Data were analysed using inductive thematic analysis, supported by inter-rater verification and member checking.

Findings reveal considerable variation in teachers' conceptualizations of CT. While teachers consistently valued CT, few articulated a unified definition. Instead, participants described overlapping criteria including interpretation, reasoning, evaluation, and communication. Teachers emphasized that CT is not limited to problem solving and may differ across mathematical domains. Structural constraints including curriculum pacing, limited instructional time, and assessment practices were identified as barriers to supporting students' CT. Overall, CT in mathematics is understood by teachers as a multidimensional and context-dependent construct rather than a single transferable skill. By foregrounding teacher perspectives, this study highlights tensions between curricular intentions and classroom enactment, contributing to ongoing efforts to clarify theoretical understandings of CT in mathematics education.

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ENHANCING EARLY MATHEMATICS LEARNING THROUGH HANDS-ON ACTIVITIES: AN ACTION RESEARCH ON INTEGRATING COOKING INTO COUNTING STRATEGIES

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We report a series of research lessons in K2 (children with ages 4-5) where a teacher embedded cognitive development of numeracy skills into a series of hands-on cooking activities incorporating real-life contexts. An action research was conducted by a kindergarten teacher who designed and implemented interactive activities in the theme of cooking integrating counting skills. "To be aware of the relationship between mathematics and life" (CDC, 2017, p.40), the objectives of the research lessons were to investigate the effectiveness of cooking activities in developing K2 children's counting skills, and to identify the mathematical concepts developed in the activities. To "create real-life learning contexts and select suitable objects and teaching aids" (CDC, 2017, p. 41), the intervention with cooking activities consisting of eleven 30-minute lessons, was conducted in three consecutive weeks. Various types of food, e.g., spaghetti with tomato sauce, sushi rolls, pizza, and etc., were made with guidance by the teachers in the lessons that cooking activities involving counting the numbers of specific ingredients in recipes, were depicted. Different from the common perceptual unit items to count, such as marbles and fingers, the quantities of objects to be counted were told to the children as the requirements of the ingredients, e.g., ten pieces of pineapples, in the activities that children were asked to cut and to obtain the corresponding numbers of the food by wooden knives (for safely reasons). The activities allowed children to making sense of numeracy with hands-on experience. In early mathematics lessons, teachers usually guide children to count objects one-by-one by one-to-one correspondence (Clements & Sarama, 2020). In the observation, children were found to firstly focus on counting the numbers of the 'cuts' (action) [$n - 1$], but not the cut objects (pieces of food) [n]. One finding in the action research shows that children were aware of the difference between the number of cuts ($n - 1$) and the obtained cut objects (n). Some children were verbally counting the actions (the numbers of cut) one-by-one and repeated to count the numbers of cut food while they obtained a quantity more. This observation not only provided an insight for the teacher to pedagogically design the ways to count through 'one-to-one correspondence', but also highlighted the conceptual differences between counting of *actions* and *objects*.

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MOVEMENT AS A BRIDGE – CREATING CONNECTIONS IN ANALYTIC GEOMETRY THROUGH AUGMENTED REALITY

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Analytic geometry connects geometry and algebra. To bridge this gap developing an understanding of mathematical concepts, operations and relationships – known as conceptual knowledge (Kilpatrick et al., 2001) – is of great importance. Teaching often focuses on procedural schemes, like calculating intersection points and lines. To enable a comprehension-oriented approach, the iOS app MalAR (transl. Learning Mathematics with Augmented Reality), offers a three-dimensional visualisation of analytic geometry. Learners view objects of analytic geometry via a device camera and explore them through body movements. According to the theory of embodiment, abstract concepts emerge from bodily interaction with the environment (e.g. Wilson, 2002). Mathematical thinking is therefore grounded in physical experience. Through physical movement, MalAR may provide a bridge between procedural and conceptual knowledge. This leads to the following research question:

RQ: How can a movement-based Augmented Reality (AR) application support the connection of procedural and conceptual knowledge in analytic geometry?

After a brief introduction to the MalAR app, 25 mathematics education students individually solved two typical textbook tasks in analytic geometry in written form, utilizing MalAR to support their solution process, in June 2024. Participants were video-recorded to capture their body movements, while simultaneous screen recordings documented the interaction with and visualization of virtual objects in MalAR. Video analysis of this exploratory study revealed that all participants moved actively during the task. An evaluation of the written solutions, combined with the screen recordings, revealed that by checking their answers through displaying the relevant objects in MalAR, the majority of learners were able to identify errors in their results, which prompted them to revise their written work and frequently led to correct solutions. These findings suggest that learners can successfully connect procedural calculation with spatial-conceptual understanding. A larger follow-up study in summer 2026 will focus on conceptual tasks, in order to examine the influence of AR on procedural and conceptual knowledge in more detail.

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TEACHERS' IDENTIFICATIONS OF SYMMETRIES IN PLANE FIGURES

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Symmetry is an important concept in mathematics, often learned in relation to the properties of geometric figures. Symmetry may also be used as a tool in problem solving and as such, can increase understanding of the mathematics involved in the problem (Leikin, 2003). The current study focuses on teachers' knowledge of reflection and rotation symmetries as attributes of two-dimensional figures. Our main question is: Are there differences between teachers' ability to identify reflection and rotational symmetries of different figures?

A two-part questionnaire was handed out to 27 practicing teachers. Each part included the following nine figures:

On the first part, teachers were asked: Is the figure symmetrical? Explain. After teachers answered this part, they were given the second part with the exact same figures and were asked: (1) For each figure, determine if the figure has reflection symmetry and if it does, draw the lines of reflection. (2) Determine if the figure has rotational symmetry and if it does, state the angles of rotation. Note that specific symmetries are not mentioned in the first part, while in the second part they are.

Results showed that all teachers recognized that the hand is asymmetrical, but around 20% believed that the Yinyang is symmetrical, perhaps not considering the coloring. Among the reflection only figures, on the first part of the questionnaire, an average of 59% of teachers noted that the figures had reflection symmetry, while on the second part, 87% drew the correct lines of symmetry. However, for the figures with both reflection and rotational symmetry, a decrease in correct responses was noted on the second part, with teachers omitting or adding lines of symmetry. Regarding rotational symmetry, on part one, figures with rotational symmetry were recognized as symmetrical, but few specifically mentioned rotation in their explanations. On the second part, when specifically asked about rotational symmetry, only a small increase in correct responses was found. The poster will display additional findings along with some common errors found in the study.

Acknowledgement

This research was supported by The Israel Science Foundation (grant No. 2650/25).

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PROFILES OF STUDENTS WITH PRE-ALGEBRA DIFFICULTIES AND THEIR DIFFERENTIAL RESPONSIVENESS TO A STRUCTURED INTERVENTION

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Pre-algebra is a critical milestone in the transition from arithmetic to algebraic thinking, yet many students, particularly those with mathematics difficulties, struggle to acquire pre-algebra skills. Despite growing recognition of heterogeneity among students with mathematics difficulties, fine-grained evidence on profiles of pre-algebra difficulties (PAD) and their instructional responsiveness remains limited. This study examines (a) the heterogeneity of pre-algebra skill profiles among students with PAD and (b) whether students with different profiles respond differentially to a structured pre-algebra intervention.

METHODS AND ANALYTIC FRAMEWORK

Sixth-grade students (N=653) were screened using a comprehensive pre-algebra assessment, from which students with PAD were identified. Cognitive Diagnostic Models (CDMs) were applied to classify students' mastery patterns across conceptual knowledge, procedural knowledge, and procedural flexibility. In the second phase, a pretest–posttest randomized controlled design was conducted with selected PAD subtypes. A researcher-developed intervention targeted equation solving and integrated contrasting cases to support procedural learning, supplemented by explicit mathematics vocabulary instruction to strengthen conceptual understanding.

RESULTS

This study identified five PAD subtypes through CDM analyses and revealed differential responsiveness to intervention across subtypes, reflecting distinct patterns of conceptual, procedural, and flexibility-related weaknesses. Students with partial conceptual foundations showed stronger improvement in procedural flexibility, whereas students with more pervasive difficulties primarily demonstrated gains in basic procedural knowledge.

IMPLICATIONS

By integrating diagnostic profiles with intervention outcomes, this study demonstrates the potential of CDMs to support precision-oriented pre-algebra instruction. The findings underscore the importance of aligning instructional components with students' underlying knowledge profiles and offer an interpretable framework for bridging assessment and differentiated intervention in pre-algebra learning.

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EMBODIED LEARNING IN THE TOPIC OF ANGLES

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This research project investigates how Grade 5 low progress learners in Singapore learn to find unknown angles using angle properties. Analysis of student work show that while the use of Geogebra applet and measuring of angles help low progress learners to understand angle properties like "angles on a straight line add up to 180°" and "vertically opposite angles are equal", these learners struggle to apply angle properties to solve for unknown angles.

In line with research in embodied learning, which highlights the importance of bodily interactions with the physical environment and manipulatives in meaning-making by students (Stolz, 2015), carefully designed embodied experiences were planned to help the students overcome their challenge.

The research question guiding this investigation is: How does using multiple representations mediated through embodied learning support Grade 5 low progress learners in Singapore in applying angle properties to solve for unknown angles? The planned intervention was carried out on two groups of subjects. The first group comprised seven Grade 5 low progress learners while the second group comprised sixteen Grade 5 middle and low progress learners. Students' learning data in the forms of artefacts, video recordings and quiz results were collected and analysed, with insights from teacher participants' reflection on their enacted practice.

The findings show that the use of concrete objects (paper cut-outs in this case) helped low progress learners visualise that unwanted angles could be subtracted from the whole angle (180° or the vertically opposite angle) to find the required angle. The use of paper cut-outs, as opposed to teaching the concept through static diagrams, helped low progress learners connect their concept of subtraction, which was to take away, with the coupled perception and action of removing the unwanted angle(s) from the whole angle. In other words, the bodily interaction of the students with the manipulatives (paper cut-outs) and its perception enhanced their meaning-making of the angle property to be learnt. During the presentation, further results will be discussed in detail.

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SENSING THROUGH SYMBOLS: EXAMINING MULTIMODAL SENSE-MAKING IN VISUAL MATHEMATICS WRITING

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Mathematics writing is an essential practice for articulating mathematical reasoning. While text-based writing is widely studied, the role of visuals—an important representation for constructing mathematical meaning—is often overlooked. This study introduces "visual mathematics writing," a multimodal approach where students integrate visual and textual representations to communicate mathematical ideas.

METHODOLOGY

The study included 140 fourth-grade students in China. Participants completed a task requiring them to explain the mathematical relationship 5×4=5+5+5+5 using both a visual representation and written text. Performance was evaluated across three domains: (a) visual features (organization, content, clarity), (b) textual features (organization, content, grammar, vocabulary), and (c) visual-text connection (the alignment between the two modes). We also assessed students' general writing ability and mathematical (multiplication) competence.

RESULTS

A Friedman rank-sum test revealed significant differences across three feature domains ($\chi^2(2) = 97.75$, $p < .001$). Students performed significantly better on visual features (72.50%) than on textual features (52.63%) and visual-text connections (54.78%). Ordinal logistic regression indicated that mathematical competence was the strongest predictor for both visual and textual features ($p < .01$). Notably, students' performance on visual-text connections remained consistently weak regardless of their mathematical or general writing ability.

CONCLUSIONS AND IMPLICATIONS

The findings suggest that while students can construct individual representations, they struggle with "meaning-making" through multimodal integration. This aligns with the conference theme "Sensing and making sense of mathematics," highlighting that the transition between different senses (visual/spatial vs. verbal/symbolic) requires targeted instructional support.

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WHEN 4TH-GRADE STUDENTS CONVERT METRIC UNITS OF MEASUREMENT BY "ADDING" AND "REMOVING" ZEROS

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In Austrian primary education, 3rd- and 4th-grade students are expected to convert metric units of measurement, for example in word problems. However, unit conversion is considered one of the most challenging topics for primary school students. Existing approaches understand conversion as "refining and coarsening" units through multiplication or division by powers of ten, or use the metric place value table, which involves shifting digits (Franke & Ruwisch, 2010). At the same time, there is broad consensus that unit conversions should not only be carried out procedurally but connected to concrete measurement activities. When asked to explain unit conversion, many students refer to "adding" or "removing" zeros. This study therefore investigates what 4th-grade students mean when using this terminology in metric unit conversion. Data were collected through semi-structured one-to-one interviews with 35 4th-grade students from six classes in three Austrian primary schools. The interviews included open questions, conversion tasks, explanatory prompts, and the use of measurement tools. Data (video recordings, transcripts, notes, and drawings) were analyzed using qualitative content analysis according to Kuckartz (2019).

The findings show that the expressions "adding" and "removing" zeros were used by 25 students across different classes and schools and do not reflect a uniform understanding but rather different underlying strategies. Four main patterns emerged: interpreting zeros as shortcuts for multiplying or dividing by powers of ten (6 students); using the structure of the metric place value table (10 students); inferring zeros from known unit relationships (e.g., 1 m = 100 cm), for example by noticing that centimeters have two more zeros than meters and transferring this pattern (5 students); and applying or removing zeros arbitrarily (11 students).

These results indicate that similar student verbalizations can mask fundamentally different conceptions, which is highly relevant for diagnostic purposes and further highlights the need to foster conceptual understanding of unit conversion.

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ENGINEERING STUDENTS' CONCEPTIONS OF RATIONAL FUNCTIONS: AN EXPLORATORY STUDY FROM APOS THEORY

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The concept of function plays an important role in upper secondary and higher-level mathematics. However, various studies have shown that the students experience persistent difficulties in understanding, representing, and interpreting them, especially rational functions (Bossé et al., 2019). These difficulties include confusion between domain and range, transitioning between different representation registers, and a lack of understanding of asymptotic behavior (Paytan Montañez, 2024).

Based on the above, and from the perspective of APOS theory (Arnon et al., 2014), this exploratory study analyzes the conceptions of rational function held by a group of engineering student at a public university in Mexico. The interpretation is therefore carried out in terms of the mental structures of action, process, object, and schema. The research question guiding the study is: What mental schema have incoming engineering students constructed with respect to rational function?

The study was conducted with a group of 35 first-semester engineering students. A questionnaire was administered, from which questions were selected to assess: 1) conceptions of the function concept, 2) identification of rational function, and 3) the schema constructed regarding rational function. The results show that first-year engineering students have a poorly developed schema concerning rational functions, and even regarding the general concept function, highlighting the need to design sequences of activities to strengthen their conceptions.

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HIGH-STAKES MATH ASSESSMENT VIA CHATGPT: PROFICIENCY AND THINKING STRATEGY INDEX

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This research explores generative artificial intelligence (AI) in high-stakes mathematical assessment under Taiwan's competency-oriented Curriculum Guidelines of 12-year Basic Education. As AI reshapes evaluation (Zhai et al., 2021), understanding its proficiency in solving complex problems and developing difficulty prediction mechanisms is now vital. The primary objectives are to evaluate ChatGPT's performance stability across consecutive General Scholastic Ability Test (GSAT) and to validate a "Thinking Strategy" index for predicting item difficulty. Using 2022 to 2024 GSAT Mathematics A exams as benchmarks, this study categorized items by content and visual traits to analyze error patterns and applied Spearman's correlation to examine the relationship between the complexity index and official difficulty levels.

Results show ChatGPT's capabilities evolved significantly, with performance improving from "Average" to "Front" tiers between 2022 and 2024. While demonstrating high proficiency in "Algebra," its accuracy declined in "Coordinate Geometry" and visual-spatial tasks. This aligns with findings on Large Language Models' limitations in geometric logic (Frieder et al., 2023). Regarding difficulty prediction, the study confirmed a significant moderate positive correlation ($r = 0.43$) between the "Thinking Strategy" index and official ratings. Consistent with cognitive complexity frameworks (Webb, 2002), quantifying strategic steps serves as a viable tool for pre-evaluating difficulty. While AI exhibits competence, visual reasoning remains a bottleneck; nevertheless, the validated index provides a scientific foundation for automated item generation difficulty control.

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EXPLORING THE ROLE OF MATH PROFICIENCY AND MATH ANXIETY WHEN STUDENTS MEET MATH IN SCIENCE TASKS

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Many studies demonstrate that students struggle to use mathematics in unfamiliar contexts, particularly within science (Towns et al. 2020). Studies also show that students with math anxiety, i.e. negative feelings concerning mathematics, perform lower than other students in both mathematics (Ramirez et al., 2018) and other subjects, such as chemistry (Mack et al. 2019). In this poster, we present plans and preliminary results from a study examining to which extent students are able to draw on their mathematical proficiency to solve science tasks that involve mathematics, and how this is affected by their attitudes and feelings towards mathematics. We draw on data from PISA 2022, in which 3185 Norwegian students completed both the mathematics and science assessments and answered questions about their feelings toward mathematics (math anxiety). Our analyses of these data confirm previous research findings, showing a substantial correlation between math anxiety and science achievement (.31).

To further investigate these relationships, we first examined the extent to which solving the PISA science tasks (n=80) required use of mathematics. Our analysis revealed three categories: tasks involving no mathematics (n_1=47), tasks including a mathematical representation (e.g., a graph or table), but where the representation was redundant for solving the task (n_2=13), and tasks in which mathematical representations and/or operations were essential for solving the task (n_3=20). In the next phase of the study, we will investigate how math achievement and math anxiety relate to these three task categories both quantitatively and through task-based student interviews. These analyses will deepen our understanding of the relationship between mathematical proficiency, math anxiety and students' ability to solve science tasks that involve mathematics.

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PROFILING COMPUTATIONAL THINKING IN SOLUTIONS TO NON-ROUTINE MATHEMATICAL PROBLEMS

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Research on computational thinking (CT) in teacher education largely focuses on pedagogy rather than on the cognitive dimension of teachers' mathematical knowledge (Dong et al., 2024; Liu et al., 2024). This study fills the gap by seeking to understand the relationship between strategy selection in non-routine mathematical problem-solving and prospective elementary school teachers' (PSTs') engagement in CT. Analyzed were 19 PSTs' written solutions to the task: The Tarnovik Republic prints stamps in the denominations 17¢, 9¢, and 5¢; what is the largest amount of postage that cannot be made using these denominations? To examine the relationship between PSTs' problem-solving strategies and their CT structures, the data were first examined to identify similarities and differences across the set of solutions. This step led to categorizing PSTs into four groups: *optimizers*, *evaluators*, *sifters*, and *explorers*. The *optimizers* solved the problem by looking for a consecutive run of reachable amounts, realizing that subsequent amounts are possible by adding multiples of 5 to reachable amounts. The *evaluators* focused on checking possibilities. Even though they might have identified consecutive runs, they continued checking and testing, at times jumping to higher numbers. *Sifters* documented a very organized approach, breaking the search space into manageable blocks (e.g., 11-20, 21-30, etc.) and using these blocks as filters to represent possibilities. Finally, *explorers* mostly used trial-and-error, demonstrating scattered approaches and random testing of different denominations. Drawing on descriptions of CT practices in the existing literature (abstraction, decomposition, pattern recognition, algorithmic thinking, and debugging), each solution was examined for evidence of engagement in these practices and assessed (using a developed 4-point rubric) for the extent to which each practice was utilized. Patterns of CT engagement were analyzed for each PST group, revealing varying engagement with CT across PSTs with different solution profiles; for instance, optimizers showed high levels of abstraction, while sifters stressed decomposition.

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EMBODIED (RE)CONFIGURATIONS: A NEW MATERIALIST ANALYSIS OF INTELLECTUAL AUTHORITY IN MATHEMATICS CLASSROOMS

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This study explores intellectual authority evidenced in embodied mathematical activity, addressing the research question: How do intellectual authority imbalances emerge and unfold during children embodied mathematical activity? I argue that children often reconfigure the materiality of mathematical activity to contest intellectual authority. I define intellectual authority as a relationship in which "one person… tends to obey, act on, or accept without question the statements or commands of another person" (Amit & Fried, 2005, p. 147). This definition brings attention to whose mathematical ideas are taken as worthy in the classroom. Theoretically, tenets from new materialism informed this study, challenging views of mathematics learning as purely cognitive to, instead, consider mathematical concepts as inseparable from objects, gestures, and spatial arrangements (de Freitas & Sinclair, 2013). Fifty-six children in three second-grade classrooms in the United States participated in this study. Data sources included three one-hour video-recorded lessons, one in each classroom. In all three lessons, students arranged a set of items in multiple arrays. In the first classroom, children organized twelve 8 in x 8 in wooden boards. In the second, 20 children arranged themselves as arrays. In the third, children arranged 16 toy trees. Critical multisemiotic discourse analysis (van Leeuwen, 2008) focused on small group. I identified reactions—i.e., students' responses to socio-material entanglements, and possible meanings associated with these (re)actions vis-à-vis intellectual authority. Findings revealed a continuum from shared to contested intellectual authority reflected in material entanglements. Examples of shared intellectual authority include children engaging in a quiet impromptu choreography where they coordinated the placement of wooden boards to complete an array. Examples of contested intellectual authority include an exasperated child grabbing classmates by the arm to relocate them in the array, and a child rotating the toy trees to switch rows and columns and win an argument about the number of rows in the array. These findings call for attention to intellectual authority beyond spoken language to also consider the materiality of tasks.

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FROM PROBLEM SOLVING TO PROBLEM POSING IN MATHEMATICS: THE ROLE OF CONNECTIONS IN STRUCTURING THE PROCESS

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Mathematical Problem-Posing Tasks (MPPTs) can be conceptualized as a cyclical process structured around four phases: Orientation, Connection, Generation, and Reflection (Cai & Rott, 2024). This research examines how this phase-based model can be enacted with prospective secondary teachers when the purpose guiding the problem-posing process is made explicit, enabling the transformation of mathematical ideas into new problems. The study involved 29 undergraduate students enrolled in a Mathematics degree program and taking a course in Mathematics Education, working in a technology-rich environment (GeoGebra and AI tools). Each MPPT was preceded by solving the original problem through paper-and-pencil strategies and technological exploration, grounding the posing process in prior mathematical activity. Two successive cycles were implemented. The first was strongly structured: participants were explicitly required to identify the situation, the prompt, and the objective of the MPPT, drawing on Baumanns' (2022) categorization of problem-posing tasks (generating new problems, reformulating a given problem for problem solving, reformulating a given problem for investigation, and constructing tasks for others). The second cycle was comparatively more open, allowing participants to appropriate the phase structure with reduced external guidance.

The analysis highlights a continuity between the resolution of the original problem and the subsequent Connection phase. Rather than functioning as a separate stage, the Connection phase emerged as an explicit reorganization of mathematical ideas that had already begun to develop during problem solving. This continuity suggests that grounding problem posing in prior problem-solving activity is not merely preparatory, but structurally constitutive of the Connection phase within the problem-posing cycle.

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This research was partially funded by the research project PID2022-139007NBI00, approved by the Spanish Ministry of Science and Innovation (MCIN).

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FROM DESIGN TO USE: ENACTING A THEORY-INFORMED APP FOR PART-WHOLE UNDERSTANDING

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Part-whole understanding is a central prerequisite for early arithmetic learning and can be supported through a structural approach that systematically emphasises relations between parts and wholes (Björklund et al., 2021). However, translating these structural principles into digital learning environments remains challenging (Drijvers & Sinclair, 2024). Against this background, a theory-informed app prototype was developed to support primary school students' part-whole understanding by implementing systematically structured decomposition tasks across enactive, iconic, and symbolic representations. The present study reports on a pilot study of this prototype and addresses the following research question: What insights for the further development of the prototype can be derived? The pilot study was conducted with first-grade students (N = 97) in a pre-post design including a control group receiving business-as-usual textbook-based part-whole instruction. In addition to assessing part-whole understanding, usage data and classroom observations were analysed to inform further development. No additional learning gains in part-whole understanding were found for the intervention group. However, variation within the intervention group suggests that gains depend on the extent to which students completed the intended decomposition tasks. Classroom observations further indicate that some students engaged superficially, and that limited ICT routine and the need for additional analogue support challenged systematic task processing. These findings suggest that further development should focus on ensuring that all intended decompositions are systematically processed, supported by more specific feedback on missing decompositions and by a closer integration of digital and analogue representations.

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A STUDY ON METHODS OF DIAGRAM FORM REVISION IN PATTERN GENERALIZATION

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In pattern generalization, some studies have focused on the process through which the mathematical structure embedded in an object is derived. Since this process proceeds progressively through the mutual coordination of numerical calculations and diagrams, the revision of diagram forms is sometimes positioned within it. However, few studies have examined how diagram forms are revised. The purpose of this paper is to identify methods of diagram form revision in pattern generalization. The revision of diagram forms refers to reconsidering the conventional rules that constitute a diagram and reconstructing it according to a different set of construction rules (Hoffmann, 2005). The research method proceeds according to the following steps: (1) Based on factual, contextual, and symbolic generalization (Radford, 2003), we design a hypothetical process of generalization concerning combinations with repetition. (2) Using the semiotic triadic relation (representamen, object, interpretant) as an analytical framework, we analyze the process of revising diagram forms. (3) By examining multiple revisions of diagram forms, we identify methods common to all processes. Combinations with repetition are chosen because dealing with a more complex phenomenon than those presented in previous studies was considered more likely to generate processes of diagram revision. Through two revisions of the diagram form, we designed a process leading to the construction of the general term for combinations with repetition. As a result of the study, the following methods of revision were identified: (1) When students cannot discern the mathematical structure, they revise the diagram form. (2) Students apply the revised diagram to a simple case. (3) Using numerical evidence as a basis, students examine the validity of the revised diagram. (4) From the perspective of simplicity, students refine the revised diagram. This result suggests that revising diagram forms can serve as a catalyst for deriving mathematical structures.

Acknowledgement

This study was partially supported by the 2026 Grant-in-Aid for Scientific Research (Encouragement Research).

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EMBODIED GROUNDING, JUDGMENT, AND AFFECT IN FRACTION MEANING-MAKING

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This study examines how mathematical symbols acquire meaning as students construct and revise judgments through embodied grounding and affective shifts. Drawing on symbol grounding theory (Lakoff & Núñez, 2000), research on rational number conceptions (Behr et al., 1983), and the process–object shift in mathematical thinking (Sfard, 1991), we investigate how students' fraction meaning-making unfolds as they justify their judgments. Participants were fifth-grade students solving a comparison task involving 1/2, 2/3, and 3/5. The study consisted of two phases. In Phase 1, students judged magnitudes using any method. In Phase 2, they were asked to explain their reasoning using diagrams. At three moments in each phase—seeing, solving, and finishing—students reported their affect using three facial-expression categories and briefly explained their choices. Written responses and drawings were analyzed qualitatively to examine how judgment, grounding, and affect interacted. This design allowed us to trace how grounding and affect shaped students' judgment processes.

One focal case illustrates how shifts in grounding were closely linked to changes in affect. In this case, diagram-based grounding enabled the student to reorganize their understanding of quantitative relations and stabilize the final judgment. Across cases, grounding played different roles: it sometimes confirmed existing judgments, sometimes triggered conceptual reorganization accompanied by increased anxiety, and sometimes failed to connect to judgment, resulting in sustained uncertainty.

These findings suggest that embodied grounding does not automatically support understanding but functions differently depending on how it is integrated into students' evolving judgment processes. By incorporating affect into symbol-grounding research, this study proposes a framework for describing mathematical meaning-making as a dynamic interaction of judgment, grounding, and affect over time. More broadly, the results suggest that diagrammatic or concrete representations may initially intensify anxiety while simultaneously enabling conceptual reorganization.

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QUESTION-FIRST IS NOT A FREE LUNCH

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In math word problems, it is often recommended to present the instructions for the task at the beginning; however, research results regarding its effectiveness are divided (Thévenot et al., 2007; Böswald & Schukajlow, 2022). We hypothesize an aptitude-by-demand interaction: the question-first cue helps mainly with high-demand items but becomes redundant or even misleading for low-demand items and for more experienced solvers (expertise reversal; Kalyuga, 2007). Grade 6-7 students (N = 572) solved four word problems (two low-demand; two high-demand) with counterbalanced question positions (Before/After; ABAB/BABA). We fitted a linear probability model with item and grade fixed effects and student-clustered robust standard errors. Item demand moderated the effect: for low-demanded items, the question-first effect was +0.31 pp (95% CI [-2.57, +3.20], p = .833), whereas for high-demanded items, it was +5.81 pp (95% CI [+1.56, +10.05], p < .01); the interaction was +5.50 pp (95% CI [+0.31, +10.69], p < .05). In the student subsamples, the overall effect was +3.6 pp; it concentrated on high-demanded items (error-prone: +9.9 pp; error-free: +5.3 pp) and was near zero for low-demanded items (error-prone: -1.4 pp; error-free: +1.1 pp). The results support demand-dependence: question-first produces a clear gain on high-demand items but not on low-demand items. These patterns align with the goal-cue/situation-model benefit under high integration demand (Thévenot et al., 2007) and with redundancy/expertise reversal when the goal is easily inferred (Kalyuga, 2007). Because assessment type is a proxy (two items per type), future work should manipulate specific demand features and collect process measures to test the mechanisms.

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A MIXED-METHOD STUDY ON (PRE-SERVICE) TEACHERS' BELIEFS ABOUT THE USE OF AI IN MATHEMATICS EDUCATION

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The rapid spread of Artificial Intelligence (AI) into everyday life and mathematics classrooms raises pressing questions about how (pre-service) teachers conceptualize its pedagogical, ethical, and assessment-related implications. Standard Likert questionnaires, although widespread in belief research, tend to capture decontextualized agreement ratings and are vulnerable to social desirability, which may mask internal tensions and contradictions in teachers' views (Safrudiannur & Rott, 2020). This poster presents a mixed-method study that combines Q methodology with qualitative content analysis to investigate pre-service and in-service teachers' beliefs about using AI in mathematics education. First, 32 participants completed a Q-sort of 29 theoretically grounded, context-rich statements on opportunities and risks of AI in mathematics classrooms, arranged on a forced normal distribution from strong disagreement to strong agreement (Lim-Ratnam et al., 2022). Second, immediately following the Q-sorting, guideline-based interviews probed 10 participants' reasoning for extreme placements, the classroom situations they imagine, and perceived trade-offs. Interview transcripts were analyzed using qualitative content analysis, linking coded segments to specific Q-sort positions. This makes it possible to reconstruct not only which constellations of statements participants endorsed, but also how they justify and qualify these endorsements in concrete teaching scenarios. Results reveal a pattern of reflective openness combined with pronounced ambivalence. While AI is perceived as offering instructional support and opportunities for differentiation, concerns about competence loss, misinformation, and ethical risks remain salient. Importantly, participants across groups emphasized the irreplaceable pedagogical authority of the teacher. The poster contrasts the study's design with conventional Likert-scale instruments for measuring teacher beliefs about AI. While Likert items may elicit independent ratings of broadly desirable goals, the Q-plus-interview approach here required participants to prioritize between competing, context-rich statements and articulate their situational reasoning.

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TEACHING HOW TO TEACH MATHEMATICAL MODELLING IN PRIMARY SCHOOL

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Mathematical modelling in primary school can help young learners make sense of mathematics by linking it to their everyday lives. Hands-on, experience-based work on modelling problems engages pupils' senses and invites embodied engagement with representations. Implementing such experiences, however, poses specific demands on teachers. We therefore developed and evaluated a seminar in primary teacher education to foster the pre-service teachers' professional competence for teaching mathematical modelling. Its content and design adapt insights from the far more extensive secondary-school research base to primary classrooms (for an overview see Greefrath, 2025). Our poster visualises the seminar concept, illustrates it with exemplary tasks and sample lesson plans, and presents results of the evaluative intervention study. Pre-service teachers' pedagogical content knowledge for teaching mathematical modelling in primary schools (modPCK) is conceptualised based on a competence model for the secondary level (Borromeo Ferri & Blum, 2010; Greefrath et al., 2021). With the aim to develop modPCK, the seminar combines theoretical block sessions with practical components, enabling students to experience modelling and teaching modelling themselves, in line with the conference theme. Across 38 participants, modPCK increased significantly over the seminar (3 measurements; rmANOVA, Greenhouse–Geisser; F (1.28,47.34) = 37.50, $p < .001$, $\eta^2 = .50$). We also observed positive effects on volitional competence aspects and interrelations between both. Building on the conceptualisation of modPCK and its implementation in the seminar, next methodological steps are (a) scaling up to larger cohorts and longer time spans and (b) developing methods to assess how it is enacted in teaching practice. In teacher education practice, the presented seminar is transferable and can be adapted, implemented, and refined to support broader integration of mathematical modelling.

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MAKING SENSE OF MANIPULATIVES IN SYMBOLICALLY STRUCTURED ENVIRONMENTS: A COLLABORATIVE DESIGN WITH LOWER-SECONDARY TEACHERS

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A "critical issue" about manipulatives (here, non-digital objects handled by students for mathematics education) is the limited interest they receive at secondary school, from teaching guides and from research (Bartolini & Martignone, 2020, p. 367). In the Swiss canton of Vaud, too, manipulatives are scarce in teaching recommendations for lower-secondary level (age 11). In fact, sensory experiences (including bodies with manipulatives) are excluded from mathematics, in favour of formal language.

This research explores how manipulatives, bodies and formal language can coexist at the lower-secondary level. It builds on Symbolically Structured Environments (SSEs, Coles & Sinclair, 2019), in which symbols stand for actions or distinctions with manipulatives. SSEs ground in the theory of inclusive materialism. Using this framework, we explore how lower-secondary concepts can exist as assemblages of manipulatives, bodies and formal language. We study how couplings between gestures, diagrams, formal language and manipulatives disturb binary norms about formal language and sensory experiences, through creative acts (de Freitas & Sinclair, 2014).

Using SSEs as a basis, we conduct a collaborative design of manipulatives with four lower-secondary teachers. Through five group meetings, we define the mathematical content, generate ideas of manipulatives, prototype them, test them if asked for and share our process. To capture this exploration, each meeting is videotaped and each person keeps a design diary. From the unscripted design process, we expect creative acts to emerge in couplings between gestures, design drawings, formal language and manipulatives. The method for analysing the videos and the diary notes will be presented in more details in the poster, together with the first research results.

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ANALYSING PROBLEM-SOLVING TASKS USED FOR SMALL GROUP WORK IN "THINKING CLASSROOMS"

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A task can be seen as a classroom activity creating a context for students to engage in thinking about subject matter (Tekkumru-Kisa et al., 2020). As such, tasks are key to mathematics teaching and learning. While Tekkumru-Kisa et al. (2020) characterise tasks according to their cognitive demand, Yeo (2017) characterises their openness based on five variables: goal, method, task complexity, answer, and extension. Tasks are also a key feature in the teaching approach "thinking classrooms" (Liljedahl, 2018), which focuses on small-group problem solving and investigative work, applying low-threshold problem-solving tasks with multiple solutions.

In the current study, we investigate small-group problem solving in upper secondary mathematics classrooms implementing the thinking classrooms approach. Contrary to what is found in previous research, the applied problem-solving tasks typically have only one solution, targeting specific curricular goals. Correspondingly, tasks might be developed to help students develop conceptual understanding or to prepare them for developing knowledge of standard procedures and methods. This might be an artefact of the upper secondary mathematics curriculum or a result of how thinking classrooms and problem-solving tasks are perceived by the participating teachers. The aim of the poster is to present and discuss different approaches to analysing the observed problem-solving tasks. The purpose of using multiple frameworks (e.g. Tekkumru-Kisa et al., 2020; Yeo, 2017) is to develop a task-analysis framework that can help identify and unpack the embedded potential and characteristics of the applied problem-solving tasks beyond that of multiple solutions, showcasing different task aspects to compare these with teacher intentions. The poster will present sample tasks from the ongoing study and show how using different frameworks highlight different aspects of the tasks.

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INTEGRATING FRACTION REPRESENTATIONS USING A NOVEL AUGMENTED REALITY SPIRAL DESIGN

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Fractions are notoriously difficult concepts for learners to understand. One reason may be that there are many different aspects of fractions (e.g., part-whole, measurement, etc.) which are often taught separately, using different visual representations (e.g., area models, number lines, etc.). Siegler et al. (2010) stressed that understanding fractions as magnitudes is crucial and should be supported using number lines. However, fractions are usually introduced to learners using area models such as a pizza. Despite their accessibility for younger learners, these models may delay understanding of fraction magnitudes due to whole number bias. To enhance conceptual understanding, we therefore seek to obviate switching between area and number line representations by integrating them into a single model. Accordingly, we propose a novel, theoretical 3D solution in the form of a spiral staircase rendered in Augmented Reality (AR). When viewed from above, a spiral staircase reduces to a pizza model: the stairs become adjacent sectors of a circle and thus resemble pizza slices. However, when viewed side-on, the pole through its center becomes a vertical number line. This overcomes a major limitation of the pizza model, whose slices are bounded by the whole, as steps (slices) representing improper fractions/mixed numbers can also be incorporated. Moreover, a digital as opposed to physical model allows a fraction's denominator (total steps in a full turn) and numerator (shaded steps) to be instantly changed. Additionally, rendering it in AR as opposed to a 2D screen enables full embodied interaction, enhanced 3D visualization and repositioning or rescaling for situated cognition in any learning environment. In practical terms, viewing the AR spiral staircase via a smartphone screen also enables implementation at scale due to ease of access and minimal cost. Future design iterations will develop the current prototype for empirical comparison to area and number line models. To corroborate our theoretical framing, we would expect it to better enhance performances while qualitative data should provide insights into its ability to facilitate deep and lasting conceptual understanding of fractions. If it can reduce whole number bias and delays in understanding fraction magnitude – enabling both representations to be introduced simultaneously and coherently – there are potentially far-reaching implications for classroom practice and curriculum design.

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CHANGES OF STUDENTS' ATTITUDES TOWARD STATISTICS THROUGH THE TEACHING AND LEARNING PROCESS OF SIMULATION-BASED STATISTICAL INFERENCE

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The Understanding that an inference can be made about a population from a sample is a fundamental aspect of statistical inference. However, when students focus on sampling representativeness, they may believe that a sample is the same as the population. One way to facilitate understanding of the relationship between a sample and the population is to model a real-world situation and simulate it based on that model (Braham, H. M. and Ben-Zvi, D., 2017). The purpose of this study is to design the teaching and learning process of simulation-based statistical inference for 11th graders and to identify students' changes through the teaching and learning process.

The sample comprises 30 students of grade 11. During the learning session, students utilize the Common Online Data Analysis Platform (CODAP), which allows simulations to be run on tablet devices accessed via a browser. The learning activity involves comparing predicted sample distributions before the simulation with the actual sample distributions after the simulation. Students' behaviours are grasped on their descriptions in the learning handouts. The effect of the designed teaching and learning process is verified through pre-survey and post-survey. The Survey of Attitudes Toward Statistics (SATS-36) (VanHoof et al., 2011) is used to investigate attitudes in each survey. The SATS-36 is measured six subscales: affect, cognitive competence, value, difficulty, interest, and effort. Responses to each question are quantified on a seven-point scale. Several questions regarding sample size and sample distribution are used to investigate informal statistical inference ability. From pre- to post-survey there were changes of attitude scores. The difference of affect was 0.05, cognitive competence was 0.20, value was -0.06, difficulty was -0.17, interest was -0.12, and effort was -0.44. These results showed that the score of cognitive competence increased and the score of effort decreased. It appeared that as students developed statistical inference ability, they tended to require less effort to understand statistics.

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CYCLICAL UNDERSTANDING OF TIME IN EARLY CHILDHOOD EDUCATION

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Time is abstract and intangible. It can only be experienced through the passage of events and made visible by measuring instruments such as clocks. In psychology, research focuses on the influence of time on people, their understanding of the concept and their awareness of it. Attention is paid to temporal orientation, subjective perception and experience, and ultimately the ontogenesis of the concept from early childhood to adulthood. Two concepts of time interact here: the *linear understanding of time*, involving the notion of a beginning and an end, and the *cyclical understanding of time*, involving the notion of periodic recurrence. Friedman (1978) first categorised studies into the following three aspects of time: *Logical time* refers to Newton's scheme, in which events are located on a time continuum. *Conventional time* refers to culturally determined systems and symbols for organising time. *Experiential time* refers to the experience and perception of duration and succession. In mathematics education, the emphasis lies on fostering an appreciation of time as a quantifiable entity characterised by distinct attributes, including benchmark knowledge and the capacity for comparison. Thus, a comprehensive understanding of time requires an appreciation of all various aspects of time, not just its conventional characteristics. Previous studies (e.g. Friedman, 1978; McCormack & Hoerl, 2017) have mostly considered the development of the understanding of time in isolation. The focus has clearly been on the linear aspect of time and the quantitative paradigm from a methodological perspective. This project, however, focuses on the qualitative assessment of the cyclical understanding of time in early education and its possible typology, because this forms the basis for understanding time-measuring instruments considered essential in our culture, such as clocks and calendars. A qualitative assessment of this cyclical understanding requires the temporal aspects to be differentiated further. In this study, 45 German first and second graders were interviewed and videotaped. Qualitative content analysis is used to evaluate the transcripts of the videos. This poster illustrates how a cyclical understanding of time can be evaluated in children's statements using a deductive categorisation system, and how this can be broadened through inductive categories.

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DIFFERENT MEANINGS OF THE CONCEPT OF A VARIABLE MEDIATED BY SIMULTANEOUS EQUATIONS EXPRESSED WITH ICONS IN TWO FINNISH ELEMENTARY SCHOOL MATHEMATICS TEXTBOOK SERIES

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New uses of variables, in addition to that of the unknown, may pose a major obstacle for students making the transition from arithmetic to algebra (Ely & Adams, 2012). Nevertheless, early algebra experiments have shown that even kindergarten and elementary school children have the capacity for algebraic thinking with, for example, varying variables (Blanton et al., 2017). In this research, the tasks in the two main mathematics textbook series for elementary schools in Finland, *Milli* and *Oivaltaja*, were analysed in a search for tasks that could be used to prepare and support students' learning of the different meanings of the concept of a variable in school algebra. During this analysis, an interesting type of task, simultaneous equations of several variables, which were written with icons instead of letter symbols, was found. Those 148 tasks were analysed by a deductive content analysis using a framework developed by the author and her colleague (Partanen & Tolvanen, 2019).

Although all the simultaneous equations and equations with icons were first classified to mediate the meaning of the *unknown* in the context of equations and inequalities, in the solution processes of the equations, also the meanings of the *generalised number*, *varying variable* and *generalised quantity* were needed in the reasoning. The author discusses expanding the potential of the type of task. For example, the laws of operations of real numbers could be discovered, or varying variables in functional relationships could be introduced through them. The results and discussion show that advanced algebraic ideas about the meanings and uses of variables could be made available to elementary school students through mediation by the tasks of this format.

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ECEC EDUCATORS' AWARENESS OF 4- TO 5-YEAR-OLDS' MATHEMATICAL SKILLS

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This study aims to explore ECEC educators' awareness of 4- to 5-year-olds' mathematical skills, because it is critical in supporting age-appropriate (Clements & Sarama, 2021), wide-ranging (Gonulates & Gilbert, 2023), and regular (Parviainen et el., 2023) mathematical learning in ECEC. It is not, however, evident how comprehensive educators' awareness of children's mathematical skills is. In this study, educators' awareness of mathematical skills is conceptualised as ability to identify age and developmentally appropriate core mathematical skills and is explored in relation to Parviainen's (2019) theoretical framework of early mathematical skills development. Web survey data were collected from ECEC educators (N = 54) who participated in EarlyMath research project. Data were analysed with the qualitative, theory driven, content analysis. Ethical review was conducted for the research project and the ethical principles (e.g., voluntary nature of research participation and careful data analysis) were followed. The findings demonstrate that ECEC educators were best aware of 4- to 5-year- olds' numerical skills and fairly of abilities for spatial thinking and mathematical thinking and reasoning. However, many of the educators failed to identify core skills comprehensively and description of different sub skills, developmental level or age-appropriate number range was often missing. The findings suggest that ECEC educators benefit professional development for their awareness of 4- to 5-year-olds' mathematical skills.

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CROCHET AND AI GEN IN MATHEMATICAL EXPLORATION BY UNDERGRADUATE MATHEMATICS STUDENTS

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The notion of humans with media, which emphasizes the idea that knowledge is produced by the collective of human and non-human beings, has reshaped how learning processes are conceived in Mathematics Education (Borba & Villareal, 2005). In this framework, emerging digital media such as Generative Artificial Intelligence (AI Gen) invite renewed discussions about how mathematical ideas are experienced in hybrid collectives, while aesthetic perspectives in mathematics education highlight the role of perception and sensorial engagement in such processes (Sinclair, 2006). In this study we present the initial results of an ongoing research that aims to investigate how undergraduate mathematics students explore tasks involving crochet and mathematics using AI Gen applications. Methodologically, we conducted teaching experiments with three pairs of students from a public university in Brazil (4 sessions of 2 hours each pair). The sessions and the computer screen were recorded, in addition to written records, with the aim of triangulating the data.

The experiments involved learning crochet, identifying mathematical patterns (e.g., symmetries, arithmetic and geometric progressions, hyperbolic geometry), and analyzing the AI Gen's capacity to recognize and to generate and analyze images of these structures. In the episodes, the students reformulated prompts to obtain more precise responses, evaluated and sometimes rejected incomplete or inadequate outputs and revised their own calculations during the interactions, demonstrating that AI Gen acted not only as a verification tool but as an epistemic participant in the collective process. The students compared their reasoning with the AI Gen outputs, and with the crochet pieces, indicating that both the digital and material elements shaped mathematical interpretations. These episodes suggest that the combination of manual work and AI Gen technologies can create multimodal conditions for mathematical exploration, promoting aesthetic engagement and reshaping how future teachers experience and interpret mathematical ideas.

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MATHEMATIZING BODY MOVEMENTS THROUGH THE LENS OF THE OBSERVER USING YOLOv8

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In early years, body movements are central to growth and show what kind of learning is happening, because verbalisation is less common (due to the developmental process) on this context whatever method or methodology is applied in mathematical education research, the observer plays a crucial role in determining what mathematics the participant is bringing forth (Ramirez, 2025).Hence, based on enactivist approach where body, is understood as an adaptively autonomous and sense-making system (Thompson, 2017, p. xxvi) showing itself as a constant problem-solving, a way to making sense of the world that also can be see it in mathematical terms. On this ongoing research about children 4-5 years old, mathematics and body movements, we sought to find an analytical tool that would allow us to explore body movements from the observer's perspective, what aspect to consider on it and how to mathematise what is observed through the movements. This involved identifying methodological criteria to describe the process of mathematisation of body movements, when this is a manifestation of mathematics learning. To do so, we adopt a quantitative-computational methodological approach based on computer vision and mathematical modelling of human body movement named YOLOv8. We analyse a video of a girl (4 years old) maintaining her balance by placing her feet on one rope and her hands on another. As a result, we propose their estimation of the joints because how proprioception is involves, a temporal evolution of vertical position (time/speed) and recognise the coordination of the observer/researcher and his own mathematization. Finally in this poster, we will present the girl's motion analysis using YOLOv8 based on human position estimation, their graphic and their mathematization, seeing a detailed account and the multiple recognitions prompt. Pursuing reflective questions on methodological challenge posed by observer body movements and mathematics.

Acknowledgment: FONDECYT INICIACION ID11251892

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DESIGNING TEACHER EDUCATION FOR MATHEMATICAL PROMISE: ITERATIVE DEVELOPMENT IN PRACTICE

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Teacher education plays a central role in preparing pre-service teachers (PSTs) to meet the needs of mathematically promising students (MPS). While research on giftedness and differentiation is well established, fewer studies examine how mathematical promise can be positioned and operationalised within teacher education programmes. In this study, mathematical promise is conceptualised as a domain-specific, dynamic, and context-dependent phenomenon that develops through opportunity, challenge, and support, requiring deliberate pedagogical consideration. This poster reports on findings from a study investigating what design insights emerged through the iterative development of a seminar and subsequent practicum addressing mathematical promise in teacher education. The study adopts an educational design research approach (McKenney & Reeves, 2018) and comprises three iterative cycles (two completed, one ongoing). Each cycle includes a targeted seminar on MPS followed by a practicum in which PSTs were encouraged to address MPS in their mathematics teaching. Data was generated through voluntary semi-structured interviews and written reflections from participating PSTs (n = 8 across two cycles). Guided by cultural-historical activity theory (CHAT), the analysis conceptualised tensions as systemic contradictions within and between activity systems. PSTs' reported experiences constituted the primary data informing the redesign between cycles. Successive iterations sought to address identified contradictions and strengthen alignment between conceptual input and classroom enactment. The analysis led to the formulation of emerging design guidelines, reflecting implications at both programme level: (1) systematic integration of mathematical promise across the teacher education programme, and intervention level: (2) explicit engagement with tensions between inclusive ideals and upward differentiation, (3) a shift from quantitative differentiation towards qualitatively different forms of mathematical challenge, (4) prioritisation of instructional judgement over ready-made strategies, and (5) structured translation between conceptual understandings and classroom action. Taken together, these findings indicate that preparing PSTs to meet the needs of MPS requires more than the addition of an isolated seminar. Rather, they point to the need for coherent integration and a repositioning of mathematical promise across the teacher education programme, alongside the cultivation of instructional judgement within inclusive frameworks. A third cycle is currently ongoing and will allow further testing, refinement, and consolidation of these emerging design guidelines.

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BEYOND THE SCREEN: A META-ANALYSIS OF DIGITALLY ENRICHED MATHEMATICS LEARNING IN PRIMARY SCHOOL

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The effectiveness of digitally enriched learning may depend on two aspects: the implementation of digital learning environments and the ways students use the learning opportunities provided by these environments, both of which may further be shaped by a variety of contextual factors (Reinhold et al., 2024). Previous meta-analyses have reported positive average effects of digital mathematics learning, but have rarely examined systematic moderators, and only limited attention has been given to primary education. This meta-analysis addresses the questions: Do digitally enriched approaches in primary mathematics instruction lead to greater learning gains than non-digital instruction? Do the design of digital tools, characteristics of the studies, the instructional context, and the implementation and use of technology moderate these effects?

Peer-reviewed studies published between 2014 and 2024 were included if they investigated digitally enriched mathematics instruction in Grades 1–4, reported cognitive outcomes, and used (quasi-)experimental pre-post designs with control groups. After a multi-stage screening of 1,076 records, 49 studies with 315 effects were included. To examine potential moderators, theoretically grounded variables were coded describing the learning environment, study design, contextual characteristics, and the interaction of technology implementation and use. Effect sizes were estimated using multilevel random-effects models; moderators were tested using mixed-effects meta-regressions.

Results show a robust positive effect of digitally enriched instruction on students' mathematics achievement compared to non-digital instruction ($g = 0.46$). Moderator analyses indicate that technology use and instructional design, as well as the mathematical content domain, are associated with differences in effect sizes, whereas many commonly discussed features show no systematic effects. The findings suggest that digital tools can substantially support mathematics learning in primary school, but their effectiveness depends on how they are implemented and used in classroom practice.

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REFRAMING NUMBER SENSE THROUGH REPRESENTATIONAL WORK

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Number sense is framed as structural understanding of number grounded in relationships, representation, and operational meaning (McIntosh et al., 1992). Although curricula emphasize such understanding, teachers' engagement with mathematics often remains predominantly procedural. This study examines how supporting primary teachers' conceptual understanding of number and operations influences instructional practice.

The study draws on McIntosh et al.'s (1992) number sense framework and Ball et al.'s (2008) model of Mathematical Knowledge for Teaching. Eight teachers (grades 3–6) participate in an eight-month research-formation process combining workshops and classroom experimentation. Data include journals, observations, transcripts, and interviews. A qualitative thematic analysis examines shifts in teachers' number sense and practice. Data collection is ongoing; findings are based on analyses to date.

Preliminary findings indicate that transformation first emerges in teachers' interpretation of mathematical understanding. Timed and representational tasks revealed a gap between procedural fluency and structural reasoning, prompting destabilization and recognition of rule-based reliance. Teachers began reinterpreting students' apparent success as procedural rather than conceptual. Representational tasks functioned as diagnostic tools, making structural gaps visible and reshaping judgments of mastery.

Professional change thus appears to originate in interpretive practice before stabilizing at the level of belief. Representational work mediates this shift and is reflected in greater use of manipulation and representations, less reliance on workbook-driven instruction, and more sustained attention to structurally demanding concepts.

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THE LONGITUDINAL RECONFIGURATION OF PRE-SERVICE TEACHERS' ASSESSMENT CONCEPTIONS IN MATHEMATICS

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Formative assessment is widely promoted in mathematics education, yet tensions persist between formative ideals and performance-driven practices. While teachers' assessment beliefs have been studied, less is known about how pre-service teachers' conceptions evolve longitudinally and how this evolution relates to their developing understanding of mathematics as a disciplinary practice. This study examines how assessment conceptions are reconfigured over a two-year teacher education program and how practicum mediates this process. It asks: How do initial assessment conceptions evolve across coursework and practicum, and how is this evolution linked to shifts in participants' epistemic views of mathematics teaching?

Grounded in research on assessment conceptions (Brown & Gao, 2015) and formative–normative functions (Laveault & Allal, 2016), conceptions are understood as interpretive frameworks connecting theory, institutional norms, and disciplinary epistemology. Assessment competence is thus conceptualized as situated professional judgment. A qualitative longitudinal collective case study was conducted with six pre-service teachers. Data were collected at two points: a questionnaire at the end of year one and interviews after the final practicum. Quantitative profiles (formative, normative, criterial) were complemented by inductive thematic analysis and cross-case comparisons. At T1, participants showed strong formative orientations (means 4.0–4.7) and low normative orientations (1.3–2.1), though traces of performance-centered beliefs persisted. By T2, four shifts emerged: assessment was reconceptualized as a cyclical regulatory process; attention moved from procedural correctness toward reasoning; professional judgment became more holistic; and practicum emerged as a key mediating space of transformation. Despite these changes, institutional constraints continued to limit full enactment of formative principles, indicating that conceptual development is negotiated rather than linear. The study theorizes assessment change as an epistemic-professional negotiation shaped by disciplinary understanding, institutional pressures, and practicum mediation within mathematics education.

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DOUBLE INSTRUMENTAL GENESIS IN PRE-SERVICE MATHEMATICS TEACHERS

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We analyse how do future mathematics teachers develop double instrumental genesis (Haspekian, 2023), in terms of personal instrumental genesis (GIPE) and professional instrumental genesis (GIPR) when constructing conics with Dynamic Geometry. We are interest in the conics teaching because offer a rich context: their teaching tensions synthetic and analytical approaches using dynamic geometry. Our framework recognizes how double instrumental geneses include usage schemes of future teachers, that is, their invariant activity organizations comprising goals, action rules, and operational invariants (Vergnaud, 1998). Research method is based on a naturalistic case study (Yin, 2009). Over 5.5 months, data were collected from Víctor (pseudonym), a final-year student for math teacher in Colombia. Our data set includes videos of peer collaboration, planning and advising sessions, implementation in classroom, and a final interview. We are using a mixed coding procedure using grounded theory coding yielded 15 episodes and 11 usage schemes.

First results. In his lesson plan to teach conics, Víctor adopted an analytic-static approach using GeoGebra to type canonical equations to display graphics, without using dragging or sliders. It was remarkable that one of his theorem-in-action was "*the intersection of two parabolas is an ellipse*". Later, in an advisor session, an experimented teacher observed Victor's difficulties to understand conics construction, generating a restructuring of his scheme, showing that social schemes require non-immediate accommodation. In the class observed, Víctor progressed from a static displayer to managing technological errors and fostering peer-work.

Final remarks: (a) social schemes orient and constrain both geneses; (b) instrumentation without instrumentalization limits DT didactic creativity; and (c) authentic practicum is the primary catalyst for GIPR evolution. Teacher education should design explicit cycles linking GIPE and GIPR, promoting the use of Digital Technologies in dual situations: personal and professional.

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INDONESIAN PRE SERVICE TEACHERS' EXPECTATION AND CHALLENGE ON STEM EDUCATION IN MATHEMATICS CLASSROOM

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It is undeniable that STEM education has become highly influential in modern curricula. Therefore, it is critical to prepare the pre-service mathematics teachers for the implementation of STEM Education integrated approaches. This study explores the expectations and perceived challenges of pre-service teachers regarding the integration of STEM into mathematics classrooms. We proposed that the nine STEM teacher identities from Holincheck and Galanti (2026) were adjusted into two dimensions: expectation and challenge. These dimensions reflect on how pre-service mathematics teachers perceive STEM Education, both in their current roles and through their future as professional educators. In the expectations dimension, it is characterized by task perception, self-image, recognition, and motivation. While in the challenges dimension, it comprises self-efficacy, competence, performance, teaching interest, and content interest. Using a qualitative approach, data were collected through semi-structured interviews and reflective journals from six third-year pre-service education students who had already taken a STEM Education course. The findings reveal high expectations for enhancing student engagement and demonstrating the real-world relevance of mathematics. This expectation elicits the characteristics of task-perception, motivation, and self-image. However, these expectations are tempered by significant challenges, including a lack of pedagogical content knowledge, limited time for interdisciplinary planning, and a perceived pressure to prioritize standardized testing over project-based learning. These challenges highlight the characteristics of competence, performance, and task-perception. The study concludes that while pre-service teachers are intrinsically motivated to adopt STEM, teacher education programs must provide more explicit, hands-on training in interdisciplinary integration to bridge the gap between expectation and practice.

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TRANSFORMATION OF TEACHERS' MATHEMATICAL KNOWLEDGE THROUGH THE DESIGN OF AN EDUCATIONAL ARTIFACT

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Research about educational artifacts has typically focused on digital artifacts or on the impact of using an artifact on student learning. There is less research on the design process of the artifact and its impact on its designer. This project proposes to analyse the transformation of teachers' mathematical knowledge through the design of a material educational artifact.

Two theoretical frameworks support this empirical research. First, the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) describes a four-pole model to assist teacher designers. There is a constant interaction between the artifact, its usage patterns, the mathematical target and staging the scenario of the lesson built to achieve the expected student knowledge. Second, embodied and situated cognition, and more specifically the ACT-IN model (Versace et al., 2018) describes how memory traces of past sensorimotor experiences are gradually transformed during recurrent multimodal exposures in similar situations. Knowledge arises from the coupling of present experience with memory traces of past experiences. Explicit and tacit knowledge (Polanyi, 1966) evolve on a continuum through the reactivation of traces.

Our hypothesis: Teachers' mathematical knowledge will be transformed by the design process of an artifact intented to solve a mathematical educational problem.

We will propose a design process teacher training course with a pre- and post-questionnaire focusing on explicit knowledge. An explanatory interview, supported by the images of the evolution of the artifact's design, will focus on the tacit knowledge. We expect to show that designing an educational artifact helps teachers to expand their understanding of a mathematical concept, through the artifact under construction.

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MATHEMATICAL LEARNING IN MOTION: EMBODIED COGNITION AND MOTION TRACKING IN KINDERGARTEN

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The learning of basic mathematical skills in kindergarten is fundamental for mathematical development. A central aspect is the development of number sense (Jordan et al., 2006). However, many children experience difficulties in acquiring basic mathematical skills in early development. Recent studies show that learning processes can be effectively supported by embodied learning activities (Abrahamson et al., 2020). In this research project, a mathematical learning environment for kindergarten children is being developed that follows the embodied cognition approach. We present an exploratory pilot study that aims to investigate embodied learning processes of kindergarten children. In particular, we work with a mathematical learning environment that focuses on promoting number sense, particularly the cardinal and ordinal number concept. The empirical research question is: *How does embodied learning facilitate the acquisition of mathematical skills in kindergarten children?*

To methodologically capture the embodied learning processes, we use a marker-based motion capture system (Qualisys) in addition to video recordings, since motion capture systems have shown to offer opportunities to investigate embodied learning processes and are increasingly used in educational settings (Reuter & Schindler, 2023). The methodological research question is: *What are opportunities and challenges of motion capture systems to investigate young learners' embodied mathematics learning?* The learning environment is embedded in a space context, where the kindergarten children are space travellers and the planets represent different learning goals and kinds of embodied learning activities (realized, e.g., through jumping, turning). This pilot study is the first step in our embodied research project for young learners. It will give first insights into the empirical as well as the methodological research questions that serve as springboard for the further developments.

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UNDERSTANDING STUDENTS' USE OF MATHEMATICAL VISUALIZATIONS IN TEXTS THROUGH READING ACTIONS

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Visualizations are used in mathematics textbooks for various reasons, e.g., to convey essential information or to support comprehension. We know that experienced readers use all available semiotic resources (Fang, 2023), and that gaze patterns are useful when examining how readers interpret and integrate visualizations with other semiotic resources (Schreiter & Vogel, 2023). In this ongoing study, we investigate strategies for comprehending mathematical texts that include visualizations containing essential information. In contrast to previous research, we connect reading comprehension to students' use of visualizations by employing eye-tracking as the methodological approach. In this study, 28 upper secondary students read two mathematical expository texts, each including symbolic expressions, visualizations and verbal language. Each text was followed by a set of reading comprehension questions and the participants' gaze patterns, written answers and voices were recorded. For this presentation, we analysed data from the 5 best- and worst-performing students based on their comprehension scores. Each visit to a visualization found in the visualized gaze patterns was categorized as one or more "reading actions". We define a reading behaviour as a sequence of reading actions, which, in turn, is an up to 5 second event in the gaze pattern. An example of a reading action is "link", which is recognized by a back-and-forth movement between visualization and verbal language or symbols typically forming a triangular shape in the visualized gaze pattern. Reading actions are preferably presented visually and will be elaborated on in the presentation. Analyzing students reading behaviour as a sequence of reading actions makes it possible to characterize different readers in detail. It also enables drawing conclusions about which reading behaviours lead to better reading comprehension, with potential implications for teaching and textbook design. Early indications from our analysis suggest that greater use of the "link" action when reading is connected to higher reading comprehension. Further research is needed to determine whether the students use these actions intentionally and, if so, for what purpose.

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STUDENTS' USE OF VISUAL REPRESENTATIONS WHEN WORKING ON WHITEBOARDS DURING PROBLEM SOLVING

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Visual representations play a central role in students' mathematical thinking, learning, and communication: students can engage with visual representations in many different forms, for instance drawings and graphs. Being able to utilize, interpret, create, and translate visual representations is an important skill for students as this can often be a key strategy in problem-solving activities (cf. Arcavi, 2003). Understanding how students use visual representations when engaging in problem solving is thus important. In this study, we investigate students' problem solving in upper secondary classrooms that adhere to the "Thinking classrooms" approach (Liljedahl, 2018), where students work in groups of three on vertical whiteboards to solve mathematical problems. The use of whiteboards allows students to create and utilize visual representations, prompted or on their own accord, as well as re-draw, add, and change their representations, due to the non-permanent nature of the whiteboards. There is limited empirical research from classrooms using this teaching approach, and accordingly, little knowledge surrounding how students utilize visual representations as they work jointly on problems in this setting. In this ongoing study, we aim to understand what kind of visual representations students use when working jointly on problem-solving activities. Drawing on video data from an on-going study on "Thinking classrooms" from Norwegian upper secondary mathematics classrooms, we utilize video observation and photos of students' vertical boards to study student use of visual representations during problem solving. To analyze the data, we use an abductive coding approach, focusing on 1) what kind of visualizations are used, and 2) the role of the visualization in the students' problem-solving process, distinguishing between prompted representations and those created on students' own accord. Preliminary findings suggest that the visual representations act as anchors, facilitating both discussions and the problem-solving processes. At the same time, representations could also confuse the students, for example when graphs are poorly drawn or incorrect. The poster will present illustrative examples of visualization representations from the observed classrooms, and our analysis (and coding schema) of said representations.

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ENHANCING PEDAGOGY STUDENTS' STATISTICAL THINKING THROUGH PROBLEM-POSING BASED LEARNING (P-PBL) INSTRUCTION

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Statistical education plays a fundamental role in undergraduate teacher preparation (Frischemeier & Biehler, 2017). In Mexico, graduates of Pedagogy programs often become elementary school teachers responsible for teaching mathematics. Research has documented deficiencies in prospective mathematics teachers' abilities related to data interpretation, decision-making, and critical evaluation of information (Rodríguez-Alveal & Díaz-Levicoy, 2025). This master's research, grounded in the Statistical Thinking Framework (STF), examines the impact of implementing the Problem-Posing Based Learning (P-PBL) instructional model on the development of statistical thinking in undergraduate Pedagogy students enrolled in a Descriptive Statistics course. The analysis considers progression across four levels of statistical thinking –idiosyncratic, transitional, quantitative and analytical– within four data-related constructs: description, organization, representation and analysis (Jones et al., 2000). This qualitative research follows an instrumental case study design to examine the statistical thinking of a specific group of 36 pedagogy students who participate in a problem-posing instructional intervention, designed according to the P-PBL instructional model, consisting of four phases: 1) the teacher introduces a problem-posing situation, 2) provides a problem-posing prompt, 3) students pose problems, and 4) the teacher selects problems for students to solve and discuss (Cai, 2022). The poster to be presented at the 49th PME Conference will showcase the problem-posing situations and prompts developed for the intervention, as well as preliminary results from the pilot study.

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A CROSS-CULTURAL ANALYSIS OF DIAGRAMS AND FORMULAS IN TEXTBOOK PROOFS

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The study of cultural differences in mathematics classrooms has long shown that proof and proving are embedded in institutional norms and in the semiotic and linguistic resources through which reasoning is communicated and validated (Miyakawa & Shinno, 2021). Recent textbook research extends this work by comparing how proving is linguistically framed across countries, documenting differences in the repertoires and uses of proof-related words that signal what kind of justification is expected, as well as differences in how proofs are organised and presented as stepwise procedures (Hakamata et al., 2025; Fischer & Oldenburg, in review for publication). From a cognitive–semiotic perspective, this challenges the assumption that formulas and diagrams form a language-independent code. Duval (2006) argues that mathematical objects are accessible only through semiotic representations and that understanding relies on coordinating at least two registers and on conversions between them, often supported by natural language and the way written text frames symbolic treatments. Consequently, even when the same formulas or diagrams appear across textbooks, their interpretability may differ depending on how explicitly the text guides learners in linking representations and in understanding their role in the argument (Duval, 2006).

Within the project "From **A** to **B**", we examine cultural differences across **A**ustria, Germany, Portugal, and **B**razil in proving activities in textbooks. We present first results showing cross-country differences in the use of diagrams and formulas in school textbooks when presenting the same geometry proof at the secondary level.

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A SYSTEMATIC LITERATURE REVIEW OF QUALITATIVE RESEARCH ON SHADOW EDUCATION IN MATHEMATICS

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Shadow education is a broad concept encompassing instructional support offered outside the regular school context. As Bray (2024) summarizes, research in this area is largely *quantitative*, often measuring the amount of tutoring received and its association with academic outcomes such as mathematics performance, with mixed results. Against this background, we present a systematic literature review of *qualitative* research that can contribute to a deeper understanding of the role of shadow education in mathematics. The following research questions were addressed: RQ1: How are the effects of the support characterized in the studies? RQ2: How is the relationship between the support and the formal schooling described?

A systematic literature search was conducted in the databases Educational Database, ERIC, Scopus, and Web of Science, resulting in 31 articles. The study was limited to studies on students aged 14–19, and data were analysed using thematic analysis.

The literature review includes digital support in asynchronous (apps, learning platforms, videos) and synchronous (online one-to-one tutoring) formats. In contrast, non-digital support was limited to synchronous formats such as tutoring programs. The effects of the support (RQ1) are characterized as *affective* (engagement, motivation, confidence), *learning-related* (understanding, student agency), *social* (relations, collaboration), and *psychological* (reduced stress, belonging). The relation between support and formal schooling (RQ2) is described as *complementary*, where support supplements regular classroom instruction, as *contrasting*, offering alternative ways to learn mathematics, and as *compensatory*, compensating for inadequate regular classroom instruction. In addition, the relationship is understood through more complex frameworks, such as *seamless learning*, which emphasizes continuity across learning settings, and *transboundary learning*, focusing on how skills, strategies, and identities developed with support can be transferred back into school.

This literature review will be a foundation for future qualitative research on students' use of shadow education in mathematics. In the poster presentation, the results are further elaborated and their implications highlighted.

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CATEGORICAL, ORDINAL, AND INTERVAL TIME REASONING

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Time teaching and learning has been reduced to clock reading and basic elapsed time calculations in elementary school (Burny et al., 2013; Kamii & Russell, 2012); though, time reasoning is a skill that extends across K-12 education and beyond. We described a framework for time ideas in K-12 education to situate mathematics content together with other critical time ideas across content domains (see Earnest & Smith, under review). This framework includes three facets of time that reflect different conceptualizations: *Categorical* (limited categories of experience; e.g., yesterday versus tomorrow signifying any event that has happened versus has yet to happen), *Ordinal* (logical sequences of experience; e.g., breakfast then lunch then dinner), and *Interval* (the mathematization of experiences; e.g., longer than or five minutes). We interviewed 4-, 5-, and 6-year-olds from diverse families and communities across the United States (n=32). Using background from parental surveys and interviews, each child was asked to reflect on the time of their lived experiences (e.g., the duration of swim lessons or when karate is).

In this poster, we present instances where participants articulated the time of their experience through the three facets of time. For example, 4-year-old Sam (non-gendered pseudonym) was asked when the fireworks at a theme-park started. Rather than stating a specific clock-time, a fact that may have been inaccessible to Sam, they recounted a sequence of events over the day stating each event while tracking on their fingers. Beginning with "right now, that is like ten" as they put up their first finger, Sam continued until reaching the fireworks at the fifth finger count. We posit this ordinal framing of time reflected Sam's construction of what time, itself, is. We propose that children's lived experiences with categorical, ordinal, and interval time are critical aspects of one's overall time understandings that, eventually, include standard tools and measures for time.

Additional information: This work was supported by a Spencer Research Grant on Education: Small.

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IDENTIFYING ELEMENTARY STUDENTS' MATHEMATICS VALUES AND AFFECT THROUGH STUDENT-GENERATED METAPHORS

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Globally, large-scale assessments have repeatedly shown that students in high-performing education systems do not necessarily report more positive emotions toward mathematics (OECD, 2023). According to Control-Value Theory (CVT), anxiety is cognition-based and arises from students' appraisals of perceived control and subjective value in learning situations (Pekrun, 2006). In addition, emotions in mathematics learning are not merely internal states, but are also embodied, situated, and shaped by social and cultural contexts (Hannula, 2012). Yet, these deeper beliefs and emotional roots are not easy to capture through direct questionnaire items because of social desirability concerns and defense mechanisms. Therefore, this study adopts metaphors as an alternative and creative lens to help upper-primary students express their perceptions of mathematics more naturally. By eliciting students' metaphorical descriptions, the study aims to uncover the values, emotions, and attitudes embedded in their mathematics learning experiences. Specifically, this study addresses the following research questions: (a) How do students express their mathematics-related values through metaphors? (b) How do students express their affective experiences toward mathematics learning through metaphors? We analysed responses from 333 Taiwanese upper-primary students using thematic analysis and sentiment coding. Students were asked to complete the sentence: "Mathematics is like ____ because ____." Results indicate that most students described mathematics as formula-based and practice-driven, reflecting an exam-oriented learning culture. The analysis also indicated that students' affect toward mathematics was predominantly negative, while also revealing mixed emotions in mathematics learning. These findings suggest that metaphors provide a useful lens for understanding how students view mathematics learning, including the emotions and underlying beliefs embedded in their responses.

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CLASS-LEVEL VARIATION IN RELATIONAL UNDERSTANDING OF THE EQUAL SIGN

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The equal sign is a significant semiotic mediator in the transition from arithmetic to algebra. While developmental perspectives often regard the shift from operational to relational understanding of the equal sign as a universal milestone, this study, following Kieran's (2022) framework, conceptualizes relational understanding as a socially and pedagogically mediated habit of mind. The Finnish national curriculum provides minimal explicit guidance regarding the equal sign, leaving student success largely dependent on the individual teachers' instructional practices. This study evaluates Finnish students' relational mastery of the equal sign and its class-level heterogeneity in Grades 3 and 5. This cross-sectional study (N=972; G3, n=518, 67% of local cohort; G5, n=454, 65% of local cohort) used a 12-item instrument designed to evaluate early algebraic reasoning, with four items focusing on the relational dimension of the equal sign. These four items (equivalence: 8+4=_+5, additive decomposition: 4=_+_=…, symbol definition, and symmetry) showed low internal consistency ($\alpha = 0.49$), which is interpreted as a substantive finding reflecting the item-dependent and fragmented nature of students' relational conceptions (Stephens et al., 2013). Across all four items, trends were consistent; for example, equivalence task mastery increased from 34% (G3) to 52% (G5). For this item, results revealed extensive inter-class variation, with mastery ranging from 0% to 64% in Grade 3 and from 27% to 100% in Grade 5. Intra-class fragmentation – defined as the class-size-weighted mean range between a class's highest- and lowest-performing items – widened from 33.0 percentage points (G3, 95% CI [27.5, 38.5]) to 41.2 (G5, 95% CI [34.1, 48.3]). While these t-distribution-derived findings indicate a trend toward increased item-dependent mastery, the change was not statistically significant. These results reveal highly uneven relational mastery across instructional environments, suggesting significant sensitivity to class-level factors. To ensure algebra readiness, findings advocate for explicit focus on the relational properties of the equal sign to reduce instructional variance and provide an equitable foundation for all students.

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THE BINARY SYSTEM AS AN OPPORTUNITY FOR SENSE-MAKING: INTEGRATING CONCRETE MANIPULATIVES AND HISTORY OF MATHEMATICS

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The binary system is the foundation of modern digital technology, yet students frequently struggle to bridge its abstract logic with physical reality. In this work, we combine the history of mathematics with a constructivist manipulative-based pedagogy to provide cognitive tools for mathematical sense-making. Our research addresses two primary questions: (1) How does historical context influence students' construction of meaning and technological awareness? (2) How do manipulatives assist students translate theoretical binary concepts into concrete understanding while enhancing their logical reasoning?

To investigate these questions, we employed pre-intervention questionnaires, post-intervention performance tests, satisfaction surveys and reflective worksheets designed to facilitate collaborative discourse and to help making sense of binary systems and mathematics. This interdisciplinary pathway integrates three components: Interlocking Blocks (IB) as manipulatives to physically represent binary quantities; game-based learning, introducing “Lucas' Fan”, a 19th-century mathematical “magic trick” based on binary numbers; and historical narrative, exploring George Boole’s biography to foster historical empathy and intrinsic motivation.

Preliminary results from an Italian 9th-grade scientific high school (N=27) indicate that this synergy between tactile manipulation and historical context successfully shifts the learning focus from procedural execution (the “how”) to conceptual understanding (the “why”). While historical narrative humanizes mathematical concepts by revealing their origins, physical interaction mediates the transition from concrete experience to abstract reasoning. Ultimately, the structural correspondence among these diverse educational artifacts promotes recursive “spiral learning” consolidating mathematical knowledge and offering a robust pathway for making sense of mathematics.

Acknowledgments

DT acknowledges the support of the Department of Excellence grant MatMod@TOV (2023-27). ES acknowledges the Project EDVANCE by the European Union.

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MAKING SENSE OF PROBABILITY: GRUNDVORSTELLUNGEN IN PRE-SERVICE TEACHERS' EXPLANATIONS

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Constituting meaning is one focus of the so-called *Grundvorstellungen*, i. e. mental representations that "characterize mathematical concepts or procedures and their possible interpretations in real-life situations" (vom Hofe & Blum, 2016, p. 230). Regarding the concept of probability, Malle and Malle (2003) listed four *Grundvorstellungen* which students should develop for a mathematically sound understanding: probability as (1) a measure of an expectation, quantified with a value from 0 to 1, (2) a share of favourable among all possible outcomes, (3) a relative frequency in a long series of trials, and (4) a subjective trust. The poster investigates the question: Which of these *Grundvorstellungen* do pre-service teachers rely on when spontaneously explaining probability? To answer this question, pre-service secondary teachers (PSTs) were asked: "How would you explain to somebody what is meant by 'probability'?". The 22 written answers were analysed using qualitative content analysis (Mayring, 2022). As starting categories deduced from the theory, the four *Grundvorstellungen* mentioned above were used. During the analysis process and the revision of the classification, the first and third category were both split into two categories to capture one aspect each: probability as (1a) an expectation, and (1b) a quantifiable value; (3a) a frequency, and (3b) related to many repetitions of a random experiment. One additional category was added: probability as (5) chance or "how probable" something is. The findings show varying frequencies of the categories, ranging from 0 mentions of category (4) to 9 mentions of category (1a). While in most answers, only one category was identified, nine answers contained more than one category. Only in three answers, categories stemming from different *Grundvorstellungen* were found. The poster will display the co-occurrences of the categories and the underlying *Grundvorstellungen*. Further research could investigate why many PSTs spontaneously activated only one or two of the above mentioned *Grundvorstellungen* and how they can be supported to develop a broader understanding of probability.

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FROM EPISTEMIC LONELINESS TO PROFESSIONAL BELONGING: NEGOTIATING MATHEMATICAL ARGUMENTATION IN A DIGITAL COMMUNITY OF TEACHERS

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Professional loneliness among primary mathematics teachers is often conceptualized in emotional or organizational terms (Maslach, 1982). This study reframes it epistemically, as an erosion of professional recognition. Mathematics teachers' identities develop dynamically through participation in communities (Wenger, 1998; Gee, 2000). Drawing on Godino and Batanero (1999), we distinguish between personal mathematical objects and practices—intuitive, locally developed ways of engaging with mathematical concepts and argumentation—and institutional mathematical objects and practices—normatively validated mathematical knowledge, forms of reasoning, and argumentation endorsed by the professional community. Professional loneliness emerges when these mathematical objects and related practices remain misaligned and teachers' personal meanings are not recognized as professionally legitimate. We examine whether a Digital Community of Teachers can mediate the gap between personal and institutional mathematical objects of argumentation. The methodological design mirrors this theoretical movement: as a researcher, I deliberately moved from eliciting teachers' individual meanings to observing and supporting their collective negotiation of socially and institutionally validated meanings. Eighteen primary mathematics teachers from across Italy participated in a Digital CoT hosted on Google Classroom. Data collection was structured to trace a progression from the personal to the social-institutional dimension: an initial asynchronous questionnaire and individual Zoom interviews elicited teachers' personal mathematical objects and argumentation practices; subsequent asynchronous Padlet tasks and synchronous focus groups supported the public negotiation of meanings within the community. A thematic analysis focused on teachers' meanings of mathematical argumentation, experiences of epistemic professional loneliness, discursive shifts, and identity markers, paying particular attention to how personal meanings were re-positioned, validated, or transformed through collective interaction.

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FUNCTIONS OF PROOF IN MATHEMATICAL PROBLEM POSING STARTING FROM A PROOF PROBLEM

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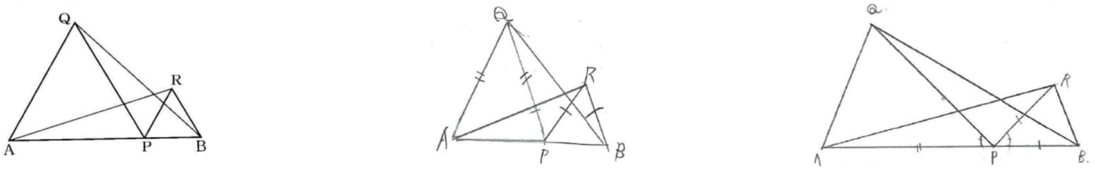
Research on the diverse processes of problem posing has become increasingly important in mathematics education (e.g., Baumanns & Rott, 2022). Specifically, while it has been suggested that during the process of mathematical problem posing starting from a proposition, utilising its proof facilitates deeper understanding (Brown & Walter, 2005), this specific intersection has received limited attention in recent research. Given this context, this study aims to examine the process of problem posing starting from a proof problem, focusing specifically on the diverse functions of proof. A qualitative case study was conducted through a lesson designed for 8th graders, involving problem posing starting from a geometrical proof problem ('prove AR = QB' in Figure 1). This study analysed how students utilised the proof during the process.

Figure 1: Initial proposition Figure 2: Posed problem Figure 3: Refined proposition



The analysis identified three aspects of utilising proof in the problem-posing process: (a) Implicit verification: Students verified the truth or falsity of a proposition posed by changing the initial hypotheses (e.g., Figure 2) by 'mimicking' the initial proof and seeing whether it remained applicable. Notably, students often concluded a proposition was false simply because the initial proof could not be applied—a process distinct from formal verification. (b) Adjustive discovery: Students adjusted the hypotheses of a tentatively posed 'defective' proposition so that the initial proof could be applied, thereby inventing a true proposition (Figure 3). This differs from immediate discovery, as it involves iterative refinement of the posed proposition through the lens of the proof. (c) Distinguishing explanation: Students identified specific conditions that separate true and false propositions (e.g., PA = PQ, PB = PR, $\angle$APQ = $\angle$BPR) through focusing on the logic of the proof. In the presentation, detailed results will be discussed.

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WHY DO STUDENTS STRUGGLE ON THE NUMBER LINE WITH FRACTIONS?

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Understanding fractions as quantities on the number line remains a persistent challenge for elementary and middle-grade students. The number line can support the reconceptualization of fractions as numbers; however, research shows that this representation is particularly difficult for many students to grasp (Tunç-Pekkan, 2015). This study investigates how 7th-grade students interpret and locate fractions on number lines, with a particular focus on identifying reasoning patterns and difficulties. Developing understanding requires reorganizing prior whole-number knowledge (Steffe & Olive, 2010), yet students often rely on whole-number reasoning that impedes their ability to view fractions as magnitudes (Braithwaite & Siegler, 2018). This study draws on Fraction Scheme Theory (Steffe & Olive, 2010) as a framework for both collecting data and analyzing students' reasoning. This perspective supports an examination of how students interpret number-line representations and whether their reasoning reflects part-whole thinking, measure-based reasoning, or other intermediate conceptualizations. A structured interview protocol consisting of 22 fraction problems designed with Fraction Scheme Theory—previously validated set—was administered in one-on-one, video-recorded sessions with 30 seventh-grade students in Türkiye, with analytic focus on six number-line items. Results showed a sharp decline in correct responses after the first number-line item, a pattern not evident in other fraction representations also. Our analysis revealed recurring reasoning tendencies such as treating "1" as the complete number line drawing on the paper, treating n/m as two separate numbers, counting tick marks instead of spaces, and searching for improper fractions within the unit interval of 0-1, having two different locations for "n/n" and "1"—all indicating difficulty interpreting fractions as measures on an extended number line. These patterns suggest that students' part–whole fraction schemes do not readily transfer to measure-based reasoning, and that mediating representations like rectangles may better support the coordination of units and the iteration required for fractions as numbers.

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SUPPORTING ACTIVE LEARNING: INSIGHTS FROM COMMUNITY COLLEGE INSTRUCTORS' USE OF SHARED COURSE MATERIALS

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Evidence-based practices spread slowly because reforms often ignore instructors' beliefs, identities, and contexts. Remillard and Bryans (2004) showed that these factors shape how instructors interpret and enact curriculum, suggesting that providing materials alone does not ensure active learning and implementation conditions matter. Therefore, we were interested in identifying the factors that influence how instructors adopt and use departmentally provided course materials designed to support active learning in a multi-section mathematics course.

Many U.S. students begin at community colleges before transferring to four-year institutions to complete a Bachelor's degree. Chasen et al. (2023) note that these colleges are understudied in curricular research and report that their instructors use active learning more often. To explore how such instructors interpret and implement curricular materials and active learning, we interviewed five community college math faculty teaching a first-year, non-algebra course for liberal arts students.

We analyzed their responses using Reinholz and Apkarian's (2018) Four Frames for Systemic Change: structures, symbols, power, and people. From our preliminary analysis, we identified the following recommendations for lead instructors 1) designate a centralized and streamlined electronic storage platform, 2) increase clarity in instructional intent through conversation rather than in writing, and 3) develop materials flexible enough for instructors at varying stages of active-learning adoption to adapt them to their comfort level.

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THE INFLUENCE OF MATH TEACHERS' BEHAVIOR ON HIGH SCHOOL STUDENTS' MATH ANXIETY: A SYSTEMATIC REVIEW

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Math anxiety among 15-year-old students is increasing worldwide (OECD, 2024) and hinders students from learning mathematics (e.g. Barroso et al., 2021). It is associated with avoidance behaviors, such as passivity (OECD, 2024) and selecting curricula with the least mathematics (Ahmed, 2018), creating a cycle of reduced practice, weaker skills, and increased anxiety. Research suggests that changes in teacher behavior could mitigate this cycle (Finlayson, 2014). This study examines the following research question: What are factors in math teachers' behavior that influence high school students' math anxiety, according to existing research?

Following the PRISMA guidelines (Moher et al., 2009), 745 records retrieved from Scopus, Web of Science, ScienceDirect, and ERIC were screened, resulting in the identification of 25 eligible studies. The most frequently examined teacher-related factor is perceived teacher support, associated with lower levels of students' math anxiety. The findings suggest that teachers can reduce math anxiety by providing both practical and emotional support while allowing students to develop independent problem-solving skills. Conversely, behavior such as failure-based punishment, stereotype threat, and poor classroom management are associated with increased levels of student math anxiety. Finally, the literature does not provide a clear conclusion on the effect of cognitive activation. More research is needed to investigate this and additional teacher-related factors that influence math anxiety in high school.

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HOW A PRIMARY SCHOOL TEACHER STIMULATES STUDENTS' MATHEMATICAL ATTITUDE: A CASE STUDY

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Students' and teachers' mathematical attitude, as consisting of emotional dispositions, beliefs about mathematics, and behavior related to mathematics, is positively related to students' mathematics achievement (e.g., Levine & Pantoja, 2021; Ruffell et al., 1998). In the Netherlands, one of the newly implemented core goals is concerned with stimulating students' mathematical attitude development. In the literature, few educational approaches to stimulate students' mathematical attitude development are described. Prior research shows that teachers play an important role in the development of students' mathematical attitude (Guillaume & Kirtman, 2010). The central question of this study is: How does a primary school teacher stimulate students' mathematical attitude development in the context of a professional learning community (PLC)? In this case study, we follow Jan, as a case of the six teacher participants in a year-long PLC focused on exploring how to stimulate students' mathematical attitude development. Jan works four days a week as a teacher in Grade 6 of an inner-city primary school. The PLC came together for six sessions prepared and led by the first author. Each PLC session followed the same structure: Check-in, reflection, designing an activity, and wrapping up. The activities teachers designed, focused on real-world connection, problem solving, and open-ended questions. Between sessions, teachers taught the designed activities and reflected. Data consisted of written preparations, audio recorded reflections, and observations of PLC sessions and classroom teaching. These were thematically analyzed, deductively and inductively. Findings indicate that Jan translates his knowledge of mathematical attitude into educational activities, in which he centralizes mathematical problems which allow for uncertainty and are connected to real-world contexts. Students' expressions of mathematical attitude, during the activities, became more closely aligned to Jan's own conceptualization. Lastly, the findings show that Jan made a development in more consciously implementing the design principles and describing the connections between characteristics of the design and students' mathematical attitude.

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COMPUTATIONAL THINKING FROM FACETS TO COGNITIVE COMPONENTS: PROPOSAL FOR A TIERED FRAMEWORK

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Formulating stepwise algorithmic solutions is a key 21st-century competence with underlying thought processes subsumed as *computational thinking* (CT), a cognitive ability (Kallia et al., 2021; Tsarava et al., 2022). Dohn (2025) challenges viewing CT as a fixed set of transferable skills, arguing that CT skills as performed activities are inherently situated within their contexts. This aligns with challenges in assessing CT, which often offers limited insight into learners' thought processes. The *research aim* of this theoretical contribution is to propose a tiered framework for CT and to explore its potential as an analytical lens for integrating theoretical and empirical work. The *proposed framework* distinguishes three analytically related levels: *(1) CT facets*: overarching dimensions of CT as a cognitive ability (e.g. abstraction; Tsarava et al., 2022); *(2) CT activities*: domain-specific (Kallia et al., 2021) and situated (Dohn, 2025) enactments of CT thought processes in concrete learning situations (e.g. abstracting when programming figures), drawing on facets; and *(3) CT components*: cognitive elements constituting these activities (von Monkiewitsch & Büscher, 2025). Locating studies within this framework suggests that apparent conceptual tensions may concern different analytical levels rather than conflicting conceptions of CT. Empirical findings may be most interpretable at the *activity* level, while *facets* may offer a higher-level descriptive language and *components* may render learners' CT thought processes more diagnosable. Thus, the framework offers a basis for synthesizing diverse perspectives and bridging theoretical, empirical, and assessment-oriented research.

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EYE-MOVEMENT MODELLING FOR ADOLESCENTS' LEARNING FROM GEOMETRIC WORKED EXAMPLES

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Previous research has highlighted that learning geometric problem solving through worked examples (WE) is effective (Barbieri et al., 2023). Understanding geometric WEs involves integrating text, symbols, and diagrams, relying on appropriate reading strategies. Building on prior research (Lee & Wu, 2018; Wu, 2024), three strategies were developed: (1) identifying symbols with diagrams, (2) recognizing propositions with diagrams, and (3) abstracting the solution plan. To examine whether visual guiding illustrates these strategies, the control group viewed a strategy video, whereas the EMME group viewed the same video augmented with an expert's eye-movement modelling example (Xie et al., 2021). Sixty-eight ninth graders were randomly assigned to two groups (34 per group). The study employed two separate 2 (group) × 2 (ability) ANCOVAs, with ability determined by a prior knowledge test (high vs. low). The dependent variables were post-test scores and the percentage of fixation duration (FD) on the WE page relative to total FD (WE + item pages), with pretest measures as covariates. The homogeneity of regression slopes assumption was satisfied. After controlling for pretest scores, a marginally significant interaction emerged between group and ability, $F(1, 63) = 3.70$, $p = .059$, $\eta^2 = .055$. Follow-up analyses showed that low-ability students in the EMME group outperformed controls, $F(1, 63) = 8.64$, $p = .005$, $\eta^2 = .121$, whereas no significant difference was found among high-ability students. After controlling for the pretest percentage of FD on the WE page, EMME students showed a marginally greater allocation of FD to the WE page than controls, $F(1, 63) = 2.81$, $p = .099$, $\eta^2 = .043$. The findings suggest that incorporating EMME into geometry WE instruction promotes deeper engagement with worked examples and enhances learning outcomes for low-ability students.

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THE IMPACT OF MODEL-BASED PROBLEM SOLVING EMPOWERED BY PROBLEM POSING ON STUDENTS' MATHEMATICS PERFORMANCE

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Researchers have developed evidence-based practices to support math problem solving of students with learning disabilities/difficulties in mathematics (LDM) which disproportionately include students from linguistically and culturally diverse backgrounds. However, when the word problems presented to students do not represent their lived experiences and cultural contexts, engagement and learning are limited, even within "evidence-based" interventions. This study directly addresses the increasing diversity in today's classrooms, including marginalized students who may not be as responsive to traditional instruction and therefore at risk of failing the school curriculum. The purpose of this study was to explore the impact of a conceptual model-based problem-solving and problem-posing intervention (COMPS+PP, Yang & Xin, 2022) integrating evidence-based practices from mathematics and special education. The primary research question was: What was the impact of COMPS+PP on improving mathematics problem solving of students with LDM? A multiple probe baseline across participants design was used to explore the impact of the COMPS+PP program on students' word problem solving performance. Four 3rd graders with LDM participated the intervention four times a week Monday through Thursday with each session lasting for 30 mins for a total of 10-12 sessions. Participants initially received the instruction from COMPS-RtI© computer tutor in which students learned to represent and solve a range of one-step additive word problems using one cohesive mathematical model equation (P + P = W). After students constructed the meaning of "part and part makes up the whole" (i.e., P + P = W), a "What-If-Not" strategy was used to engage students in posing real-world problems that is relevant to their culture and lives. They then engaged in solving the problems they created themselves or those created by their peers. Participants improved their word problem solving performance from an average of 33% correct during the baseline to an average of 88 % correct post COMPS+PP intervention. The video footage showed that participants eagerly volunteered and actively competed for opportunities to come to the whiteboard to create their own problems. The problem-posing activities substantially changed the dynamics of the learning process, transforming students from disengaged or passively engaged learners to active and enthusiastic participants.

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MIDDLE SCHOOL STUDENT'S RELATIONAL THINKING STRATEGIES: OPERATIONS WITH FRACTIONS

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Relational thinking involves transforming mathematical expressions by drawing on the fundamental properties of numbers and operations, rather than merely computing a result by following a fixed sequence of procedures (Carpenter, Levi, Franke, & Koehler-Zeringue, 2005). It serves as a bridge between arithmetic and algebraic thinking (Kızıltoprak & Köse, 2017). However, a review of the literature reveals that studies examining student's relational thinking strategies within the context of fraction operations remain limited. The study aimed to examine the relational thinking strategies used by a seventh-grade student in fraction operations. This study, employing a qualitative research design, was conducted with a single student. Data were collected through interviews with the student based on a researcher-designed worksheet consisting of seven tasks. The tasks involved true/false sentences, fraction addition and subtraction, addition and subtraction with mixed numbers, fraction multiplication and division, the four operations with decimals and equations with two unknowns. Thematic analysis was conducted. Four main categories of strategies were identified: (a) use of fundamental properties of operations, (b) use of relationships among numbers, fractions, and decimal representations, (c) explaining relationships after determining the unknown, and (d) result-oriented operational processes. The findings indicated that the student predominantly relied on strategies grounded in the fundamental properties of operations rather than procedural algorithms. In particular, the student focused on recognizing relationships between quantities on both sides of an equation, transforming numbers and fractions accordingly, and restructuring operations. In some tasks, multiple relational thinking strategies were employed simultaneously. In several instances, the student initially determined the unknown through calculation, and then proposed an alternative solution based on relational thinking. In addition, several strategies not previously reported in the literature, such as whole decomposition, completing to the whole, and decomposing mixed numbers to perform subtraction were identified. These findings extend current understandings of relational thinking in fraction contexts.

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THE ROLE OF AI-DRIVEN AUTOMATED FEEDBACK IN NOVICE MATHEMATICS TEACHERS' PROFESSIONAL GROWTH: A LONGITUDINAL STUDY

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This study applies the Interconnected Model of Professional Growth (IMPG) to investigate how AI-driven automated feedback (via the AIME platform) influences the professional growth of three novice high school mathematics teachers in China over one year. Findings indicate that automated feedback acts as a catalyst for reflective practice. The study contributes to understanding how AI tools can be designed and integrated to support sustainable, situated professional development.

Novice teacher growth is situated, iterative, and individualized. AI-driven automated feedback tools (e.g., AIME) offer scalable, data-informed insights for teacher reflection, but their sustained impact on professional growth is under-explored. This study applies the Interconnected Model of Professional Growth (IMPG) framework to investigate how AI-driven automated feedback serves as an stimulus to influence the professional growth of novice high school mathematics teachers in China.

This research employs a longitudinal, mixed-methods design to examine the learning pathways of three novice high school mathematics teachers from a public school in Southern China who regularly engage with the automatic feedback from the AIME platform over a one-year period. Data collection occurred in three phases, including baseline interviews, followed by monthly cycles of lesson recording, pre-/post-lesson interviews, reflective journals, and researcher-led debrief sessions.

Findings indicate that automated feedback acts as a catalyst prompting adjustments in pedagogical understanding (Personal Domain) and encouraging experimentation in instructional strategies (Domain of Practice). This research contributes theoretically by extending the IMPG framework into the context of AI-enhanced professional development. Practically, it provides nuanced insights into how automated feedback tools foster meaningful and sustainable professional growth for novice teachers.

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MATHEMATICS AS A PRIORITY NEED IN INCLUSIVE TEACHING OF STUDENTS WITH VISUAL IMPAIRMENTS

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Introduction: Teaching students with visual impairments (VI) requires significant pedagogical adaptations to ensure equitable access to learning (Ferreira & Sefotho, 2020). In Switzerland, compulsory education is structured around a shared curriculum that defines learning objectives and professional expectations. However, most teaching materials are primarily developed for sighted learners, and teachers report insufficient access to resources and practical guidelines to support individualized adaptations (Hayes & Proulx, 2023).

Aim: This study aimed to identify teachers' priority support needs across curriculum subjects in French-speaking Switzerland.

Method: An online self-report questionnaire was administered to mainstream and specialized teachers involved in the education of students with VI in French-speaking Switzerland. A total of 54 completed questionnaires were analysed using descriptive methods to compare perceived needs across subject areas.

Results: Findings revealed differences between subjects. Mathematics was identified as the area with the highest level of reported need, followed by science. Teachers highlighted challenges related to visually complex content, symbolic representations, and the limited availability of adapted teaching materials.

Conclusion and implication: These results underscore the necessity of strengthening subject-specific support for teachers, particularly in mathematics. They also lay the groundwork for a second research phase aimed at documenting existing practices and developing adapted instructional resources to enhance curriculum accessibility for students with VI.

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